Approximating the divergence of electromagnetic fields by edge elements

Patrick Ciarlet

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POEMS, ENSTA ParisTech, France

Outline

- **Maxwell equations and a priori regularity of the fields**
- Discretization and error estimates on the divergence of the fields
- Variational formulations
- Numerical illustrations
- Conclusion and perspectives

Stationary/static problem

- Let Ω be a *Lipschitz, polyhedral domain* with connected boundary $\partial\Omega$.
- Given source terms $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ $(\operatorname{div} \boldsymbol{f} = 0)$ and $\varrho \in H^{-1}(\Omega)$, solve:

$$\begin{cases} \textit{Find } \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ with } \mathbf{curl}\, \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ s.t.} & \\ \mathbf{curl}\, \left(\mu^{-1}\mathbf{curl}\, \boldsymbol{E}\right) = \boldsymbol{f} & \text{in } \Omega\ ; \\ \operatorname{div} \varepsilon\, \boldsymbol{E} = \varrho & \text{in } \Omega\ ; \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

NB. With coefficients $\varepsilon, \mu > 0$ a.e. ; $\varepsilon, \varepsilon^{-1}, \mu, \mu^{-1} \in L^\infty(\Omega)$.

- The problem is well-posed in $\boldsymbol{H}_0(\mathbf{curl}\,; \Omega)$:

$$\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\varrho\|_{H^{-1}(\Omega)}.$$

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Helmholtz decomposition

- Let $\mathcal{V}_N(\Omega,\varepsilon) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}\,,\Omega) \,|\, \mathrm{div}\,\varepsilon\boldsymbol{v} = 0\}$.
- According to the Helmholtz decomposition of $\boldsymbol{H}_0(\mathbf{curl}\,,\Omega)$ e.g. [Monk'03]:

$$\boldsymbol{E} = \boldsymbol{E}_0 + \nabla\phi,\ \boldsymbol{E}_0 \in \mathcal{V}_N(\Omega,\varepsilon),\ \phi \in H^1_0(\Omega).$$

NB. The decomposition is orthogonal wrt $(\varepsilon\,\cdot\,|\cdot) + (\mu^{-1}\mathbf{curl}\,\cdot\,|\mathbf{curl}\,\cdot)$.

- One may characterize $\boldsymbol{E}_0$ and $\nabla\phi$ separately:

$$\begin{cases} \textit{Find } \boldsymbol{E}_0 \in \mathcal{V}_N(\Omega,\varepsilon) \textit{ s.t.} & \mathbf{curl}\,\left(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_0\right) = \boldsymbol{f} & \text{in } \Omega\ ; \\ \textit{Find } \phi \in H^1_0(\Omega) \textit{ s.t.} & \mathrm{div}\,\varepsilon\,\nabla\phi = \varrho & \text{in } \Omega. \end{cases}$$

- In what follows, we focus on $\boldsymbol{E}_0$; $\nabla\phi$ can be handled similarly [Jr-Wu-Zou'14, §§3-4].

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Regularity of the fields

- $\boldsymbol{E}_0 \in \mathcal{V}_N(\Omega,\varepsilon) \subset \mathcal{X}_N(\Omega,\varepsilon) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}\,,\Omega) \,|\, \mathrm{div}\,\varepsilon\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$.

- $\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_0 \in \mathcal{X}_T(\Omega,\mu) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}\,,\Omega) \,|\, \mathrm{div}\,\mu\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \mu\boldsymbol{v}\cdot\boldsymbol{n}_{|\partial\Omega} = 0\}$.

- `Theorem` [Costabel-Dauge-Nicaise'99]: Consider $\varepsilon, \mu^{-1} \in W^{1,\infty}(\Omega)$.
  If Ω is *convex* then $\mathcal{X}_N(\Omega,\varepsilon) \subset \boldsymbol{H}^1(\Omega)$ and $\mathcal{X}_T(\Omega,\mu) \subset \boldsymbol{H}^1(\Omega)$.
  If Ω is *non-convex* then $\exists \delta_{max}^{Dir}, \delta_{max}^{Neu} \in]1/2,1[$ s.t.

$$\mathcal{X}_N(\Omega,\varepsilon) \subset \cap_{0\le\delta<\delta_{max}^{Dir}} \boldsymbol{H}^\delta(\Omega), \quad \text{and} \quad \mathcal{X}_T(\Omega,\mu) \subset \cap_{0\le\delta<\delta_{max}^{Neu}} \boldsymbol{H}^\delta(\Omega)\,.$$

- Following [Jr-Wu-Zou'14], let $\varepsilon, \mu^{-1} \in W^{1,\infty}(\Omega)$.

  To fix ideas, suppose that Ω is non-convex and define $\delta_{max} := \min(\delta_{max}^{Dir}, \delta_{max}^{Neu})$.

  Choose a regularity exponent $\delta \in]1/2, \delta_{max}[$.

  NB. If Ω is convex, then $\delta = 1$.

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Edge element discretization

- Let $(\mathcal{T}_h)_h$ be a shape regular family of tetrahedral meshes of Ω.
- Define $\mathcal{X}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}\,;\Omega) \,|\, \boldsymbol{v}_{h|K} = \mathbf{a}_K + \mathbf{b}_K \times \mathbf{x},\ \forall K \in \mathcal{T}_h\}$.
- Assume$^{(\star)}$ $\forall h,\ \|\boldsymbol{E}_0 - \boldsymbol{E}_{0,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim \inf_{\boldsymbol{v}_h \in \mathcal{X}_h} \|\boldsymbol{E}_0 - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)}$.
- Edge element interpolation ($\delta \in]1/2, \delta_{max}[$), cf. [Alonso-Valli'99], [Jr-Zou'99]:

$$\|\boldsymbol{E}_0 - \boldsymbol{E}_{0,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h^\delta \,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

- QUESTION: *What of* $\|\mathrm{div}\,\varepsilon(\boldsymbol{E}_0 - \boldsymbol{E}_{0,h})\|$ *?*

  From the above: $\|\mathrm{div}\,\varepsilon(\boldsymbol{E}_0 - \boldsymbol{E}_{0,h})\|_{H^{-1}(\Omega)} \lesssim h^\delta \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$.

- Define $Q_h := \{q_h \in H^1_0(\Omega) \,|\, q_{h|K} \in P_1(K),\ \forall K \in \mathcal{T}_h\}$.

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On the divergence - 1

- *What of* $\|\mathrm{div}\,\varepsilon(\boldsymbol{E}_0 - \boldsymbol{E}_{0,h})\|$*?*

- Define $\mathcal{V}_h := \{\boldsymbol{v}_h \in \mathcal{X}_h \,|\, (\varepsilon\,\boldsymbol{v}_h|\nabla q_h) = 0,\ \forall q_h \in Q_h\}$.
- `Theorem` [$\mathcal{V}_h$] [Jr-Wu-Zou'14]: Consider $\varepsilon \in W^{1,\infty}(\Omega)$.
  Assume that $(\mathcal{T}_h)_h$ is quasi-uniform.
  Given $\delta \in]1/2, \delta_{max}^{Dir}[$ and $s \in]1/2, 1]$, it holds

$$\forall \boldsymbol{v}_h \in \mathcal{V}_h, \quad \|\mathrm{div}\,\varepsilon\boldsymbol{v}_h\|_{H^{-s}(\Omega)} \lesssim h^{s+\delta-1}\|\mathbf{curl}\,\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}.$$

- Assume$^{(\star\star)}$ $\boldsymbol{E}_{0,h} \in \mathcal{V}_h$.

  `Corollary`: $\|\mathrm{div}\,\varepsilon(\boldsymbol{E}_0 - \boldsymbol{E}_{0,h})\|_{H^{-s}(\Omega)} \lesssim h^{s+\delta-1}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$.

- Comments:
  - Given $\boldsymbol{v}_h \in \mathcal{X}_h$: $\mathrm{div}\,\varepsilon\boldsymbol{v}_h \in \cap_{1/2<s\leq 1} H^{-s}(\Omega)$.
  - Quasi-uniformity assumption can be removed, use [Li-Melenk-Wohlmuth-Zou'10].
  - The assumptions$^{(\star)}$ and $^{(\star\star)}$ are tied to the (discrete) variational formulations.

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On the divergence - 2

- *One can relax the regularity assumption...*

- `Theorem` [Bonito-Guermond-Luddens'13]: Consider $\varepsilon, \mu^{-1} \in PW^{1,\infty}(\Omega)$.
  Then $\exists \delta_{max}^{Dir}, \delta_{max}^{Neu} > 0$ s.t.

$$\mathcal{X}_N(\Omega, \varepsilon) \subset \cap_{0 \le \delta < \delta_{max}^{Dir}} \boldsymbol{PH}^{\delta}(\Omega), \quad \text{and} \quad \mathcal{X}_T(\Omega, \mu) \subset \cap_{0 \le \delta < \delta_{max}^{Neu}} \boldsymbol{PH}^{\delta}(\Omega)\,.$$

  Define $\delta_{max} := \min(\delta_{max}^{Dir}, \delta_{max}^{Neu})$.

- Interpolation ($\delta \in]0, \delta_{max}[$), cf. [Jr'16]: $\|\boldsymbol{E}_0 - \boldsymbol{E}_{0,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h^{\delta}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$.

- Divergence estimates can still be found, even though $\delta_{max}^{Dir} < 1/2$ is possible...

- `Theorem` [$\mathcal{V}_h$]: Consider $\varepsilon \in PW^{1,\infty}(\Omega)$. Choose conforming meshes $(\mathcal{T}_h)_h$.
  Given $\underline{\delta} \in]0, \delta_{max}^{Dir}[$ and $s \in]1 - \underline{\delta}, 1]$, it holds

$$\forall \boldsymbol{v}_h \in \mathcal{V}_h, \quad \|\mathrm{div}\, \varepsilon \boldsymbol{v}_h\|_{H^{-s}(\Omega)} \lesssim h^{s+\underline{\delta}-1} \|\mathbf{curl}\, \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}.$$

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On the divergence - 3

- *The "discrete compactness property"*:

  The family $(\mathcal{V}_h)_h$ satisfies the discrete compactness property if: for all sequences $(\boldsymbol{v}_h)_h \in (\mathcal{V}_h)_h$ s.t. $\|\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim 1$, there exists a subsequence that converges in $\boldsymbol{L}^2(\Omega)$.

  How to derive such a result when $\varepsilon \in PW^{1,\infty}(\Omega)$?

1. Choose $\underline{\delta} \in]0, \min(1/2, \delta^{Dir}_{max})[$ and fix $s \in]1-\underline{\delta}, 1]$. Observe that:

$$\forall h,\ \boldsymbol{v}_h \in \mathcal{X}_{N,-s}(\Omega,\varepsilon) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}\,;\Omega) \,|\, \mathrm{div}\,\varepsilon\,\boldsymbol{v} \in H^{-s}(\Omega)\} \quad \text{(Edge elements)};$$

$$\|\mathrm{div}\,\varepsilon\boldsymbol{v}_h\|_{H^{-s}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \lesssim 1 \quad \text{(Theorem } [\mathcal{V}_h]\text{)}.$$

2. `Theorem` [Bonito-Guermond'11]$^+$: Consider $\varepsilon \in PW^{1,\infty}(\Omega)$.
   Given $s \in]1/2, 1[$, $\mathcal{X}_{N,-s}(\Omega,\varepsilon)$ is compactly imbedded into $\boldsymbol{L}^2(\Omega)$.
   $\boldsymbol{v} \mapsto (\|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\varepsilon\boldsymbol{v}\|^2_{H^{-s}(\Omega)})^{1/2}$ defines a norm on $\mathcal{X}_{N,-s}(\Omega,\varepsilon)$;
   this norm is equivalent to the full norm.
3. One concludes that $(\mathcal{V}_h)_h$ satisfies the discrete compactness property!

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  - **mixed VF**
  - **perturbed VF**
- Numerical illustrations
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Variational formulation: mixed

- Given source terms $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ ($\operatorname{div} \boldsymbol{f} = 0$) and $\varrho \in L^2(\Omega)$, solve:

$$\begin{cases} \textit{Find } \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ with } \mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ s.t.} & \\ \mathbf{curl}\,\left(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}\right) = \boldsymbol{f} & \text{in } \Omega\,; \\ \operatorname{div}\varepsilon\,\boldsymbol{E} = \varrho & \text{in } \Omega\,; \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

- The discrete mixed variational formulation uses edge elements for the field and P_1 elements for the multiplier (with $p_h = 0$):

$$\begin{cases} \textit{Find } (\boldsymbol{E}'_h, p_h) \in \mathcal{X}_h \times Q_h \textit{ s.t.} \\ \forall \boldsymbol{v}_h \in \mathcal{X}_h, \quad (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}'_h|\mathbf{curl}\,\boldsymbol{v}_h) + (\varepsilon\,\boldsymbol{v}_h|\nabla p_h) = (\boldsymbol{f}|\boldsymbol{v}_h) \\ \forall q_h \in Q_h, \quad (\varepsilon \boldsymbol{E}'_h|\nabla q_h) = -(\varrho|q_h). \end{cases}$$

- Given $\delta \in]1/2, \delta_{max}[$, one obtains [Chen-Du-Zou'00]:

$$\|\boldsymbol{E} - \boldsymbol{E}'_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h^\delta\,\{\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\varrho\|_{L^2(\Omega)}\}.$$

Variational formulation: perturbed - 1

- Given source terms $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ ($\operatorname{div} \boldsymbol{f} = 0$) and $\varrho \in L^2(\Omega)$, solve:

$$\left\{ \begin{array}{ll} \textit{Find } \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ with } \mathbf{curl}\, \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ s.t.} & \\ \mathbf{curl}\, \left(\mu^{-1}\mathbf{curl}\, \boldsymbol{E}\right) = \boldsymbol{f} & \text{in } \Omega\,; \\ \operatorname{div} \varepsilon\, \boldsymbol{E} = \varrho & \text{in } \Omega\,; \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{array} \right.$$

- To take into account the condition on the divergence, choose a *perturbed variational formulation* (with $\gamma(h) > 0$ "small", see below), replacing the exact form by

$$a_h(\boldsymbol{v}, \boldsymbol{v}') := (\mu^{-1}\mathbf{curl}\, \boldsymbol{v} | \mathbf{curl}\, \boldsymbol{v}') + \gamma(h)(\varepsilon\, \boldsymbol{v} | \boldsymbol{v}') \text{ for } \boldsymbol{v}, \boldsymbol{v}' \in \boldsymbol{H}_0(\mathbf{curl}\,; \Omega).$$

- If $\varrho \neq 0$, solve two discrete variational formulations

  1. *Find* $\phi_h \in Q_h$ *s.t.* $\forall q_h \in Q_h,\quad (\varepsilon \nabla \phi_h | \nabla q_h) = -(\varrho | q_h)$.
  2. *Find* $\boldsymbol{E}_h \in \mathcal{X}_h$ *s.t.* $\forall \boldsymbol{v}_h \in \mathcal{X}_h,\quad a_h(\boldsymbol{E}_h, \boldsymbol{v}_h) = (\boldsymbol{f} | \boldsymbol{v}_h) + \gamma(h)(\varepsilon\, \nabla \phi_h | \boldsymbol{v}_h)$.

By construction, $\boldsymbol{E}_h - \nabla \phi_h \in \mathcal{V}_h$.

Variational formulation: perturbed - 2

- Assumption$^{(\star\star)}$ is fulfilled. Assumption$^{(\star)}$ writes:

$$\forall h,\ \|\boldsymbol{E}_0 - \boldsymbol{E}_{0,h}\|_{a_h} \lesssim \inf_{\boldsymbol{v}_h \in \mathcal{X}_h} \|\boldsymbol{E}_0 - \boldsymbol{v}_h\|_{a_h} + \sqrt{\gamma(h)}\,\|\boldsymbol{E}_0\|_{\boldsymbol{L}^2(\Omega)}.$$

- `Theorem` [Jr-Wu-Zou'14]$^+$: Consider $\varepsilon, \mu^{-1} \in W^{1,\infty}(\Omega)$.
Let $0 < \gamma(h) \lesssim h^{2\delta_{max}}$. Given $\delta \in]1/2, \delta_{max}[$ and $s \in]1/2, 1]$, it holds

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h^{\delta}\{\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\varrho\|_{L^2(\Omega)}\}\,;$$
$$\|\mathrm{div}\,\varepsilon(\boldsymbol{E} - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim h^{s+\delta-1}\{\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\varrho\|_{L^2(\Omega)}\}.$$

- If Ω is *convex*: let $0 < \gamma(h) \lesssim h^2$. Given $s \in]1/2, 1]$, it holds

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h\,\{\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\varrho\|_{L^2(\Omega)}\}\,;$$
$$\|\mathrm{div}\,\varepsilon(\boldsymbol{E} - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim h^{s}\{\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\varrho\|_{L^2(\Omega)}\}.$$

- See the numerical illustrations for the *practical choice* of $\gamma(h)(\lesssim h^2)$.

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Variational formulations: a comparison

- A comparison of $\boldsymbol{E}'_h$ (mixed VF) with $\boldsymbol{E}_h$ (perturbed VF):
  - Given $\delta \in]1/2, \delta_{max}[$, one obtains by direct computations:

$$\|\mathbf{curl}\,(\boldsymbol{E}'_h - \boldsymbol{E}_h)\|_{\boldsymbol{L}^2(\Omega)} \lesssim h^{\delta_{max}+\delta/2} \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\varrho\|_{L^2(\Omega)} \right) ;$$

  - By construction, $\boldsymbol{E}'_h - \boldsymbol{E}_h \in \mathcal{V}_h$ so, given $s \in]1/2, 1]$, `Theorem` $[\mathcal{V}_h]$ yields:

$$\|\mathrm{div}\,\varepsilon(\boldsymbol{E}'_h - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim h^{s+3\delta/2+\delta_{max}-1} \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\varrho\|_{L^2(\Omega)} \right) .$$

  - It follows that (`Theorem` [Bonito-Guermond'11]$^{+}$):

$$\|\boldsymbol{E}'_h - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,,\Omega)} \lesssim h^{\delta_{max}+\delta/2} \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\varrho\|_{L^2(\Omega)} \right) .$$

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- **Numerical illustrations**
  - **estimates on** $\|\mathrm{div}\,\varepsilon(\boldsymbol{E} - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)}$
  - **sensitivity to** $\gamma(h)$
  - **mixed VF vs. perturbed VF**
- Conclusion and perspectives

Numerics [1]

- `Numerical example` [1]: stationary/static problem in 3D [Jr-Wu-Zou'14].
  $\varepsilon = \mu = 1$ in the unit cube Ω, with *smooth solution*.

- $(\mathcal{T}_h)_h$ built from an initial mesh refined uniformly (5 levels).
- Solver based on the *perturbed VF*: one has to solve two *direct problems*, one in Q_h, one in $\mathcal{X}_h$ (with $\gamma(h) = h^2$).
- Pb in $\mathcal{X}_h$ is solved iteratively (bi-CGSTAB, with the [Hiptmair-Xu'07] preconditioner).
- Computations have been carried out with COMSOL Multiphysics.
- One can choose $\delta = 1$ for the convergence rates. So, one expects

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h$$
$$\|\mathrm{div}\,(\boldsymbol{E} - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim h^s \text{ for } s \in]1/2, 1].$$

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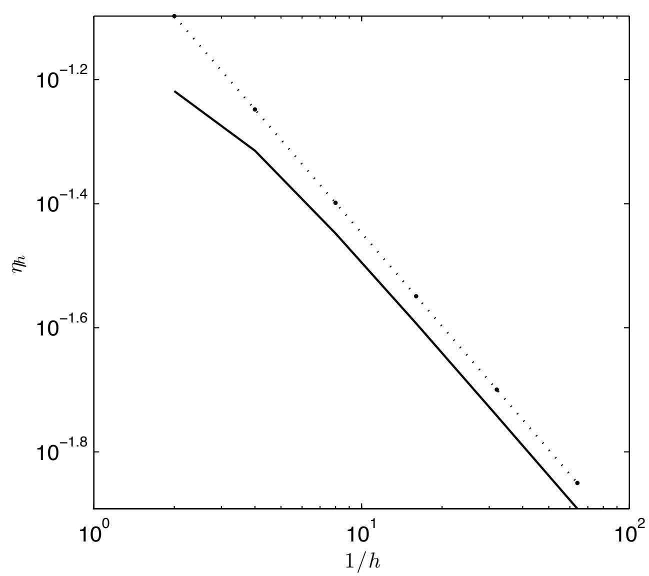
Numerics [1] – results

- $\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h$ is observed.
- For the $\|\mathrm{div}\,(\boldsymbol{E}-\boldsymbol{E}_h)\|_{H^{-s}(\Omega)}$ error, one has, cf. [Jr-Wu-Zou'14]:

$$\|\mathrm{div}\,(\boldsymbol{E}-\boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim h^s(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}+\|\varrho\|_{L^2(\Omega)})+h^{s-1/2}\Big(\sum_{f\in\mathcal{F}_h}\|[\boldsymbol{E}_h\cdot\boldsymbol{n}]\|^2_{L^2(f)}\Big)^{1/2}.$$

So, one has to observe that $\eta_h := \left(\sum_{f\in\mathcal{F}_h}\|[\boldsymbol{E}_h\cdot\boldsymbol{n}]\|^2_{L^2(f)}\right)^{1/2} \lesssim h^{1/2}$.



solid line: η_h ;
dotted line: slope -1/2.

Numerics [2]

- `Numerical example` [2]: stationary/static problem in 2D by K. Brodt (2015-16).
  Given a source term $\vec{f} \in \vec{L}^2(\Omega)$ ($\mathrm{div}\, \vec{f} = 0$), solve:

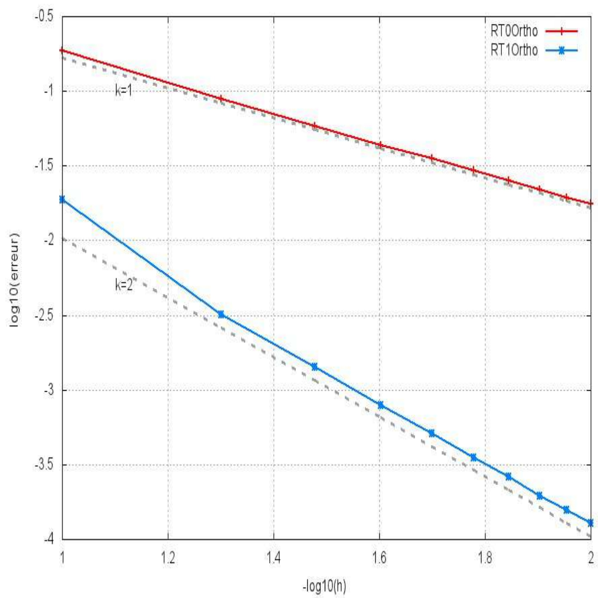
$$\begin{cases} \textit{Find } \vec{E} \in \vec{H}_0(\mathrm{curl}\,, \Omega) \textit{ s.t.} \\ \vec{\mathrm{curl}} \left(\mu^{-1} \mathrm{curl}\, \vec{E} \right) = \vec{f} & \text{in } \Omega\ ; \\ \mathrm{div}\, \varepsilon\, \vec{E} = 0 & \text{in } \Omega. \end{cases}$$

  Above, Ω is the unit square, and $\Omega_1 :=]0, \frac{1}{2}[\times]0,1[$, $\Omega_2 :=]\frac{1}{2}, 1[\times]0,1[$.
  The parameters are set to: $(\varepsilon_1, \mu_1) = (\frac{5}{3}, \frac{5}{3})$, resp. $(\varepsilon_2, \mu_2) = (\frac{10}{3}, 5)$.
  The solution $\vec{E}$ is *piecewise smooth*.

- The meshsize h varies from 0.1 to 0.01.
- Computations have been carried out with Freefem++ (and a direct solver from UMFPACK library), using either first order or <u>second order</u> edge finite elements.
- For the perturbed VF, the theory suggests the use of $\gamma(h) = h^2$ for first order FE, resp. of $\gamma(h) = h^4$ for <u>second order</u> FE.
- We study the sensitivity of the perturbed VF to the value of γ.

Numerics [2] – results

- The $\|\cdot\|_{\vec{L}^2(\Omega)}$ relative error (with $\gamma = 10^{-5}$ fixed):



red line: 1st order ;
blue line: 2nd order ;
dashed lines: slopes -1 and -2.

Numerics [3]

- `Numerical example` [3]: stationary/static problem in 3D by K. Brodt (2015-16). ε piecewise-constant, $\mu = 1$ in the unit cube Ω, with *singular solution* ($\delta_{max} = 0.45$).

- The meshsize h varies from 0.136 to 0.024.
- Computations have been carried out with Freefem++ (and a direct solver from UMFPACK library), using first order edge finite elements.
- For the perturbed VF, the theory suggests the use of $\gamma(h) = h^{0.9}$.
- However, numerical experiments suggest that it is more stable to use

$$\gamma(h) = (h_{min})^{0.9} \leq h^{0.9}, \text{ where } h_{min} = \min_{k \in \mathcal{T}_h} h_K.$$

  The results are reported for this value.
- We compare the *mixed and perturbed VFs*.

Numerics [3] – results

- CPU times, in seconds (using a 1.6GHz i5-4200U Core, with 6GB RAM):

h	mixed VF	perturbed VF
0.136	0.003	0.001
0.049	0.055	0.039
0.033	0.339	0.186
0.024	4.165	0.756

NB. Speed-up ≈ 5.5 for $h = 0.024$.

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Conclusions

- Discrete ε-divergence free elements have "small" $\|\mathrm{div}\,\varepsilon \cdot \|_{H^{-s}(\Omega)}$ ($s \in]1/2, 1]$)...
- The properties of the solutions to the time-harmonic/time-dependent Maxwell problems can be analyzed similarly, cf. [Jr-Wu-Zou'14].
- The perturbed approach can be applied to magnetostatics with "optimized" $\gamma(h)$.
- Numerical experiments suggest that one can use a posteriori/adaptive strategies with the perturbed approach ($\gamma(h) = (h_{min})^{2\delta_{max}}$).
- Numerical experiments suggest that one can solve problems with sign-changing coefficients with the perturbed approach.