MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2001; **24**:9–30
MOS subject classification: 46 E 35; 83 C 50

On traces for functional spaces related to Maxwell's equations Part I: An integration by parts formula in Lipschitz polyhedra

A. Buffa[1,*,†] and P. Ciarlet, Jr.[2]

[1] *Dipartimento di Matematica, Universita degli Studi di Pavia, Via Ferrata 1, 27100 Pavia, Italy*
[2] *ENSTA/UMA, 32, boulevard Victor, 75739 Paris Cedex 15, France*

Communicated by P. Colli

SUMMARY

The aim of this paper is to study the tangential trace and tangential components of fields which belong to the space $\mathbf{H}(\mathbf{curl}, \Omega)$, when Ω is a polyhedron with Lipschitz continuous boundary. The appropriate functional setting is developed in order to suitably define these traces on the whole boundary and on a part of it (for partially vanishing fields and general ones.) In both cases it is possible to define *ad hoc* dualities among tangential trace and tangential components. In addition, the validity of two related integration by parts formulae is provided. Copyright © 2001 John Wiley & Sons, Ltd.

1. INTRODUCTION

The aim of this paper is to give a precise meaning to the following integration by parts formula

$$\int_\Omega \{\mathbf{curl}\,\mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{curl}\,\mathbf{u}\}\,\mathrm{d}\Omega = \langle \mathbf{u} \wedge \mathbf{n}, \mathbf{v} \rangle$$

Here, Ω denotes an open subset of $\mathbb{R}^3$ and its boundary is called Γ. This formula is well known for $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$ (see (1) below) and $\mathbf{u} \in H^1(\Omega)^3$ (see for instance [14]): the brackets then stand for the duality product between $H^{1/2}(\Gamma)^3$ and its dual, $H^{-1/2}(\Gamma)^3$; this holds for any Lipschitz domain. In the case when both fields belong to $\mathbf{H}(\mathbf{curl}, \Omega)$, a similar formula has been derived, with $\langle \cdot, \cdot \rangle$ as the duality product $H^{-1/2}_{\mathrm{div}_\Gamma}(\Gamma) - H^{-1/2}_{\mathrm{curl}_\Gamma}(\Gamma)$, when the boundary of the domain is sufficiently regular (cf. [24]). In the more general case (a Lipschitz domain and both fields in $\mathbf{H}(\mathbf{curl}, \Omega)$), Sheen [21] has already proved that such a formula is valid, with brackets meaningful in Lip(Γ), the space of Lipschitz functions defined on Γ, and its dual.

In this paper, we introduce some Sobolev spaces defined on the boundary and we derive another valid formula. In Section 2, we first investigate the range of the tangential trace mapping from $H^1(\Omega)^3$. For that, we need to characterize precisely the space $H^{1/2}(\Gamma)$. Then, in Section 3, we

* Correspondence to: A. Buffa, Dipartimento di Matematica, Universita degli Studi di Pavia, Via Ferrata 1, 27100 Pavia, Italy
† E-mail: annalisa@dragon.ian.pv.cnr.it

Contract/grant sponsor: I.A.N. -C.N.R. Pavia, Italy

 Received 3 July 2000

introduce the tangential operators on the boundary, which allows to define the range of the tangential trace and that of the tangential components from $\mathbf{H}(\mathbf{curl}, \Omega)$. In Section 4, we obtain integration by parts formulae for fields in $\mathbf{H}(\mathbf{curl}, \Omega)$, or for fields in $\mathbf{H}(\mathbf{curl}, \Omega)$ which vanish on a part of the boundary.

In a companion paper [8], we prove that Hodge decompositions can be obtained on the boundary of such domains, these decompositions being similar to those described in [12] in the case of regular domains. The two papers are closely related, in particular when we prove that the space $H^{3/2}(\Gamma)$, which is introduced with two different definitions hereafter and in [8], is unique (cf. Theorem 3.4).

Recall that if the domain Ω is **regular**, all the definitions here below make sense and are correct (see [24]).

Let us set

$$\mathbf{H}(\mathbf{curl}, \Omega) := \{\mathbf{u} \in L^2(\Omega)^3 \colon \mathbf{curl}\, \mathbf{u} \in L^2(\Omega)^3\}, \quad \|\cdot\|_{0,\mathrm{curl}} \text{ the graph norm} \tag{1}$$

$$\begin{aligned} \mathbf{H}^{-1/2}_{\mathrm{div}_\Gamma}(\Gamma) &:= \{\boldsymbol{\lambda} \in H^{-1/2}(\Gamma)^3 \colon \boldsymbol{\lambda} \cdot \mathbf{n} = 0,\ \mathrm{div}_\Gamma \boldsymbol{\lambda} \in H^{-1/2}(\Gamma)\} \\ &:= \{\mathbf{u} \wedge \mathbf{n} \colon \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)\} \\ \mathbf{H}^{-1/2}_{\mathrm{curl}_\Gamma}(\Gamma) &:= \{\boldsymbol{\lambda} \in H^{-1/2}(\Gamma)^3 \colon \boldsymbol{\lambda} \cdot \mathbf{n} = 0,\ \mathrm{curl}_\Gamma \boldsymbol{\lambda} \in H^{-1/2}(\Gamma)\} \\ &:= \{\mathbf{n} \wedge (\mathbf{u} \wedge \mathbf{n}) = \mathbf{u}_T \colon \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)\} \end{aligned} \tag{2}$$

In the case in which Ω is only a Lipschitz polyhedron, i.e. its boundary is not smooth, several problems occur; namely in definition (2) neither the quantity $\boldsymbol{\lambda} \cdot \mathbf{n}$ nor the differential operator div_Γ are meaningful anymore. In order to give a good and useful definition of the trace space, that is the equivalent of (2) in the case of a piecewise smooth boundary, we need some preliminaries. Note that, for general Lipschitz domains, the characterization of traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ has been given by Tartar in [22]. However, from this paper, it is also clear that the definition of differential operators on Lipschitz manifolds in the context of Sobolev spaces, is, in general, an 'ill-posed problem'. Thus, the need for an 'intermediate' characterization on a class of Lipschitz domains with piecewise smooth boundaries.

In the following we assume that Ω is a Lipschitz polyhedron not necessarily convex. All the results carry out to the case of a Lipschitz curvilinear polyhedron, that is a Lipschitz polyhedron with curved faces (cf. [10]). Whenever the results are derived in a different way for the Lipschitz curvilinear polyhedron case, an explicit mention is added.

2. PRELIMINARIES

The boundary Γ is split in N (open) faces $(\Gamma_j)_{j=1,\dots,N}$, $\Gamma = \bigcup_j \bar{\Gamma}_j$. Let us denote by $(e_{ij})_{i,j \in \{1,\dots,N\}}$ its (open) edges: when Γ_i and Γ_j are two adjacent faces, e_{ij} is the 'common' edge. Additionally, Γ_{ij} stands for the open set $\Gamma_i \cup \Gamma_j \cup e_{ij}$. Its vertices are $(S_k)_{k=1,\dots,k}$.

Let $\mathbf{n}$ denote the unit outward normal to Ω. Finally, let $\boldsymbol{\tau}_{ij}$ be a unit vector parallel to e_{ij} and $\mathbf{n}_j = \mathbf{n}_{|\Gamma_j}$; $\boldsymbol{\tau}_i = \boldsymbol{\tau}_{ij} \wedge \mathbf{n}_i$. The couple $(\boldsymbol{\tau}_i, \boldsymbol{\tau}_{ij})$ is an orthonormal basis of the plane generated by Γ_i; $(\boldsymbol{\tau}_i, \boldsymbol{\tau}_{ij}, \mathbf{n}_i)$ is an orthonormal basis of $\mathbb{R}^3$.

For elements φ of $L^2(\Gamma)$, one adopts the notation $\varphi_j = \varphi_{|\Gamma_j}$. This notation is used whenever the restriction to a face is considered, that is as regards to any functional space in which the restriction to a face is allowed.

In this paper, boldface characters are used for all vector fields and some vector spaces, such as for instance $L^2(\Omega)^3$ which is denoted by $\mathbf{L}^2(\Omega)$.

Let us set:

$$\mathbf{L}^2_t(\Gamma) := \{\boldsymbol{\varphi} \in \mathbf{L}^2(\Gamma)^3 : \boldsymbol{\varphi} \cdot \mathbf{n}_{|\Gamma} = 0\}, \quad \langle \cdot, \cdot \rangle_t \text{ its scalar product}$$

$$\mathbf{H}^{1/2}_{-}(\Gamma) := \{\boldsymbol{\lambda} \in \mathbf{L}^2_t(\Gamma) : \boldsymbol{\lambda}_j \in \mathbf{H}^{1/2}(\Gamma_j),\ 1 \leqslant j \leqslant N\}$$

Note that in the remainder of the paper, $\mathbf{L}^2_t(\Gamma)$ is identified with the space of *two dimensional*, tangential, square integrable, vector fields. The consequence of this choice is that, *on the boundary* Γ, one deals with two-dimensional vector fields whereas, in Ω, three-dimensional ones are considered. Of course, the same identification holds for all the spaces derivating from $\mathbf{L}^2_t(\Gamma)$, e.g. $\mathbf{H}^{1/2}_{-}(\Gamma)$.

Definition 2.1. Let us define the 'tangential components trace' mapping $\pi_\tau : \mathscr{D}(\bar{\Omega})^3 \to \mathbf{H}^{1/2}_{-}(\Gamma)$ and the 'tangential trace mapping' $\gamma_\tau : \mathscr{D}(\bar{\Omega})^3 \to \mathbf{H}^{1/2}_{-}(\Gamma)$ as $\mathbf{u} \mapsto \mathbf{n} \wedge (\mathbf{u} \wedge \mathbf{n})_{|\Gamma}$ and $\mathbf{u} \mapsto \mathbf{u} \wedge \mathbf{n}_{|\Gamma}$, respectively.

On the one hand, it is surely true that π_τ and γ_τ can be extended to linear continuous mappings from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^{1/2}_{-}(\Gamma)$. On the other, it is proved in what follows that these mappings are not surjective and that their ranges are different subspaces of $\mathbf{H}^{1/2}_{-}(\Gamma)$.

Without loss of generality, let us focus our attention on the mapping π_τ and deduce the properties related to the mapping γ_τ (this rule is applied throughout the paper.)

Since one deals with polyhedrons, given a function $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega)$, the definition of $\pi_\tau \boldsymbol{\varphi}$ can be understood face by face:

$$\pi_{\tau,j}\boldsymbol{\varphi} := \boldsymbol{\varphi}_j - (\boldsymbol{\varphi}_j \cdot \mathbf{n}_j)\mathbf{n}_j, \quad \forall \boldsymbol{\varphi} \in \mathbf{H}^1(\Omega)$$

One gets then that an equivalent definition of π_τ is

$$\pi_\tau : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}_{-}(\Gamma), \quad \pi_\tau \boldsymbol{\varphi}(\mathbf{x}) = \pi_{\tau,j}\boldsymbol{\varphi}(\mathbf{x}) \quad \text{a.e. } \mathbf{x} \in \Gamma_j,\ \forall j$$

Now it is easy to see that the range of this mapping is a true subspace of $\mathbf{H}^{1/2}_{-}(\Gamma)$. For that, a preliminary result is needed.

Proposition 2.2.

$$\varphi \in H^{1/2}(\Gamma) \Leftrightarrow \begin{cases} \varphi \in H^{1/2}(\Gamma_i), & \forall i \in \{1, \ldots, N\} \text{ and} \\ \displaystyle\int_{\Gamma_i}\int_{\Gamma_j} \frac{|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \, d\sigma(\mathbf{x}) \, d\sigma(\mathbf{y}) < \infty, & \forall i \neq j \text{ s.t. } \bar{\Gamma}_i \cap \bar{\Gamma}_j \neq \emptyset \end{cases} \tag{3}$$

Proof. This result was first stated in [17]. Let us recall from [16, 17] that φ is in $H^{1/2}(\Gamma)$ if and only if

$$\varphi \in L^2(\Gamma) \quad \text{and} \quad |\varphi|_{1/2,\Gamma} := \left\{ \int_\Gamma \int_\Gamma \frac{|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \, d\sigma(x) \, d\sigma(y) \right\}^{1/2} < \infty \tag{4}$$

with $\mathrm{d}\sigma$ a measure on Γ. Now, if $\varphi \in H^{1/2}(\Gamma)$, φ belongs to $H^{1/2}(\Gamma_i)$, for $i \in \{1, \ldots, N\}$. In addition, owing to (4), the terms in the right-hand side of (3) are bounded. Conversely, as $\varphi \in H^{1/2}(\Gamma_i)$, $\forall i \in \{1, \ldots, N\}$, one obtains in particular that $\varphi \in L^2(\Gamma)$. Next, following [3], let us write

$$|\varphi|^2_{1/2,\Gamma} = \sum_{i=1}^{N} |\varphi|^2_{1/2,\Gamma_i} + \sum_{i \neq j} \int_{\Gamma_i} \int_{\Gamma_j} \frac{|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \, \mathrm{d}\sigma(\mathbf{x}) \, \mathrm{d}\sigma(\mathbf{y})$$

If $\bar{\Gamma}_i \cap \bar{\Gamma}_j = \emptyset$, $\mathrm{dist}(\Gamma_i, \Gamma_j) = C_{ij} > 0$. Then, in this case,

$$\begin{aligned} \int_{\Gamma_i} \int_{\Gamma_j} \frac{|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \, \mathrm{d}\sigma(\mathbf{x}) \, \mathrm{d}\sigma(\mathbf{y}) &\leqslant \frac{1}{C_{ij}^3} \int_{\Gamma_i} \int_{\Gamma_j} |\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2 \, \mathrm{d}\sigma(\mathbf{x}) \, \mathrm{d}\sigma(\mathbf{y}) \\ &\leqslant \frac{2}{C_{ij}^3} \{ |\Gamma_j| \, \|\varphi\|^2_{0,\Gamma_i} + |\Gamma_i| \, \|\varphi\|^2_{0,\Gamma_j} \} \end{aligned}$$

Therefore, there exists a constant $C(\Gamma) > 0$ which depends only on the geometry of Γ such that

$$|\varphi|^2_{1/2,\Gamma} \leqslant \sum_{i=1}^{N} |\varphi|^2_{1/2,\Gamma_i} + \sum_{i \neq j, \bar{\Gamma}_i \cap \bar{\Gamma}_j \neq \emptyset} \int_{\Gamma_i} \int_{\Gamma_j} \frac{|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \, \mathrm{d}\sigma(\mathbf{x}) \, \mathrm{d}\sigma(\mathbf{y}) + C(\Gamma) \, \|\varphi\|^2_{0,\Gamma}$$

which allows to conclude that φ belongs to $H^{1/2}(\Gamma)$. □

In the above equations, there are two types of terms, depending whether the intersection is an edge or a vertex. In the latter case, we prove hereunder that the corresponding term in (3) is automatically finite. Before that, let us state a very simple result which is used on several occasions throughout this paper, the simple proof of which is omitted.

Proposition 2.3. Fix a point M of Γ, and then let $\mathscr{V}$ be a neighbourhood of M. Then there exists a plane P and a bi-Lipschitz continuous transform Π from $\mathscr{V}$ to $\Pi(\mathscr{V}) \subset P$.

Now, as a Lipschitz mapping preserves H^1 regularity (cf. [15]), it is certainly enough for $H^{1/2}$ regularity: thus it is (locally) equivalent to consider $H^{1/2}$ regularity on the boundary or in $\mathbb{R}^2$. Then one has

Proposition 2.4. Let Γ_i and Γ_j be two adjacent faces, which share a 'common' vertex. Assume φ on Γ is given such that $\varphi \in H^{1/2}(\Gamma_i)$ and $\varphi \in H^{1/2}(\Gamma_j)$. Then,

$$\int_{\Gamma_i} \int_{\Gamma_j} \frac{|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \, \mathrm{d}\sigma(\mathbf{x}) \, \mathrm{d}\sigma(\mathbf{y}) < \infty$$

Proof. Owing to the above remark, in order to prove this statement it is enough to consider the case of a scalar function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that $\varphi_{|\{0 < x_1, x_2 < 1\}} \in H^{1/2}(\{0 < x_1, x_2 < 1\})$ and $\varphi_{|\{-1 < y_1, y_2 < 0\}} \in H^{1/2}(\{-1 < y_1, y_2 < 0\})$ and to prove that

$$\int_{0 < x_1, x_2 < 1} \int_{0 < y_1, y_2 < 1} \frac{|\varphi(\mathbf{x}) - \varphi(-\mathbf{y})|^2}{\|\mathbf{x} + \mathbf{y}\|^3} \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} < +\infty$$

Now, by triangle inequality, it is sufficient to prove that

$$\int_{0 < x_1, x_2 < 1} \left(\int_{0 < y_1, y_2 < 1} \frac{\mathrm{d}\mathbf{y}}{\|\mathbf{x} + \mathbf{y}\|^3} \right) |\varphi(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} < +\infty \tag{5}$$

By direct estimation (or see [15, p. 19]) it can be shown that

$$c \frac{1}{\|\mathbf{x}\|} \leqslant \int_{0 < y_1, y_2 < 1} \frac{\mathrm{d}\mathbf{y}}{\|\mathbf{x} + \mathbf{y}\|^3} \leqslant C \frac{1}{\|\mathbf{x}\|}, \quad c, C \in \mathbb{R}$$

The integral bounding (5) reads then

$$\int_{0 < x_1, x_2 < 1} \frac{|\varphi(\mathbf{x})|^2}{\|\mathbf{x}\|} \, \mathrm{d}\mathbf{x} \tag{6}$$

In order to prove that the quantity (6) is bounded, some very fine imbedding theorems are required, such as the ones recalled by Tartar in [23]. Let $L^{r,q}(\mathbb{R}^2)$ $1 \leqslant r, q \leqslant \infty$ be the family of Lorentz spaces (see [4] and also [23]). In [23] it is proven that $H^{1/2}(\mathbb{R}^2) \hookrightarrow L^{4,2}(\mathbb{R}^2)$ (where $\hookrightarrow$ stands for continuously imbedded.) Now, let us use the easy characterization of $L^{r,\infty}(\mathbb{R}^2)$ (see [4, pp. 6–8]), which reads

$$f \in L^{r,\infty}(\mathbb{R}^2) \Leftrightarrow \alpha^r \int_{\mathbb{R}^2} 1_{\{\mathbf{x} : |f(\mathbf{x})| > \alpha\}} \, \mathrm{d}\mathbf{x} \leqslant C$$

where C is a constant; the function ψ defined as $\psi(\mathbf{x}) = 1/\sqrt{\|\mathbf{x}\|}$ belongs to $L^{4,\infty}(\mathbb{R}^2)$, with $C \geqslant \pi$.

Since multiplication acts on the family of Lorentz spaces in the following way:

$$f \in L^{a,b}(\mathbb{R}^2), \quad g \in L^{c,d}(\mathbb{R}^2) \quad \text{then } fg \in L^{q,r}(\mathbb{R}^2) \quad \text{with } \frac{1}{q} = \frac{1}{a} + \frac{1}{c}, \ \frac{1}{r} = \frac{1}{b} + \frac{1}{d}$$

one directly gets that $\varphi/\sqrt{\|\mathbf{x}\|} \in L^{2,2}(\mathbb{R}^2) = L^2(\mathbb{R}^2)$. □

As a conclusion, one obtains

Theorem 2.5.

$$\varphi \in H^{1/2}(\Gamma) \Leftrightarrow \begin{cases} \varphi \in H^{1/2}(\Gamma_i), \quad \forall i \in \{1, \dots, N\} \text{ and} \\ \displaystyle\int_{\Gamma_i} \int_{\Gamma_j} \frac{|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} \, \mathrm{d}\sigma(\mathbf{x}) \, \mathrm{d}\sigma(\mathbf{y}) < \infty, \quad \forall i \neq j \text{ s.t. } \bar{\Gamma}_i \cap \bar{\Gamma}_j = e_{ij} \end{cases} \tag{7}$$

Owing to this theorem, we are able to define a suitable subspace of $\mathbf{H}_{-}^{1/2}(\Gamma)$ which contains the range $\pi_\tau(\mathbf{H}^1(\Omega))$. For that, assume $\boldsymbol{\varphi} = \mathbf{n} \wedge (\mathbf{u} \wedge \mathbf{n})_{|\Gamma}$, for some $\mathbf{u} \in \mathbf{H}^1(\Omega)$, and let us focus on a given term (7) for Γ_i and Γ_j such that they have a common edge e_{ij}. $\boldsymbol{\varphi}$ is, by definition, parallel to Γ_i (respectively Γ_j) when its restriction to Γ_i (resp. Γ_j) is considered. Recall that $(\boldsymbol{\tau}_i, \boldsymbol{\tau}_{ij})$ is an orthonormal basis of the plane generated by Γ_i while $(\boldsymbol{\tau}_i, \boldsymbol{\tau}_{ij}, \mathbf{n}_i)$ is an orthonormal basis or $\mathbb{R}^3$. One has

$$\mathbf{u} = u_i\boldsymbol{\tau}_i + u_{ij}\boldsymbol{\tau}_{ij} + u_n\mathbf{n}_i \quad \text{and} \quad \boldsymbol{\varphi} = \varphi_i\boldsymbol{\tau}_i + \varphi_{ij}\boldsymbol{\tau}_{ij} + \varphi_n\mathbf{n}_i \tag{8}$$

Let α be the angle between $\boldsymbol{\tau}_i$ and $\boldsymbol{\tau}_j$ ($c := \cos\alpha$, $s := \sin\alpha$):

$$\boldsymbol{\tau}_j = c\boldsymbol{\tau}_i - s\mathbf{n}_i \tag{9}$$

This leads to the expressions

$$\boldsymbol{\varphi}_i = u_i\boldsymbol{\tau}_i + u_{ij}\boldsymbol{\tau}_{ij}$$

$$\boldsymbol{\varphi}_j = c(cu_i - su_n)\boldsymbol{\tau}_i + u_{ij}\boldsymbol{\tau}_{ij} - s(cu_i - su_n)\mathbf{n}_i$$

Equation (7) applied to $\mathbf{u}$ allows to 'control' quantities by $\|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\|^2$, for $\mathbf{x} \in \Gamma_i$ and $\mathbf{y} \in \Gamma_j$. On the one hand, there holds

$$|\varphi_{ij}(\mathbf{x}) - \varphi_{ij}(\mathbf{y})|^2 = |u_{ij}(\mathbf{x}) - u_{ij}(\mathbf{y})|^2 \leqslant \|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\|^2$$

On the other, $\varphi_n(\mathbf{x}) = 0$: another condition can then only be obtained with $\varphi_i(\mathbf{x}) = u_i(\mathbf{x})$ and a linear combination of the components of $\boldsymbol{\varphi}(\mathbf{y})$, in order to provide a term in $u_i(\mathbf{y})$. But, as $s \neq 0$ by definition, it is easily seen that any linear combination introduces in its turn a term in $u_n(\mathbf{y})$: this term cannot be controlled in any way.

Equation (7) thus implies a single condition

$$\mathcal{N}_{i,j}^{\|}(\boldsymbol{\varphi}) := \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\boldsymbol{\varphi}_i \cdot \boldsymbol{\tau}_{ij}(\mathbf{x}) - \boldsymbol{\varphi}_j \cdot \boldsymbol{\tau}_{ij}(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3}\, \mathrm{d}\sigma(\mathbf{x})\, \mathrm{d}\sigma(\mathbf{y}) < \infty$$

Now, let $(\psi_i, \psi_j) \in H^{1/2}(\Gamma_i) \times H^{1/2}(\Gamma_j)$. Let us adopt the notation:

$$\psi_i \stackrel{1/2}{=} \psi_j \text{ at } e_{ij} \Leftrightarrow \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\psi_i(\mathbf{x}) - \psi_j(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3}\, \mathrm{d}\sigma(\mathbf{x})\, \mathrm{d}\sigma(\mathbf{y}) < \infty$$

Therefore, if $\mathscr{I}_j$ stands for the set of indices i such that the faces Γ_j and Γ_i have a common edge e_{ij}, we have proved that the range of π_τ is included in

$$\mathbf{H}_{\|}^{1/2}(\Gamma) := \{\boldsymbol{\psi} \in \mathbf{H}_{-}^{1/2}(\Gamma) \colon \boldsymbol{\psi}_j \cdot \boldsymbol{\tau}_{ij} \stackrel{1/2}{=} \boldsymbol{\psi}_i \cdot \boldsymbol{\tau}_{ij} \text{ at } e_{ij}\ \forall j, \forall i \in \mathscr{I}_j\}$$

It is now clear that $\mathbf{H}_{\|}^{1/2}(\Gamma)$ is not a closed subspace of $\mathbf{H}_{-}^{1/2}(\Gamma)$. The rest of the paragraph is now devoted to proving that $\mathbf{H}_{\|}^{1/2}(\Gamma)$ is a Hilbert space (with a suitable norm) and that it is indeed the range of π_τ.

Proposition 2.6. The space $\mathbf{H}_{\|}^{1/2}(\Gamma)$ is a Hilbert space when endowed with the following norm:

$$\|\boldsymbol{\psi}\|_{\|,1/2,\Gamma}^2 := \sum_{j=1}^{N} \|\boldsymbol{\psi}\|_{1/2,\Gamma_j}^2 + \sum_{j=1}^{N}\sum_{i \in \mathscr{I}_j} \mathcal{N}_{i,j}^{\|}(\boldsymbol{\psi}) \tag{10}$$

Proof. Let $\{\boldsymbol{\varphi}^k\}_{k\in\mathbb{N}} \subset \mathbf{H}_\parallel^{1/2}(\Gamma)$ be a Cauchy sequence with respect to norm (10). Let us show then that it converges in $\mathbf{H}_\parallel^{1/2}(\Gamma)$. By standard arguments one gets that there exists a limit $\boldsymbol{\varphi} \in \mathbf{H}_-^{1/2}(\Gamma)$. Let us focus now the attention on two faces Γ_i and Γ_j such that $i \in \mathscr{I}_j$. Using the notation introduced in (8), it is clear that $\boldsymbol{\varphi}^k \cdot \boldsymbol{\tau}_{ij} \in H^{1/2}(\Gamma_{ij})$. By uniqueness of the limit one obtains that $\boldsymbol{\varphi} \cdot \boldsymbol{\tau}_{ij} \in H^{1/2}(\Gamma_{ij})$ which means, in particular, that $\mathscr{N}_{i,j}^\parallel(\boldsymbol{\varphi}) < \infty$. □

Let us define also

$$\mathscr{N}_{ij}^\perp(\boldsymbol{\varphi}) := \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\boldsymbol{\psi}_i \cdot \boldsymbol{\tau}_i(\mathbf{x}) - \boldsymbol{\psi}_j \cdot \boldsymbol{\tau}_j(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|^3}\, \mathrm{d}\sigma(\mathbf{x})\, \mathrm{d}\sigma(\mathbf{y})$$

and the related functional space:

$$\mathbf{H}_\perp^{1/2}(\Gamma) := \{\boldsymbol{\psi} \in \mathbf{H}_-^{1/2}(\Gamma)\colon \boldsymbol{\psi}_i \cdot \boldsymbol{\tau}_i \overset{1/2}{=} \boldsymbol{\psi}_j \cdot \boldsymbol{\tau}_j \text{ at } e_{ij}\ \forall i \in \mathscr{I}_j, \forall j\}$$

which is a Hilbert space when endowed with the natural norm:

$$\|\boldsymbol{\psi}\|_{\perp,1/2,\Gamma}^2 := \sum_{j=1}^N \|\boldsymbol{\psi}\|_{1/2,\Gamma_j}^2 + \sum_{j=1}^N \sum_{i\in\mathscr{I}_j} \mathscr{N}_{i,j}^\perp(\boldsymbol{\psi})$$

With the same argument, it is not hard to see that $\gamma_\tau\colon \mathbf{H}^1(\Omega) \to \mathbf{H}_\perp^{1/2}(\Gamma)$.

Proposition 2.7. The mapping $\pi_\tau\colon \mathbf{H}^1(\Omega) \to \mathbf{H}_\parallel^{1/2}(\Gamma)$ (resp. $\gamma_\tau\colon \mathbf{H}^1(\Omega) \to \mathbf{H}_\perp^{1/2}(\Gamma)$) such that $\mathbf{u} \mapsto \mathbf{n} \wedge (\mathbf{u} \wedge \mathbf{n})_{|\Gamma}$ (resp. $\mathbf{u} \mapsto \mathbf{u} \wedge \mathbf{n}_{|\Gamma}$) is linear continuous and surjective. As a consequence, there exists a continuous lifting mapping $\mathscr{R}_\tau$ (resp. $\mathscr{R}_\gamma$) from $\mathbf{H}_\parallel^{1/2}(\Gamma)$ (resp. $\mathbf{H}_\perp^{1/2}(\Gamma)$) to $\mathbf{H}^1(\Omega)$.

Proof. By standard arguments, for every $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega)$, one gets that $\boldsymbol{\varphi}_{|\Gamma} \in \mathbf{H}^{1/2}(\Gamma)$. This assures that the mapping π_τ is continuous from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}_-^{1/2}(\Gamma)$.

We prove now surjectivity by means of the construction of a compatible normal component at every face Γ_j. This construction yields a function of $\mathbf{H}^{1/2}(\Gamma)$ and it is then extended in Ω with a standard argument.

We construct over Γ a particular partition of unity. Let us consider three sets of Lipschitz functions $\{\chi_{S_k}\}_k, \{\chi_{e_{ij}}\}_{i,j\in\mathscr{I}_i, j>i}$ and $\{\chi_{\Gamma_i}\}_i$ such that

(1) $\sum_{k=1}^K \chi_{S_k} + \sum_{i=1}^N \sum_{j\in\mathscr{I}_i, j>i} \chi_{e_{ij}} + \sum_{i=1}^N \chi_{\Gamma_i} \equiv 1$ on Γ.
(2) $\operatorname{supp}\{\chi_{S_k}\} \subset \hat{\Gamma}_k :=$ interior of the union of the closed faces $\bar{\Gamma}_i$ having S_k as a vertex.
(3) $\operatorname{supp}\{\chi_{e_{ij}}\} \subset \Gamma_{ij}$.
(4) $\operatorname{supp}\{\chi_{\Gamma_i}\} \subset \Gamma_i$.

By means of this partition of unity, we are left with the construction of the normal component in three different and independent situations: in a neighbourhood of a vertex, of an edge or inside a face.

(a) Of course, inside a face the normal component can be chosen equal to zero.
(b) In a neighbourhood of an edge e_{ij}.

Let us set $i=1$ and $j=2$. Let $\boldsymbol{\varphi}\in\mathbf{H}_{\|}^{1/2}(\Gamma_{ij})$: we want to construct a function $\mathbf{u}\in\mathbf{H}^{1/2}(\Gamma_{12})$ such that $\mathbf{n}\wedge(\mathbf{u}\wedge\mathbf{n})\equiv\boldsymbol{\varphi}$. According to notation (8) this means that

$$\mathbf{u}_1=\varphi_1\boldsymbol{\tau}_1+\varphi_{12}\boldsymbol{\tau}_{12}+u_1\mathbf{n}_1,\quad \mathbf{u}_2=\varphi_2\boldsymbol{\tau}_2+\varphi_{12}\boldsymbol{\tau}_{12}+u_2\mathbf{n}_2$$

where u_1 and u_2 are the unknowns of the problem. Using now (9), the regularity of $\boldsymbol{\varphi}$, and the equality $\mathbf{n}_2=c\mathbf{n}_1+s\tau_1$, one gets that $\mathbf{u}\in\mathbf{H}^{1/2}(\Gamma_{12})$ if and only if

$$-s\varphi_2+cu_2\overset{1/2}{=}u_1\quad\text{and}\quad c\varphi_2+su_2\overset{1/2}{=}\varphi_1\quad\text{at } e_{12} \tag{11}$$

It is immediate to see that the two conditions in (11) are compatible. It is enough to choose u_2 according to the second constraint, and, afterwards, u_1 according to the first one.

(c) In a neighbourhood of a vertex.

By extension, we have a cone with a polygonal (convex or not) transverse section. Hereafter, let us suppose that this section is a triangle (the general case can be treated with the same argument.) Let S be the vertex, Γ_i for $i=1,2,3$ the three incoming faces, and, respectively, $\{e_{ij}\}_{i,j=1,2,3}$ the three incoming edges (with the convention $e_{ij}=e_{ji}$.) Let us set $\hat{\Gamma}=(\Gamma_1\cup\Gamma_2\cup\Gamma_3)\cup(e_{12}\cup e_{23}\cup e_{31})\cup\{S\}$: a function $\boldsymbol{\varphi}\in\mathbf{H}_{\|}^{1/2}(\hat{\Gamma})$ is provided and we want to construct a function $\mathbf{u}\in\mathbf{H}^{1/2}(\hat{\Gamma})$ such that $\mathbf{n}\wedge(\mathbf{u}\wedge\mathbf{n})\equiv\boldsymbol{\varphi}$.

Using the notation introduced in (8), one gets

$$\boldsymbol{\tau}_1=c_1\boldsymbol{\tau}_3-s_1\mathbf{n}_3,\quad \boldsymbol{\tau}_2=c_2\boldsymbol{\tau}_1-s_2\mathbf{n}_1,\quad \boldsymbol{\tau}_3=c_3\boldsymbol{\tau}_2-s_3\mathbf{n}_2$$

Thus, we directly obtain the compatibility conditions:

$$\begin{aligned}
&\text{(C1)}\ \ \varphi_1\overset{1/2}{=}c_2\varphi_2+s_2u_2\quad\text{and}\quad\text{(C2)}\ u_1\overset{1/2}{=}c_2u_2-s_2\varphi_2\ \text{at } e_{12}\\
&\text{(C3)}\ \ \varphi_2\overset{1/2}{=}c_3\varphi_3+s_3u_3\quad\text{and}\quad\text{(C4)}\ u_2\overset{1/2}{=}c_3u_3-s_3\varphi_3\ \text{at } e_{23}\\
&\text{(C5)}\ \ \varphi_3\overset{1/2}{=}c_1\varphi_1+s_1u_1\quad\text{and}\quad\text{(C6)}\ u_3\overset{1/2}{=}c_1u_1-s_1\varphi_1\ \text{at } e_{13}
\end{aligned} \tag{12}$$

where u_i for $i=1,2,3$ are the unknowns. Let $u_i^{(1)}\in H^{1/2}(\Gamma_i)$, $i=1,2,3$ be three functions which verify (C5), (C1), (C3), respectively (which are independent conditions.) One can then obtain another set of three functions $u_i^{(2)}\in H^{1/2}(\Gamma_i)$, by a *decoupled version* of constraints (C2), (C4), (C6), namely

$$u_1^{(2)}\overset{1/2}{=}c_2u_2^{(1)}-s_2\varphi_2\ \text{at } e_{12},\quad u_2^{(2)}\overset{1/2}{=}c_3u_3^{(1)}-s_3\varphi_3\ \text{at } e_{23},$$

$$u_3^{(2)}\overset{1/2}{=}c_1u_1^{(1)}-s_1\varphi_1\ \text{at } e_{13}$$

Now, let ξ_{ij} be a function such that (the existence of such a function is proved by direct construction in Proposition 1.8 below), for all $\phi\in H^{1/2}(\Gamma_i)$

$$\xi_{ij}\phi\in H^{1/2}(\Gamma_i),\quad \xi_{ij}|_{e_{ij}}=1,\ \xi_{ij}|_{e_{i\ell}}=0,\ \ell\neq j$$

A sample of functions u_i which verify the set of constraints (12) is then

$$u_1 = \xi_{13} u_1^{(1)} + \xi_{12} u_1^{(2)}, \quad u_2 = \xi_{21} u_2^{(1)} + \xi_{23} u_2^{(2)}, \quad u_3 = \xi_{32} u_3^{(2)} + \xi_{31} u_3^{(2)}$$

Now, we have proven that π_τ is linear, continuous and surjective from $\mathbf{H}^1(\Omega)$ onto $\mathbf{H}_\parallel^{1/2}(\Gamma)$. Its kernel is $\ker(\pi_\tau) = \{\mathbf{u} \in \mathbf{H}^1(\Omega)\colon \mathbf{n} \wedge (\mathbf{u} \wedge \mathbf{n})_{|\Gamma} = 0\}$. Thus π_τ is linear, continuous and bijective from $\mathbf{H}^1(\Omega)/\ker(\pi_\tau)$ onto $\mathbf{H}_\parallel^{1/2}(\Gamma)$.

The same arguments can be applied to the mapping γ_τ. □

Proposition 2.8. Let us identify Γ_i with $\Gamma_0 := \{\mathbf{x} = (x_1, x_2)\colon\ 0 < x_1 < 1,\ 0 < x_2 < 1\}$. Set $\partial\Gamma_0^1 := \{(0, x_2)\colon\ 0 < x_2 < 1\}$, and $\partial\Gamma_0^2 := \{(x_1, 0)\colon\ 0 < x_1 < 1\}$. Then, the function χ defined by $\chi(\mathbf{x}) = (1 - x_1/x_2)^+$ is such that $\chi_{|\partial\Gamma_0^1} = 1$, $\chi_{|\partial\Gamma_0^2} = 0$, and $\chi\phi \in H^{1/2}(\Gamma_0)$ for all $\phi \in H^{1/2}(\Gamma_0)$.

Proof. As $\chi \in L^\infty(\Gamma_0)$, $\chi\phi \in L^2(\Gamma_0)$. There remains to prove that

$$\int_{\Gamma_0} \int_{\Gamma_0} \frac{|\chi\phi(\mathbf{x}) - \chi\phi(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3}\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} < \infty$$

It is clear that

$$|\chi(\mathbf{x})\phi(\mathbf{x}) - \chi(\mathbf{y})\phi(\mathbf{y})|^2 \leqslant 2|\chi(\mathbf{x}) - \chi(\mathbf{y})|^2\, |\phi(\mathbf{x})|^2 + 2|\chi(\mathbf{y})|^2\, |\phi(\mathbf{x}) - \phi(\mathbf{y})|^2$$

Then,

$$\int_{\Gamma_0} \int_{\Gamma_0} \frac{|\chi\phi(\mathbf{x}) - \chi\phi(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3}\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \leqslant 2 \int_{\mathbf{x} \in \Gamma_0} |\phi(\mathbf{x})|^2\, \mathrm{d}\mathbf{x}$$
$$\int_{\mathbf{y} \in \Gamma_0} \frac{|\chi(\mathbf{x}) - \chi(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3}\, \mathrm{d}\mathbf{y} + 2\, \|\chi\|_{\Gamma_0, \infty}^2\, |\phi|_{1/2, \Gamma_0}^2$$

Here, we want to use the same technique as is Proposition 2.4. For a given $\mathbf{x} \in \Gamma_0$, one has to evaluate

$$I(\mathbf{x}) = \int_{\mathbf{y} \in \Gamma_0} \frac{(\chi(\mathbf{x}) - \chi(\mathbf{y}))^2}{\|\mathbf{x} - \mathbf{y}\|^3}\, \mathrm{d}\mathbf{y}$$

in terms of $\|\mathbf{x}\|$. Basically, we would like to obtain that there exists a constant C independent of $\mathbf{x}$ such that

$$I(\mathbf{x}) \leqslant \frac{C}{\|\mathbf{x}\|} \tag{13}$$

as we know that $\phi/\sqrt{\|\mathbf{x}\|}$ belongs to $L^2(\Gamma_0)$. To reach that goal, it is convenient to split Γ_0 into two parts, that is $\Gamma_0^+ := \{\mathbf{x} \in \Gamma_0\colon\ x_1 < x_2\}$, and $\Gamma_0^- := \{\mathbf{x} \in \Gamma_0\colon\ x_1 > x_2\}$. Then $I(\mathbf{x})$ is the sum of

$$I^+(\mathbf{x}) = \int_{\mathbf{y} \in \Gamma_0^+} \frac{(\chi(\mathbf{x}) - \chi(\mathbf{y}))^2}{\|\mathbf{x} - \mathbf{y}\|^3}\, \mathrm{d}\mathbf{y} \quad \text{and} \quad I^-(\mathbf{x}) = \int_{\mathbf{y} \in \Gamma_0^-} \frac{(\chi(\mathbf{x}) - \chi(\mathbf{y}))^2}{\|\mathbf{x} - \mathbf{y}\|^3}\, \mathrm{d}\mathbf{y}$$

To further take advantage of the explicit form of $\chi(\mathbf{x})$, it is also convenient to consider the cases $\mathbf{x}\in\Gamma_0^+$ and $\mathbf{x}\in\Gamma_0^-$ separately. Therefore, one has to carry out the computations in four cases:

Case 1. $I^+(\mathbf{x})$ for $\mathbf{x}\in\Gamma_0^+$.
Case 2. $I^+(\mathbf{x})$ for $\mathbf{x}\in\Gamma_0^-$.
Case 3. $I^-(\mathbf{x})$ for $\mathbf{x}\in\Gamma_0^+$.
Case 4. $I^-(\mathbf{x})$ for $\mathbf{x}\in\Gamma_0^-$.

Case 1. In this case, one has

$$I^+(\mathbf{x}) = \int_{y_2=0}^{y_2=1}\int_{y_1=0}^{y_1=y_2}\frac{(\chi(\mathbf{x})-\chi(\mathbf{y}))^2}{\|\mathbf{x}-\mathbf{y}\|^3}\,\mathrm{d}y_1\,\mathrm{d}y_2$$
$$= \int_{y_2=0}^{y_2=1}\mathrm{d}y_2\int_{y_1=0}^{y_1=y_2}\frac{(x_1/x_2-y_1/y_2)^2}{((x_1-y_1)^2+(x_2-y_2)^2)^{3/2}}\,\mathrm{d}y_1$$

In the integral of variable y_1 (y_2 frozen), let us perform the change of variable $\tau = y_1/y_2$. Then, one obtains

$$I^+(\mathbf{x}) = \int_{y_2=0}^{y=1}\mathrm{d}y_2\int_{\tau=0}^{\tau=1}\frac{(x_1/x_2-\tau)^2}{((x_1-y_2\tau)^2+(x_2-y_2)^2)^{3/2}}\,y_2\,\mathrm{d}\tau$$
$$= \int_{\tau=0}^{\tau=1}\left(\frac{x_1}{x_2}-\tau\right)^2\mathrm{d}\tau\int_{y_2=0}^{y_2=1}\frac{y_2\,\mathrm{d}y_2}{((x_1-y_2\tau)^2+(x_2-y_2)^2)^{3/2}} \tag{14}$$

Let us define θ_τ the angle between $\mathbf{x}$ and $\mathbf{x}-\mathbf{u}(\tau)$, with $\mathbf{u}(\tau)$ the vector of components $(\tau, 1)$: integrating the integral in the variable y_2, one gets after some elementary computations

$$I^+(\mathbf{x}) = \int_{\tau=0}^{\tau=1}\left(\frac{x_1}{x_2}-\tau\right)^2\frac{\|\mathbf{x}\|(1-\cos\theta_\tau)}{(x_1-x_2\tau)^2}\,\mathrm{d}\tau = \frac{\|\mathbf{x}\|}{x_2^2}\int_{\tau=0}^{\tau=1}(1-\cos\theta_\tau)\,\mathrm{d}\tau$$

Now, $\mathbf{x}$ is in Γ_0^+, so

$$\|\mathbf{x}\|^2 < 2x_2^2,\quad \text{i.e. } \frac{1}{x_2^2} < \frac{2}{\|\mathbf{x}\|^2}$$

Case 1 is, therefore, completed as

$$I^+(\mathbf{x}) < \frac{4}{\|\mathbf{x}\|}$$

The other three cases can be fixed in a similar manner. One finds

$$I^+(\mathbf{x}) < \frac{5+\sqrt{5}}{\|\mathbf{x}\|}\ \text{(Case 2)},\quad I^-(\mathbf{x}) < \frac{4}{\|\mathbf{x}\|}\ \text{(Case 3)},\quad I^-(\mathbf{x}) = 0\ \text{(Case 4)}$$

Therefore, (13) holds with $C = \max(4+4, 5+\sqrt{5}) = 8$.

In the remainder of the paper, let us call $\mathbf{H}_\parallel^{-1/2}(\Gamma)$ the dual space of $\mathbf{H}_\parallel^{1/2}(\Gamma)$ (with $\mathbf{L}_t^2(\Gamma)$ as the pivot space), and $\langle\cdot,\cdot\rangle_{\parallel,1/2,\Gamma}$ the duality product between $\mathbf{H}_\parallel^{-1/2}(\Gamma)$ and $\mathbf{H}_\parallel^{1/2}(\Gamma)$. Using the definition of the dual norm, given $\boldsymbol{\lambda}\in\mathbf{H}_\parallel^{-1/2}(\Gamma)$, one has

$$\|\boldsymbol{\lambda}\|_{\parallel,-1/2,\Gamma} = \sup_{\boldsymbol{\varphi}\in\mathbf{H}_\parallel^{1/2}(\Gamma)}\frac{\langle\boldsymbol{\lambda},\boldsymbol{\varphi}\rangle_{\parallel,1/2,\Gamma}}{\|\boldsymbol{\varphi}\|_{\parallel,1/2,\Gamma}}. \tag{15}$$

3. TANGENTIAL OPERATORS AND TRACE MAPPINGS FOR **H(curl**, Ω)

3.1. Characterization of ∇_Γ *and* div_Γ *for polyhedral domains*

Since we deal with polyhedrons, as before, the definition of the tangential gradient operator, ∇_Γ, is given face by face as follows:

$$\nabla_{\Gamma_j} u = \pi_{\tau, j}(\nabla u) \quad \forall u \in H^2(\Omega)$$

One has $\nabla_{\Gamma_j}: H^2(\Omega) \to \mathbf{H}^{1/2}(\Gamma_j)$. Let us define now the operator ∇_Γ as

$$\nabla_\Gamma: H^2(\Omega) \to \mathbf{H}_-^{1/2}(\Gamma) \quad \nabla_\Gamma u(\mathbf{x}) = \nabla_{\Gamma_j} u(\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \Gamma_j \tag{16}$$

and then, the following equality holds:

$$\nabla_\Gamma u = \pi_\tau(\nabla u) \tag{17}$$

In the same way, the tangential curl operator $\mathbf{curl}_\Gamma$ can be defined. Namely, let us set

$$\mathbf{curl}_\Gamma u = \gamma_\tau(\nabla u) \tag{18}$$

Proposition 3.1. Let $H^{3/2}(\Gamma)$ be defined as

$$H^{3/2}(\Gamma) := \{u_{|\Gamma} \text{ with } u \in H^2(\Omega)\}$$

then it is a Hilbert space endowed with the norm

$$\|\varphi\|_{3/2,\Gamma} = \inf_{u \in H^2(\Omega): u_{|\Gamma} = \varphi} \|u\|_2$$

Proof. It is enough to observe that the space $H^2(\Omega) \cap H_0^1(\Omega)$ is a closed subspace of $H^2(\Omega)$. The proof is then completely standard. □

From identity (17) and the results of the previous section, one gets

Proposition 3.2. The operator ∇_Γ defined in (16) is a linear continuous mapping from $H^{3/2}(\Gamma)$ to $\mathbf{H}_\parallel^{1/2}(\Gamma)$.

Definition 3.3. Let us define $\mathrm{div}_\Gamma: \mathbf{H}_\parallel^{-1/2}(\Gamma) \to H^{-3/2}(\Gamma)$ the adjoint operator of $-\nabla_\Gamma$ so that

$$\langle \mathrm{div}_\Gamma \boldsymbol{\lambda}, \varphi \rangle_{3/2,\Gamma} = -\langle \boldsymbol{\lambda}, \nabla_\Gamma \varphi \rangle_{\parallel, 1/2, \Gamma} \quad \forall \varphi \in H^{3/2}(\Gamma),\ \boldsymbol{\lambda} \in \mathbf{H}_\parallel^{-1/2}(\Gamma)$$

In the companion paper [8], an alternate definition of $H^{3/2}(\Gamma)$ is used, namely

$${}_2H^{3/2}(\Gamma) := \{p \in H^1(\Gamma): \nabla_\Gamma p \in \mathbf{H}_\parallel^{1/2}(\Gamma)\}$$

It is a Hilbert space when equipped with the graph norm

$${}_2\|p\|_{3/2,\Gamma} = \{\|p\|_{1,\Gamma}^2 + \|\nabla_\Gamma p\|_{\parallel,1/2,\Gamma}^2\}^{1/2}$$

Fortunately, one has

Theorem 3.4. $H^{3/2}(\Gamma) = {}_2H^{3/2}(\Gamma)$ and, in addition, $\|\cdot\|_{3/2,\Gamma}$ and ${}_2\|\cdot\|_{3/2,\Gamma}$ are equivalent norms.

Proof. This result was stated by Grisvard without proof in [17]. For the sake of completeness, we report here a proof.

(i) $H^{3/2}(\Gamma) \subset {}_2H^{3/2}(\Gamma)$. By definition, p belongs to $H^1(\Gamma)$ if and only if $p \in L^2(\Gamma)$, $\nabla_\Gamma p \in \mathbf{L}_t^2(\Gamma)$. Thus, ${}_2H^{3/2}(\Gamma) := \{p \in L^2(\Gamma)\colon \nabla_\Gamma p \in \mathbf{H}_\parallel^{1/2}(\Gamma)\}$, as $\mathbf{H}_\parallel^{1/2}(\Gamma)$ is a subset of $\mathbf{L}_t^2(\Gamma)$.

Now, given $\varphi \in H^{3/2}(\Gamma)$, one has $\varphi \in L^2(\Gamma)$, and $\nabla_\Gamma \varphi \in \mathbf{H}_\parallel^{1/2}(\Gamma)$ (owing to Proposition 2.7 and (17)). Thus, φ is an element of ${}_2H^{3/2}(\Gamma)$.

(ii) ${}_2H^{3/2}(\Gamma) \subset H^{3/2}(\Gamma)$. Let $p \in {}_2H^{3/2}(\Gamma)$. Owing to Proposition 2.7, as $\nabla_\Gamma p \in \mathbf{H}_\parallel^{1/2}(\Gamma)$, there exists $\mathbf{x} \in \mathbf{H}^1(\Omega)$ such that $\pi_\tau \mathbf{x} = \nabla_\Gamma p$. In addition, as $p \in H^{1/2}(\Gamma)$, there exists $u \in H^1(\Omega)$ which is the solution of

$$\Delta u = 0 \quad \text{in } \Omega, \quad u_{|\Gamma} = p$$

$\mathbf{w} := \nabla u$ belongs to $\mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega)$. Thus $\mathbf{y} := \mathbf{x} - \mathbf{w}$ also belongs to $\mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega)$. Let us prove that $\mathbf{y} \wedge \mathbf{n}_{|\Gamma} = 0$, which is equivalent by density (see [19, p. 22]) to

$$\langle \mathbf{y} \wedge \mathbf{n}, \mathbf{z} \rangle_{1/2,\Gamma} = 0, \quad \forall \mathbf{z} \in \prod_j \mathscr{D}(\Gamma_j)^3$$

Let $H_{00}^{1/2}(\Gamma_j)$ and its dual $H_{00}^{-1/2}(\Gamma_j)$ be defined in the usual way (cf. [18, vol. I, p. 72] or [16], in Section 1.2). Then, as $\mathbf{w} \wedge \mathbf{n}_{|\Gamma} \in \mathbf{H}^{-1/2}(\Gamma)$, $\mathbf{w} \wedge \mathbf{n}_{|\Gamma_j} \in \mathbf{H}_{00}^{-1/2}(\Gamma_j)$ for all j, and thus

$$\begin{aligned} \langle \mathbf{w} \wedge \mathbf{n}, \mathbf{z} \rangle_{1/2,\Gamma} &= \sum_{j=1}^{N} \langle \mathbf{w}_j \wedge \mathbf{n}_j, \mathbf{z}_j \rangle_{1/2,00,\Gamma_j} = \sum_{j=1}^{N} \langle \nabla_{\Gamma_j} u \wedge \mathbf{n}_j, \mathbf{z}_j \rangle_{1/2,00,\Gamma_j} \\ &= \sum_{j=1}^{N} \int_{\Gamma_j} (\nabla_{\Gamma_j} p \wedge \mathbf{n}_j) \cdot \mathbf{z}_j \, \mathrm{d}\sigma = \int_\Gamma (\mathbf{x} \wedge \mathbf{n}_j) \cdot \mathbf{z} \, \mathrm{d}\sigma \end{aligned}$$

owing to the boundary condition defining u. Thus, actually $\mathbf{y}$ belongs to $\mathbf{X} := \mathbf{H}_0(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega)$. According to [6], if N_d is the orthogonal of $\Delta(H^2(\Omega) \cap H_0^1(\Omega))$ in $L^2(\Omega)$

$$\exists \mathbf{y}_R \in \mathbf{X} \cap \mathbf{H}^1(\Omega), \quad \exists s \in \mathscr{S}_d := \{\lambda \in H_0^1(\Omega)\colon \Delta\lambda \in N_d\} \quad \text{such that } \mathbf{y} = \mathbf{y}_R + \nabla s$$

Collecting the above results, one gets $\nabla(u + s) = \mathbf{x} - \mathbf{y}_R$, i.e. $u' := (u + s)$ is an element of $H^2(\Omega)$, which, by construction, satisfies to $u'_{|\Gamma} = u_{|\Gamma} + s_{|\Gamma} = p$.

To conclude, one has to check that the norms are equivalent. Owing to the open mapping theorem (see for instance [7]), it is enough to prove that

$$\exists C > 0, \quad \forall p \in {}_2H^{3/2}(\Gamma), \quad \|p\|_{3/2,\Gamma} \leqslant C\, {}_2\|p\|_{3/2,\Gamma}$$

This can be readily achieved by following the skeleton of the proof (ii), with the help of a closure result of [10], which states that $\mathbf{X} \cap \mathbf{H}^1(\Omega)$ is closed in $\mathbf{X}$ and that $\|\cdot\|_1$ and $\|\cdot\|_{0,\mathrm{curl},\mathrm{div}}$ are equivalent norms in $\mathbf{X} \cap \mathbf{H}^1(\Omega)$. □

Remark 3.5. It is not hard to see that there also holds:

$$ {}_2H^{3/2}(\Gamma) := \{p \in H^1(\Gamma): \mathbf{curl}_\Gamma\, p \in \mathbf{H}_\perp^{1/2}(\Gamma)\} $$

Remark 3.6. This proof can be extended easily to the more general case of a curvilinear Lipschitz polyhedron. For that, two results have to be generalized.

The first one is the decomposition $\mathbf{y} = \mathbf{y}_R + \nabla s$. This is carried out by simply resuming the proof of Proposition 5.1 in [6]. It uses the results of [2, 14, 20, 11] which are valid in the case of curvilinear Lipschitz polyhedra.

The second one states that $\mathbf{X} \cap \mathbf{H}^1(\Omega)$ is closed in $\mathbf{X}$ and that $\|\cdot\|_1$ and $\|\cdot\|_{0,\mathrm{curl},\mathrm{div}}$ are equivalent norms on $\mathbf{X}\cap\mathbf{H}^1(\Omega)$. Corollary 2.5 of [10] can still be applied.

Corollary 3.7. $H^{3/2}(\Gamma) := \{p \in H^1(\Gamma): \nabla_\Gamma p \in \mathbf{H}_-^{1/2}(\Gamma)\}$. In addition, an equivalent norm on $H^{3/2}(\Gamma)$ is $p \mapsto \{\|p\|_{1,\Gamma}^2 + \sum_j \|p\|_{3/2,\Gamma_j}^2\}^{1/2}$.

To prove this corollary, the following technical lemma is helpful.

Lemma 3.8. Let $u: \mathbb{R}^2 \to \mathbb{R}$ be a function which belongs to[‡] $H^1(\mathbb{R}^2)$ such that $u_{|\mathbb{R}^2_-} \in H^{3/2}(\mathbb{R}^2_-)$ and $u_{|\mathbb{R}^2_+} \in H^{3/2}(\mathbb{R}^2_+)$. One gets that $\partial u/\partial y \in H^{1/2}(\mathbb{R}^2)$.

Proof. Let $\mathbf{w} = \nabla u$; $\mathbf{w}^- := \mathbf{w}_{|\mathbb{R}^2_-}$; $\mathbf{w}^+ := \mathbf{w}_{|\mathbb{R}^2_+}$.

According to the assumptions, one has $\mathbf{w}^- \in \mathbf{H}^{1/2}(\mathbb{R}^2_-)$, $\mathbf{w}^+ \in \mathbf{H}^{1/2}(\mathbb{R}^2_+)$ and $w_y^- = w_y^+$ in $H^{-1/2}(\{x=0\})$. By a standard reflection argument, we can assume that $\mathbf{w}^+ \equiv 0$ and prove that $w_y^- \in H_{00}^{1/2}(\mathbb{R}^2_-)$. Indeed, if we define $\tilde{u}$ as any $H^{3/2}$-extension of $u_{|\mathbb{R}^2_+}$ to $\mathbb{R}^2$, one has $(u-\tilde{u}) \in H^1(\mathbb{R}^2)$, $(u-\tilde{u})_{|\mathbb{R}^2_+} = 0$ and $(u-\tilde{u})_{|\mathbb{R}^2_-} \in H^{3/2}(\mathbb{R}^2_-)$.

For $0 \leqslant s \leqslant 1$, let

$$ \mathbf{V}^s := \{\mathbf{z} \in \mathbf{H}^s(\mathbb{R}^2_-): \mathbf{curl}\,\mathbf{z} = 0,\ z_y = 0 \text{ at } x = 0\} $$

Note that, for any s, $\mathbf{V}^s$ is a closed subspace of $\mathbf{H}^s(\mathbf{curl}, \mathbb{R}^2_-) = \{u \in \mathbf{H}^s(\mathbb{R}^2_-), \mathbf{curl}\,\mathbf{u} \in H^s(\mathbb{R}^2_-)\}$. Using standard interpolation theory (see [9, p. 57]), one gets

$$ [\mathbf{V}^1, \mathbf{V}^0]_\theta = \mathbf{V}^s, \quad \theta \in (0,1),\ s = 1-\theta $$

Now, let us consider the mapping $\mathscr{Y}^s: \mathbf{V}^s \to H_0^s(\mathbb{R}^2_-)$ defined as $\mathbf{z} \mapsto z_y$, for some s: surely $\mathscr{Y}^1$ and $\mathscr{Y}^0$ are linear and continuous. Owing to the classical interpolation theory, one also gets that $\mathscr{Y}^{1/2}: \mathbf{V}^{1/2} \to H_{00}^{1/2}(\mathbb{R}^2_-)$ is linear and continuous. □

Proof of Corollary 3.7. It is clear that $H^{3/2}(\Gamma)$ is a subset of $\{p \in H^1(\Gamma): \nabla_\Gamma p \in \mathbf{H}_-^{1/2}(\Gamma)\}$.

Now, let $q \in \{p \in H^1(\Gamma): \nabla_\Gamma p \in \mathbf{H}_-^{1/2}(\Gamma)\}$, and let Γ_{ij} a set of two adjacent faces. Owing to the lemma, we know that $\nabla_\Gamma q \cdot \boldsymbol{\tau}_{ij} \in H^{1/2}(\Gamma_{ij})$: thus $\mathscr{N}_{i,j}^{\|}(\nabla_\Gamma q) < \infty$. In other words, q belongs to ${}_2H^{3/2}(\Gamma)$. The equivalence of the norms stems from the fact that $\mathscr{N}_{i,j}^{\|}(\nabla_\Gamma q)$ is bounded (up to a constant) by $\|q\|_{1,\Gamma}$ and $\|\nabla_{\Gamma_i} q\|_{1/2,\Gamma_i}$, $\|\nabla_{\Gamma_j} q\|_{1/2,\Gamma_j}$. □

‡ $\mathbb{R}^2_-$ (resp. $\mathbb{R}^2_+$ is the open half-plane containing all points of $\mathbb{R}^2$ with $x < 0$ (resp. $x > 0$).

3.2. Trace mappings for $\mathbf{H}(\mathbf{curl},\Omega)$

Let us set

$$\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma):=\{\boldsymbol{\lambda}\in\mathbf{H}_{\|}^{-1/2}(\Gamma)\colon \mathrm{div}_\Gamma(\boldsymbol{\lambda})\in H^{-1/2}(\Gamma)\}$$

Theorem 3.9. The mapping $\gamma_\tau\colon\mathbf{H}(\mathbf{curl},\Omega)\to\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ is linear and continuous.

Proof. Let us work on smooth functions and then apply the density of the set $\mathscr{D}(\bar{\Omega})^3$ in $\mathbf{H}(\mathbf{curl},\Omega)$. Let $\mathbf{u}\in\mathscr{D}(\bar{\Omega})^3$ and $\boldsymbol{\lambda}$ be its tangential trace. For every $\mathbf{v}\in\mathbf{H}^1(\Omega)$, the following integration by parts holds:

$$\int_\Omega\{\mathbf{curl}\,\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{curl}\,\mathbf{u}\}\,\mathrm{d}\Omega=\int_\Gamma\boldsymbol{\lambda}\,(\cdot\,\pi_\tau(\mathbf{v})\,\mathrm{d}\sigma \tag{19}$$

Let us prove first that $\boldsymbol{\lambda}\in\mathbf{H}_{\|}^{-1/2}(\Gamma)$. In Proposition 2.7, the surjectivity of π_τ: $\mathbf{H}^1(\Omega)\to\mathbf{H}_{\|}^{1/2}(\Gamma)$ is provided. Using then the integration by parts (19) in the right-hand side of (15) and by Cauchy–Schwarz inequality, one obtains

$$\|\boldsymbol{\lambda}\|_{\|,-1/2,\Gamma}\leqslant C\|\mathbf{u}\|_{0,\mathrm{curl}} \tag{20}$$

Let $\varphi\in H^2(\Omega)$; the following holds:

$$\begin{aligned}\int_\Omega\mathbf{curl}\,\mathbf{u}\cdot\nabla\varphi\,\mathrm{d}\Omega&=-\int_\Gamma\mathbf{u}\wedge\mathbf{n}\cdot\nabla\varphi\,\mathrm{d}\sigma\\&=-\langle\mathbf{u}\wedge\mathbf{n},\nabla_\Gamma\varphi\rangle_{\|,1/2,\Gamma}=\langle\mathrm{div}_\Gamma(\mathbf{u}\wedge\mathbf{n}),\varphi\rangle_{3/2,\Gamma}\end{aligned} \tag{21}$$

Now, as $\varphi_{|\Gamma}$ belongs to $H^{1/2}(\Gamma)$, there exists v in $H^1(\Omega)$ such that $v_{|\Gamma}=\varphi_{|\Gamma}$ and $\|v\|_1\leqslant C\|\varphi\|_{1/2,\Gamma}$, with a constant C independent of φ_Γ. By construction, $v_0:=v-\varphi\in H_0^1(\Omega)$. Then (21) yields

$$\begin{aligned}\langle\mathrm{div}_\Gamma(\mathbf{u}\wedge\mathbf{n}),\varphi_\Gamma\rangle_{3/2,\Gamma}&=\int_\Omega\mathbf{curl}\,\mathbf{u}\cdot\nabla(\varphi+v_0)\,\mathrm{d}\Omega\leqslant\|\mathbf{u}\|_{0,\mathrm{curl}}\|v\|_1\\&\leqslant C\|\mathbf{u}\|_{0,\mathrm{curl}}\|\varphi\|_{1/2,\Gamma}\end{aligned}$$

This last inequality means that the functional $\mathrm{div}_\Gamma(\mathbf{u}\wedge\mathbf{n})$ can be extended to a linear continuous functional on $H^{1/2}(\Gamma)$, as $H^2(\Omega)_{|\Gamma}$ is dense in $H^{1/2}(\Gamma)=H^1(\Omega)_{|\Gamma}$. Moreover, one gets

$$\mathrm{div}_\Gamma(\mathbf{u}\wedge\mathbf{n})\in H^{-1/2}(\Gamma)\quad\forall\mathbf{u}\in\mathscr{D}(\bar{\Omega})^3\quad\text{and}\quad\|\mathrm{div}_\Gamma(\mathbf{u}\wedge\mathbf{n})\|_{-1/2,\Gamma}\leqslant C\|\mathbf{u}\|_{0,\mathrm{curl}} \tag{22}$$

By density of $\mathscr{D}(\bar{\Omega})^3$ in $\mathbf{H}(\mathbf{curl},\Omega)$ we conclude that (20) and (22) are true for every function $\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega)$. □

Using the above result, the following Green formula holds true:

$$\int_\Omega\{\mathbf{curl}\,\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{curl}\,\mathbf{u}\}\,\mathrm{d}\Omega=\langle\gamma_\tau(\mathbf{u}),\pi_\tau(\mathbf{v})\rangle_{\|,1/2,\Gamma}\quad\forall\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega),\quad\mathbf{v}\in\mathbf{H}^1(\Omega)$$

(Again, $\langle\cdot,\cdot\rangle_{\|,1/2,\Gamma}$ is the duality product $\mathbf{H}_{\|}^{-1/2}(\Gamma)-\mathbf{H}_{\|}^{1/2}(\Gamma)$ with $\mathbf{L}_t^2(\Gamma)$ as pivot space.)

During the course of the proof, notice that we have rephrased one of the results of [1], i.e. that there exists a constant C which depends only on the geometry such that

$$\|\mathbf{u}\wedge\mathbf{n}\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \leqslant C\|\mathbf{u}\|_{0,\mathrm{curl}}, \quad \forall \mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega)$$

with a slightly different definition of the space of tangential traces.

Moreover, for every $\mathbf{u}\in\mathscr{D}(\bar{\Omega})^3$ and $\varphi\in H^2(\Omega)$ one has

$$-\int_\Gamma \mathbf{u}\wedge\mathbf{n}\cdot\nabla_\Gamma\varphi\,\mathrm{d}\sigma = \int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\nabla\varphi\,\mathrm{d}\Omega = \int_\Gamma \mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}\varphi\,\mathrm{d}\sigma$$

Using the definition of div_Γ given above, and the density of $H^2(\Omega)_{|\Gamma}$ in $H^{1/2}(\Gamma)$, one reaches

$$\langle \mathrm{div}_\Gamma(\mathbf{u}\wedge\mathbf{n}),\varphi\rangle_{1/2,\Gamma} = \langle \mathbf{curl}\,\mathbf{u}\cdot\mathbf{n},\varphi\rangle_{1/2,\Gamma} \quad \forall\mathbf{u}\in\mathscr{D}(\bar{\Omega})^3, \quad \varphi\in H^{1/2}(\Gamma)$$

By density of $\mathscr{D}(\bar{\Omega})^3$ in $\mathbf{H}(\mathbf{curl},\Omega)$, we have

$$\mathrm{div}_\Gamma(\mathbf{u}\wedge\mathbf{n}) = \mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}, \quad \forall\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega)$$

We pass now to the characterization of the range of the mapping π_τ in the context of $\mathbf{H}(\mathbf{curl},\Omega)$ vector fields. Let $\mathrm{curl}_\Gamma : \mathbf{H}_\perp^{-1/2}(\Gamma)\to H^{-3/2}(\Gamma)$ be the adjoint operator of the vector $\mathbf{curl}_\Gamma$ defined in (18). Let us set

$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) := \{\boldsymbol{\lambda}\in\mathbf{H}_\perp^{-1/2}(\Gamma)\colon \mathrm{curl}_\Gamma(\boldsymbol{\lambda})\in H^{-1/2}(\Gamma)\}.$$

Then, the corresponding trace theorem holds:

Theorem 3.10. The mapping π_τ: $\mathbf{H}(\mathbf{curl},\Omega)\to\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ is linear and continuous.

2.3. *On a part of the boundary*

The aim of this subsection is now to extend the results stated in the previous subsections to the case of spaces and trace mappings defined on a part of and no more on the whole boundary.

Let $\bar{\Gamma}_+$ be a collection of closed faces of Γ such that $\bar{\Gamma}_+$ is connected. Γ_+ is then an open subset of Γ with a piecewise regular boundary $\partial\Gamma_+$. Let $\Gamma_- = \Gamma\backslash\bar{\Gamma}_+$ and $\mathscr{I}_+$ be the set of indices j such that $\Gamma_j\subseteq\Gamma_+$. We denote by $\partial\mathscr{I}_+\subseteq\mathscr{I}_+$ the set of indices j corresponding to the faces which share at least one edge with $\partial\Gamma_+$. Finally, we denote by $\boldsymbol{\tau}_+$ the unit tangent vector to $\partial\Gamma_+$, $\boldsymbol{\tau}_{+j} = \boldsymbol{\tau}_{|\bar{\Gamma}_j\cap\partial\Gamma_+}$, and by $\mathbf{v}_+$ the outward normal vector defined as $\mathbf{v}_{|\bar{\Gamma}_j\cap\partial\Gamma_+} = \mathbf{v}_{+j}$ for any $j\in\partial\mathscr{I}_+$, with $\mathbf{v}_{+j} = \boldsymbol{\tau}_{+j}\wedge\mathbf{n}_j$ (the orientation of $\boldsymbol{\tau}_{+j}$ is such that $\mathbf{v}_{+j}$ is an outward normal).

Let

$$H_{00}^{1/2}(\Gamma_+) := \{\varphi\in H^{1/2}(\Gamma_+)\colon \tilde{\varphi}\in H^{1/2}(\Gamma)\}$$

Above, $\tilde{\varphi}$ is the prolongation by zero to Γ. This space is endowed with its usual norm (cf. [16] or [18]) and we denote by $H_{00}^{-1/2}(\Gamma_+)$ its dual space with $L^2(\Gamma_+)$ as pivot space.

We set

$$\mathbf{H}_\|^{1/2}(\Gamma_+) := \{\mathbf{u}_{|\Gamma_+},\ \mathbf{u}\in\mathbf{H}_\|^{1/2}(\Gamma)\} \tag{21}$$

$$\mathbf{H}_{\|,00}^{1/2}(\Gamma_+) := \{\mathbf{u}\in\mathbf{H}_\|^{1/2}(\Gamma_+)\colon \tilde{\mathbf{u}}\in\mathbf{H}_\|^{1/2}(\Gamma)\} \tag{24}$$

$$H^{3/2}(\Gamma_+) := H^{3/2}(\Gamma)_{|\Gamma_+}, \quad H_0^{3/2}(\Gamma_+) = H^{3/2}(\Gamma_+)\cap H_0^1(\Gamma_+) \tag{25}$$

Proposition 3.11. The spaces defined in (23) and (24) are Hilbert spaces endowed with the norms

$$\|\boldsymbol{\psi}\|^2_{1/2,\|,\Gamma_+} := \sum_{j\in\mathcal{I}_+} \|\boldsymbol{\psi}\|^2_{1/2,\Gamma_j} + \sum_{j\in\mathcal{I}_+} \sum_{i\in\mathcal{I}_+\cap\mathcal{I}_j} \mathcal{N}^{\|}_{ij}(\boldsymbol{\psi})) \tag{26}$$

$$\|\boldsymbol{\psi}\|^2_{1/2,\|,00,\Gamma_+} := \|\boldsymbol{\psi}\|^2_{1/2,\,\|,\,\Gamma_+} + \int_{\Gamma_+} \frac{|\boldsymbol{\psi}(\mathbf{x})\cdot\boldsymbol{\tau}_+|^2}{d(\mathbf{x},\partial\Gamma_+)}\,\mathrm{d}\sigma \tag{27}$$

Finally, $H^{3/2}(\Gamma_+)$ is closed when endowed with the norm:

$$\|u\|^2_{3/2,\Gamma_+} := \sum_{j\in\mathcal{I}_+} \|u\|^2_{3/2,\Gamma_j} + \|u\|^2_{1,\Gamma_+} \tag{28}$$

and $H^{3/2}_0(\Gamma_+)$ is a closed subspace of $H^{3/2}(\Gamma_+)$. Moreover, for any $\varphi\in H^{3/2}_0(\Gamma_+)$, $\tilde{\varphi}\in H^{3/2}(\Gamma)$.

Proof. The first statement is a straightforward recalling of definition (10) and using standard results for the space $H^{1/2}_{00}(\Gamma)$. For what concerns the space $H^{3/2}(\Gamma_+)$, it is enough to realize that Corollary 3.7 states precisely that the trace of a function in $H^2(\Omega)$ belongs to $H^{3/2}(\Gamma_j)$ for any j and it is globally $H^1(\Gamma)$. The space $H^{3/2}(\Gamma_+)$ is then closed with respect to norm (28). The last statement comes again using Theorem 3.4 and reminding that Γ_+ is a collection of faces, i.e., in the space $H^{3/2}(\Gamma)$ there is no compatibility conditions on the normal derivatives through the edges. □

Remark 3.12. Let $H^{3/2}_{00}(\Gamma_+) := \{v\in H^{3/2}_0(\Gamma_+)\colon \tilde{v}\in H^{3/2}(\Gamma)\}$. Here, one has $H^{3/2}_{00}(\Gamma_+) := H^{3/2}_0(\Gamma_+)$. In the general case, when Γ_+ is not a collection of faces, $H^{3/2}_{00}(\Gamma_+)$ is a proper subspace of $H^{3/2}_0(\Gamma_+)$. However, the theory can be easily extended to this case.

We denote by $\mathbf{H}^{-1/2}_{\|}(\Gamma_+)$, $\mathbf{H}^{-1/2}_{\|,00}(\Gamma_+)$ and $H^{-3/2}(\Gamma_+)$ the dual spaces (the last one is the dual space of $H^{3/2}_0(\Gamma_+)$). The duality products read $\langle\cdot,\cdot\rangle_{\|,1/2,\Gamma_+}$, $\langle\cdot,\cdot\rangle_{\|,1/2,00,\Gamma_+}$ and $\langle\cdot,\cdot\rangle_{3/2,\Gamma_+}$, respectively.

We are now in the position to define the differential operators we do need for our theory:

Proposition 3.13. The operators

$$\nabla_{\Gamma_+}: H^{3/2}(\Gamma_+) \to \mathbf{H}^{1/2}_{\|}(\Gamma_+) \quad \nabla_{\Gamma_+}: H^{3/2}_0(\Gamma_+) \to \mathbf{H}^{1/2}_{\|,00}(\Gamma_+) \tag{29}$$

are linear and continuous.

The operator $\mathrm{div}_{\Gamma_+}: \mathbf{H}^{-1/2}_{\|,00}(\Gamma_+) \to H^{-3/2}(\Gamma_+)$ can be defined as the adjoint of $-\nabla_{\Gamma_+}$ by the following:

$$\langle \mathrm{div}_{\Gamma_+}\boldsymbol{\lambda}, v\rangle_{3/2,\Gamma_+} = -\langle\boldsymbol{\lambda}, \nabla_{\Gamma_+} v\rangle_{\|,1/2,00,\Gamma_+} \boldsymbol{\lambda}\in\mathbf{H}^{-1/2}_{\|,00}(\Gamma_+),\ v\in H^{3/2}_0(\Gamma_+) \tag{30}$$

Using (30), Definition 3.3 and recalling that for any $\varphi\in H^{3/2}_0(\Gamma_+)$, $\tilde{\varphi}\in H^{3/2}(\Gamma)$, we have that

$$\forall\boldsymbol{\lambda}\in\mathbf{H}^{-1/2}_{\|}(\Gamma),\quad \mathrm{div}_\Gamma(\boldsymbol{\lambda})_{|\Gamma_+} = \mathrm{div}_{\Gamma_+}(\boldsymbol{\lambda}_{|\Gamma_+}) \quad \text{in } H^{-3/2}(\Gamma_+)$$

In the same way and with self-explanatory notations, we can define the operators $\mathbf{curl}_{\Gamma_+}: H^{3/2}(\Gamma_+) \to \mathbf{H}^{1/2}_{\perp}(\Gamma_+)$, $\mathbf{curl}_{\Gamma_+}: H^{3/2}_0(\Gamma_+) \to H^{1/2}_{\perp,00}(\Gamma_+)$ and its adjoint curl_{Γ_+}: $\mathbf{H}^{-1/2}_{\perp,00}(\Gamma_+) \to H^{-3/2}(\Gamma_+)$. Accordingly, it holds

$$\forall\boldsymbol{\lambda}\in\mathbf{H}^{-1/2}_{\perp}(\Gamma),\quad \mathrm{curl}_\Gamma(\boldsymbol{\lambda})_{|\Gamma_+} = \mathrm{curl}_{\Gamma_+}(\boldsymbol{\lambda}_{|\Gamma_+}) \quad \text{in } H^{-3/2}(\Gamma_+)$$

The following trace theorem can now be stated:

Proposition 3.14. Let

$$\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_+}, \Gamma_+) := \{\boldsymbol{\lambda} \in \mathbf{H}_{\parallel}^{-1/2}(\Gamma_+)\colon \mathrm{div}_{\Gamma_+} \boldsymbol{\lambda} \in H^{-1/2}(\Gamma_+)\} \tag{31}$$

$$\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_{\Gamma_+}, \Gamma_+) := \{\boldsymbol{\lambda} \in \mathbf{H}_{\perp}^{-1/2}(\Gamma_+)\colon \mathrm{curl}_{\Gamma_+} \boldsymbol{\lambda} \in H^{-1/2}(\Gamma_+)\} \tag{32}$$

Moreover, let $H^1(\partial\Gamma_+)$ denote the standard H^1 space on the Lipschitz manifold $\partial\Gamma_+$; $H^{-1}(\partial\Gamma_+)$ is its dual space (with $L^2(\partial\Gamma_+)$ as pivot) and $\langle \cdot, \cdot \rangle_{1,\partial\Gamma_+}$ is the corresponding duality pairing.

Let the operators t_ν: $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_+}, \Gamma_+) \cap \mathscr{D}(\bar{\Omega})_{|\Gamma_+} \to H^{-1}(\partial\Gamma_+)$ and t_τ: $\mathbf{H}_{\perp}^{-1/2}$ $(\mathrm{curl}_{\Gamma_+}, \Gamma_+) \cap \mathscr{D}(\bar{\Omega})_{|\Gamma_+} \to H^{-1}(\partial\Gamma_+)$ be defined by the mappings $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda} \cdot \nu_{+|\partial\Gamma_+}$ and $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda} \cdot \boldsymbol{\tau}_{+|\partial\Gamma_+}$, respectively. They can be extended to linear and continuous operators from $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_+}, \Gamma_+)$ and $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_{\Gamma_+}, \Gamma_+)$, respectively.

Moreover, if $\boldsymbol{\lambda} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, we have that $t_\nu(\boldsymbol{\lambda}_{|\Gamma_+}) + t_\nu^-(\boldsymbol{\lambda}_{|\Gamma_+}) = 0$ at $\partial\Gamma_+$, where t_ν^- denotes the same mapping as t_ν, but on the side Γ_-.

Proof. We focus our attention only on the operator t_ν since a similar proof, carried out by suitably replacing spaces and operators, works for t_τ. Let $\mathbf{u} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_+}, \Gamma_+) \cap \mathscr{D}(\bar{\Omega})_{|\Gamma_+}$ and $\psi \in H^1(\partial\Gamma_+)$. Its extension, which belongs to $H^{3/2}(\Gamma_+)$, is still denoted by ψ. The following chain holds true:

$$\begin{aligned}\langle \mathbf{u}, \nabla_{\Gamma_+} \psi \rangle_{\parallel,1/2,\Gamma_+} + \langle \mathrm{div}_{\Gamma_+} \mathbf{u}, \psi \rangle_{1/2,\Gamma_+} &= \int_{\Gamma_+} (\mathbf{u} \cdot \nabla_{\Gamma_+} \psi + \mathrm{div}_{\Gamma_+} \mathbf{u}\, \psi)\, \mathrm{d}\sigma \\ &= \langle t_\nu \mathbf{u}, \psi \rangle_{1,\partial\Gamma_+}\end{aligned} \tag{33}$$

By density, since the left-hand side of (33) is bounded with respect to the norms of ψ in $H^{3/2}(\Gamma_+)$ and of $\mathbf{u}$ in $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_+}, \Gamma_+)$, the continuity of t_ν is proved.

The last statement is nothing but a jump relation and its validity comes directly from (33). □

Let us now set

$$\mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega) := \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)\colon \mathbf{u} \wedge \mathbf{n}_{|\Gamma_-} = 0 \text{ in } \mathbf{H}_{00}^{-1/2}(\Gamma)\}$$

Theorem 3.15. Let us set

$$\mathbf{H}_{\parallel,00}^{-1/2}(\mathrm{div}_{\Gamma_+}, \Gamma_+) := \{\boldsymbol{\lambda} \in \mathbf{H}_{\parallel,00}^{-1/2}(\Gamma_+)\colon \mathrm{div}_{\Gamma_+} \boldsymbol{\lambda} \in H_{00}^{-1/2}(\Gamma_+)\}$$

$$\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_+}^0, \Gamma_+) := \{\boldsymbol{\lambda} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_+}, \Gamma_+)\colon t_\nu(\boldsymbol{\lambda}) = 0\}$$

The mapping γ_τ^+: $\mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\parallel,00}^{-1/2}(\mathrm{div}_{\Gamma_+}, \Gamma_+)$ and its restriction $\gamma_\tau^{+,0}\colon \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_+}^0, \Gamma_+)$ defined as $\mathbf{u} \mapsto \gamma_\tau(\mathbf{u})_{|\Gamma_+}$ are linear and continuous.

Proof. Let $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$. The fact that $\gamma_\tau^+(\mathbf{u}) \in \mathbf{H}_{\parallel,00}^{-1/2}(\Gamma_+)$ and $\gamma_\tau^{+,0}(\mathbf{v}) \in \mathbf{H}_{\parallel}^{-1/2}(\Gamma_+)$ is straightforward from Theorem 3.9 and the standard results in functional analysis. We have only to analyse the divergence operators. Applying the same reasoning as in the proof of Theorem 3.9, and recalling definition (30) of div_{Γ_+}, one can prove

$$\mathrm{div}_{\Gamma_+}(\gamma_\tau^+(\mathbf{u})) = \mathbf{curl}\, \mathbf{u} \cdot \mathbf{n}_{|\Gamma_+}$$

Now, by a standard argument, for any $\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega)$, $\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}_{|\Gamma_+}\in H_{00}^{-1/2}(\Gamma_+)$. Moreover, if $\mathbf{u}\in\mathbf{H}_{0,\Gamma_-}(\mathbf{curl},\Omega)$, $\gamma_\tau^+(\mathbf{u})$ is replaced by $\gamma_\tau^{+,0}(\mathbf{u})$ and we have that

$$\mathrm{div}_\Gamma(\widetilde{\gamma_\tau^{+,0}}(\mathbf{u})) = \mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}_{|\Gamma} \text{ in } H^{-1/2}(\Gamma) \text{ and}$$

$$\mathrm{div}_{\Gamma_+}(\gamma_{\tau_+}^{+,0}(\mathbf{u})) = \mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}_{|\Gamma_+} \text{ in } H_{00}^{-1/2}(\Gamma_+) \tag{34}$$

where $\tilde{}$ denotes the trivial extension. Now, $\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}_{|\Gamma_+}\in H^{-1/2}(\Gamma_+)$ since $\mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}_{|\Gamma_-}=0$ and we deduce that $\mathrm{div}_{\Gamma_+}(\gamma_\tau^{+,0}(\mathbf{u}))\in H^{-1/2}(\Gamma_+)$. Using now the last statement of Proposition 3.14, with the first equality in (34), we deduce that $t_v(\gamma_\tau^{+,0}(\mathbf{u}))=0$. □

The corresponding theorem related to the tangential components trace mapping is the following:

Theorem 3.16. Let us set

$$\mathbf{H}_{\perp,00}^{-1/2}(\mathrm{curl}_{\Gamma_+},\Gamma_+) := \{\boldsymbol{\lambda}\in\mathbf{H}_{\perp,00}^{-1/2}(\Gamma_+)\colon \mathrm{curl}_{\Gamma_+}\boldsymbol{\lambda}\in H_{00}^{-1/2}(\Gamma_+)\}$$

$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_{\Gamma_+}^0,\Gamma_+) := \{\boldsymbol{\lambda}\in\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_{\Gamma_+},\Gamma_+)\colon t_\tau(\boldsymbol{\lambda})=0\}$$

The mappings $\pi_\tau^+\colon \mathbf{H}(\mathbf{curl},\Omega)\to\mathbf{H}_{\perp,00}^{-1/2}(\mathrm{curl}_{\Gamma_+},\Gamma_+)$ and its restriction $\pi_\tau^{+,0}\colon\mathbf{H}_{0,\Gamma_+}(\mathbf{curl},\Omega)\to\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_{\Gamma_+}^0,\Gamma_+)$ defined as $\mathbf{u}\mapsto\pi_\tau(\mathbf{u})_{|\Gamma_+}$ are linear and continuous.

4. INTEGRATION BY PARTS FORMULAE

4.1. General case

First of all, let us set

$$\mathbf{H}_0(\mathrm{div},\Omega) := \{\mathbf{u}\in\mathbf{L}^2(\Omega)\colon \mathrm{div}\,\mathbf{u}\in L^2(\Omega) \quad \mathbf{u}\cdot\mathbf{n}_{|\Gamma}=0\}$$

$$\mathbf{H}_0(\mathrm{div}\,0,\Omega) := \{\mathbf{u}\in\mathbf{H}_0(\mathrm{div},\Omega)\colon \mathrm{div}\,\mathbf{u}=0\}$$

The following Hodge decomposition holds for every function $\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega)$ [2, Section 3.5]:

$$\mathbf{u} = \Phi + \nabla p \quad \text{with } \Phi\in\mathbf{H}_0(\mathrm{div}\,0,\Omega) \quad \text{and} \quad \mathbf{curl}\,\Phi = \mathbf{curl}\,\mathbf{u} \tag{35}$$

where p is uniquely defined up to a constant as the solution of the following variational problem: Find $p\in H^1(\Omega)/\mathbb{R}$ such that

$$\int_\Omega \nabla p\cdot\nabla q\,\mathrm{d}\Omega = \int_\Omega \mathbf{u}\cdot\nabla q\,\mathrm{d}\Omega \quad \forall q\in H^1(\Omega)/\mathbb{R}$$

By definition, Φ belongs to $\mathbf{Y}=\mathbf{H}(\mathbf{curl},\Omega)\cap\mathbf{H}_0(\mathrm{div}\,0,\Omega)$ and we use an *ad hoc* regularity result. Owing to the splitting of $\mathbf{Y}$ (see Remark 4.2) and a regularity result drawn from [11, Corollary 23.5], if Ω is a curvilinear bounded polyhedron, one has $\mathbf{Y}\hookrightarrow\mathbf{H}^{1/2+\sigma}(\Omega)$, for some $\sigma>0$. σ is the maximum exponent for the regularity of the Neumann problem associated with the Laplace equation in Ω.

Let us start by considering two vector fields, $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that their decompositions (35) read

$$\mathbf{u} = \Phi + \nabla p, \quad \mathbf{v} = \Psi + \nabla q \quad \text{with } \Phi, \Psi \in \mathbf{Y}, \ p, q \in H^2(\Omega)$$

At the end, we shall use a density argument to get rid of the requested extra-regularity. For these $H^{1/2+\sigma}$-regular vector fields, there clearly holds

$$\int_\Omega \{\mathbf{curl}\,\mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{curl}\,\mathbf{u}\} \, d\Omega = \langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{v}) \rangle_t \tag{36}$$

Now, analysing the term in the right-hand side of (36), using formulae (17) and (18), one has

$$\langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{v}) \rangle_t = \langle \gamma_\tau(\Phi), \pi_\tau(\Psi) \rangle_t + \langle \gamma_\tau(\Phi), \nabla_\Gamma q \rangle_{\parallel, 1/2, \Gamma} + \langle \pi_\tau(\Psi), \mathbf{curl}_\Gamma p \rangle_{\perp, 1/2, \Gamma} \tag{37}$$

From (37), we deduce

$$\int_\Omega \{\mathbf{curl}\,\mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{curl}\,\mathbf{u}\} \, d\Omega$$
$$= \langle \gamma_\tau(\Phi), \pi_\tau(\Psi) \rangle_t - \langle \mathrm{div}_\Gamma(\gamma_\tau(\Phi)), q \rangle_{3/2, \Gamma} + \langle \mathrm{curl}_\Gamma(\pi_\tau(\Psi)), p \rangle_{3/2, \Gamma} \tag{38}$$

Applying now Theorems 3.9 and 3.10 to the functions Φ and Ψ, respectively, one gets that both $\mathrm{div}_\Gamma(\gamma_\tau(\Phi))$ and $\mathrm{curl}_\Gamma(\pi_\tau(\Psi))$ belong to $H^{-1/2}(\Gamma)$. This means that the $H^{-3/2} - H^{3/2}$ dualities can be replaced by the more convenient $H^{-1/2} - H^{1/2}$ ones. Namely, (38) reads as follows:

$$\int_\Omega \{\mathbf{curl}\,\mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{curl}\,\mathbf{u}\} \, d\Omega$$
$$= \langle \gamma_\tau(\Phi), \pi_\tau(\Psi) \rangle_t - \langle \mathrm{div}_\Gamma(\gamma_\tau(\Phi)), q \rangle_{1/2, \Gamma} + \langle \mathrm{curl}_\Gamma(\pi_\tau(\Psi)), p \rangle_{1/2, \Gamma} \tag{39}$$

By the standard argument of the density of $H^2(\Omega)$ in $H^1(\Omega)$, the integration by parts formula holds for any fields $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$.

It is then possible to define the following duality for any $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$:

$${}_\gamma\langle \gamma_\tau(\mathbf{u}), \pi_\tau(\mathbf{v}) \rangle_\pi = \langle \gamma_\tau(\Phi), \pi_\tau(\Psi) \rangle_t - \langle \mathrm{div}_\Gamma(\gamma_\tau(\Phi)), q \rangle_{1/2, \Gamma} + \langle \mathrm{curl}_\Gamma(\pi_\tau(\Psi)), p \rangle_{1/2, \Gamma} \tag{40}$$

The three dualities on the right-hand side are well defined and continuous with respect to the $\mathbf{H}(\mathbf{curl}, \Omega)$ norms.

Remark 4.1. The duality

$$(\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma))' = \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$$

where $\mathbf{L}_t^2(\Gamma)$ is the pivot space is proven in the companion paper [8].

Remark 4.2. The definition of duality (40) between the tangential trace and tangential components of vectors fields which belong to $\mathbf{H}(\mathbf{curl}, \Omega)$ relies on decomposition (35). Let us mention that the orthogonality of this decomposition does not play a key role in the definition of (40).

Let us consider, for example, the decomposition in 'regular' and 'singular' part inferred from [6]. Starting from (35), Φ belongs to the space $\mathbf{Y}$. According to Proposition 6.1 of [6], in the general case of a curvilinear Lipschitz polyhedron,[§] if N_n is the orthogonal of $\Delta(\{\mu\in H^2(\Omega); \partial\mu/\partial n_{|\Gamma}=0\})$ in $L^2(\Omega)$ (cf. [11, p. 198] and $\mathscr{S}_n := \{\lambda\in H^1(\Omega)\colon \Delta\lambda\in N_n,\ \partial\lambda/\partial n_{|\Gamma}=0\}$, one has $\mathbf{Y}=(\mathbf{Y}\cap\mathbf{H}^1(\Omega))\oplus\nabla\mathscr{S}_n$. Therefore,

$$\exists\Phi_R\in\mathbf{Y}\cap\mathbf{H}^1(\Omega),\quad \exists s\in\mathscr{S}_n\quad \text{such that } \Phi=\Phi_R+\nabla s$$

Thus, if $p' := p+s$, one can write

$$\forall\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega),\quad \exists\Phi_R\in\mathbf{Y}\cap\mathbf{H}^1(\Omega),\quad p'\in H^1(\Omega)\ \text{such that } \mathbf{u}=\Phi_R+\nabla p' \tag{41}$$

Let then $\mathbf{u}=\Phi_R+\nabla p'$ and $\mathbf{v}=\Psi_R+\nabla q'$ with $\Phi_R, \Psi_R\in\mathbf{Y}\cap\mathbf{H}^1(\Omega)$ and $p', q'\in H^1(\Omega)$, duality (40) can be equivalently defined as

$${}_\gamma\langle\gamma_\tau(\mathbf{u}),\pi_\tau(\mathbf{v})\rangle_\pi=\langle\pi_\tau(\Psi_R),\gamma_\tau(\Phi_R)\rangle_t-\langle\mathrm{div}_\Gamma(\gamma_\tau(\Phi_R)),q'\rangle_{1/2,\Gamma}+\langle\mathrm{curl}_\Gamma(\pi_\tau(\Psi_R)),p'\rangle_{1/2,\Gamma}$$

4.2. Partially vanishing fields

Let us split the boundary Γ in Γ_+ an open arbitrary connected subset with a piecewise smooth boundary, and $\Gamma_-=\Gamma\backslash\bar{\Gamma}_+$. Let

$$H^1_{0,\Gamma_-}(\Omega) := \{\varphi\in H^1(\Omega)\colon \varphi_{|\Gamma_-}=0\}$$

$$\mathbf{H}_{0,\Gamma_+}(\mathrm{div},\Omega) := \{u\in\mathbf{H}(\mathrm{div},\Omega)\colon \mathbf{u}\cdot\mathbf{n}_{|\Gamma_+}=0 \text{ in } H_{00}^{-1/2}(\Gamma_+)\}$$

$$\mathbf{H}_{0,\Gamma_+}(\mathrm{div}\,0,\Omega) := \mathbf{H}(\mathrm{div}\,0,\Omega)\cap\mathbf{H}_{0,\Gamma_+}(\mathrm{div},\Omega)$$

Then, the following Green formulae associated with the spaces $\mathbf{H}(\mathrm{div},\Omega)$ and $H^1(\Omega)$ are valid

$$\int_\Omega\{\mathrm{div}\,\mathbf{u}\,\varphi+\mathbf{u}\cdot\nabla\varphi\}\,\mathrm{d}\Omega=\langle\mathbf{u}\cdot\mathbf{n},\varphi\rangle_{1/2,00,\Gamma_+}\quad \forall\mathbf{u}\in\mathbf{H}(\mathrm{div},\Omega),\ \varphi\in H^1_{0,\Gamma_-}(\Omega)$$

$$\int_\Omega\{\mathrm{div}\,\mathbf{u}\,\varphi+\mathbf{u}\cdot\nabla\varphi\}\,\mathrm{d}\Omega=\langle\mathbf{u}\cdot\mathbf{n},\varphi\rangle_{1/2,\Gamma_-}\quad \forall\mathbf{u}\in\mathbf{H}_{0,\Gamma_+}(\mathrm{div},\Omega),\ \varphi\in H^1(\Omega)$$

By analogy, we look for the same integration by parts formulae which generalize the Green formula (39).

As in the previous subsection, we need to introduce Hodge decompositions of the vector fields involved. Let $\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{v}\in\mathbf{H}_{0,\Gamma_-}(\mathbf{curl},\Omega)$. Regarding the field $\mathbf{u}$, we consider the decomposition given in (41). For what concerns now the vector field $\mathbf{v}$ we use another *ad hoc* Hodge decomposition. Namely, let $\mathbf{Y}_m=\mathbf{H}_{0,\Gamma_-}(\mathrm{curl},\Omega)\cap\mathbf{H}_{0,\Gamma_+}(\mathrm{div}\,0,\Omega)$. Recall that the following decomposition holds (see [13, p. 968]): $\mathbf{L}^2(\Omega)=\nabla H^1_{0,\Gamma_-}(\Omega)\oplus\mathbf{H}_{0,\Gamma_+}(\mathrm{div}\,0,\Omega)$. Therefore,

$$\forall\mathbf{v}\in\mathbf{H}_{0,\Gamma_-}(\mathbf{curl},\Omega),\quad \exists\Psi\in\mathbf{Y}_m,\quad q\in H^1_{0,\Gamma_-}(\Omega)\ \text{such that } \mathbf{v}=\Psi+\nabla q \tag{42}$$

We want now to decompose again the function Ψ as

$$\Psi=\Psi_R+\nabla q',\quad \Psi_R\in\mathbf{H}^1(\Omega)\cap\mathbf{H}_{0,\Gamma_-}(\mathbf{curl},\Omega),\quad p'\in H^1_{0,\Gamma_-}(\Omega)$$

It is well known that there exist a $\zeta\in\mathbf{H}^1(\Omega)$ and a $p\in H^1(\Omega)$ such that $\Psi=\zeta+\nabla p$. Since $\Psi\wedge\mathbf{n}=0$ on Γ_-, we deduce that $\zeta\wedge\mathbf{n}_{|\Gamma_-}=-(\mathbf{curl}_\Gamma p)_{|\Gamma_-}$. As a consequence, $(\mathbf{curl}_\Gamma p)_{|\Gamma_-}\in\mathbf{H}^{1/2}_\perp(\Gamma_-)$

[§] One simply has to resume the proof given in [6] for a Lipschitz polyhedron, and use in particular the theory developed in [5] which is valid in the curvilinear case.

and, by means of Theorem 3.4, we deduce that actually $p \in H^{3/2}(\Gamma_-)$. It is not hard to see that there exists a function $p_R \in H^2(\Omega)$ such that $p_{R|\Gamma_-} = p$. Now, it is enough to take $\Psi_R = \zeta + \nabla p_R$ and $p' = p - p_R$.

As a consequence, we have that $\mathbf{v} \in \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega)$ can be decomposed as $\mathbf{v} = \Psi_R + \nabla q'$, $\Psi_R \in \mathbf{H}^1(\Omega) \cap \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega)$ and $q' \in H^1_{0,\Gamma_-}(\Omega)$. Of course, this decomposition is not orthogonal nor direct but it is the one we need in what follows.

Let us concentrate now on the integration by parts formula and, as in the case of the whole boundary, let us first consider some regularized functions before the application, at the end, of a suitable density argument.

Let $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{v} \in \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega)$ be two functions such that the following 'regularized' decompositions hold:

$$\begin{aligned} \mathbf{u} &= \Phi_R + \nabla p', \quad \Phi_R \in \mathbf{Y} \cap \mathbf{H}^1(\Omega), \quad p' \in H^2(\Omega) \\ \mathbf{v} &= \Psi_R + \nabla q', \quad \Psi_R \in \mathbf{H}^1(\Omega) \cap \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega), \quad q' \in H^2(\Omega) \cap H^1_{0,\Gamma_-}(\Omega) \end{aligned} \tag{43}$$

Let us denote $\langle \cdot, \cdot \rangle_{t,\Gamma_+}$ the scalar product in $\mathbf{L}^2_t(\Gamma_+)$. Then one has

$$\int_\Omega \{\mathbf{curl}\, \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{curl}\, \mathbf{u}\} \, \mathrm{d}\Omega = \langle \gamma_\tau^+(\mathbf{u}), \pi_\tau^{+,0}(\mathbf{v}) \rangle_{t,\Gamma_+} \tag{44}$$

Now, analysing the term in the right-hand side of (44), using decomposition (43), similarly as in the case of the whole boundary, one has

$$\begin{aligned} \langle \gamma_\tau^+(\mathbf{u}), \pi_\tau^{+,0}(\mathbf{v}) \rangle_{t,\Gamma_+} &= \langle \gamma_\tau^+(\Phi_R), \pi_\tau^{+,0}(\Psi_R) \rangle_{t,\Gamma_+} \\ &\quad + \langle \gamma_\tau^+(\Phi_R), \nabla_{\Gamma_+} q' \rangle_{\|,1/2,00,\Gamma_+} + \langle \pi_\tau^{+,0}(\Psi_R), \mathbf{curl}_{\Gamma_+} p' \rangle_{\perp,1/2,\Gamma_+} \end{aligned}$$

In particular, passing now to the adjoint operators in the last two duality pairings, using Theorems 3.15 and 3.16 for $\gamma_\tau^+(\Phi_R)$ and $\pi_\tau^{+,0}(\Psi_R)$, and by standard manipulations, the following duality can be defined:

$$\begin{aligned} {}_{\gamma,00,\Gamma_+}\langle \gamma_\tau^+(\mathbf{u}), \pi_\tau^{+,0}(\mathbf{v}) \rangle_{\pi,\Gamma_+} &:= \langle \gamma_\tau^+(\Phi_R), \pi_\tau^{+,0}(\Psi_R) \rangle_{t,\Gamma_+} \\ &- \langle \mathrm{div}_{\Gamma_+}(\gamma_\tau^+(\Phi_R)), q' \rangle_{1/2,00,\Gamma_+} + \langle \mathrm{curl}_{\Gamma_+}(\pi_\tau^{+,0}(\Psi_R)), p' \rangle_{1/2,\Gamma_+} \end{aligned} \tag{45}$$

Of course, the following integration by parts holds $\forall \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$, $\mathbf{v} \in \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega)$:

$$\int_\Omega \{\mathbf{curl}\, \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{curl}\, \mathbf{u}\} \, \mathrm{d}\Omega = {}_{\gamma,00,\Gamma_+}\langle \gamma_\tau^+(\mathbf{u}), \pi_\tau^{+,0}(\mathbf{v}) \rangle_{\pi,\Gamma_+} \tag{46}$$

Remark 4.3. As in the previous subsection, we would like to define the duality also by means of the orthogonal, standard Hodge decomposition. Let again $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$, $\mathbf{v} \in \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega)$. $\mathbf{u}$ is decomposed by means of (35) and $\mathbf{v}$ by means of (42).

Unfortunately, the definition of duality (45) by means of the above decompositions can be given in the case of regular domains but not in the case of a (curvilinear) Lipschitz polyhedron. In the latter case, the regularity results associated with the space $\mathbf{Y}_m$ is only $\mathbf{Y}_m \hookrightarrow \mathbf{H}^{1/4+\eta}(\Omega)$ for some positive η (see [13, 11]). In particular, this means that the $\mathbf{L}^2_t(\Gamma_+)$ scalar product between $\gamma_\tau^+(\Phi)$ and $\pi_\tau^{+,0}(\Psi)$—or any other duality product—is not allowed anymore.

However, a 'mixed' formulation involving (41) and (42) still works.

ACKNOWLEDGEMENTS

The authors are pleased to thank Prof. G. Savarè for some helpful discussions on Sobolev and Lorentz spaces. This work has been partially supported by I.A.N.-C.N.R. Pavia, Italy.

REFERENCES

1. Alonso A, Valli A. Some remarks on the characterization of the space of tangential traces of $\mathbf{H}(rot;\Omega)$ and the construction of an extension operator. *Manuscr. Math.* 1996; **89**: 159–178.
2. Amrouche C, Bernardi C, Dauge M, Girault V. Vector potentials in three-dimensional non-smooth domains. *Math. Meth. in the Appl. Sci.* 1998; **21**: 823–864.
3. Assous F, Ciarlet P Jr., Sonnendrücker E, Raviart PA. A characterization of the singular part of the solution to Maxwell's equations in a polyhedral domain. *Math. Meth. in the Appl. Sci.* 1999; **22**: 485–499.
4. Bergh J, Löfström J. *Interpolation Spaces. An introduction*. Springer: Berlin, 1976.
5. Birman MS, Solomyak MZ. Construction in a piecewise smooth domain of a function of the class H^2 from the value of the conormal derivative. *J. Math. Sov.* 1990; **49**: 1128–1136.
6. Bonnet-Ben Dhia AS, Hazard C, Lohrengel S. A singular field method for the solution of Maxwell's equations in polyhedral domains. *SIAM J. Appl. Math.* 1999; **59**: 2028–2044.
7. Brezis H. *Analyse fonctionnelle. Théorie et applications*, Masson: Paris, 1983.
8. Buffa A, Ciarlet Jr, P. On traces for functional spaces related to Maxwell's equations. Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications. *Math. Meth. in the Appl. Sci.*
9. Cessenat M. (1996), Mathematical Methods in Electromagnetism. Linear Theory and Applications. *Series of Advances in Mathematics for Applied Sciences*, vol. 41. World Scientific: Singapore.
10. Costabel M, Dauge M. Maxwell and Lamé eigenvalues on polyhedral domains. *Math. Meth. in the Appl. Sci.* 1999; **22**: 243–258.
11. Dauge M. Elliptic Boundary Value Problems on Corner Domains. Lecture Notes in Mathematics. Springer: Berlin, 1988.
12. De la Bourdonnaye A. Décomposition de $H_{div}^{-1/2}(\Gamma)$ et nature de l'opérateur de Steklov–Poincaré du problème extérieur de l'électromagnétisme. *C. R. Acad. Sci. Paris, Sér I* 1993; **316**: 369–372.
13. Fernandes P, Gilardi G. Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions. *Math. Models Meth. Appl. Sci.* 1997; **7**: 957–991.
14. Girault V, Raviart PA. Finite Element Methods for Navier–Stokes Equations. *Springer Series in Computational Mathematics*, Springer: Berlin, 1986.
15. Grisvard P. Elliptic Problems in Nonsmooth Domains. *Monographs and Studies in Mathematics*, vol. 24. Pitman: London, 1985.
16. Grisvard P. *Singularities in Boundary Value Problems*, RMA 22. Masson: Paris, 1992.
17. Grisvard P. Théorèmes de traces relatifs à un polyèdre. *C. R. Acad. Sci. Paris, Sér. A* 1974; **278**: 1581–1583.
18. Lions JL, Magenes E. *Problèmes aux Limites non Homogènes et Applications*. Dunod: Paris, 1968.
19. Lohrengel S. Etude mathématique et résolution numérique des équations de Maxwell dans un domaine non régulier. Thése de l'Université Paris VI, Paris, France, 1998.
20. Nec̆as J. *Les Méthodes Directes en Théorie des Équations Elliptiques*. Masson: Paris, 1967.
21. Sheen D. A generalized Green's theorem *Appl. Math. Lett.* 1992; **5**: 95–98.
22. Tartar L. On the characterization of traces of a Sobolev space used for Maxwell's equation. *Proceedings of a Meeting held in Bordeaux, in honour of Michel Artola*, November 1997.
23. Tartar L. Imbedding theorems of Sobolev spaces into Lorentz spaces. *Boll. UMI* 1998.
24. Terrasse I. Résolution mathématique et numérique des équations de Maxwell instationnaires par une méthode de potentiels retardés. Thése de l'Ecole Polytechnique, CMAP, 91128 Palaiseau Cedex, France, 1993.

