Augmented variational formulations for solving Maxwell's equations

Patrick Ciarlet

ciarlet@ensta.fr

ENSTA & CNRS URA 853

Time-dependent Maxwell equations

In a bounded Lipschitz polyhedron Ω. The time interval is $[0, T]$, with $T > 0$.

- *Maxwell's equations in the vacuum*

$$\left\{\begin{array}{l} \text{Find } (\mathbf{E}(t), \mathbf{B}(t)) \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \text{ such that} \\ \frac{\partial \mathbf{E}}{\partial t} - c^2\, \mathbf{curl}\, \mathbf{B} = -\frac{1}{\varepsilon_0} \mathbf{J} \; ; \\ \frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\, \mathbf{E} = 0 \; ; \\ \operatorname{div} \mathbf{E} = \frac{1}{\varepsilon_0} \rho \; ; \\ \operatorname{div} \mathbf{B} = 0 \; ; \\ \mathbf{E}(0) = \mathbf{E}_0 \; , \; \mathbf{B}(0) = \mathbf{B}_0 \; . \end{array}\right.$$

$$\Big(\frac{\partial \mathbf{J}}{\partial t} \in L^2(0, T; \mathbf{L}^2(\Omega)),\ \rho \in \mathcal{C}^0(0, T; L^2(\Omega)) \; ; \; \frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{J} = 0.$$

$$\mathbf{E}_0 \in \mathbf{H}(\mathbf{curl}\,, \Omega),\ \operatorname{div} \mathbf{E}_0 = \frac{1}{\varepsilon_0} \rho(0) \; ; \mathbf{B}_0 \in \mathbf{H}(\mathbf{curl}\,, \Omega),\ \operatorname{div} \mathbf{B}_0 = 0. \Big)$$

Time-dependent Maxwell equations (2)

The boundary is split as $\partial\Omega = \bar{\Gamma}_C \cup \bar{\Gamma}_A$.
The enclosing material around Γ_C is a perfect conductor.
A Silver-Müller boundary condition is imposed on Γ_A: 1st order absorbing condition ($\mathbf{e}^\star = 0$), or incoming/outgoing EM wave ($\mathbf{e}^\star \neq 0$).

- *Boundary conditions*

$$\left\{ \begin{array}{l} \mathbf{E} \times \mathbf{n} = 0 \text{ on } \Gamma_C \ ; \\ (\mathbf{E} - c\mathbf{B} \times \mathbf{n}) \times \mathbf{n} = \mathbf{e}^\star \times \mathbf{n} \text{ on } \Gamma_A . \end{array} \right.$$

$$\left(\frac{\partial \mathbf{e}^\star}{\partial t} \in L^2(0,T;\mathbf{L}^2(\Gamma_A)). \right)$$

- Consequences: some *"Additional" Boundary conditions*

$$\left. \begin{array}{l} \mathbf{B} \cdot \mathbf{n} = 0 \ ; \\ c^2 \, (\mathbf{curl}\, \mathbf{B}) \times \mathbf{n} = \frac{1}{\varepsilon_0} \mathbf{J} \times \mathbf{n} \end{array} \right\} \text{ on } \Gamma_C .$$

$$\left. \begin{array}{l} c^2 \, (\mathbf{curl}\, \mathbf{E}) \times \mathbf{n} = c \, \frac{\partial}{\partial t}(\mathbf{e}^\star_T) - c \, \frac{\partial}{\partial t} \mathbf{E}_T \\ c^2 \, (\mathbf{curl}\, \mathbf{B}) \times \mathbf{n} = \frac{1}{\varepsilon_0} \mathbf{J} \times \mathbf{n} + \frac{\partial}{\partial t}(\mathbf{e}^\star \times \mathbf{n}) - c \, \frac{\partial}{\partial t} \mathbf{B}_T \end{array} \right\} \text{ on } \Gamma_A .$$

Static problems

- *Electrostatic-like equations*

$$\left\{\begin{array}{l}\text{Find } \mathbf{E} \in \mathbf{L}^2(\Omega) \text{ such that}\\ \mathbf{curl}\,\mathbf{E} = \mathbf{f},\ \operatorname{div}\mathbf{E} = g \text{ in } \Omega\,;\\ \mathbf{E} \times \mathbf{n}_{|\partial\Omega} = \mathbf{e}^\star \times \mathbf{n}.\end{array}\right.$$

$$\left(\mathbf{f} \in \mathbf{H}(\operatorname{div}^0, \Omega),\, g \in L^2(\Omega),\, \mathbf{e}^\star \in \mathbf{H}^{1/2}(\partial\Omega)\colon \mathbf{f}\cdot\mathbf{n}_{|\partial\Omega} = \operatorname{div}_\Gamma(\mathbf{e}^\star \times \mathbf{n}).\right)$$

NB. $\mathbf{f} = 0$ and $\mathbf{e}^\star = 0$ for the electrostatic equations.

- *Magnetostatic-like equations*

$$\left\{\begin{array}{l}\text{Find } \mathbf{B} \in \mathbf{L}^2(\Omega) \text{ such that}\\ \mathbf{curl}\,\mathbf{B} = \mathbf{f},\ \operatorname{div}\mathbf{B} = g \text{ in } \Omega\,;\\ \mathbf{B} \cdot \mathbf{n}_{|\partial\Omega} = b.\end{array}\right.$$

$$\left(\mathbf{f} \in \mathbf{H}(\operatorname{div}^0, \Omega),\, g \in L^2(\Omega),\, b \in L^2(\partial\Omega)\colon \int_\Omega g\, d\Omega = \int_{\partial\Omega} b\, d\Gamma.\right)$$

NB. $g = 0$ and $b = 0$ for the magnetostatic equations.

Essential boundary conditions

- Remark:
  - $\exists \tilde{\mathbf{e}} \in \mathbf{H}^1(\Omega)$ such that $\tilde{\mathbf{e}} \times \mathbf{n}_{|\partial\Omega} = \mathbf{e}^\star \times \mathbf{n}$.
  - $\exists \psi \in H^1(\Omega)$ such that $\Delta\psi = c(b)$ in Ω, $\dfrac{\partial \psi}{\partial n}_{|\partial\Omega} = b$.

  Consequently, it is possible to solve both static problems with homogeneous boundary conditions, ie. $\mathbf{e}^\star = 0$ and $b = 0$.
- *Define*

$$\left\{ \begin{array}{l} \mathbf{X}_E^0 := \mathbf{H}_0(\mathbf{curl}\,,\Omega) \cap \mathbf{H}(\mathrm{div}\,,\Omega)\ ; \\ \mathbf{X}_B^0 := \mathbf{H}(\mathbf{curl}\,,\Omega) \cap \mathbf{H}_0(\mathrm{div}\,,\Omega). \end{array} \right.$$

- Hypothesis: the norm associated to

$$(\cdot,\cdot)_{X^0} : \ (\mathbf{u},\mathbf{v}) \mapsto (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0 + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})_0$$

  is a norm on $\mathbf{X}_E^0$ and $\mathbf{X}_B^0$, which is *equivalent* to the full norm.
  From [Weber'80], [Fernandes-Gilardi'97], [Amrouche-Bernardi-Dauge-Girault'98]:
  Assume for instance that Ω *is simply connected*, and that $\partial\Omega$ *is connected*.

Essential boundary conditions (2)

- *Define the variational problem* (P^0) *in* $\mathbf{X}_E^0$

$$\left\{ \begin{array}{l} \text{Find } \mathbf{E} \in \mathbf{X}_E^0 \text{ such that} \\ (\mathbf{E}, \mathbf{v})_{X^0} = (\mathbf{f}, \mathbf{curl}\, \mathbf{v})_0 + (g, \operatorname{div} \mathbf{v})_0, \quad \forall \mathbf{v} \in \mathbf{X}_E^0. \end{array} \right.$$

- Theorem:
  $\exists!\mathbf{E} \in \mathbf{X}_E^0$ *solution to problem* (P^0).
  In addition, $\mathbf{E}$ *is the only solution to the electrostatic problem.*

Essential boundary conditions (2 proof)

- *Define the variational problem* (P^0) *in* $\mathbf{X}_E^0$

$$\left\{\begin{array}{l}\text{Find } \mathbf{E} \in \mathbf{X}_E^0 \text{ such that} \\ (\mathbf{E}, \mathbf{v})_{X^0} = (\mathbf{f}, \mathbf{curl}\,\mathbf{v})_0 + (g, \operatorname{div}\mathbf{v})_0, \quad \forall \mathbf{v} \in \mathbf{X}_E^0.\end{array}\right.$$

- Theorem:
  $\exists! \mathbf{E} \in \mathbf{X}_E^0$ *solution to problem* (P^0).
  In addition, $\mathbf{E}$ *is the only solution to the electrostatic problem.*

  Proof:

  (i) Existence and uniqueness of the solution to problem (P^0) is ok.

  (ii) $\forall g' \in L^2(\Omega)$: $\exists! \phi \in H_0^1(\Omega)$ such that $\Delta\phi = g'$.
  As $\mathbf{v} = \nabla\phi \in \mathbf{X}_E^0$, there holds $(\operatorname{div}\mathbf{E}, g')_0 = (g, g')_0$, $\forall g'$. $\operatorname{div}\mathbf{E} = g$ follows.

  (iii) $\mathbf{f} \in \mathbf{H}_0(\operatorname{div}^0, \Omega)$: according to Thm 3.6 p. 48 of [Girault-Raviart'86],
  $\exists! \mathbf{w} \in \mathbf{X}_E^0$ such that $\operatorname{div}\mathbf{w} = 0$, and $\mathbf{curl}\,\mathbf{w} = \mathbf{f}$.
  $\mathbf{v} = \mathbf{E} - \mathbf{w} \in \mathbf{X}_E^0$ yields $\|\mathbf{curl}\,(\mathbf{E} - \mathbf{w})\|_0 = 0$, so $\mathbf{curl}\,\mathbf{E} = \mathbf{f}$.

  (iv) Now, if the electrostatic problem has two solutions, it is clear that the difference satisfies (P^0) with homogeneous r.h.s, so it is zero; uniqueness follows.

Natural boundary conditions

- *Define*

$$\begin{cases} \mathbf{X}_E := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}\,, \Omega) \cap \mathbf{H}(\mathrm{div}\,, \Omega) \ : \ \mathbf{v} \times \mathbf{n}_{|\partial\Omega} \in \mathbf{L}^2_t(\partial\Omega)\}\ ; \\ \mathbf{X}_B := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}\,, \Omega) \cap \mathbf{H}(\mathrm{div}\,, \Omega) \ : \ \mathbf{v} \cdot \mathbf{n}_{|\partial\Omega} \in L^2(\partial\Omega)\}. \end{cases}$$

  NB. Following [Costabel'90], one finds that $\mathbf{X}_E = \mathbf{X}_B$.

- Hypothesis: the norm associated to

$$(\cdot,\cdot)_{X_E} \ : \ (\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0 + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})_0 + (\mathbf{u} \times \mathbf{n}, \mathbf{v} \times \mathbf{n})_{0,\partial\Omega}$$

  is a norm on $\mathbf{X}_E$, which is *equivalent* to the full norm.
  From [Fernandes-Gilardi'97]:
  Assume for instance that Ω *is simply connected*, and that $\partial\Omega$ *is connected*.
  NB. For the magnetostatic problem set in $\mathbf{X}_B$, replace the scalar product by

$$(\cdot,\cdot)_{X_B} \ : \ (\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0 + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})_0 + (\mathbf{u} \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{n})_{0,\partial\Omega}.$$

Natural boundary conditions (2)

- *Define the variational problem* (P) *in* $\mathbf{X}_E$

$$\begin{cases} \text{Find } \mathbf{E} \in \mathbf{X}_E \text{ such that} \\ (\mathbf{E}, \mathbf{v})_{X_E} = (\mathbf{f}, \mathbf{curl}\, \mathbf{v})_0 + (g, \operatorname{div} \mathbf{v})_0, \quad \forall \mathbf{v} \in \mathbf{X}_E. \end{cases}$$

- Theorem:
  $\exists! \mathbf{E} \in \mathbf{X}_E$ *solution to problem* (P).
  In addition, $\mathbf{E}$ *is the only solution to the electrostatic problem.*

Natural boundary conditions (2 proof)

- *Define the variational problem (P) in $\mathbf{X}_E$*

$$\begin{cases} \text{Find } \mathbf{E} \in \mathbf{X}_E \text{ such that} \\ (\mathbf{E}, \mathbf{v})_{X_E} = (\mathbf{f}, \mathbf{curl}\,\mathbf{v})_0 + (g, \operatorname{div}\mathbf{v})_0, \quad \forall \mathbf{v} \in \mathbf{X}_E. \end{cases}$$

- Theorem:
  $\exists!\mathbf{E} \in \mathbf{X}_E$ *solution to problem (P).*
  In addition, $\mathbf{E}$ is the only solution to the electrostatic problem.

  Proof:

  (i) Existence and uniqueness of the solution to problem (P) is ok.

  (ii) same as problem (P^0).

  (iii) $\mathbf{f} \in \mathbf{H}_0(\operatorname{div}^0, \Omega)$: according to Thm 3.6 p. 48 of [Girault-Raviart'86], $\exists!\mathbf{w} \in \mathbf{X}_E^0$ such that $\operatorname{div}\mathbf{w} = 0$, and $\mathbf{curl}\,\mathbf{w} = \mathbf{f}$.
  $\mathbf{v} = \mathbf{E} - \mathbf{w} \in \mathbf{X}_E$ yields $\|\mathbf{curl}\,(\mathbf{E} - \mathbf{w})\|_0^2 + \|\mathbf{E} \times \mathbf{n}\|_{0,\partial\Omega}^2 = 0$,
  so $\mathbf{curl}\,\mathbf{E} = \mathbf{f}$ and $\mathbf{E} \times \mathbf{n}_{|\partial\Omega} = 0$.

  (iv) same as problem (P^0).

Numerical experiments

With E. Jamelot.

- Discretization of problems (P^0) and (P):
  P_1 Lagrange finite element, conforming in X^0 or X.
- 2D experiments.
- *Comparison* between
  - a method based on scalar potentials (discretized by the P_1 Lagrange FE);
  - two discretizations of problem (P^0);
  - one discretization of problem (P).
- Case of a smooth solution in an L-shaped domain.
- Case of a singular solution in an L-shaped domain.

First conclusions

- As $\mathbf{X}_E^0 \cap \mathbf{H}^1(\Omega)$ *is not dense in* $\mathbf{X}_E^0$, when Ω is not convex, a discretization of problem (P^0) with the conforming P_1 Lagrange FE requires a Singular Complement.
  - Ok in 2D cartesian geometries (with F. Assous, E. Garcia, J. Segré, E. Sonnendrücker; see also [Hazard-Lohrengel'02]).
  - Ok in 2D axisymmetric geometries (with F. Assous, S. Labrunie, J. Segré).
  - Under way in 21/2D prismatic geometries (with S. Kaddouri, J. Zou).
  - Under way in 21/2D axisymmetric geometries (with S. Labrunie).

- Cf. [Ciarlet-Hazard-Lohrengel'98], [Costabel-Dauge'98]: $\mathbf{H}^1(\Omega) = \mathbf{X}_E \cap \mathbf{H}^1(\Omega)$ *is dense in* $\mathbf{X}_E$, so the discretization converges to the exact solution. Nevertheless, the convergence is *poor* on the boundary, when the domain is not convex.

Continuous vs. discrete formulations

- *1st order Maxwell's equations* + IC + Charge conservation equation

$$\frac{\partial \mathbf{E}}{\partial t} - c^2\,\mathbf{curl}\,\mathbf{B} = -\frac{1}{\varepsilon_0}\mathbf{J} \Longrightarrow \frac{\partial}{\partial t}\mathrm{div}\,\mathbf{E} \overset{cce}{=} \frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial t} \Longrightarrow \mathrm{div}\,\mathbf{E} = \frac{1}{\varepsilon_0}\rho,$$ thanks to the IC.

- *2nd order Maxwell's equations* + IC + Charge conservation equation

$$\left\{ \begin{array}{l} \dfrac{\partial^2 \mathbf{E}}{\partial t^2} + c^2\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = -\dfrac{1}{\varepsilon_0}\dfrac{\partial \mathbf{J}}{\partial t}\ ;\ \dfrac{\partial \mathbf{E}}{\partial t}(0) = \mathbf{E}_1 \\ \left(\mathbf{E}_1 := c^2\,\mathbf{curl}\,\mathbf{B}_0 - \dfrac{1}{\varepsilon_0}\mathbf{J}(0)\ ;\ \mathrm{div}\,\mathbf{E}_1 \overset{cce}{=} \dfrac{\partial \rho}{\partial t}(0)\right) \end{array} \right. .$$

$$\Longrightarrow \frac{\partial^2}{\partial t^2}\mathrm{div}\,\mathbf{E} \overset{cce}{=} \frac{1}{\varepsilon_0}\frac{\partial^2 \rho}{\partial t^2} \Longrightarrow \mathrm{div}\,\mathbf{E} = \frac{1}{\varepsilon_0}\rho,$$ thanks to the two IC.

Continuous vs. discrete formulations (2)

- *2nd order Maxwell's equations: variational formulation*

$$\frac{d^2}{dt^2}(\mathbf{E},\mathbf{v})_0 + c^2(\mathbf{curl}\,\mathbf{E},\mathbf{curl}\,\mathbf{v})_0 = -\frac{1}{\varepsilon_0}\frac{d}{dt}(\mathbf{J},\mathbf{v})_0,\ \forall \mathbf{v}\in \mathbf{H}_0(\mathbf{curl}\,,\Omega).$$

- *Scalar potentials:* $q \in H^1_0(\Omega)$, $\mathbf{v} = \nabla q \in \mathbf{H}_0(\mathbf{curl}\,,\Omega)$

$$\frac{d^2}{dt^2}(\mathrm{div}\,\mathbf{E}, q)_0 = -\frac{1}{\varepsilon_0}\frac{d}{dt}(\mathrm{div}\,\mathbf{J}, q)_0 = \frac{1}{\varepsilon_0}\frac{d^2}{dt^2}(\rho, q)_0,\ \forall q\in H^1_0(\Omega).$$

- *Edge FE (1st Nédélec's family)* + P_1 Lagrange FE for the scalar potentials

$$-\frac{d^2}{dt^2}(\mathbf{E}_h, \nabla q_h)_0 = \frac{1}{\varepsilon_0}\frac{d^2}{dt^2}(\rho, q_h)_0,\ \forall q_h.$$

  The divergence constraint is *weakly* enforced.

- P_1 *Lagrange finite element, conforming* in X^0 or X: no such discrete scheme.

1st augmented Variational Formulations

- *Define the variational problem* (Q^0) *in* $\mathbf{X}_E^0 \times L^2(\Omega)$

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{E}, p) \in \mathbf{X}_E^0 \times L^2(\Omega) \text{ such that} \\ (\mathbf{E}, \mathbf{v})_{X^0} + (p, \operatorname{div} \mathbf{v})_0 = (\mathbf{f}, \mathbf{curl}\, \mathbf{v})_0 + (g, \operatorname{div} \mathbf{v})_0, \quad \forall \mathbf{v} \in \mathbf{X}_E^0 \\ (\operatorname{div} \mathbf{E}, q)_0 = (g, q)_0, \quad \forall q \in L^2(\Omega) \end{array} \right. .$$

- Theorem:
  $\exists!(\mathbf{E}, p) \in \mathbf{X}_E^0 \times L^2(\Omega)$ *solution to problem* (Q^0).
  In addition, $p = 0$ *and* $\mathbf{E}$ *is the only solution to the electrostatic problem.*
- *Define the variational problem* (Q) *in* $\mathbf{X}_E \times L^2(\Omega)$

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{E}, p) \in \mathbf{X}_E \times L^2(\Omega) \text{ such that} \\ (\mathbf{E}, \mathbf{v})_{X_E} + (p, \operatorname{div} \mathbf{v})_0 = (\mathbf{f}, \mathbf{curl}\, \mathbf{v})_0 + (g, \operatorname{div} \mathbf{v})_0, \quad \forall \mathbf{v} \in \mathbf{X}_E \\ (\operatorname{div} \mathbf{E}, q)_0 = (g, q)_0, \quad \forall q \in L^2(\Omega) \end{array} \right. .$$

- Theorem:
  $\exists!(\mathbf{E}, p) \in \mathbf{X}_E \times L^2(\Omega)$ *solution to problem* (Q).
  In addition, $p = 0$ *and* $\mathbf{E}$ *is the only solution to the electrostatic problem.*

1st augmented Variational Formulation (proof)

- Theorem:
  $\exists!(\mathbf{E}, p) \in \mathbf{X}_E^0 \times L^2(\Omega)$ solution to problem (Q^0).
  In addition, $p = 0$ and $\mathbf{E}$ is the only solution to the electrostatic problem.

  Proof:

  (i) Existence and uniqueness of the solution to problem (P) stem from the *Babuska-Brezzi theory* (cf. for instance [Girault-Raviart'86]).

  The *inf-sup* condition is proved as follows:
  $\forall q \in L^2(\Omega)$, $\exists! \phi \in H_0^1(\Omega)$ such that $\Delta\phi = q$.
  As $\mathbf{v} = \nabla\phi \in \mathbf{X}_E^0$ with $||\mathbf{v}||_{X^0} = ||q||_0$, there holds $\dfrac{(\mathrm{div}\,\mathbf{v}, q)_0}{||\mathbf{v}||_{X^0}} = ||q||_0$, so the *inf-sup* condition follows with a unit constant.

  (ii) Take ϕ as the solution to: find $\phi \in H_0^1(\Omega)$ such that $\Delta\phi = p$.
  $\mathbf{v} = \nabla\phi \in \mathbf{X}_E^0$ yields $(\mathrm{div}\,\mathbf{E}, p)_0 + ||p||_0^2 = (g, p)_0$, so $p = 0$.

  (iii) To conclude, it is enough to note that problem (Q^0) reduces to problem (P^0)...

NB. The same proof works for problem (Q).

Discretization & Numerical experiments

- Discretization of problem (Q^0), cf. [Assous-Degond-Heintzé-Raviart-Segré'93]: $P_2 - iso - P_1$ Taylor-Hood finite element.
- The proof of the *uniform discrete inf-sup condition* can be found in [Ciarlet-Girault'02].
- 2D experiments, *with F. Assous, E. Garcia, S. Labrunie, J. Segré*:
  - 2D cartesian Vlasov-Maxwell system ;
  - 2D axisymmetric Maxwell equations...

NB. The same discretization scheme works for problem (Q).

- *Conclusion:* the same as for the direct approaches (without Lagrange multiplier).

A difficulty

- In X_B, a convergence problem was encountered for the 2D time-harmonic Maxwell equations (cf. [Costabel-Dauge-Martin'99]):
  with $(\mathbf{B},\mathbf{v})_{X_B}$ *replaced by the sum of* $(\mathbf{B},\mathbf{v})_{X^0}$ *and a penalized term on the boundary, i.e.*

  $$(\mathbf{B},\mathbf{v})_{X^0} + s\,(\mathbf{B}\cdot\mathbf{n},\mathbf{v}\cdot\mathbf{n})_{0,\partial\Omega},\ s \text{ varying},$$

  the numerical method failed to capture the (first) singular eigenvector...
  To avoid this problem, M. Costabel and M. Dauge advocate the use of weighted regularization techniques.

- In what follows, we investigate the introduction of a 2nd Lagrange multiplier, with values on the boundary.

2nd augmented Variational Formulation in B

- *Define the variational problem* (R_B) *in* $\mathbf{X}_B \times L^2_0(\Omega) \times L^2(\partial\Omega)$

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{B}, p, \lambda_B) \in \mathbf{X}_B \times L^2_0(\Omega) \times L^2(\partial\Omega) \text{ such that} \\ (\mathbf{B}, \mathbf{v})_{X_B} + (p, \operatorname{div} \mathbf{v})_0 + (\lambda_B, \mathbf{v} \cdot \mathbf{n})_{0,\partial\Omega} = \\ \qquad\qquad (\mathbf{f}, \mathbf{curl}\, \mathbf{v})_0 + (g, \operatorname{div} \mathbf{v})_0 + (b, \mathbf{v} \cdot \mathbf{n})_{0,\partial\Omega}, \quad \forall \mathbf{v} \in \mathbf{X}_B \\ (\operatorname{div} \mathbf{B}, q)_0 = (g, q)_0, \quad \forall q \in L^2_0(\Omega) \\ (\mathbf{B} \cdot \mathbf{n}, \mu)_0 = (b, \mu)_{0,\partial\Omega}, \quad \forall \mu \in L^2(\partial\Omega) \end{array} \right. .$$

- Theorem:
  $\exists!(\mathbf{B}, p, \lambda_B) \in \mathbf{X}_B \times L^2_0(\Omega) \times L^2(\partial\Omega)$ *solution to problem* (R_B).
  In addition, $p = 0$, $\lambda_B = 0$ *and* $\mathbf{B}$ *is the only solution to the magnetostatic problem.*

NB. Problem (R_B) satisfies the *inf-sup* condition.

2nd augmented Variational Formulation in E

- *Define the variational problem* (R_E) *in* $\mathbf{X}_E \times L^2(\Omega) \times \mathbf{L}_t^2(\partial\Omega)$

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{E}, p, \vec{\lambda}_E) \in \mathbf{X}_E \times L^2(\Omega) \times \mathbf{L}_t^2(\partial\Omega) \text{ such that} \\ (\mathbf{E}, \mathbf{v})_{X_E} + (p, \operatorname{div} \mathbf{v})_0 + (\vec{\lambda}_E, \mathbf{v}_T)_{0,\partial\Omega} = \\ \qquad\qquad (\mathbf{f}, \mathbf{curl}\, \mathbf{v})_0 + (g, \operatorname{div} \mathbf{v})_0 + (\mathbf{e}_T^\star, \mathbf{v}_T)_{0,\partial\Omega}, \quad \forall \mathbf{v} \in \mathbf{X}_E \\ (\operatorname{div} \mathbf{E}, q)_0 = (g, q)_0, \quad \forall q \in L^2(\Omega) \\ (\mathbf{E}_T, \vec{\mu})_0 = (\mathbf{e}_T^\star, \vec{\mu})_{0,\partial\Omega}, \quad \forall \vec{\mu} \in \mathbf{L}_t^2(\partial\Omega) \end{array} \right. .$$

- Proposition:
  The tangential trace space $(\mathbf{X}_E)_T$ *is dense in* $\mathbf{L}_t^2(\partial\Omega)$.
- Theorem:
  $\exists!(\mathbf{E}, p, \vec{\lambda}_E) \in \mathbf{X}_E \times L^2(\Omega) \times \mathbf{L}_t^2(\partial\Omega)$ *solution to problem* (R_E).
  In addition, $p = 0$, $\vec{\lambda}_E = 0$ *and* $\mathbf{E}$ *is the only solution to the electrostatic problem.*

NB. Problem (R_E) does not satisfies the *inf-sup* condition.

Discretization

- In [Assous-Degond-Heintzé-Raviart-Segré'93], it turns out that such an approach has been used, *in a formal way*...
  - The $P_2 - iso - P_1$ Taylor-Hood FE is used to discretize the pairs $(\mathbf{E}, p)/(\mathbf{B}, p)$.
  - Unfortunately, the constraint on the boundary, ie. $\forall \mu \in L^2(\partial\Omega)$ or $\forall \vec{\mu} \in \mathbf{L}_t^2(\partial\Omega)$ is *removed* in such a way that the discretized EM field must verify

$$(\mathbf{E}_h, \mathbf{B}_h) \in \mathbf{X}_E^0 \times \mathbf{X}_B^0.$$

  - In a non-convex domain, $\mathbf{X}_E^0 \cap \mathbf{H}^1(\Omega)$ *is not dense in* $\mathbf{X}_E^0$ (and the same with index $_B$). This leads to the *failure* of the resulting numerical scheme for both fields, since the singularities of the EM field can not be approximated.
- So work is under way to find good candidates to discretize the third unknown:
  - either to get a relevant discretized saddle-point VF with three unknowns ;
  - or to remove the constraint on the boundary and, at the same time, still enable the numerical approximation of the singular electromagnetic fields.

Conclusion & Perspectives

- Treating the boundary condition as essential (in $\mathbf{X}^0$, with a Singular Complement):
  - Ok in 2D (cartesian/axisymmetric) domains for (Vlasov-)Maxwell systems.
  - Under way in 21/2D (prismatic/axisymmetric) domains (the Laplace pb is solved).
  - In 3D, an idea would be to decouple corner singularities from edge singularities, and then use 21/2D techniques...

    NB. Alternatives, without Singular Complement, can also be considered...

- Treating the boundary condition as natural (in $\mathbf{X}$):
  - The direct approach is ok (poor approximation of the boundary values).
  - The approach with a single Lagrange multiplier (on the divergence) is ok (*idem*).
  - The approach with two Lagrange multipliers is ok for the continuous problem; possible discretizations are investigated..