A practical tool to solve indefinite problems: T-coercivity

A.-S. Bonnet-Bendhia, L. Chesnel,

Patrick Ciarlet

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POEMS, UMR 7231 CNRS-ENSTA-INRIA

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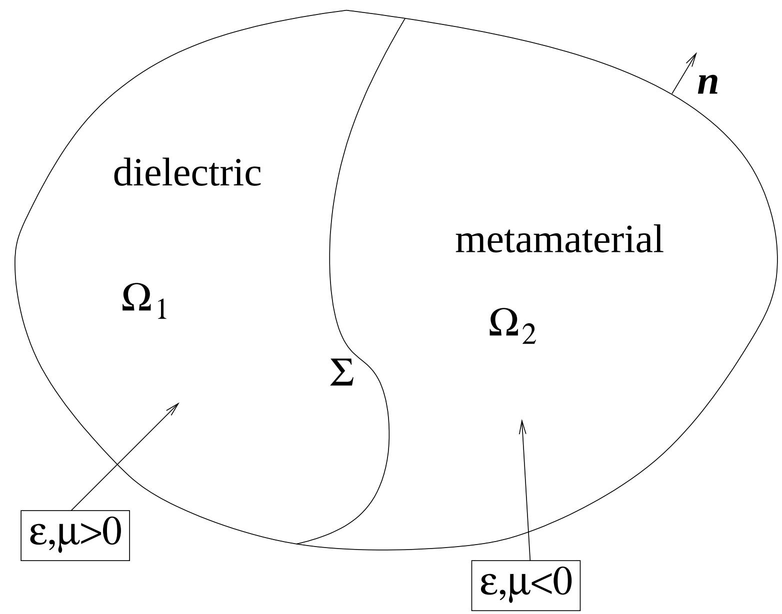
- Numerical examples.
- Conclusion.

Motivation

- *Goal*: Solve numerically a time-harmonic problem in electromagnetism, set in a heterogeneous medium like below.

The domain Ω:



At a given frequency ω, the (negative) metamaterial is modelled as a material with real, strictly negative, electric permittivity ε and magnetic permeability μ.

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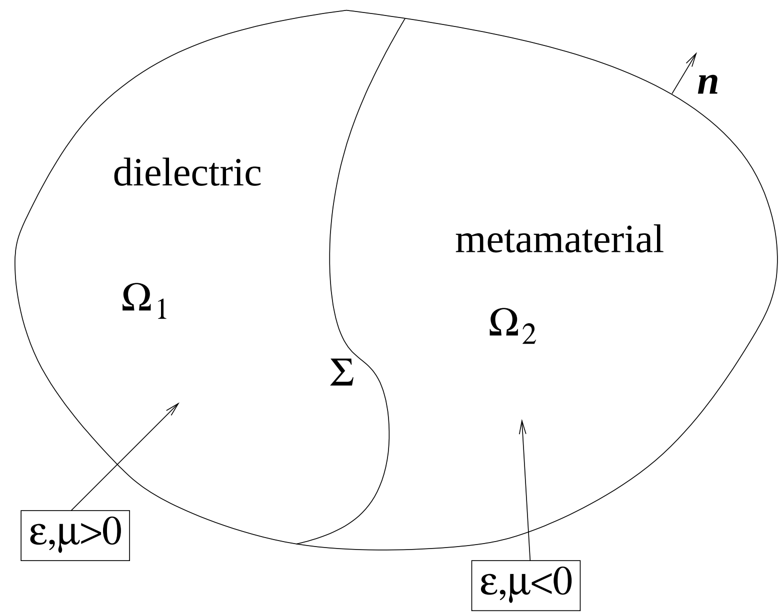

$$\begin{cases} \varepsilon := \varepsilon_{effective}(\omega) & < 0 \\ \mu := \mu_{effective}(\omega) & < 0 \end{cases}$$

in a (negative) metamaterial, in some frequency ranges.

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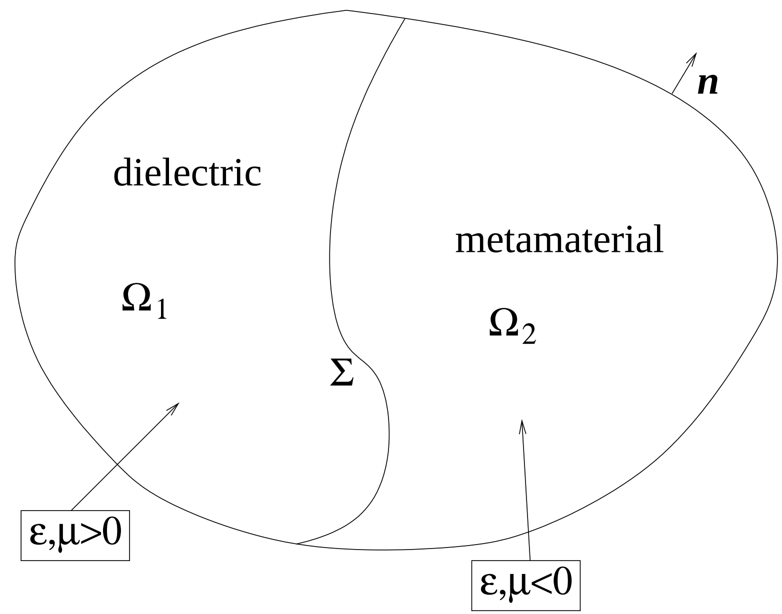

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in a (negative) metamaterial, in some frequency ranges.

NB. In general, $\varepsilon_{effective}(\omega) = \varepsilon' + \imath\,\varepsilon''$, $(\varepsilon', \varepsilon'') \in \mathbb{R}^2$, and it can happen that $|\varepsilon''| << |\varepsilon'|$, so we neglect the imaginary part ; similarly for $\mu_{effective}(\omega)$.

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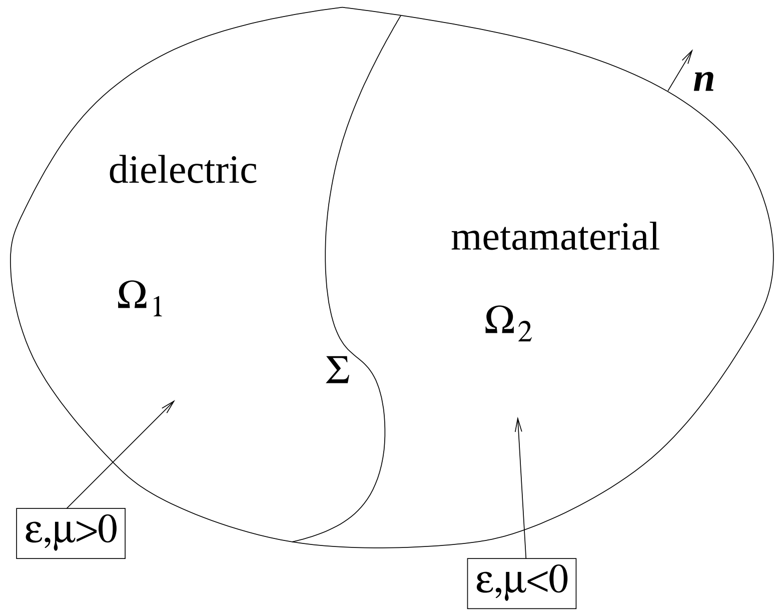

- *Possible practical applications*:
  - perfect lens [Pendry'00], [Maystre-Enoch'04],
  - photonic traps [Genov-Zhang-Zhang'09], etc.

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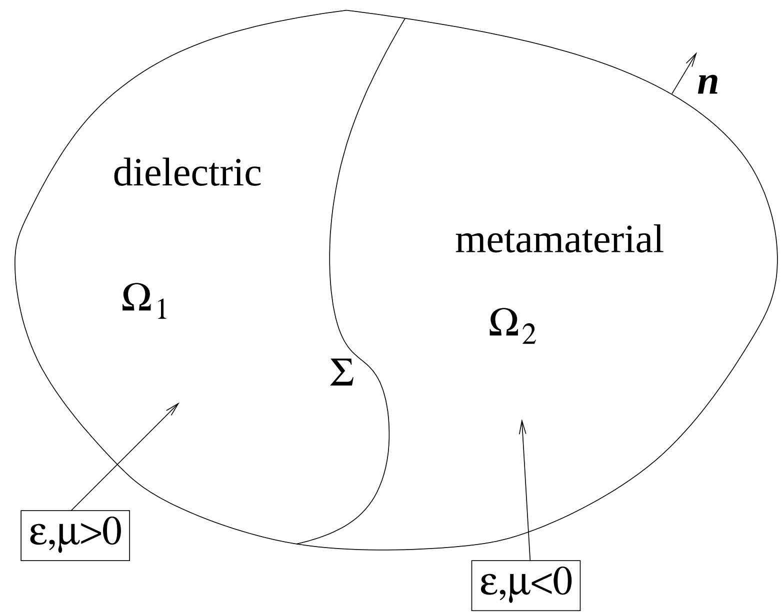

- *Questions*:
  - Is the problem to be solved *well-posed*?
  - How to *compute* a numerical approximation of the solution?

Maxwell problem (electric field)

- Given $\omega > 0$ and source terms $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, $\varrho \in H^{-1}(\Omega)$ ($\operatorname{div} \boldsymbol{J} - \imath\omega\varrho = 0$).

$$\begin{cases} \textit{Find } \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ with } \mathbf{curl}\, \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ s.t.} & \\ \mathbf{curl}\, \left(\mu^{-1}\mathbf{curl}\, \boldsymbol{E}\right) - \omega^2\varepsilon\, \boldsymbol{E} = \imath\omega\boldsymbol{J} & \text{in } \Omega\ ; \\ \operatorname{div} \varepsilon\, \boldsymbol{E} = \varrho & \text{in } \Omega\ ; \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

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- First, solve (assume that $\partial\Omega$ is connected):

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- Second, set $\boldsymbol{K} := \imath\omega\boldsymbol{J} + \omega^2\varepsilon\,\mathbf{grad}\,\varphi$, and solve:

$$(P_E) \quad \begin{cases} \textit{Find } \boldsymbol{E}' \in \boldsymbol{L}^2(\Omega) \textit{ with } \mathbf{curl}\,\boldsymbol{E}' \in \boldsymbol{L}^2(\Omega) \textit{ s.t.} & \\ \mathbf{curl}\,\left(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}'\right) - \omega^2\varepsilon\,\boldsymbol{E}' = \boldsymbol{K} & \text{in } \Omega\,; \\ \operatorname{div}\varepsilon\,\boldsymbol{E}' = 0 & \text{in } \Omega\,; \\ \boldsymbol{E}' \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

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- The electric field $\boxed{\boldsymbol{E} := \boldsymbol{E}' + \mathbf{grad}\, \varphi}$ solves the Maxwell problem.

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- Second, in $\boldsymbol{H}_0(\mathbf{curl}\,;\Omega) := \{\boldsymbol{F} \in \boldsymbol{L}^2(\Omega) \,|\, \mathbf{curl}\, \boldsymbol{F} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{F} \times \boldsymbol{n}_{|\partial\Omega} = 0\}$:

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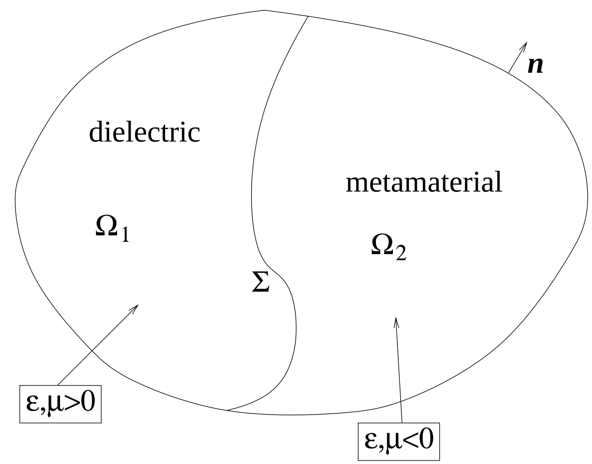
Indefinite problems

At the crossing of the interface, both ε and μ exhibit a sign-shift.



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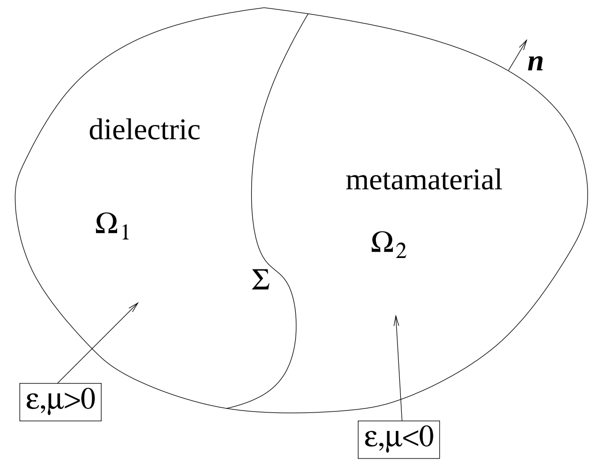

- The permittivity ε has a sign-shift:

The form $(\psi, \psi') \mapsto \int_\Omega \varepsilon \,\mathbf{grad}\, \psi \cdot \mathbf{grad}\, \psi' \, d\Omega$ is indefinite.

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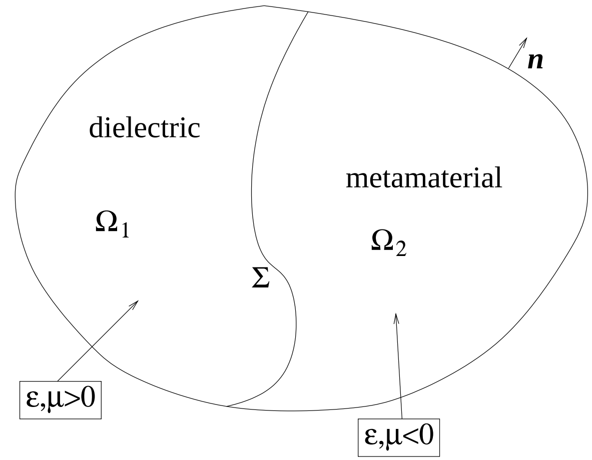

- The permeability μ has a sign-shift:

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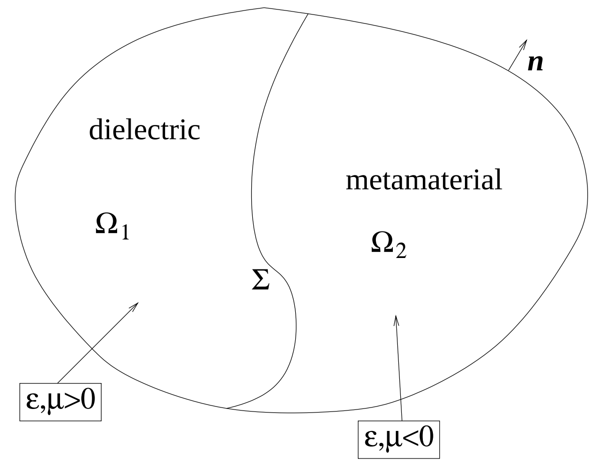

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In addition, what are the properties of the "electric" functional space

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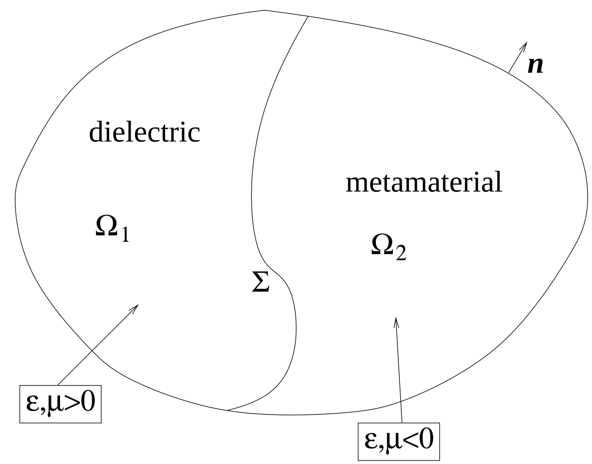

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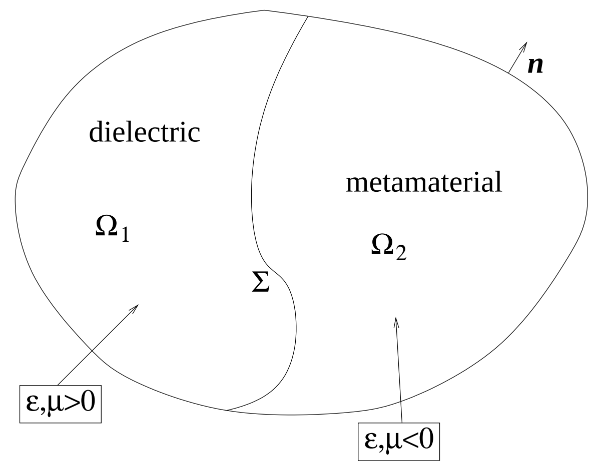

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Then $(\boldsymbol{F}, \boldsymbol{F}') \mapsto \omega^2 \int_\Omega \varepsilon\, \boldsymbol{F} \cdot \overline{\boldsymbol{F}'} \, d\Omega$ is treated as a compact perturbation term.

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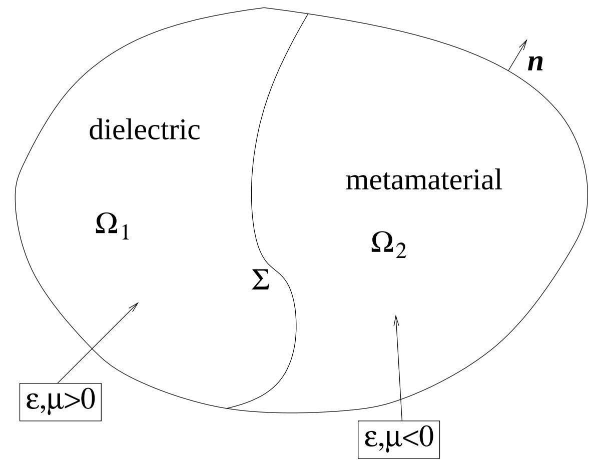

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From now on, we focus mainly on the indefiniteness.

Indefinite problems-2

- To fix ideas, we consider Problem (P_φ), ie. the Variational Formulation

$$(VF_\varphi) \begin{cases} \textit{Find } \varphi \in H_0^1(\Omega) \textit{ s.t.} \\ \forall \psi \in H_0^1(\Omega), \quad \int_\Omega \varepsilon \, \mathbf{grad}\, \varphi \cdot \overline{\mathbf{grad}\, \psi} \, d\Omega = -\langle \varrho, \psi \rangle. \end{cases}$$

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NB. $(F, F') \mapsto \omega^2 \displaystyle\int_{\Omega_\perp} \varepsilon\, F \overline{F'}\, d\Omega_\perp$ is obviously a compact perturbation term.

Abstract setting

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[Hadamard] The Problem (VF) is *well-posed* if, and only if, for all f, it has one and only one solution u, with continuous dependence:

$$\exists C > 0, \; \forall f \in V', \; \|u\|_V \leq C \, \|f\|_{V'}.$$

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[Lax-Milgram] OK provided that $a(\cdot,\cdot)$ is coercive!

Abstract setting-2

- [Banach-Necas-Babuska] Introduce the two conditions

$$(BNB_1) \qquad \exists \alpha' > 0,\ \forall v \in V,\ \sup_{w \in V \setminus \{0\}} \frac{|a(v,w)|}{\|w\|_V} \geq \alpha' \, \|v\|_V .$$

$$(BNB_2) \qquad \forall w \in V:\ \{\forall v \in V,\ a(v,w) = 0\} \implies \{w = 0\}.$$

NB. Condition (BNB_1) is called an *inf-sup condition*, or a *stability condition*.

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- `Theorem (Well-posedness)` The two assertions below are equivalent:
  (i) the Problem (VF) is well-posed ;
  (ii) the form $a(\cdot,\cdot)$ satisfies conditions (BNB_1) and (BNB_2).

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NB. Condition (BNB_1) is called an *inf-sup condition*, or a *stability condition*.

- `Definition (T-coercivity)` The form $a(\cdot,\cdot)$ is $\mathtt{T}$-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V),\ \text{bijective},\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha}\, \|v\|_V^2.$$

Abstract setting-2

- [Banach-Necas-Babuska] Introduce the two conditions

$$(BNB_1) \qquad \exists \alpha' > 0,\ \forall v \in V,\ \sup_{w \in V \setminus \{0\}} \frac{|a(v,w)|}{\|w\|_V} \geq \alpha' \, \|v\|_V .$$

$$(BNB_2) \qquad \forall w \in V:\ \{\forall v \in V,\ a(v,w) = 0\} \implies \{w = 0\}.$$

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- `Theorem (Well-posedness)` The three assertions below are equivalent:
  (i) the Problem (VF) is well-posed ;
  (ii) the form $a(\cdot,\cdot)$ satisfies conditions (BNB_1) and (BNB_2).
  (iii) the form $a(\cdot,\cdot)$ is `T`-coercive.

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Abstract setting-3

- Solve the *coercive+compact* Variational Formulation

$$(VF_{c+c}) \begin{cases} \textit{Find } u \in V \textit{ s.t.} \\ \forall v \in V,\ a_0(u,v) + c(u,v) = \langle f, v \rangle\ , \end{cases}$$

with $a_0(\cdot,\cdot)$ and $c(\cdot,\cdot)$ two continuous sesquilinear forms over $V \times V$:

(c_1) The form $a_0(\cdot,\cdot)$ is $\mathtt{T}$-coercive ;

(c_2) The operator $\mathtt{C} \in \mathcal{L}(V)$ associated to $c(\cdot,\cdot)$ is compact.

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  NB. The operator associated to $(a_0 + c)(\cdot,\cdot)$ is Fredholm of index 0 (and injective).

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Practical T-coercivity

- In our case (Problem (VF_φ)):
  - Ω, Ω_1 and Ω_2 are domains of $\mathbb{R}^d$, $d \geq 1$: $\Omega_1 \cap \Omega_2 = \emptyset$, $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$;

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- Introduce $V_k := \{v_k \in H^1(\Omega_k) \,|\, v_{k\,|\Gamma_k} = 0\}$, $k = 1, 2$:

$$\boxed{V = \{v \,|\, v_{|\Omega_k} \in V_k,\ k = 1, 2,\ \text{Matching}_\Sigma(v_{|\Omega_1}, v_{|\Omega_2}) = 0\}},$$

with $\text{Matching}_\Sigma(v_1, v_2) := v_{1\,|\Sigma} - v_{2\,|\Sigma}$.

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$$\forall v_2 \in V_2, \sigma_2^- \|\mathbf{grad}\, v_2\|^2_{\boldsymbol{L}^2(\Omega_2)} \leq -a_2(v_2, v_2) \leq \sigma_2^+ \|\mathbf{grad}\, v_2\|^2_{\boldsymbol{L}^2(\Omega_2)}.$$

NB. We assume $0 < \sigma_k^- \leq \sigma_k^+ < \infty$, $k = 1, 2$.

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Practical T-coercivity-2

- First try:

$$\forall v \in H_0^1(\Omega), \quad \mathtt{T}_- \, v := \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{cases} .$$

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$$\forall v \in H_0^1(\Omega), \quad \mathtt{T} \, v := \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 + 2R_1 \, v_1 & \text{in } \Omega_2 \end{cases} .$$

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$$\mathtt{T}^2\, v = \begin{cases} (\mathtt{T}v)_1 = v_1 & \text{in } \Omega_1 \\ -(\mathtt{T}v)_2 + 2R_1\, (\mathtt{T}v)_1 = -(-v_2 + 2R_1\, v_1) + 2R_1\, v_1 = v_2 & \text{in } \Omega_2 \end{cases}.$$

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Can one achieve T-coercivity?

Practical T-coercivity-3

- Computations:

$$\begin{aligned} |a(v, \mathtt{T}v)| &= |a_1(v_1, v_1) - a_2(v_2, v_2) + 2a_2(v_2, R_1\, v_1)| \\ &\geq |a_1(v_1, v_1) - a_2(v_2, v_2)| - 2|a_2(v_2, R_1\, v_1)| \\ &\geq \sigma_1^- \, \|v_1\|_{V_1}^2 - a_2(v_2, v_2) - 2|a_2(v_2, R_1\, v_1)| \end{aligned}$$

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Practical T-coercivity-3

- Computations: let $\eta > 0$, apply Young's inequality

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|a(v, \mathtt{T}v)| &= |a_1(v_1, v_1) - a_2(v_2, v_2) + 2a_2(v_2, R_1\, v_1)| \\
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&\geq \sigma_1^- \, \|v_1\|_{V_1}^2 - a_2(v_2, v_2) - 2|a_2(v_2, R_1\, v_1)| \\
&\geq \sigma_1^- \, \|v_1\|_{V_1}^2 - a_2(v_2, v_2) + \eta a_2(v_2, v_2) + \eta^{-1} a_2(R_1\, v_1, R_1\, v_1) \\
&\geq (\sigma_1^- - \eta^{-1}\sigma_2^+ \, |||R_1|||^2)\|v_1\|_{V_1}^2 - (1-\eta)a_2(v_2, v_2).
\end{aligned}$$

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\end{aligned}
$$

- To obtain $|a(v, \mathtt{T}v)| \geq \underline{\alpha}(\eta)\, \|v\|_V^2$ for some $\eta > 0$, one needs

$$
\frac{\sigma_1^-}{\sigma_2^+} > |||R_1|||^2.
$$

Practical T-coercivity-4

- Third try: let $R_2 \in \mathcal{L}(V_2, V_1)$ s.t. for all $v_2 \in V_2$, Matching$_\Sigma(R_2 v_2, v_2) = 0$.

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- Conclusion: to achieve T-coercivity, one needs

$$\frac{\sigma_1^-}{\sigma_2^+} > \left(\inf_{R_1} |||R_1|||\right)^2 \quad \text{or} \quad \frac{\sigma_2^-}{\sigma_1^+} > \left(\inf_{R_2} |||R_2|||\right)^2.$$

Optimality of T-coercivity

- Study of an elementary setting:
  - piecewise constant coefficient σ ;

    in this case, $\sigma_1^- = \sigma_1^+ = \sigma_1$, and $\sigma_2^- = \sigma_2^+ = |\sigma_2|$;

    define the *contrast* $\kappa_\sigma = \dfrac{\sigma_2}{\sigma_1} \in]-\infty, 0[$.

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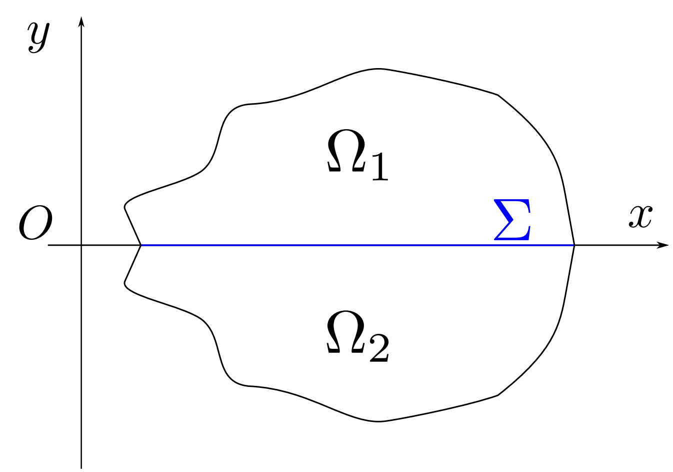
- First case: $\sigma_1 \neq -\sigma_2$, or $\kappa_\sigma \neq -1$, in a symmetric geometry.

Sample symmetric geometry:



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  Let $R_1 \in \mathcal{L}(V_1, V_2)$ s.t. for all $v_1 \in V_1$, $R_1 v_1(x,y) = v_1(x,-y)$, a.e. in Ω_2.

  One finds $|||R_1||| = 1$.

  To achieve T-coercivity, one needs $\dfrac{\sigma_1}{|\sigma_2|} > 1$.

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  To achieve $\mathtt{T}$-coercivity, one needs $\dfrac{\sigma_1}{|\sigma_2|} > 1$.

  Let $R_2 \in \mathcal{L}(V_2, V_1)$ s.t. for all $v_2 \in V_2$, $R_2 v_2(x, y) = v_2(x, -y)$, a.e. in Ω_1.

  One finds $|||R_2||| = 1$.

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  To achieve $\mathtt{T}$-coercivity, one needs $\dfrac{\sigma_1}{|\sigma_2|} > 1$.

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  To achieve $\mathtt{T}$-coercivity, one needs $\dfrac{|\sigma_2|}{\sigma_1} > 1$.

- <u>Conclusion</u>: Problem (VF_φ) is well-posed when $\kappa_\sigma \neq -1$.

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Optimality of T-coercivity-2

- Study of an elementary setting:
  - piecewise constant coefficient σ.

Optimality of T-coercivity-2

- Study of an elementary setting:
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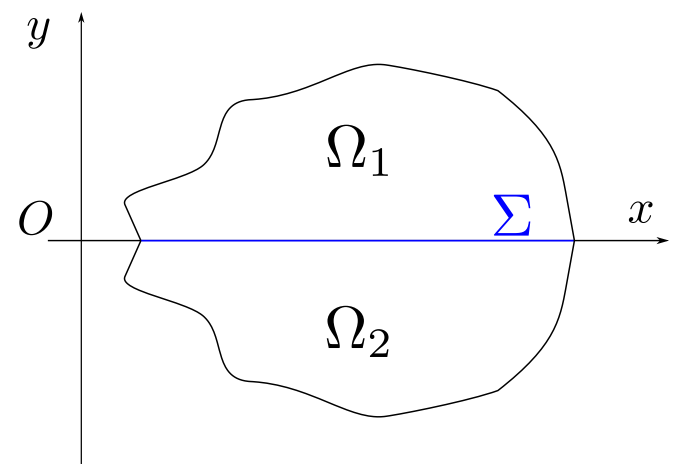
- <u>Second case</u>: $\sigma_1 = -\sigma_2$, or $\kappa_\sigma = -1$, in a symmetric geometry.

Sample symmetric geometry:



Optimality of T-coercivity-2

- Study of an elementary setting:
  - piecewise constant coefficient σ.

- Second case: $\sigma_1 = -\sigma_2$, or $\kappa_\sigma = -1$, in a symmetric geometry.

  Let $g \in \mathcal{C}_0^\infty(\Sigma)$ and solve for $k = 1, 2$

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  Define u by $u_{|\Omega_k} = U_k$, $k = 1, 2$: $\text{Matching}_\Sigma(u_{|\Omega_1}, u_{|\Omega_2}) = 0$, so $u \in H^1_0(\Omega)$.

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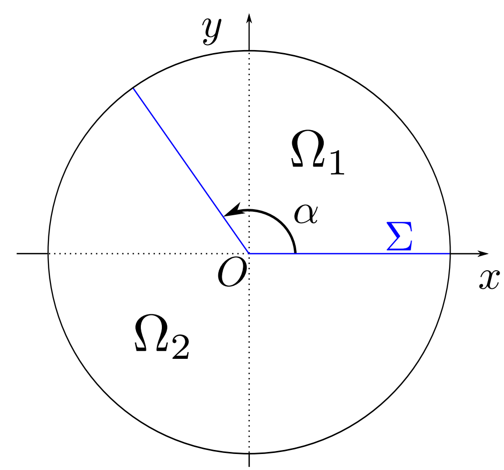

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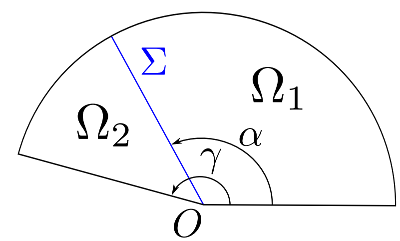

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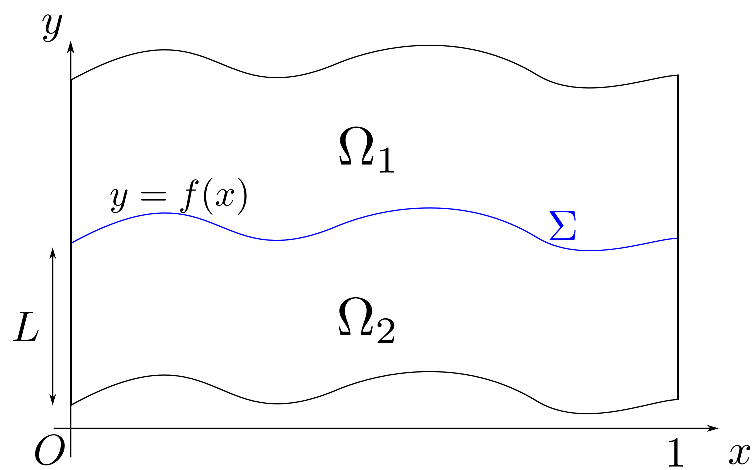

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- Handle general geometries by *localization*.
  - Build a partition of unity, and use the T-coercivity results locally.
  - *A priori* estimate: there exists an interval I_Σ of $]-\infty, 0[$ s.t. if $\kappa_\sigma \notin I_\Sigma$, then

$$\exists C > 0,\ \forall v \in H_0^1(\Omega),\ \|v\|_{H_0^1(\Omega)} \leq C\,\{\|\mathrm{div}\,\sigma\,\mathbf{grad}\,v\|_{H^{-1}(\Omega)} + \|v\|_{L^2(\Omega)}\}.$$

  - Use Peetre's Lemma to conclude.

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  - In this case, the associated operator is Fredholm of index 0.
  - The interval I_Σ always contains -1.
  - If the interface is $\mathcal{C}^1$ without endpoints, then $I_\Sigma = \{-1\}$.
  - Problem (VF_{E_z}) can be solved similarly.

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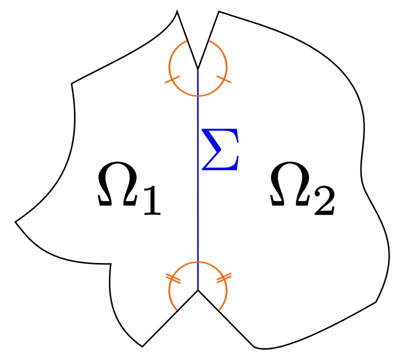
Locally symmetric geometry:
$\kappa_\sigma \neq -1$.



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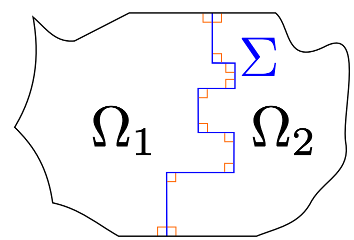

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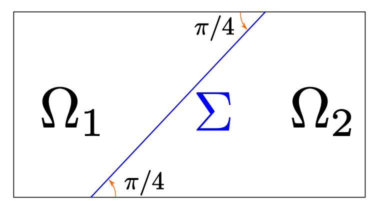
Boundary vertices with angles $\pi/4$ and $3\pi/4$: $\kappa_\sigma \notin [-3, -1/3]$.



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(direct computations: line singularity)

Locally symmetric geometry:
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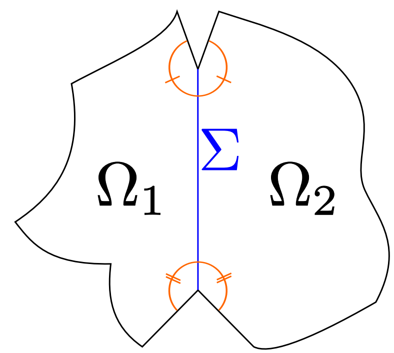

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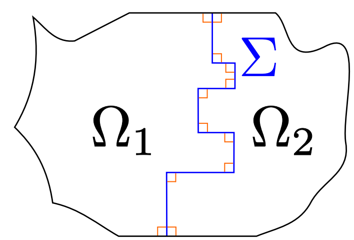

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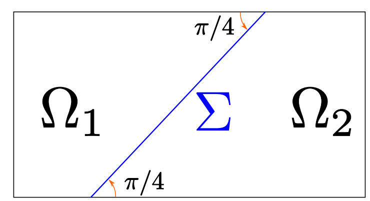

Numerical experiments

- In a symmetric domain, made up of two adjacent squares.
- An *exact* piecewise smooth solution of Problem (VF_φ) is available.
- Two contrasts: $\kappa_\sigma \in \{-2, -1.001\}$.
- Discretization using P_1 Lagrange FE.
- We study below the influence of the meshes.

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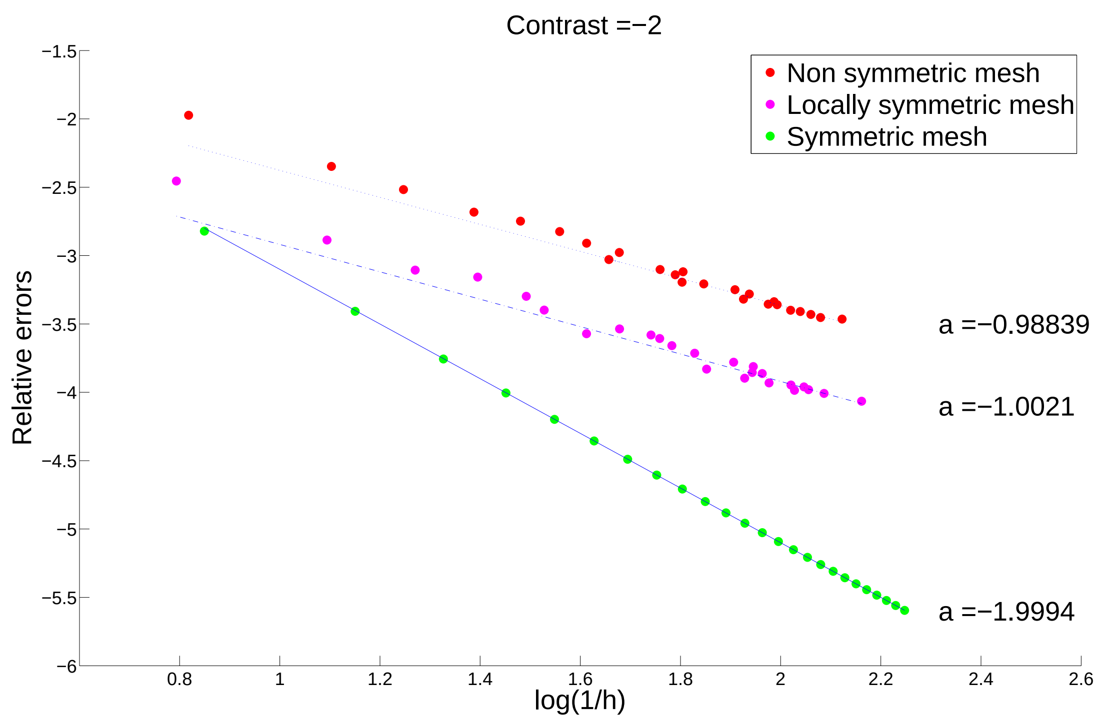

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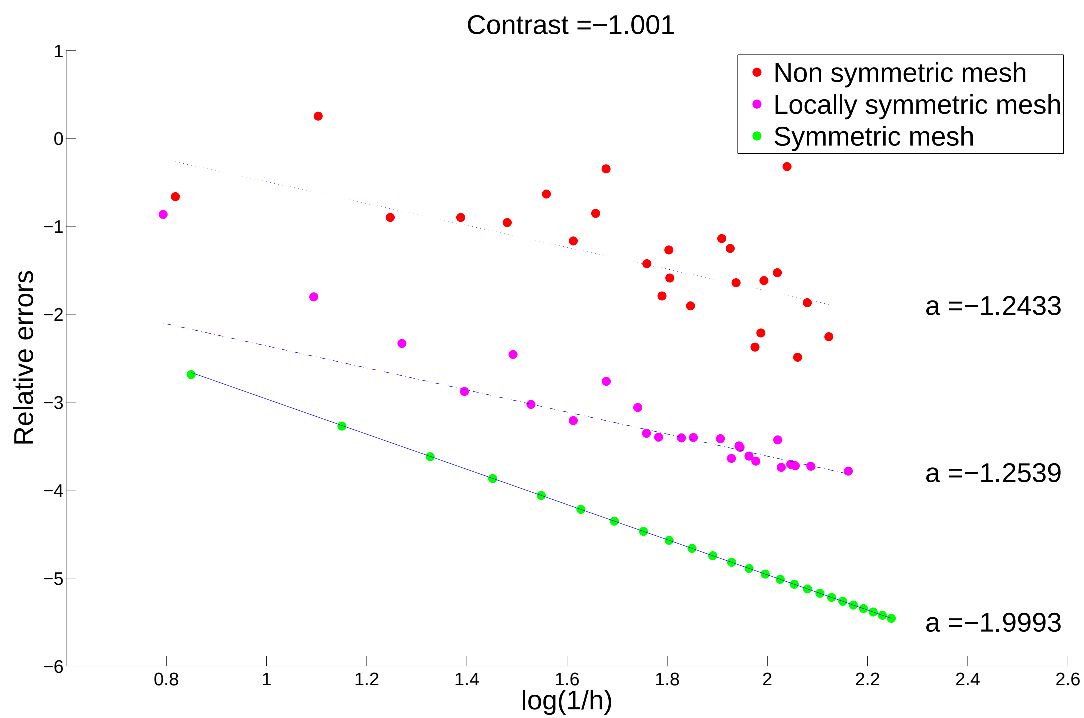

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Numerical experiments-2

- In the unit cube, split in two halves (with $\Sigma := \{\frac{1}{2}\}\times]0,1[\times]0,1[$).
- Piecewise constant ε, μ.
- An *exact* piecewise smooth solution of Maxwell's equations is available.
- Discretization of the *augmented* formulation [Jr'05]

  (set in $\boldsymbol{X}_\varepsilon(\Omega) = \{\boldsymbol{F} \in \boldsymbol{H}_0(\mathbf{curl}\,;\Omega) \,|\, \mathrm{div}\,\varepsilon\,\boldsymbol{F} \in L^2(\Omega)\}$.)

$$\begin{cases} \text{Find } \boldsymbol{E}' \in \boldsymbol{X}_\varepsilon(\Omega) \text{ s.t.} \\ \forall \boldsymbol{F} \in \boldsymbol{X}_\varepsilon(\Omega), \quad \displaystyle\int_\Omega \mu^{-1}(\mathbf{curl}\,\boldsymbol{E}' \cdot \overline{\mathbf{curl}\,\boldsymbol{F}} + \varepsilon^{-2}\mathrm{div}\,\varepsilon\,\boldsymbol{E}'\overline{\mathrm{div}\,\varepsilon\,\boldsymbol{F}})\,d\Omega \\ \qquad\qquad\qquad\qquad -\omega^2 \displaystyle\int_\Omega \varepsilon\,\boldsymbol{E}' \cdot \overline{\boldsymbol{F}}\,d\Omega = \int_\Omega \boldsymbol{K} \cdot \overline{\boldsymbol{F}}\,d\Omega\,. \end{cases}$$

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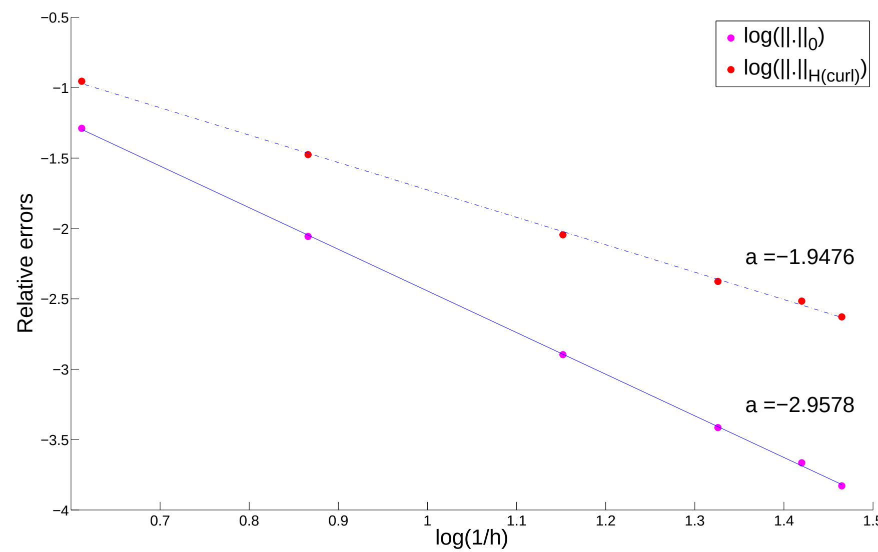

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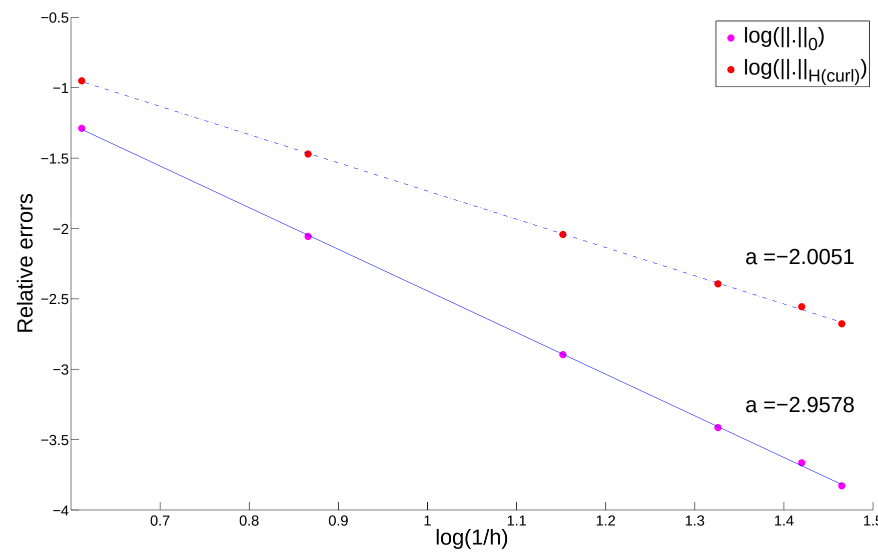

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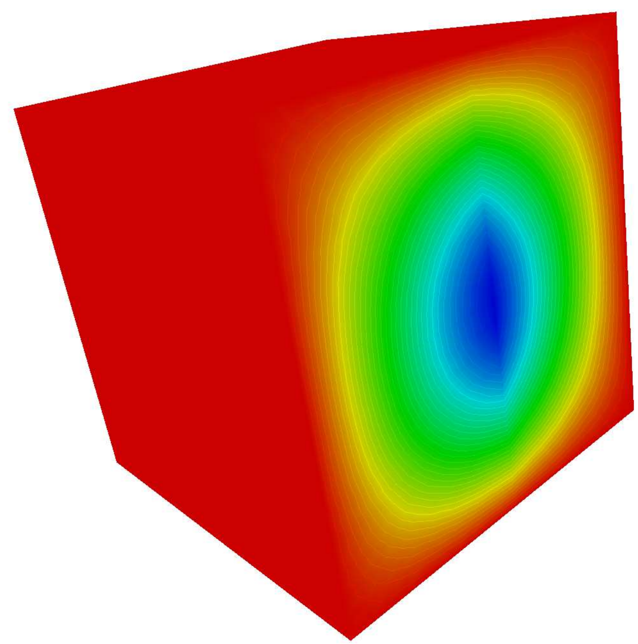

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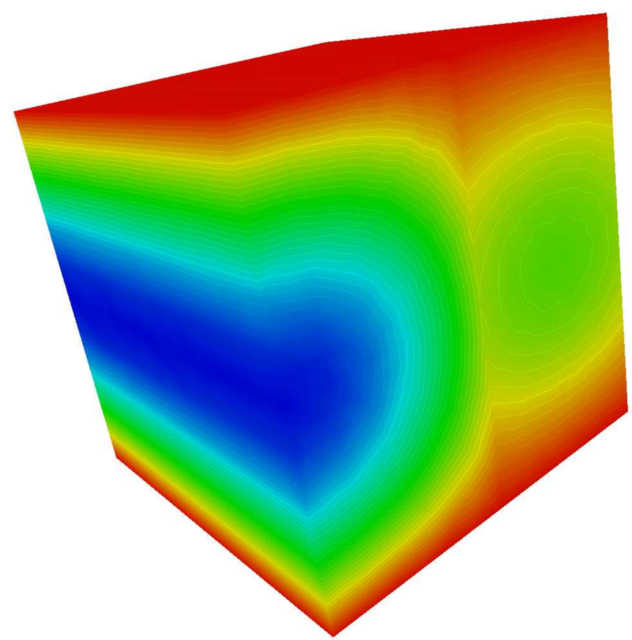

Conclusions/Perspectives

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  - numerical analysis when $\mathtt{T}$-coercivity applies (cf. [BonnetBenDhia-Jr-Zwölf'10], [Nicaise-Venel'11], DG-approach [Chung-Jr'11], etc.) ;
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- In the *critical* cases: are models derived from physics still relevant?
  - re-visit models (homogenization, multi-scale numerics, etc.).

    (METAMATH Project, submitted to ANR ; coordinator S. Fliss (POEMS)).