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On traces for functional spaces related to Maxwell's equations Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications

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SUMMARY

Hodge decompositions of tangential vector fields defined on piecewise regular manifolds are provided. The first step is the study of L^2 tangential fields and then the attention is focused on some particular Sobolev spaces of order $-\frac{1}{2}$. In order to reach this goal, it is required to properly define the first order differential operators and to investigate their properties. When the manifold Γ is the boundary of a polyhedron Ω, these spaces are important in the analysis of tangential trace mappings for vector fields in $\mathbf{H}(\mathbf{curl}, \Omega)$ on the whole boundary or on a part of it. By means of these Hodge decompositions, one can then provide a complete characterization of these trace mappings: general extension theorems, from the boundary, or from a part of it, to the inside; definition of suitable dualities and validity of integration by parts formulae. Copyright © 2001 John Wiley & Sons, Ltd.

1. INTRODUCTION

This paper is concerned with the characterization and the properties of tangential vector fields on non-smooth manifolds. In the case of regular manifolds the functional theory related to tangential fields has been completely developed. Sobolev spaces and tangential differential operators have been defined and analysed (e.g. in [1, 13, 4]). When the manifold Γ is not smooth, several problems occur: the definition of Sobolev spaces is possible only for low values of regularity exponent ($H^s(\Gamma)$, for $|s| \leqslant 1$), and the definition of differential operators, in this context, is far from being obvious.

In this paper, we consider Γ as the boundary of a Lipschitz polyhedron Ω not necessarily convex. We use the same approach as Grisvard [8, 9]: we suitably define Sobolev spaces and differential operators, working face by face, and then we analyse the matching conditions at edges and vertices. Our attention is devoted to the definition of $H^s(\Gamma)$, for $1 < |s| \leqslant \frac{3}{2}$, and of first-order

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differential operators, namely the gradient, the divergence, the scalar and vector curls. It is proven that all these operators verify properties which are necessary in order to develop a standard functional theory, for instance that every curl-free vector is a gradient and so on.

Using these tools and the same argument as in [6] for the regular case, we start by proving a first Hodge decomposition:

$$\mathbf{L}^2_t(\Gamma) = \nabla_\Gamma H^1(\Gamma) \overset{\perp}{\oplus} \mathbf{curl}_\Gamma H^1(\Gamma)$$

Then, taking advantage of the characterization of traces for $\mathbf{H}(\mathbf{curl}, \Omega)$, which has been given in the companion paper [3], we provide a Hodge decomposition (always as a gradient and a **curl** sum) of both the tangential trace ($\mathbf{u} \wedge \mathbf{n}$) and tangential components ($\mathbf{n} \wedge (\mathbf{u} \wedge \mathbf{n})$) for a vector field $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ ($\mathbf{n}$ is here the unit outer normal to Ω on Γ).

By means of this decomposition it is not hard to prove the surjectivity of both trace mappings (tangential trace and tangential components). Moreover, lifting mappings from the boundary, and from a part of it, to Ω, are provided, and we prove that the tangential trace and the tangential components can be put in duality as in the regular case (see e.g. [4]).

Finally, all our results can be extended with no additional efforts to curvilinear Lipschitz polyhedron, that is, to a class of piecewise regular manifold (cf. [5]), but not to the general case of Lipschitz manifolds. In this more general case, the characterization of traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ has been given by Tartar [12] and, from this paper, it is clear that the definition of differential operators on Lipschitz manifolds in the context of Sobolev spaces, is, in general, an 'ill-posed problem'.

The outline of the paper is the following. In Section 2, we make precise our notation and the functional framework, in Section 3, we provide the Hodge decomposition in the $\mathbf{L}^2_t(\Gamma)$ context; in Section 4 we introduce differential operators in more 'regular' and, respectively, less 'regular' spaces, and we study their properties. In Section 5, we obtain the Hodge decomposition in the spaces of traces of $\mathbf{H}(\mathbf{curl}, \Omega)$ and we analyse its direct consequences: existence of a continuous extension mapping, the duality between the space of the tangential trace and that of tangential components and the related integration by parts formula. Finally, in Section 6, we extend the same results to the case of an open manifold and we deduce the characterization of trace mappings on the part of the manifold, and we establish a suitable duality among spaces and the validity of the integration by parts formula.

2. PRELIMINARIES

Let Ω be a Lipschitz polyhedron not necessarily convex, Γ its boundary and $\mathbf{n}$ the unit outward normal to Ω on Γ. All along this paper, we assume that Γ is simply connected and connected. When Γ is formed of several connected components, all our statements will apply to each connected component. Moreover, Γ is split in N (open) faces $(\Gamma_j)_{j=1,\dots,N}$, $\Gamma = \bigcup_j \bar{\Gamma}_j$. Let us denote by $(e_{ij})_{i,j\in\{1,\dots,N\}}$ its (open) edges: when Γ_i and Γ_j are two adjacent faces, e_{ij} denotes the 'common' edge. Let $\mathscr{I}_j$ stand for the set of indices i such that the faces Γ_i and Γ_j have a common edge. Additionally, let Γ_{ij} be the open set $\Gamma_i \cup \Gamma_j \cup e_{ij}$. The vertices are denoted by $(S_k)_{k=1,\dots,K}$.

Finally, let $\boldsymbol{\tau}_{ij}$ be a unit vector parallel to e_{ij} and $\mathbf{n}_j = \mathbf{n}_{|\Gamma_j}$; $\boldsymbol{\tau}_i = \boldsymbol{\tau}_{ij} \wedge \mathbf{n}_i$. The couple $(\boldsymbol{\tau}_i, \boldsymbol{\tau}_{ij})$ is an orthonormal basis of the plane generated by Γ_i; $(\boldsymbol{\tau}_i, \boldsymbol{\tau}_{ij}, \mathbf{n}_i)$ is an orthonormal basis of $\mathbb{R}^3$.

For elements φ of $L^2(\Gamma)$, we adopt the notation $\varphi_j = \varphi_{|\Gamma_j}$. This notation is used whenever the restriction to a face is considered, that is as regards any functional space, in which the restriction to a face is allowed.

Boldface characters are used for all vector fields and some vector spaces, such as for instance $L^2(\Omega)^3$ which is denoted by $\mathbf{L}^2(\Omega)$.

On Γ, the Sobolev spaces $H^s(\Gamma)$ with $|s| < 1$ are standardly defined (cf. [11]). Let us set

$$\mathbf{H}(\mathbf{curl}, \Omega) := \{\mathbf{u} \in \mathbf{L}^2(\Omega)\colon \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega)\}, \quad \|\cdot\|_{0,\mathrm{curl}} \text{ the graph norm}$$

$$\mathbf{L}^2_t(\Gamma) := \{\boldsymbol{\varphi} \in \mathbf{L}^2(\Gamma)\colon \boldsymbol{\varphi} \cdot \mathbf{n}_{|\Gamma} = 0\}, \quad \langle \cdot, \cdot \rangle_t \text{ its scalar product}$$

$$H^1(\Gamma) := \{\varphi \in L^2(\Gamma)\colon \varphi_j \in H^1(\Gamma_j) \quad \text{and} \quad \varphi_{j|e_{ij}} = \varphi_{i|e_{ij}} \text{ in } H^{1/2}(e_{ij}),\ \forall i \in \mathscr{I}_j,\ \forall j\} \tag{1}$$

$$\mathbf{H}^{1/2}_{-}(\Gamma) := \{\boldsymbol{\lambda} \in \mathbf{L}^2_t(\Gamma)\colon \boldsymbol{\lambda}_j \in \mathbf{H}^{1/2}(\Gamma_j),\ \forall j\}$$

They are Hilbert spaces when endowed with the respective graph norms. The definition of $H^1(\Gamma)$ stems from the fact that H^1-regularity is preserved by (bi-) Lipschitz mappings (cf. [11]). Let $H^{-1}(\Gamma)$ be the dual space of $H^1(\Gamma)$.

Note that in the remainder of the paper, $\mathbf{L}^2_t(\Gamma)$ is identified with the space of *two-dimensional*, tangential, square integrable, vector fields. The consequence of this choice is that, *on the boundary* Γ, one deals with two-dimensional vector fields whereas, in Ω, three-dimensional ones are considered. Of course, the same identification holds for all the spaces derivated from $\mathbf{L}^2_t(\Gamma)$.

Definition 2.1. Let $\nabla_\Gamma\colon H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$ and $\mathbf{curl}_\Gamma\colon H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$ be defined by

$$(\nabla_\Gamma \varphi)_j = \nabla_2(\varphi_j) \quad (\mathbf{curl}_\Gamma \varphi)_j = \mathbf{curl}_2(\varphi_j) \quad \forall j \tag{2}$$

where ∇_2 (resp. $\mathbf{curl}_2$) is the gradient (resp. vector two-dimensional **curl**) of φ as a function of two variables (these operators are of course well defined locally on each face).

Remark 2.2. The $\mathbf{curl}_\Gamma$ could be equivalently defined as $\mathbf{curl}_\Gamma \varphi = \nabla_\Gamma \varphi \wedge \mathbf{n}$.

It is obvious that these operators are linear, continuous from $H^1(\Gamma)$ to $\mathbf{L}^2_t(\Gamma)$. Their kernels are the set of constant functions due to our assumption on Γ, i.e. Γ is connected. These properties stem from the definition of $H^1(\Gamma)$ which has been obtained via a bi-Lipschitz mapping of Γ on a smooth manifold.

Their adjoint operators are then defined, in a standard way, as

Definition 2.3. Let us define $\mathrm{div}_\Gamma\colon \mathbf{L}^2_t(\Gamma) \to H^{-1}(\Gamma)/\mathbb{R}$ and $\mathrm{curl}_\Gamma\colon \mathbf{L}^2_t(\Gamma) \to H^{-1}(\Gamma)/\mathbb{R}$ as the adjoint operators of $-\nabla_\Gamma$ and $\mathbf{curl}_\Gamma$, respectively. In particular, the following duality pairings hold:

$$\langle \mathrm{div}_\Gamma \boldsymbol{\lambda}, \varphi \rangle_{1,\Gamma} = -\int_\Gamma \boldsymbol{\lambda} \cdot \nabla_\Gamma \varphi \, \mathrm{d}\sigma, \quad \forall \varphi \in H^1(\Gamma), \quad \forall\, \boldsymbol{\lambda} \in \mathbf{L}^2_t(\Gamma)$$

$$\langle \mathrm{curl}_\Gamma \boldsymbol{\lambda}, \varphi \rangle_{1,\Gamma} = \int_\Gamma \boldsymbol{\lambda} \cdot \mathbf{curl}_\Gamma \varphi \, \mathrm{d}\sigma, \quad \forall \varphi \in H^1(\Gamma), \quad \forall\, \boldsymbol{\lambda} \in \mathbf{L}^2_t(\Gamma)$$

It is clear that div_Γ and $\mathbf{curl}_\Gamma$ are, respectively, linear and continuous operators from $\mathbf{L}^2_t(\Gamma)$ to $H^{-1}(\Gamma)$.

3. HODGE DECOMPOSITION OF $\mathbf{L}_t^2(\Gamma)$

In this section, we shall analyse the properties and the validity of standard relationships among gradient, divergence and curl operators in the $\mathbf{L}_t^2(\Gamma)$ context. Keeping in mind that we have to recover all the well-known results in the case of smooth domains (cf. [4] or [6]), in the context of non-smooth domains, let us start by the following important result:

Proposition 3.1. The following identities hold:

$$\mathrm{Ker}(\mathrm{curl}_\Gamma) = \mathrm{Im}\,(\nabla_\Gamma), \quad \mathrm{Ker}(\mathrm{div}_\Gamma) = \mathrm{Im}\,(\mathbf{curl}_\Gamma) \tag{3}$$

They are equivalent to the following:

- $\mathrm{curl}_\Gamma(\nabla_\Gamma\varphi) = 0, \quad \mathrm{div}_\Gamma(\mathbf{curl}_\Gamma\varphi) = 0 \ \ \forall\varphi \in H^1(\Gamma)$;
- for every $\boldsymbol{\psi} \in \mathbf{L}_t^2(\Gamma)$ such that $\mathrm{curl}_\Gamma\boldsymbol{\psi} = 0$ (resp. $\mathrm{div}_\Gamma\boldsymbol{\psi} = 0$) there exists a function $\alpha \in H^1(\Gamma)$ such that $\boldsymbol{\psi} = \nabla_\Gamma\alpha$ (resp. $\boldsymbol{\psi} = \mathbf{curl}_\Gamma\alpha$).

Remark 3.2. This proposition has an important consequence, which is that both $\mathrm{Im}\,(\nabla_\Gamma)$ and $\mathrm{Im}\,(\mathbf{curl}_\Gamma)$ are closed in $\mathbf{L}_t^2(\Gamma)$. Therefore, their adjoints div_Γ and $\mathbf{curl}_\Gamma$ are surjective operators (see e.g. [2]).

Proof. First of all note that the two identities in (3) are equivalent, up to a rotation of vector fields. We deal then only with the first one (the proof for the second one stems easily.)

Step 1. Let us start by proving that $\mathrm{Ker}(\mathrm{curl}_\Gamma) \supseteq \mathrm{Im}\,(\nabla_\Gamma)$, which is equivalent to showing that $\mathrm{curl}_\Gamma(\nabla_\Gamma\varphi) = 0 \ \forall\varphi \in H^1(\Gamma)$. From the definition of $H^1(\Gamma)$ in (1), given a function $\varphi \in H^1(\Gamma)$, one has $\varphi_{j|e_{ij}} = \varphi_{i|e_{ij}}$ in $H^{1/2}(e_{ij})$. Deriving this equality along the edge (cf. [10, p. 94]) one gets

$$\nabla_\Gamma\varphi_j \cdot \boldsymbol{\tau}_{ij} = \nabla_\Gamma\varphi_i \cdot \boldsymbol{\tau}_{ij} \quad \text{in } H_{00}^{-1/2}(e_{ij}). \tag{4}$$

where $H_{00}^{-1/2}(e_{ij})$ is the dual space of $H_{00}^{1/2}(e_{ij})$.

Moreover, it is clear that

$$\mathrm{curl}_\Gamma(\nabla_\Gamma\varphi_j) = 0 \quad \forall j \tag{5}$$

Let $\boldsymbol{\tau}'_j$ be the (clockwise) unit vector tangent to $\partial\Gamma_j$; using (4), (5) and Definition 2.3, one has, for $\eta \in H^1(\Gamma)$:

$$\begin{aligned}\langle \mathrm{curl}_\Gamma(\nabla_\Gamma\varphi), \eta\rangle_{1,\Gamma} &= \int_\Gamma \nabla_\Gamma\varphi \cdot \mathbf{curl}_\Gamma\,\eta\,\mathrm{d}\sigma \\ &= \sum_{j=1}^N \int_{\Gamma_j} \nabla_\Gamma\varphi_j \cdot \mathbf{curl}_\Gamma\,\eta_j\,\mathrm{d}\sigma = \sum_{j=1}^N \int_{\partial\Gamma_j} \nabla_\Gamma\varphi_j \cdot \boldsymbol{\tau}'_j\,\eta\,\mathrm{d}\sigma\end{aligned} \tag{6}$$

By choosing a function η which vanishes in a neighbourhood of the vertices, the last sum in (6) reads

$$\sum_{j=1}^N \int_{\partial\Gamma_j} \nabla_\Gamma\varphi_j \cdot \boldsymbol{\tau}'_j\,\eta\,\mathrm{d}\sigma = \sum_{j=1}^N \sum_{i\in\mathscr{I}_j,\, i<j} \varepsilon_{ij}\langle \nabla_\Gamma\varphi_j \cdot \boldsymbol{\tau}_{ij} - \nabla_\Gamma\varphi_i \cdot \boldsymbol{\tau}_{ij}, \eta\rangle_{1/2,00,e_{ij}} \tag{7}$$

Here, ε_{ij} is either equal to -1 or $+1$, depending on the orientation of $\boldsymbol{\tau}_{ij}$. Using (4) in (7), one gets that the right-hand side of (7) is identically equal to zero. By a density argument, which is that the set of elements of $H^1(\Gamma)$, which vanish in a neighbourhood of the vertices, is dense in $H^1(\Gamma)$, the first statement is proved.

Step 2. Let us analyse now the other inclusion $\mathrm{Ker}(\mathrm{curl}_\Gamma) \subseteq \mathrm{Im}(\nabla_\Gamma)$ and proceed by direct construction: let $\boldsymbol{\psi} \in \mathbf{L}^2_t(\Gamma)$ be such that $\mathrm{curl}_\Gamma \boldsymbol{\psi} = 0$ in $H^{-1}(\Gamma)$; we shall prove that there exists ξ in $H^1(\Gamma)$ such that $\psi = \nabla_\Gamma \xi$. Recalling the definition of curl_Γ, one has

$$0 = \langle \mathrm{curl}_\Gamma \boldsymbol{\psi}, \varphi \rangle_{1,\Gamma} = \int_\Gamma \boldsymbol{\psi} \cdot \mathbf{curl}_\Gamma \varphi \, \mathrm{d}\sigma \quad \forall \varphi \in H^1(\Gamma)$$

which is equivalent to the following:

$$\mathrm{curl}_\Gamma \boldsymbol{\psi}_j = 0 \text{ in } H^{-1}(\Gamma_j) \quad \text{and} \quad \boldsymbol{\psi}_j \cdot \boldsymbol{\tau}_{ij} = \boldsymbol{\psi}_i \cdot \boldsymbol{\tau}_{ij} \text{ in } H^{-1/2}_{00}(e_{ij}) \quad \forall i \in \mathscr{I}_j, \quad \forall j \tag{8}$$

Since every face is a regular manifold, one can apply standard results (see, e.g. [7, p. 31]) to get

$$\exists! \xi_j \in H^1(\Gamma_j) \quad \text{such that } \boldsymbol{\psi}_j = \nabla_\Gamma(\xi_j + c_j) \tag{9}$$

where $(c_j)_{1 \leqslant j \leqslant N}$ are arbitrary constants. The second equality in (8) implies then

$$\frac{\partial \xi_i}{\partial \boldsymbol{\tau}_{ij}} = \frac{\partial \xi_j}{\partial \boldsymbol{\tau}_{ij}} \quad \text{in } H^{-1/2}_{00}(e_{ij}), \quad \forall i \in \mathscr{I}_j, \quad \forall j$$

Using the theorem of 'null derivative' for distributions, one obtains then that $\xi_i = \xi_j + c_{ij}$ in $H^{1/2}(e_{ij})$ where c_{ij} is a new arbitrary constant (which depends on both c_i and c_j).

Now, it is sufficient to show that there is a choice of constants $(c_j)_{1 \leqslant j \leqslant N}$ which yields an H^1-function. Let Γ_j be a face and set $c_j = 0$: for every $i \in \mathscr{I}_j$, let us choose the value of c_i (the constant in (9)) in a way that $\xi_i = \xi_j$ in $H^{1/2}(e_{ij})$, that is $c_{ij} = 0$.

To end the proof, let us denote by i and i' two indices in $\mathscr{I}_j$ such that Γ_i and $\Gamma_{i'}$ have a common edge $e_{ii'}$ which shares a vertex, namely S_k, with e_{ij}. We show that necessarily $\xi_i = \xi_{i'}$ in $H^{1/2}(e_{ii'})$. This comes from the following set of relationships:

$$\begin{aligned} &\text{(a)} \;\; \xi_i \in H^{1/2}(e_{ij} \cup e_{ii'} \cup S_k) \quad \text{and} \quad \xi_{i'} \in H^{1/2}(e_{i'j} \cup e_{ii'} \cup S_k) \\ &\text{(b)} \;\; \xi_i = \xi_j \text{ in } H^{1/2}(e_{ij}) \quad \text{and} \quad \xi_{i'} = \xi_j \text{ in } H^{1/2}(e_{i'j}) \\ &\text{(c)} \;\; \xi_i = \xi_{i'} + c_{ii'} \quad \text{in} \quad H^{1/2}(e_{ii'}) \end{aligned} \tag{10}$$

The coupling term (an integral) at the vertex S_k (see e.g. [9, p. 17]) coming from (a) and (b) in (10) imposes that $c_{ii'} = 0$ in (c). □

Remark 3.3. If Γ is not simply connected, (3) is not true anymore. Indeed, one should add a finite-dimensional space to $\mathrm{Im}(\nabla_\Gamma)$ (resp. $\mathrm{Im}(\mathbf{curl}_\Gamma)$ to get the whole of $\mathrm{Ker}(\mathrm{curl}_\Gamma)$ (resp. $\mathrm{Ker}(\mathrm{div}_\Gamma)$). The Hodge decomposition (11) should also be modified accordingly.

We are now in the position to construct the Hodge decomposition of the space $\mathbf{L}^2_t(\Gamma)$.

Theorem 3.4. The following decomposition holds:

$$\mathbf{L}^2_t(\Gamma) = \nabla_\Gamma H^1(\Gamma) \overset{\perp}{\oplus} \mathbf{curl}_\Gamma H^1(\Gamma) \tag{11}$$

where $\overset{\perp}{\oplus}$ means that the decomposition is direct and orthogonal with respect to $\langle \cdot, \cdot \rangle_t$.

Proof. Let $\mathbf{u} \in \mathbf{L}^2_t(\Gamma)$. Let us consider the differential problem: find $p \in H^1(\Gamma)$ such that

$$\int_\Gamma (\mathbf{u} - \nabla_\Gamma p) \cdot \nabla_\Gamma q \, \mathrm{d}\sigma = 0 \quad \forall q \in H^1(\Gamma) \tag{12}$$

Since $\int_\Gamma \nabla_\Gamma p \cdot \nabla_\Gamma p \, \mathrm{d}\sigma \geqslant C \|p\|^2_{H^1(\Gamma)/\mathbb{R}}$ (C is independent of p), problem (12) admits a unique solution $p \in H^1(\Gamma)/\mathbb{R}$. Moreover, from equality (12) and Definition 2.3, one gets $\mathrm{div}_\Gamma(\mathbf{u} - \nabla_\Gamma p) = 0$. Using Proposition 3.1, we have

$$\exists! \xi \in H^1(\Gamma)/\mathbb{R} \quad \text{such that } \mathbf{u} - \nabla_\Gamma p = \mathbf{curl}_\Gamma \xi$$

The orthogonality of the decomposition comes directly from (4)–(7). □

4. SPACES RELATED TO TANGENTIAL TRACES FOR $\mathbf{H}(\mathbf{curl}, \Omega)$

4.1. Preliminaries

We start by studying a complete characterization for the space of the tangential trace (and tangential components) of $\mathbf{H}^1(\Omega)$.

In order to do that let us start with the following definition:

Definition 4.1. The 'tangential components trace' mapping $\pi_\tau : \mathscr{D}(\bar{\Omega})^3 \mapsto \mathbf{H}^{1/2}_-(\Gamma)$ and the 'tangential trace' mapping $\gamma_\tau : \mathscr{D}(\bar{\Omega})^3 \to \mathbf{H}^{1/2}_-(\Gamma)$ are defined as $\mathbf{u} \mapsto \mathbf{n} \wedge (\mathbf{u} \wedge \mathbf{n})_{|\Gamma}$ and $\mathbf{u} \mapsto \mathbf{n} \wedge \mathbf{u}_{|\Gamma}$, respectively.

On the one hand, it is true that π_τ and γ_τ can be extended to a linear continuous mapping from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^{1/2}_-(\Gamma)$. On the other, we show in what follows that these mappings are not surjective and that their ranges are two different subspaces of $\mathbf{H}^{1/2}_-(\Gamma)$.

Let us focus our attention on the analysis of the mapping π_τ and, at the end, by using the identity $\pi_\tau(\mathbf{u}) = \mathbf{n} \wedge \gamma_\tau(\mathbf{u})$ we shall recover the properties of γ_τ.

Since one deals with polyhedrons, given a function $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega)$, the definition of $\pi_\tau \varphi$ can be understood face by face:

$$\pi_{\tau,j} \boldsymbol{\varphi} := \boldsymbol{\varphi}_j - (\boldsymbol{\varphi}_j \cdot \mathbf{n}_j)\mathbf{n}_j \quad \forall \boldsymbol{\varphi} \in \mathbf{H}^1(\Omega)$$

One gets then that an equivalent definition of π_τ is

$$\pi_\tau : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}_-(\Gamma) \quad \pi_\tau \boldsymbol{\varphi}(\mathbf{x}) = \pi_{\tau,j} \boldsymbol{\varphi}(\mathbf{x}) \quad \text{for } a.e.\ \mathbf{x} \in \Gamma_j, \quad \forall j$$

In the following we prove that the range of this mapping is a true subspace of $\mathbf{H}^{1/2}_-(\Gamma)$. For that, a preliminary result is required.

Theorem 4.2.

$$\varphi \in H^{1/2}(\Gamma) \Leftrightarrow \begin{cases} \varphi \in H^{1/2}(\Gamma_i), \quad \forall i \in \{1, \ldots, N\} \text{ and} \\ \displaystyle\int_{\Gamma_i}\int_{\Gamma_j} \frac{|\varphi(\mathbf{x}) - \varphi(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|^3}\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y}) < \infty, \quad \forall i \neq j \text{ s.t. } \bar{\Gamma}_i \cap \bar{\Gamma}_j = e_{ij} \end{cases}$$

For the proof of this result, we refer to [3].

Proposition 4.3. Let $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{\varphi} = \pi_\tau \mathbf{u}$ and $\boldsymbol{\psi} = \gamma_\tau \mathbf{u}$. Moreover, let Γ_i and Γ_j be two faces with a common edge e_{ij}. The following integrals are bounded:

$$\mathcal{N}_{ij}^{\parallel}(\boldsymbol{\varphi}) := \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\boldsymbol{\varphi}_i \cdot \boldsymbol{\tau}_{ij}(\mathbf{x}) - \boldsymbol{\varphi}_j \cdot \boldsymbol{\tau}_{ij}(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|^3}\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})$$

$$\mathcal{N}_{ij}^{\perp}(\boldsymbol{\psi}) := \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\boldsymbol{\psi}_i \cdot \boldsymbol{\tau}_i(\mathbf{x}) - \boldsymbol{\psi}_j \cdot \boldsymbol{\tau}_j(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|^3}\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y})$$

For the proof, see [3].

Now, let $(\psi_i, \psi_j) \in H^{1/2}(\Gamma_i) \times H^{1/2}(\Gamma_j)$. Let us adopt the notation

$$\psi_i \overset{1/2}{=} \psi_j \text{ at } e_{ij} \Leftrightarrow \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\psi_i(\mathbf{x}) - \psi_j(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|^3}\,\mathrm{d}\sigma(\mathbf{x})\,\mathrm{d}\sigma(\mathbf{y}) < \infty \tag{13}$$

By Proposition 4.3, the range of π_τ is included in

$$\mathbf{H}_{\parallel}^{1/2}(\Gamma) := \{\boldsymbol{\varphi} \in \mathbf{H}_{-}^{1/2}(\Gamma) : \boldsymbol{\varphi}_i \cdot \boldsymbol{\tau}_{ij} \overset{1/2}{=} \boldsymbol{\varphi}_j \cdot \boldsymbol{\tau}_{ij} \text{ at } e_{ij}\ \forall i \in \mathscr{I}_j,\ \forall j\}$$

And analogously the range of γ_τ is included in

$$\mathbf{H}_{\perp}^{1/2}(\Gamma) := \{\boldsymbol{\psi} \in \mathbf{H}_{-}^{1/2}(\Gamma) : \boldsymbol{\psi}_i \cdot \boldsymbol{\tau}_i \overset{1/2}{=} \boldsymbol{\psi}_j \cdot \boldsymbol{\tau}_j \text{ at } e_{ij}\ \forall i \in \mathscr{I}_j,\ \forall j\}$$

The following two propositions are now devoted to proving that $\mathbf{H}_{\parallel}^{1/2}(\Gamma)$ and $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ are Hilbert spaces (endowed with suitable norms) and that they are indeed the range of π_τ and of γ_τ, respectively.

Proposition 4.4. The space $\mathbf{H}_{\parallel}^{1/2}(\Gamma)$ and $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ are Hilbert spaces when endowed with the following norms, respectively:

$$\|\boldsymbol{\psi}\|_{\parallel,1/2,\Gamma}^2 := \sum_{j=1}^{N} \|\boldsymbol{\psi}_j\|_{1/2,\Gamma_j}^2 + \sum_{j=1}^{N} \sum_{i \in \mathscr{I}_j} \mathcal{N}_{ij}^{\parallel}(\boldsymbol{\psi})$$

$$\|\boldsymbol{\psi}\|_{\perp,1/2,\Gamma}^2 := \sum_{j=1}^{N} \|\boldsymbol{\psi}_j\|_{1/2,\Gamma_j}^2 + \sum_{j=1}^{N} \sum_{i \in \mathscr{I}_j} \mathcal{N}_{ij}^{\perp}(\boldsymbol{\psi})$$

See [3] for the proof of this result.

Proposition 4.5. The mapping $\pi_\tau : \mathbf{H}^1(\Omega) \to \mathbf{H}_\parallel^{1/2}(\Gamma)$ such that $\mathbf{u} \mapsto \mathbf{n}(\mathbf{u} \wedge \mathbf{n})_{|\Gamma}$ is linear, continuous and surjective. Moreover, there exists a continuous lifting mapping $\mathscr{R}_\pi$ from $\mathbf{H}_\parallel^{1/2}(\Gamma)$ to $\mathbf{H}^1(\Omega)$. In the same way $\gamma_\tau : \mathbf{H}^1(\Omega) \to \mathbf{H}_\perp^{1/2}(\Gamma)$ is linear continuous and surjective. Let $\mathscr{R}_\gamma$ denote the associated continuous lifting mapping.

See [3] for the proof of this result.

In the remainder of the paper, let us call $\mathbf{H}_\parallel^{-1/2}(\Gamma)$ and $\mathbf{H}_\perp^{-1/2}(\Gamma)$ the dual spaces of $\mathbf{H}_\parallel^{-1/2}(\Gamma)$ and $\mathbf{H}_\perp^{-1/2}(\Gamma)$, respectively (with $\mathbf{L}_t^2(\Gamma)$ as the pivot space.) Moreover, let us denote by $\langle \cdot, \cdot \rangle_{\parallel,1/2,\Gamma}$ (resp. $\langle \cdot, \cdot \rangle_{\perp,1/2,\Gamma}$) the duality product between $\mathbf{H}_\parallel^{-1/2}(\Gamma)$ and $\mathbf{H}_\parallel^{1/2}(\Gamma)$ (resp. $\mathbf{H}_\perp^{-1/2}(\Gamma)$ and $\mathbf{H}_\perp^{1/2}(\Gamma)$).

4.2. *More about differential operators on* Γ

We need now to extend the definitions of the differential operators on Γ, both to more and less regular Sobolev spaces.

Let us set

$$H^{3/2}(\Gamma) := \{\varphi \in H^1(\Gamma) \text{ such that } \nabla_\Gamma \varphi \in \mathbf{H}_\parallel^{1/2}(\Gamma)\} \tag{14}$$

or equivalently

$$H^{3/2}(\Gamma) := \{\varphi \in H^1(\Gamma) \text{ such that } \mathbf{curl}_\Gamma \varphi \in \mathbf{H}_\perp^{1/2}(\Gamma)\}$$

This is a Hilbert space endowed with the natural norm

$$\|\varphi\|_{3/2,\Gamma}^2 := \sum_{j=1}^{N} \|\varphi\|_{1,\Gamma_j}^2 + \|\nabla_\Gamma \varphi\|_{\parallel,1/2,\Gamma}^2 = \sum_{j=1}^{N} \|\varphi\|_{1,\Gamma_j}^2 + \|\mathbf{curl}_\Gamma \varphi\|_{\perp,1/2,\Gamma}^2$$

For a sharper result on the definition of $H^{3/2}(\Gamma)$, see the Corollary 3.7 of [3]. In the following, let us denote by $H^{-3/2}(\Gamma)$ the dual space of $H^{3/2}(\Gamma)$ with $L^2(\Gamma)$ as the pivot space.

An immediate consequence of this definition is that the restriction of the tangential gradient, $\nabla_\Gamma : H^{3/2}(\Gamma) \to \mathbf{H}_\parallel^{1/2}(\Gamma)$ and of the tangential vector curl, $\mathbf{curl}_\Gamma : H^{3/2}(\Gamma) \to \mathbf{H}_\perp^{1/2}(\Gamma)$ are linear, continuous, injective up to a constant. Their adjoints $\mathrm{div}_\Gamma : \mathbf{H}_\parallel^{-1/2}(\Gamma) \to H^{-3/2}(\Gamma)$ and $\mathrm{curl}_\Gamma : \mathbf{H}_\perp^{-1/2}(\Gamma) \to H^{-3/2}(\Gamma)$ are then defined by: for every $\varphi \in H^{3/2}(\Gamma)$, $\boldsymbol{\lambda} \in \mathbf{H}_\parallel^{-1/2}(\Gamma)$, $\boldsymbol{\psi} \in \mathbf{H}_\perp^{-1/2}(\Gamma)$

$$\langle \mathrm{div}_\Gamma \boldsymbol{\lambda}, \varphi \rangle_{3/2,\Gamma} = -\langle \boldsymbol{\lambda}, \nabla_\Gamma \varphi \rangle_{\parallel,1/2,\Gamma}$$

$$\langle \mathrm{curl}_\Gamma \boldsymbol{\psi}, \varphi \rangle_{3/2,\Gamma} = \langle \boldsymbol{\psi}, \mathbf{curl}_\Gamma \varphi \rangle_{\perp,1/2,\Gamma}$$

We now make use of the following result which has been proved in [3].

Theorem 4.6. Let

$$\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\boldsymbol{\lambda} \in \mathbf{H}_\parallel^{-1/2}(\Gamma) : \mathrm{div}_\Gamma \boldsymbol{\lambda} \in H^{-1/2}(\Gamma)\}$$

$$\mathbf{H}_\perp^{-1/2}(\mathbf{curl}_\Gamma, \Gamma) := \{\boldsymbol{\lambda} \in \mathbf{H}_\perp^{-1/2}(\Gamma) : \mathrm{curl}_\Gamma \boldsymbol{\lambda} \in H^{-1/2}(\Gamma)\}$$

The mappings $\pi_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\gamma_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ are linear and continuous. Moreover, the following integration by parts holds true:

$$\int_\Omega \{\mathbf{curl}\, \mathbf{v} \cdot \mathbf{u} - \mathbf{curl}\ \mathbf{u} \cdot \mathbf{v}\}\, \mathrm{d}\Omega = \langle \gamma_\tau \mathbf{u}, \pi_\tau \mathbf{v} \rangle_{\parallel,1/2,\Gamma} \quad \forall \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega), \quad \mathbf{v} \in \mathbf{H}^1(\Omega)$$

Now, we certainly have that the following identities hold:

$$\nabla_\Gamma(u_{|\Gamma}) = \pi_\tau(\nabla u) \quad \text{and} \quad \mathbf{curl}_\Gamma(u_{|\Gamma}) = \gamma_\tau(\nabla u) \quad \forall u \in \mathscr{D}_e(\bar{\Omega})$$

where $\mathscr{D}_e(\bar{\Omega})$ is the subset of $\mathscr{D}(\bar{\Omega})$ made up of functions which vanish in a neighbourhood of every edge $(e_{ij})_{1 \leqslant j \leqslant N, i \in \mathscr{I}_j}$. Moreover, the right-hand sides depend only on the trace of u.

By the density of $\mathscr{D}_e(\bar{\Omega})$ in $H^1(\Omega)$ and using Theorem 4.6, we have that the tangential gradient and vector curl can be extended as linear continuous operators:

$$\nabla_\Gamma : H^{1/2}(\Gamma) \to \mathbf{H}_\perp^{-1/2}(\Gamma) \quad \text{and} \quad \mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to \mathbf{H}_\parallel^{-1/2}(\Gamma) \tag{15}$$

Accordingly their adjoints $\mathrm{div}_\Gamma : \mathbf{H}_\perp^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and $\mathrm{curl}_\Gamma : \mathbf{H}_\parallel^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are defined, for every $\varphi \in H^{1/2}(\Gamma)$, $\boldsymbol{\lambda} \in \mathbf{H}_\perp^{1/2}(\Gamma)$ and $\boldsymbol{\psi} \in \mathbf{H}_\parallel^{1/2}(\Gamma)$, by:

$$\begin{aligned} \langle \mathrm{div}_\Gamma \boldsymbol{\lambda}, \varphi \rangle_{1/2,\Gamma} &= -\langle \nabla_\Gamma \varphi, \boldsymbol{\lambda} \rangle_{\perp,1/2,\Gamma} \\ \langle \mathrm{curl}_\Gamma \boldsymbol{\psi}, \varphi \rangle_{1/2,\Gamma} &= \langle \mathbf{curl}_\Gamma \varphi, \boldsymbol{\psi} \rangle_{\parallel,1/2,\Gamma} \end{aligned} \tag{16}$$

We follow now the same line as in Section 3 to prove the equivalent of Proposition 3.1:

Proposition 4.7. The following identities hold:‡

$$\mathrm{Ker}(\mathrm{curl}_\Gamma(\mathbf{H}_\perp^{-1/2})) = \mathrm{Im}(\nabla_\Gamma(H^{1/2})), \quad \mathrm{Ker}(\mathrm{div}_\Gamma(\mathbf{H}_\parallel^{-1/2})) = \mathrm{Im}(\mathbf{curl}_\Gamma(H^{1/2})) \tag{17}$$

Remark 4.8. As a consequence $\mathrm{Im}(\nabla_\Gamma(H^{1/2}))$ is closed in $\mathbf{H}_\perp^{-1/2}$, and $\mathrm{Im}(\mathbf{curl}_\Gamma(H^{1/2}))$ is closed in $\mathbf{H}_\parallel^{-1/2}(\Gamma)$. Their adjoints are thus surjective operators (see [2]): div_Γ from $\mathbf{H}_\perp^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$, and curl_Γ from $\mathbf{H}_\parallel^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$.

Proof. Since, as before, the two identities in (17) are equivalent, we deal only with the first one. The proof is split into three steps. Let us prove that

- $\mathrm{Ker}(\mathrm{curl}_\Gamma) \supseteq \mathrm{Im}(\nabla_\Gamma)$,
- $\mathrm{Im}(\nabla_\Gamma)$ is dense in $\mathrm{Ker}(\mathrm{curl}_\Gamma)$,
- $\mathrm{Im}(\nabla_\Gamma)$ is closed in $\mathbf{H}_\perp^{-1/2}(\Gamma)$.

Step 1. Straightforward from Proposition 3.1 and the density of $H^1(\Gamma)$ in $H^{1/2}(\Gamma)$.

Step 2. For that, let us prove that any continuous linear form which vanishes on $\mathrm{Im}(\nabla_\Gamma)$ also vanishes on $\mathrm{Ker}(\mathrm{curl}_\Gamma)$. Let $\boldsymbol{\psi} \in \mathbf{H}_\perp^{1/2}(\Gamma)$ be such that

$$\langle \nabla_\Gamma p, \boldsymbol{\psi} \rangle_{\perp,1/2,\Gamma} = 0 \quad \forall p \in H^{1/2}(\Gamma) \tag{18}$$

We want to prove now that

$$\langle \mathbf{w}, \boldsymbol{\psi} \rangle_{\perp,1/2,\Gamma} = 0, \quad \forall \mathbf{w} \in \mathrm{Ker}(\mathbf{curl}_\Gamma) \tag{19}$$

and in order to reach our goal we have to characterize the set of functions $\boldsymbol{\psi} \in H_\perp^{1/2}(\Gamma)$ which satisfy (18). From (18), one infers that $\mathrm{div}_\Gamma \boldsymbol{\psi} = 0$. Using Proposition 3.1, there exists a function $\beta \in H^1(\Gamma)$ such that $\boldsymbol{\psi} = \mathbf{curl}_\Gamma \beta$. Moreover, owing to the definition of $H^{3/2}(\Gamma)$, one has $\beta \in H^{3/2}(\Gamma)$.

‡ Summing up, an alternate notation of this kind of results is the following:

$$H^{1/2}(\Gamma) \overset{\nabla_\Gamma}{\to} \mathbf{H}_\perp^{-1/2}(\Gamma) \overset{\mathrm{curl}_\Gamma}{\to} H^{-3/2}(\Gamma) \to \{0\}, \quad H^{1/2}(\Gamma) \overset{\mathbf{curl}_\Gamma}{\to} \mathbf{H}_\parallel^{-1/2}(\Gamma) \overset{\mathrm{div}_\Gamma}{\to} H^{-3/2}(\Gamma) \to \{0\}$$

Coming back to (19), $\forall \mathbf{w} \in \mathrm{Ker}(\mathrm{curl}_\Gamma)$

$$\langle \mathbf{w}, \mathbf{curl}_\Gamma \beta \rangle_{\perp,1/2,\Gamma} = \langle \mathrm{curl}_\Gamma \mathbf{w}, \beta \rangle_{3/2,\Gamma} = 0$$

Step 3. Using Peetre's Lemma (see e.g. [7, p. 18]), the embedding of $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$ being compact, one has to prove that there exists a constant C such that, for every $p \in H^{1/2}(\Gamma)$, the following inequality holds true:

$$\|p\|_{1/2,\Gamma} \leqslant C(\|p\|_{0,\Gamma} + \|\nabla_\Gamma p\|_{\perp,-1/2,\Gamma}) \tag{20}$$

Over each face, one has

$$\|p_j\|_{1,\Gamma_j} \leqslant \sqrt{2}(\|p_j\|_{0,\Gamma_j}, + \|\nabla_\Gamma p_j\|_{0,\Gamma_j}), \quad \forall p_j \in H^1(\Gamma_j)$$

Owing to [7, p. 20], there exists a constant c such that

$$\|p_j\|_{0,\Gamma_j} \leqslant c(\|p_j\|_{-1,\Gamma_j} + \|\nabla_\Gamma p_j\|_{-1,\Gamma_j}), \quad \forall p_j \in L^2(\Gamma_j)$$

Using three results of [10], which are the main interpolation theorem (p. 31), the definition of $H^{1/2}_{00}(\Gamma_j)$ (p. 72), and the duality theorem (p. 34), one readily obtains

$$\begin{aligned} \|p_j\|_{1/2,\Gamma_j} &\leqslant c(\|p_j\|_{-1/2,00,\Gamma_j} + \|\nabla_\Gamma p_j\|_{-1/2,00,\Gamma_j}) \\ &\leqslant c(\|p_j\|_{0,\Gamma_j} + \|\nabla_\Gamma p_j\|_{-1/2,00,\Gamma_j}) \end{aligned} \tag{21}$$

Let Γ_i and Γ_j be two faces with a common edge e_{ij}. In the following let p stand for the restriction of p to Γ_{ij}. Owing to Theorem 4.2, and by a localization argument, one needs only to prove that:

$$\|p\|_{1/2,\Gamma_{ij}}, \leqslant c(\|p\|_{0,\Gamma_{ij}} + \|\nabla_\Gamma p_j\|_{-1/2,00,\Gamma_{ij}})$$

Let $\tilde{\Gamma}_{ij}$ be the surface composed of two half-planes, one containing Γ_i, the other containing Γ_j, separated by a straight line containing e_{ij}. Let $\mathscr{P}$ be the plane containing Γ_j and $\mathscr{J} : \mathscr{P} \to \tilde{\Gamma}_{ij}$ be the canonical (see below), injective and piecewise orthogonal application which sends $\mathscr{P}$ in $\tilde{\Gamma}_{ij}$. It turns out that $\mathscr{J} \in W^{1,\infty}(\mathscr{P})$. Let $\tilde{p}$ denote a continuous extension of p, which belongs to $H^{1/2}(\tilde{\Gamma}_{ij})$. By standard interpolation properties in Sobolev spaces (see [11]), one gets that $\tilde{p} \circ \mathscr{J} \in H^{1/2}(\mathscr{P})$ and that $H^{1/2}$-norms of $\tilde{p}$ and $\tilde{p} \circ \mathscr{J}$ are equivalent. Owing to (21) applied to $\tilde{p} \circ \mathscr{J}$, one gets

$$\|\tilde{p}\|_{1/2,\tilde{\Gamma}_{ij}} \leqslant C(\|\tilde{p} \circ \mathscr{J}\|_{0,\mathscr{P}} + \|\nabla(\tilde{p} \circ \mathscr{J})\|_{-1/2,\mathscr{P}}) \tag{22}$$

Let us estimate now the right-hand side of (22) by the following argument.

Without loss of generality (here, the value of the diedric angle is set a $\pi/4$), let us suppose that the application $\mathscr{J}$ has the form

$$\mathscr{J} : \mathscr{P} \to \tilde{\Gamma}_{ij}(x,y) \mapsto (\tilde{x},\tilde{y},\tilde{z}) = \begin{cases} (x,y,0) & \text{if } x \leqslant 0 \\ \left(\dfrac{x}{\sqrt{2}}, y, \dfrac{x}{\sqrt{2}}\right) & \text{if } x \geqslant 0 \end{cases}$$

Now, by direct calculation of the gradient $\nabla(\tilde{p} \circ \mathscr{J})$, one has

$$\nabla(\tilde{p} \circ \mathscr{J}) = \left(\frac{\partial \tilde{p}}{\partial \tilde{x}}, \frac{\partial \tilde{p}}{\partial \tilde{y}}\right)^{\mathrm{T}} x \leqslant 0 \quad \nabla(\tilde{p} \circ \mathscr{J}) = \left(\frac{1}{\sqrt{2}}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{1}{\sqrt{2}}\frac{\partial \tilde{p}}{\partial \tilde{z}}, \frac{\partial \tilde{p}}{\partial \tilde{y}}\right)^{\mathrm{T}} x \geqslant 0 \tag{23}$$

And, on the other hand, one also has

$$\nabla_\Gamma \tilde{p} = (\nabla_\Gamma \tilde{p} \cdot \boldsymbol{\tau}_{ij}, \nabla_\Gamma \tilde{p} \cdot \boldsymbol{\tau})^{\mathrm{T}} \quad \text{where } \boldsymbol{\tau} = \boldsymbol{\tau}_i \text{ on } \Gamma_i \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\tau}_j \text{ on } \Gamma_j, \text{ that is}$$

$$\nabla_\Gamma \tilde{p}_j = \left(-\frac{\partial \tilde{p}}{\partial \tilde{y}}, \frac{\partial \tilde{p}}{\partial \tilde{x}} \right)^{\mathrm{T}}, \quad \nabla_\Gamma \tilde{p}_i = \left(-\frac{\partial \tilde{p}}{\partial \tilde{y}}, \frac{1}{\sqrt{2}} \frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{1}{\sqrt{2}} \frac{\partial \tilde{p}}{\partial \tilde{z}} \right)^{\mathrm{T}} \tag{24}$$

The fact that $\nabla_\Gamma \tilde{p} \in \mathbf{H}_\perp^{-1/2}(\tilde{\Gamma}_{ij})$ implies that

$$\nabla_\Gamma \tilde{p} \cdot \boldsymbol{\tau}_{ij} \in H^{-1/2}(\tilde{\Gamma}_i), \quad \nabla_\Gamma \tilde{p} \cdot \boldsymbol{\tau}_{ij} \in H^{-1/2}(\tilde{\Gamma}_j) \quad \text{and} \quad \nabla_\Gamma \tilde{p} \cdot \boldsymbol{\tau} \in H^{-1/2}(\tilde{\Gamma}_{ij}) \tag{25}$$

Using (23)–(25) and the equivalence of $H^{1/2}$ norms, we finally get

$$\|\nabla(\tilde{p} \circ \mathscr{J})\|_{-1/2,\mathscr{P}} \leqslant C \|\nabla_\Gamma \tilde{p}\|_{\perp,-1/2,\tilde{\Gamma}_{ij}}$$

By a localization argument, (20) holds true. □

5. HODGE DECOMPOSITION FOR $\mathbf{H}_\parallel^{-1/2}(\mathbf{div}_\Gamma, \Gamma)$

Using the definitions in the previous sections, the Laplace–Beltrami operator can be defined as

$$\Delta_\Gamma \varphi := \mathrm{div}_\Gamma \nabla_\Gamma \varphi = -\mathrm{curl}_\Gamma \, \mathbf{curl}_\Gamma \varphi \quad \forall \varphi \in H^1(\Gamma) \tag{26}$$

and it is easily proved, in a variational setting (using the Lax–Milgram theorem), that $\Delta_\Gamma : H^1(\Gamma)/\mathbb{R} \to H^{-1}(\Gamma)$ is an isomorphism. We set now

$$\mathscr{H}(\Gamma) := \{ u \in H^1(\Gamma)/\mathbb{R} \quad \text{such that } \Delta_\Gamma u \in H^{-1/2}(\Gamma) \} \tag{27}$$

We are now in a position to prove the next Theorem which allows to split a tangential field, which belongs to $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. The first is explicitly a decomposition in a 'regular' part and a 'singular' part, while the second is a direct Hodge decomposition.

Theorem 5.1. The following holds:

$$\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \mathbf{curl}_\Gamma(H^{1/2}(\Gamma)/\mathbb{R}) + \mathbf{H}_\perp^{1/2}(\Gamma) \tag{28}$$

$$\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \mathbf{curl}_\Gamma(H^{1/2}(\Gamma)/\mathbb{R}) \oplus \nabla_\Gamma(\mathscr{H}(\Gamma)) \tag{29}$$

where $\oplus$ denotes a direct sum which is orthogonal in the following sense: let $\beta \in H^{1/2}(\Gamma)$ and $\beta_n \in H^1(\Gamma)$ be a sequence such that $\beta_n \to \beta$ in $H^{1/2}(\Gamma)$, then

$$\langle \mathbf{curl}_\Gamma \beta, \nabla_\Gamma \alpha \rangle := \lim_{n \to +\infty} \int_\Gamma \mathbf{curl}_\Gamma \beta_n \cdot \nabla_\Gamma \alpha \, \mathrm{d}\sigma = 0 \quad \forall \beta \in H^{1/2}(\Gamma), \quad \alpha \in \mathscr{H}(\Gamma) \tag{30}$$

Proof. Let $\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. We prove first decomposition (28). Owing to Proposition 4.7 and its associated Remark, the operator div_Γ is surjective, that is, there exists a function $\boldsymbol{\psi} \in \mathbf{H}_\perp^{1/2}(\Gamma)$ such that

$$\mathrm{div}_\Gamma \mathbf{u} = \mathrm{div}_\Gamma \boldsymbol{\psi}.$$

Moreover, this function is defined up to the kernel of the div_Γ operator with respect to the scalar product in $\mathbf{H}_\perp^{1/2}(\Gamma)$, where the kernel of div_Γ can be characterized as $\mathrm{Im}\{\mathrm{curl}_\Gamma(H^{3/2}(\Gamma))\}$ by a proof similar to that of Proposition 3.1 and recalling Theorem 3.4 in [3].

Applying Proposition 4.7, we know that there exists a (unique) $\alpha \in H^{1/2}(\Gamma)/\mathbb{R}$ such that

$$\mathbf{u} - \boldsymbol{\psi} = \mathbf{curl}_\Gamma \alpha \tag{31}$$

In order to prove now the validity of (29), we need to apply decomposition (11) to the function $\boldsymbol{\psi}$ in (31). There exists a single $(p, q) \in (H^1(\Gamma)/\mathbb{R})^2$ such that $\boldsymbol{\psi} = \nabla_\Gamma p + \mathbf{curl}_\Gamma q$. Moreover, since $\mathrm{div}_\Gamma(\boldsymbol{\psi}) \in H^{-1/2}(\Gamma)$, we deduce $p \in \mathscr{H}(\Gamma)$. Rewriting (31) we obtain

$$\mathbf{u} = \nabla_\Gamma p + \mathbf{curl}_\Gamma \beta, \quad p \in \mathscr{H}(\Gamma), \quad \beta = q + \alpha, \quad \beta \in H^{1/2}(\Gamma)/\mathbb{R}$$

In order to prove that the sum in (29) is direct we proceed by contradiction. Let $\boldsymbol{\xi} \in \nabla_\Gamma \mathscr{H}(\Gamma) \cap \mathbf{curl}_\Gamma H^{1/2}(\Gamma)$, in particular $\boldsymbol{\xi} \in \mathbf{L}^2_t(\Gamma)$ and it belongs to $\mathrm{Im}(\mathbf{curl}_\Gamma) \cap \mathrm{Im}(\nabla_\Gamma) = \{0\}$. Finally, (30) is surely true since $\int_\Gamma \nabla_\Gamma \alpha \cdot \mathbf{curl}_\Gamma \beta \, \mathrm{d}\sigma = 0$ for any $\alpha, \beta \in H^1(\Gamma)$. □

Remark 5.2. Decomposition (29) is then the one which corresponds to the Hodge decomposition proved in [6], but in our case the space $\mathscr{H}(\Gamma)$ cannot be a posteriori replaced by $H^{3/2}(\Gamma)$. The key point is that a shift theorem for the Laplace–Beltrami operator is missing. Nevertheless, few comments are due. Let (cf. (14))

$$H^{3/2}_{reg}(\Gamma) := \{u \in H^1(\Gamma) \text{ such that } \nabla_\Gamma u \in \mathbf{H}^{1/2}_\parallel(\Gamma) \cap \mathbf{H}^{1/2}_\perp(\Gamma)\}$$

The following holds:

$$H^{3/2}_{\mathrm{reg}}(\Gamma) \subseteq \mathscr{H}(\Gamma) \subseteq H^1(\Gamma) \text{ but } H^{3/2}(\Gamma) \nsubseteq \mathscr{H}(\Gamma)$$

Actually, the condition $\mathrm{div}_\Gamma \nabla_\Gamma p \in H^{-1/2}(\Gamma)$ imposes $\nabla_\Gamma p_i \cdot \boldsymbol{\tau}_i = \nabla_\Gamma p_j \cdot \boldsymbol{\tau}_j$ at any edge e_{ij}. The space $H^{3/2}_{\mathrm{reg}}(\Gamma)$ is exactly the subspace of $H^{3/2}(\Gamma)$ of functions which verify this condition.

The result of the same type is also true for the space $\mathbf{H}^{-1/2}_\perp(\mathrm{curl}_\Gamma, \Gamma)$:

Theorem 5.3. The following holds:

$$\mathbf{H}^{-1/2}_\perp(\mathrm{curl}_\Gamma, \Gamma) = \nabla_\Gamma(H^{1/2}(\Gamma)/\mathbb{R}) + \mathbf{H}^{1/2}_\parallel(\Gamma) \tag{32}$$

$$\mathbf{H}^{-1/2}_\perp(\mathrm{curl}_\Gamma, \Gamma) = \nabla_\Gamma(H^{1/2}(\Gamma)/\mathbb{R}) \oplus \mathbf{curl}_\Gamma(\mathscr{H}(\Gamma)) \tag{33}$$

where $\oplus$ has to be regarded as in Theorem 4.1.

The following characterization theorems for the trace mappings in $\mathbf{H}(\mathbf{curl}, \Omega)$ are a direct consequence of these results.

Theorem 5.4. The mappings $\pi_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}_\perp(\mathrm{curl}_\Gamma, \Gamma)$ and $\gamma_\tau : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}_\parallel(\mathrm{div}_\Gamma, \Gamma)$ are surjective, that is, they have continuous inverses.

Proof. Since the two cases are equivalent, let us focus the attention on γ_τ. Let $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}_\parallel(\mathrm{div}_\Gamma, \Gamma)$. By Theorem 4.1, it is decomposed as:

$$\boldsymbol{\lambda} = \boldsymbol{\psi} + \mathbf{curl}_\Gamma \alpha \quad \alpha \in H^{1/2}(\Gamma)/\mathbb{R}, \quad \boldsymbol{\psi} \in \mathbf{H}^{1/2}_\perp(\Gamma)$$

Using Proposition 4.5, with the same notation, one gets that $\mathscr{R}_\gamma \boldsymbol{\psi}$ is a lifting of $\boldsymbol{\psi}$ inside Ω and it belongs to $\mathbf{H}^1(\Omega)$. On the other hand, let us denote by $\mathscr{R}_1 \alpha \in H^1(\Omega)$ a continuous lifting of α in Ω. The function:

$$\mathbf{u} = \mathscr{R}_r \boldsymbol{\psi} + \nabla(\mathscr{R}_1 \alpha)$$

verifies $\mathbf{u}\wedge\mathbf{n}_{|\Gamma}=\boldsymbol{\lambda}$ and belongs to $\mathbf{H}(\mathbf{curl},\Omega)$. γ_τ is thus linear, continuous and surjective. Up to its kernel, its inverse is continuous, owing to the open mapping theorem. □

Remark 5.5. Of course, many choices of the 'regular' part (belonging to $\mathbf{H}_\perp^{1/2}(\Gamma)$ and $\mathbf{H}_\parallel^{1/2}(\Gamma)$, respectively), in decompositions (28) and (32) are possible, but in principle, these choices cannot lead to orthogonal decompositions. For example, using Theorem 5.4 above and the decomposition of fields of $H(\text{curl},\Omega)$ of Remark 4.2 in [3], we see that the function $\boldsymbol{\psi}\in\mathbf{H}_\perp^{1/2}(\Gamma)$ can be chosen, for example, as the tangential trace of a function which belongs to $\mathbf{H}^1(\Omega)\cap\mathbf{H}_0(\text{div},\Omega)$.

Finally, another immediate consequence of this decomposition is the following identity:

$$\mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma,\Gamma)=(\mathbf{H}_\perp^{-1/2}(\text{curl}_\Gamma,\Gamma))'$$

with $\mathbf{L}_t^2(\Gamma)$ as a pivot space in a sense which is made clear subsequently. This duality has already been stated in the case of a smooth domain (see e.g. [4, p. 40]).

Definition 5.6. Let $\mathbf{u}\in\mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma,\Gamma)$ and $\mathbf{v}\in\mathbf{H}_\perp^{-1/2}(\text{curl}_\Gamma,\Gamma)$. Owing to Theorems 5.1 and 5.3 there exist $\alpha_u,\alpha_v\in\mathscr{H}(\Gamma)$ and $\beta_u,\beta_v\in H^{1/2}(\Gamma)$ such that

$$\mathbf{u}=\nabla_\Gamma\alpha_u+\mathbf{curl}_\Gamma\beta_u,\quad \mathbf{v}=\mathbf{curl}_\Gamma\,\alpha_v+\nabla_\Gamma\beta_v$$

Let us define

$$_\gamma\langle\mathbf{u},\mathbf{v}\rangle_\pi=-\langle\Delta_\Gamma\alpha_u,\beta_v\rangle_{1/2,\Gamma}+\langle\Delta_\Gamma\alpha_v,\beta_u\rangle_{1/2,\Gamma}\tag{34}$$

It is not hard to see that when $\mathbf{u}$ and $\mathbf{v}$ are smooth enough, namely $\mathbf{u},\mathbf{v}\in\mathbf{L}_t^2(\Gamma)$, the left-hand side is equal to the $\mathbf{L}_t^2(\Gamma)$ scalar product between $\mathbf{u}$ and $\mathbf{v}$. By a density argument the formula (34) is then well defined with $\mathbf{L}_t^2(\Gamma)$ as pivot space.

Note that this duality is not strictly linked to the direct decomposition that is used in Definition 5.6. Following Remark 5.5, given another, non-orthogonal decomposition,

$$\mathbf{u}=\boldsymbol{\psi}+\mathbf{curl}_\Gamma p,\quad \mathbf{v}=\boldsymbol{\varphi}+\nabla_\Gamma q,\quad \psi\in\mathbf{H}_\perp^{1/2}(\Gamma),\ \boldsymbol{\varphi}\in\mathbf{H}_\parallel^{1/2}(\Gamma),\ p,q\in H^{1/2}(\Gamma)$$

The duality can still be defined as

$$_\gamma\langle\mathbf{u},\mathbf{v}\rangle_\pi=\langle\nabla_\Gamma q,\psi\rangle_{\perp,1/2,\Gamma}+\langle\mathbf{curl}_\Gamma p,\boldsymbol{\varphi}\rangle_{\parallel,1/2,\Gamma}+\langle\boldsymbol{\psi},\boldsymbol{\varphi}\rangle_t$$

As a conclusion, the following integration by parts formula holds true:

$$\int_\Omega\{\mathbf{curl}\,\mathbf{v}\cdot\mathbf{u}-\mathbf{curl}\,\mathbf{u}\cdot\mathbf{v}\}\mathrm{d}\Omega={}_\gamma\langle\gamma_\tau(\mathbf{u}),\pi_\tau(\mathbf{v})\rangle_\pi\quad\forall\mathbf{u},\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega)$$

6. ON A PART OF Γ— TRACE THEOREMS AND HODGE DECOMPOSITIONS

The aim of this section is now to extend the results stated in Section 5 to the case of spaces and trace mappings defined on a part of and no more on the whole boundary.

As in the companion paper, [3], we need to introduce some notations and spaces. Let $\bar\Gamma_+$ be a collection of closed faces of Γ such that $\bar\Gamma_+$ is connected. Γ_+ is then an open subset of Γ with a piecewise regular boundary $\partial\Gamma_+$. Let $\Gamma_- = \Gamma\backslash\bar{\Gamma}_+$ and $\mathcal{I}_+$ be the set of indices j such that $\Gamma_j \subseteq \Gamma_+$. We denote by $\partial\mathcal{I}_+ \subseteq \mathcal{I}_+$ the set of indices j corresponding to the faces which share at least one edge with $\partial\Gamma_+$. Finally, we denote by $\boldsymbol{\tau}_+$ the unit tangent vector to $\partial\Gamma_+$, $\boldsymbol{\tau}_{+j} = \boldsymbol{\tau}_{+|\bar{\Gamma}_j\cap\partial\Gamma_+}$, and by $\mathbf{v}_+$ the outward normal vector defined as $\mathbf{v}_{+|\bar{\Gamma}_j\cap\partial\Gamma_+} = \mathbf{v}_{+j}$ for any $j\in\partial\mathcal{I}_+$, with $\mathbf{v}_{+j} = \boldsymbol{\tau}_{+j}\wedge\mathbf{n}_j$ (the orientation of $\boldsymbol{\tau}_{+j}$ is such that $\mathbf{v}_{+j}$ is an outward normal).

We start by considering the Hodge decomposition of the space $\mathbf{L}^2_t(\Gamma_+)$.

Theorem 6.1. The following holds:

$$\mathbf{L}^2_t(\Gamma_+) = \nabla_{\Gamma_+}H^1_0(\Gamma_+)\overset{\perp}{\oplus}\mathbf{curl}_{\Gamma_+}\{H^1(\Gamma_+)/\mathbb{R}\}$$
$$\mathbf{L}^2_t(\Gamma_+) = \nabla_{\Gamma_+}\{H^1(\Gamma_+)/\mathbb{R}\}\overset{\perp}{\oplus}\mathbf{curl}_{\Gamma_+}H^1_0(\Gamma_+) \tag{35}$$

Proof. We focus our attention on the first decomposition in (35), since the second one is equivalent. Given $\mathbf{u}\in\mathbf{L}^2_t(\Gamma)$, we solve:

$$\text{Find } p\in H^1_0(\Gamma_+):\int_{\Gamma_+}\nabla_{\Gamma_+}p\cdot\nabla_{\Gamma_+}q\,\mathrm{d}\sigma = \int_{\Gamma_+}\mathbf{u}\cdot\nabla_{\Gamma_+}q\,\mathrm{d}\sigma,\quad q\in H^1_0(\Gamma_+)$$

This problem admits a unique solution and, moreover, it holds $\mathrm{div}_{\Gamma_+}(\mathbf{u}-\nabla_{\Gamma_+}p) = 0$. As a consequence, by the same reasoning as in Proposition 3.1, we have that $\mathbf{u} = \nabla_{\Gamma_+}p + \mathbf{curl}_{\Gamma_+}q$, $q\in H^1(\Gamma_+)/\mathbb{R}$. □

Let $\tilde{\cdot}$ be the extension by zero to Γ. We need the spaces:

$$H^{1/2}_{00}(\Gamma_+) := \{\varphi\in H^{1/2}(\Gamma_+):\ \tilde{\varphi}\in H^{1/2}(\Gamma)\}$$
$$\mathbf{H}^{1/2}_{\parallel}(\Gamma_+) := \{\mathbf{u}_{|\Gamma_+},\ \mathbf{u}\in\mathbf{H}^{1/2}_{\parallel}(\Gamma)\} \tag{36}$$
$$\mathbf{H}^{1/2}_{\perp}(\Gamma_+) := \{\mathbf{u}_{|\Gamma_+},\ \mathbf{u}\in\mathbf{H}^{1/2}_{\perp}(\Gamma)\} \tag{37}$$
$$\mathbf{H}^{1/2}_{\parallel,00}(\Gamma_+) := \{\mathbf{u}\in\mathbf{H}^{1/2}_{\parallel}(\Gamma_+):\ \tilde{\mathbf{u}}\in\mathbf{H}^{1/2}_{\parallel}(\Gamma)\} \tag{38}$$
$$\mathbf{H}^{1/2}_{\perp,00}(\Gamma_+) := \{\mathbf{u}\in\mathbf{H}^{1/2}_{\perp}(\Gamma_+):\ \tilde{\mathbf{u}}\in\mathbf{H}^{1/2}_{\perp}(\Gamma)\} \tag{39}$$

We refer to [3] for the definition of the related norms.

Let us denote by $H^{-1/2}_{00}(\Gamma_+)$, $\mathbf{H}^{-1/2}_{\parallel}(\Gamma_+)$, $\mathbf{H}^{-1/2}_{\perp}(\Gamma_+)$, $\mathbf{H}^{-1/2}_{\parallel,00}(\Gamma_+)$ and $\mathbf{H}^{-1/2}_{\perp,00}(\Gamma_+)$ their dual spaces, respectively; and the duality products by $\langle\cdot,\cdot\rangle_{1/2,00,\Gamma_+}$, $\langle\cdot,\cdot\rangle_{\parallel,1/2,\Gamma_+}$, $\langle\cdot,\cdot\rangle_{\perp,1/2,\Gamma_+}$, $\langle\cdot,\cdot\rangle_{\parallel,1/2,00,\Gamma_+}$ and $\langle\cdot,\cdot\rangle_{\perp,1/2,00,\Gamma_+}$, respectively. By standard arguments, the differential operators defined in section 5 can be easily adapted to functions defined only on Γ_+. For example, the gradient and divergence operators read now

$$\begin{aligned}&\nabla_{\Gamma_+}: H^{1/2}_{00}(\Gamma_+)\to\mathbf{H}^{-1/2}_{\perp}(\Gamma_+)\quad\text{and}\quad\nabla_{\Gamma_+}: H^{1/2}(\Gamma_+)\to\mathbf{H}^{-1/2}_{\perp,00}(\Gamma_+)\\&\mathrm{div}_{\Gamma_+}:\mathbf{H}^{1/2}_{\perp}(\Gamma_+)\to H^{-1/2}_{00}(\Gamma_+)\quad\text{and}\quad\mathrm{div}_{\Gamma_+}:\mathbf{H}^{1/2}_{\perp,00}(\Gamma_+)\to H^{-1/2}(\Gamma_+)\end{aligned} \tag{40}$$

In a similar way, the operators $\mathbf{curl}_{\Gamma_+}$ and curl_{Γ_+} are defined.

Moreover, the following properties hold:

$$(\nabla_\Gamma\beta)_{|\Gamma_+} = \nabla_{\Gamma_+}(\beta_{|\Gamma_+})\quad\text{and}\quad(\mathbf{curl}_\Gamma\beta)_{|\Gamma_+} = \mathbf{curl}_{\Gamma_+}(\beta_{|\Gamma_+})\quad\forall\beta\in H^{1/2}(\Gamma) \tag{41}$$

Indeed, let $\mathbf{v} \in \mathbf{H}^{1/2}_{\perp,00}(\Gamma_+) \cap \Pi_{j \in \mathscr{I}_+} \mathscr{D}(\Gamma_j)^2$, the following chain of equalities proves the validity of (41):

$$\langle (\nabla_\Gamma \beta)_{|\Gamma_+}, \mathbf{v} \rangle_{\perp,1/2,00,\Gamma_+} = \langle \nabla_\Gamma \beta, \tilde{\mathbf{v}} \rangle_{\perp,1/2,\Gamma} = -\langle \mathrm{div}_\Gamma \tilde{\mathbf{v}}, \beta \rangle_{1/2,\Gamma}$$
$$= -\langle \mathrm{div}_{\Gamma_+} \mathbf{v}, \beta_{|\Gamma_+} \rangle_{1/2,\Gamma_+} = \langle \nabla_{\Gamma_+}(\beta_{|\Gamma_+}), \mathbf{v} \rangle_{\perp,1/2,00,\Gamma_+}$$

where the last inequality corresponds to the definition of div_{Γ_+}.

The proof of Proposition 4.7 applies unchanged and proves also:

Proposition 6.2. The operators $\nabla_{\Gamma_+}: H^{1/2}(\Gamma_+) \to \mathbf{H}^{-1/2}_{\perp,00}(\Gamma_+)$ and $\nabla_{\Gamma_+}: H^{1/2}_{00}(\Gamma_+) \to \mathbf{H}^{-1/2}_{\perp}(\Gamma_+)$ have closed ranges.

Of course, by the open mapping theorem, the operator div_{Γ_+} is surjective in the corresponding spaces.

Let us now prove a trace theorem. For that, let us recall

Proposition 6.3. Let

$$\mathbf{H}^{-1/2}_{\|}(\mathrm{div}_{\Gamma_+}, \Gamma_+) := \{\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}_{\|}(\Gamma_+): \mathrm{div}_{\Gamma_+} \boldsymbol{\lambda} \in H^{-1/2}(\Gamma_+)\} \tag{42}$$

$$\mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}_{\Gamma_+}, \Gamma_+) := \{\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}_{\perp}(\Gamma_+): \mathrm{curl}_{\Gamma_+} \boldsymbol{\lambda} \in H^{-1/2}(\Gamma_+)\} \tag{43}$$

Moreover, let $H^1(\partial\Gamma_+)$ denote the standard H^1 space on the Lipschitz manifold $\partial\Gamma_+$; $H^{-1}(\partial\Gamma_+)$ is its dual space (with $L^2(\partial\Gamma_+)$ as pivot) and $\langle \cdot, \cdot \rangle_{1,\partial\Gamma_+}$ is the corresponding duality pairing.

Let the operators $t_\nu: \mathbf{H}^{-1/2}_{\|}(\mathrm{div}_{\Gamma_+}, \Gamma_+) \cap \mathscr{D}(\bar{\Omega})_{|\Gamma_+} \to H^{-1}(\partial\Gamma_+)$ and $t_\tau: \mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}_{\Gamma_+}, \Gamma_+) \cap \mathscr{D}(\bar{\Omega})_{|\Gamma_+} \to H^{-1}(\partial\Gamma_+)$ be defined by the mappings $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda} \cdot \mathbf{v}_{+|\partial\Gamma_+}$ and $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda} \cdot \boldsymbol{\tau}_{+|\partial\Gamma_+}$, respectively.

They can be extended to linear and continuous operators from $\mathbf{H}^{-1/2}_{\|}(\mathrm{div}_{\Gamma_+}, \Gamma_+)$ and $\mathbf{H}^{-1/2}_{\perp}(\mathbf{curl}_{\Gamma_+}, \Gamma_+)$, respectively.

Moreover, if $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}_{\|}(\mathrm{div}_\Gamma, \Gamma)$, we have that $t_\nu(\boldsymbol{\lambda}_{|\Gamma_+}) + t_\nu^-(\boldsymbol{\lambda}_{|\Gamma_-}) = 0$ at $\partial\Gamma_+$, where t_ν^- denotes the same mapping as t_ν, but on the side Γ_-.

The proof of this proposition can be found in [3].

We set,

$$\mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+}, \Gamma_+) = \{\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}_{\|,00}(\Gamma_+): \mathrm{div}_{\Gamma_+} \boldsymbol{\lambda} \in H^{-1/2}_{00}(\Gamma_+)\} \tag{44}$$

$$\mathbf{H}^{-1/2}_{\|}(\mathrm{div}^0_{\Gamma_+}, \Gamma_+) = \{\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}_{\|}(\mathrm{div}_{\Gamma_+}, \Gamma_+): t_\nu(\boldsymbol{\lambda}) = 0\} \tag{45}$$

$$\mathscr{H}(\Gamma_+) = \{u \in H^1(\Gamma_+): \Delta_{\Gamma_+} u \in H^{-1/2}(\Gamma_+), t_\nu(\nabla_{\Gamma_+} u) = 0\} \tag{46}$$

$$\mathscr{H}_{00}(\Gamma_+) = \{u \in H^1_0(\Gamma_+): \Delta_{\Gamma_+} u \in H^{-1/2}_{00}(\Gamma_+)\} \tag{47}$$

The following holds.

Theorem 6.4. The following decompositions are valid:

$$\mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+}, \Gamma_+) = \mathbf{H}^{1/2}_{\perp}(\Gamma_+) + \mathbf{curl}_{\Gamma_+} H^{1/2}(\Gamma_+) \tag{48}$$

$$\mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+}, \Gamma_+) = \nabla_{\Gamma_+} \mathscr{H}_{00}(\Gamma_+) \oplus \mathbf{curl}_{\Gamma_+} H^{1/2}(\Gamma_+) \tag{49}$$

$$\mathbf{H}_{\|}^{-1/2}(\mathrm{div}^0_{\Gamma_+}, \Gamma_+) = \mathbf{H}^{1/2}_{\perp,00}(\Gamma_+) + \mathbf{curl}_{\Gamma_+} H^{1/2}_{00}(\Gamma_+) \tag{50}$$

$$\mathbf{H}_{\|}^{-1/2}(\mathrm{div}^0_{\Gamma_+}, \Gamma_+) = \nabla_{\Gamma_+}\mathscr{H}(\Gamma_+) \oplus \mathbf{curl}_{\Gamma_+} H^{1/2}_{00}(\Gamma_+) \tag{51}$$

Here, $\oplus$ denotes a direct sum, in the sense of Theorem 5.1.

Proof. Let $\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+}, \Gamma_+)$. Since $\mathrm{div}_{\Gamma_+}\boldsymbol{\lambda} \in H^{-1/2}_{00}(\Gamma_+)$, using Proposition 6.2, we deduce that there exists a $\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{\perp}(\Gamma_+)$ such that $\mathrm{div}_{\Gamma_+}(\boldsymbol{\lambda} - \boldsymbol{\psi}) = 0$. By using again Proposition 6.2, there exists a $\beta \in H^{1/2}(\Gamma_+)$ such that $\boldsymbol{\lambda} = \boldsymbol{\psi} + \mathbf{curl}_{\Gamma_+}\beta$: (48) holds.

Now, using Theorem 6.1, we decompose ψ as

$$\boldsymbol{\psi} = \nabla_{\Gamma_+} q + \mathbf{curl}_{\Gamma_+} p \quad q \in H^1_0(\Gamma_+),\ p \in H^1(\Gamma_+)$$

Since $\mathrm{div}_{\Gamma_+}(\boldsymbol{\psi}) \in H^{-1/2}_{00}(\Gamma_+)$, we deduce $q \in \mathscr{H}_{00}(\Gamma_+)$. We have then $\boldsymbol{\lambda} = \nabla_{\Gamma_+} q + \mathbf{curl}_{\Gamma_+}(\beta + p)$. It is easy to see that the sum is direct and then (49) is proved.

The proof of (50) and (51) could be achieved with a similar technique but we propose here an alternative. Let $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}^0_{\Gamma_+}, \Gamma_+)$. We know, by Proposition 6.3, that $\tilde{\boldsymbol{\lambda}} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma)$. Now using, Theorem 4.1, we have

$$\tilde{\boldsymbol{\lambda}} = \boldsymbol{\psi} + \mathbf{curl}_{\Gamma}\beta \quad \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{\perp}(\Gamma), \quad \beta \in H^{1/2}(\Gamma)/\mathbb{R}$$

Since $\tilde{\boldsymbol{\lambda}}_{|\Gamma_-} = 0$, we deduce $\psi_{|\Gamma_-} = -(\mathbf{curl}_{\Gamma}\beta)_{|\Gamma_-}$. By means of Theorem 3.4 in [3] and (41), we know that $\beta_{|\Gamma_-} \in H^{3/2}(\Gamma_-) := H^{3/2}(\Gamma)_{|\Gamma_-}$: there exists a function $\beta_R \in H^{3/2}(\Gamma)$ such that $\beta_{R|\Gamma_-} = \beta_{|\Gamma_-}$, we then have

$$\boldsymbol{\lambda} = (\boldsymbol{\psi} + \mathbf{curl}_{\Gamma} B_R)_{|\Gamma_+} + \mathbf{curl}_{\Gamma_+}(\beta - \beta_R)$$

where $\boldsymbol{\varphi} = (\boldsymbol{\psi} + \mathbf{curl}_{\Gamma}\beta_R)_{|\Gamma_+} \in \mathbf{H}^{1/2}_{\perp,00}(\Gamma_+)$ and $(\beta - \beta_R)_{|\Gamma_+} \in H^{1/2}_{00}(\Gamma_+)$. (50) is proved. As before, we decompose now $\boldsymbol{\varphi}$ by means of Theorem 6.1, and we get

$$\boldsymbol{\varphi} = \nabla_{\Gamma_+} p + \mathbf{curl}_{\Gamma_+} q \quad p \in \mathscr{H}(\Gamma_+),\ q \in H^{1/2}_{00}(\Gamma_+)$$

By substitution, (51) is proved. □

Remark 6.5. Of course, with self-explanatory notations, we have that:

$$\mathbf{H}^{-1/2}_{\perp,00}(\mathrm{curl}_{\Gamma_+}, \Gamma_+) = \mathbf{H}^{1/2}_{\|}(\Gamma_+) + \nabla_{\Gamma_+} H^{1/2}(\Gamma_+) \tag{52}$$

$$\mathbf{H}^{-1/2}_{\perp,00}(\mathrm{curl}_{\Gamma_+}, \Gamma_+) = \mathbf{curl}_{\Gamma_+}\mathscr{H}_{00}(\Gamma_+) \oplus \nabla_{\Gamma_+} H^{1/2}(\Gamma_+) \tag{53}$$

$$\mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}^0_{\Gamma_+}, \Gamma_+) = \mathbf{H}^{1/2}_{\|,00}(\Gamma_+) + \nabla_{\Gamma_+} H^{1/2}_{00}(\Gamma_+) \tag{54}$$

$$\mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}^0_{\Gamma_+}, \Gamma_+) = \mathbf{curl}_{\Gamma_+}\mathscr{H}(\Gamma_+) \oplus \nabla_{\Gamma_+} H^{1/2}_{00}(\Gamma_+) \tag{55}$$

The following theorem is an easy consequence of the above:

Theorem 6.6. The mapping $\gamma_\tau^+ : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+}, \Gamma_+)$ (respectively its restriction $\gamma_\tau^{+,0} : \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\|}^{-1/2}(\mathrm{div}^0_{\Gamma_+}, \Gamma_+)$) which associates to a vector field $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ (resp. to $\mathbf{u} \in \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega)$) its tangential components on Γ_+, that is $\mathbf{u} \wedge \mathbf{n}_{|\Gamma_+}$, is linear continuous and admits a continuous inverse.

Proof. The fact that both mappings are linear and continuous is straightforward. As far as surjectivity is concerned, let us start by analysing the mapping $\gamma_\tau^{+,0}$. Let $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}^0_{\Gamma_+}, \Gamma_+)$, it is decomposed as $\boldsymbol{\lambda} = \boldsymbol{\psi} + \mathbf{curl}_{\Gamma_+}\beta$ for some $\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{\perp,00}(\Gamma_+)$ and $\beta \in H^{1/2}_{00}(\Gamma_+)$. Now, the function $\mathbf{u} = \mathscr{R}_\gamma\widetilde{\boldsymbol{\psi}} + \nabla(\mathscr{R}_1\widetilde{\beta})$ is the extension of $\boldsymbol{\lambda}$ to Ω.

We consider now the case of the mapping γ_τ^+. Let $\boldsymbol{\mu} \in \mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+}, \Gamma_+)$. According to Theorem 6.4, $\boldsymbol{\mu}$ is decomposed as $\boldsymbol{\mu} = \boldsymbol{\varphi} + \mathbf{curl}_{\Gamma_+}\alpha$, with $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}_{\perp}(\Gamma_+)$ and $\alpha \in H^{1/2}(\Gamma_+)$. Now, both functions $\boldsymbol{\varphi}$ and α can be extended to the whole boundary. On the extended functions, one can apply profitably Theorem 5.4.

As usual, the continuity of the inverse stems from the open mapping theorem. □

Remark 6.7. With obvious notation, we have also obtained that the mappings

$$\pi_\tau^+ : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}_{\perp,00}(\mathrm{curl}_{\Gamma_+}, \Gamma_+), \quad \pi_\tau^{+,0} : \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}^0_{\Gamma_+}, \Gamma_+)$$

both defined as $\mathbf{u} \mapsto \pi_\tau(\mathbf{u})_{|\Gamma_+}$, are linear continuous and surjective.

We focus now our attention on the extension of the duality given in Definition 5.6 to the case of a part Γ_+ of the boundary.

The following identities are consequences of the theory developed until now:

$$\mathbf{H}^{-1/2}_{\|}(\mathrm{div}^0_{\Gamma_+}, \Gamma_+) = (\mathbf{H}^{-1/2}_{\perp,00}(\mathrm{curl}_{\Gamma_+}, \Gamma_+))'$$

$$\mathbf{H}^{-1/2}_{\|,00}(\mathrm{div}_{\Gamma_+}, \Gamma_+) = (\mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}^0_{\Gamma_+}, \Gamma_+))'$$

with $\mathbf{L}^2_t(\Gamma_+)$ as pivot space. Since the two identities are completely symmetric, let us define the duality operator only for the first one (the second one is then straightforward).

Definition 6.8. Let $\mathbf{u} \in \mathbf{H}^{-1/2}_{\|}(\mathrm{div}^0_{\Gamma_+}, \Gamma_+)$ and $\mathbf{v} \in \mathbf{H}^{-1/2}_{\perp,00}(\mathbf{curl}_{\Gamma_+}, \Gamma_+)$. Using Theorem 6.4, there exists $\alpha_u \in \mathscr{H}(\Gamma_+)$, $\alpha_v \in \mathscr{H}_{00}(\Gamma_+)$ and $\beta_u \in H^{1/2}_{00}(\Gamma_+)$, $\beta_v \in H^{1/2}(\Gamma_+)$ such that:

$$\mathbf{u} = \nabla_{\Gamma_+}\alpha_u + \mathbf{curl}_{\Gamma_+}\beta_u \quad \text{and} \quad v = \mathbf{curl}_{\Gamma_+}\alpha_v + \nabla_{\Gamma_+}\beta_v$$

Let us define

$${}_{\gamma,\Gamma_+}\langle \mathbf{u}, \mathbf{v}\rangle_{\pi,00,\Gamma_+} = -\langle \Delta_{\Gamma_+}\alpha_u, \beta_v\rangle_{1/2,\Gamma_+} + \langle \Delta_{\Gamma_+}\alpha_v, \beta_u\rangle_{1/2,00,\Gamma_+}$$

Finally, by standard argument, we conclude that the following integration by parts formula holds true:

$$\int_\Omega \{\mathbf{curl}\ \mathbf{v}\cdot\mathbf{u} - \mathbf{curl}\ \mathbf{u}\cdot\mathbf{v}\}\,\mathrm{d}\Omega = {}_{\gamma,\Gamma_+}\langle \gamma_\tau^{+,0}\mathbf{u}, \pi_\tau^+\mathbf{v}\rangle_{\pi,00,\Gamma_+}$$

$$\forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega),\ \mathbf{u} \in \mathbf{H}_{0,\Gamma_-}(\mathbf{curl}, \Omega)$$

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