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Solution of axisymmetric Maxwell equations

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SUMMARY

In this article, we study the static and time-dependent Maxwell equations in axisymmetric geometry. Using the mathematical tools introduced in (*Math. Meth. Appl. Sci.* 2002; **25**:49), we investigate the decoupled problems induced in a meridian half-plane, and the splitting of the solution in a regular part and a singular part, the former being in the Sobolev space H^1 component-wise. It is proven that the singular parts are related to singularities of Laplace-like or wave-like operators. We infer from these characterizations: (i) the finite dimension of the space of singular fields; (ii) global space and space–time regularity results for the electromagnetic field. This paper is the continuation of (*Modél. Math. Anal. Numér*. 1998; **32**:359, *Math. Meth. Appl. Sci.* 2002; **25**:49). Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: Maxwell equations; axisymmetry; conical vertices; reentrant edges; Hodge decomposition; singularities; regularity of solutions

1. INTRODUCTION

In this paper, we propose to study the static and time-dependent Maxwell equations in axisymmetric geometry. It is the continuation of Reference [1], where the mathematical tools were introduced, and will be followed by another paper where numerical developments and applications will be shown.

Though it is mathematically a two-dimensional (2D) situation, the axisymmetric case can be viewed, from a modelling point of view, as an intermediate between a full three-dimensional (3D) problem and a 2D one. Indeed, while the geometry of real devices is very rarely Cartesian, it is much more common to have an axial symmetry, at least approximately or locally. In other words, the axisymmetric geometry can be considered as a zero-order approximation of a real 3D case [2]. Nevertheless, very few mathematical analyses had been carried out so far in the framework of axisymmetric problems [3,2].

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In Reference [1], we recalled two basic principles for solving the static or time-dependent Maxwell equations in an axisymmetric domain Ω with circular edges and conical vertices. The first is the *space decomposition* principle, which states that the natural spaces associated with Maxwell's equations

$$\mathscr{X}^d = \mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega) \quad \text{and} \quad \mathscr{X}^n = \mathbf{H}(\mathbf{curl};\Omega)\cap\mathbf{H}_0(\mathrm{div};\Omega)$$

can be split as the direct sum of a *regular* part $\mathscr{X}_R^{d/n} = \mathscr{X}^{d/n}\cap\mathbf{H}^1(\Omega)$—which is closed within the natural space—and a suitably chosen *singular* part $\mathscr{X}_S^{d/n}$. In the time-dependent case, this splitting is continuous with respect to time. The second is the Curie principle (cf. Reference [2]): the solution is axisymmetric iff so are the data and initial conditions.

Namely, we proved that the axisymmetric subspaces of $\mathscr{X}^d$ and $\mathscr{X}^n$ obey the decomposition principle except, in the electric case, when some conical vertex has an aperture angle $\pi/\beta_\star$, characterized by the presence of the eigenvalue 3/4 in the spectrum of the local Laplace operator. (In this article, we shall always assume that this phenomenon does not occur.) We also proposed singular complements related to dual singularities of Laplace-like problems. These results parallel those of References [4,5], but *cannot* be considered as a mere application of them, since the domain Ω is *not* a polyhedron (nor even a 'curved polyhedron'), and specific treatments have to be designed to handle the conical vertices.

The article is organized as follows. Section 2 recalls the basic notations related to the axisymmetric geometry, and the functional-analytic framework adapted to the study of boundary-value problems in this geometry (i.e. weighted Sobolev spaces and the variational problems defined on them). Section 3 is devoted to the study of the singular solutions to the Laplace-like problems, which allows one to determine the dimension of the singular parts. In Section 4, an in-depth examination of the static Maxwell equations is performed. We focus on two topics: the decoupling, induced by the axial symmetry, of the equations into two systems posed in the meridian half-plane; and its relationship with the splitting in regular and singular parts. As a by-product, we determine the global space regularity of the electromagnetic field. This study is one of the two key ingredients needed for the understanding of the time-dependent equations. The other one is the analysis of singularities of a wave-like problem, which is carried out in Section 5. Finally, Section 6 investigates the time-dependent equations, in the spirit of Section 4; we conclude with a space–time regularity result for the electromagnetic field.

2. BASIC DEFINITIONS AND NOTATIONS

2.1. The axisymmetric geometry and operators

Let Ω be a bounded and simply connected axisymmetric domain of $\mathbb{R}^3$, Γ its perfectly conducting boundary, and $\mathbf{n}$ the unit outward normal to Γ. Ω is generated by the rotation of a polygon ω around one of its sides, denoted γ_a. The other sides are denoted γ_i, $1\leqslant i\leqslant n+1$, and generate the faces Γ_i, $1\leqslant i\leqslant n+1$, of Γ. The notations are the same as in Reference [1] and are recalled in Figure 1. We shall mostly use the cylindrical co-ordinates (r,θ,z).

In this article, we generally assume that the fields defined on Ω possess an *axial symmetry*. The definition of axial symmetry for general scalar- or vector-valued distributions is found in References [2,1]. In practice, it means that they are entirely characterized by the data

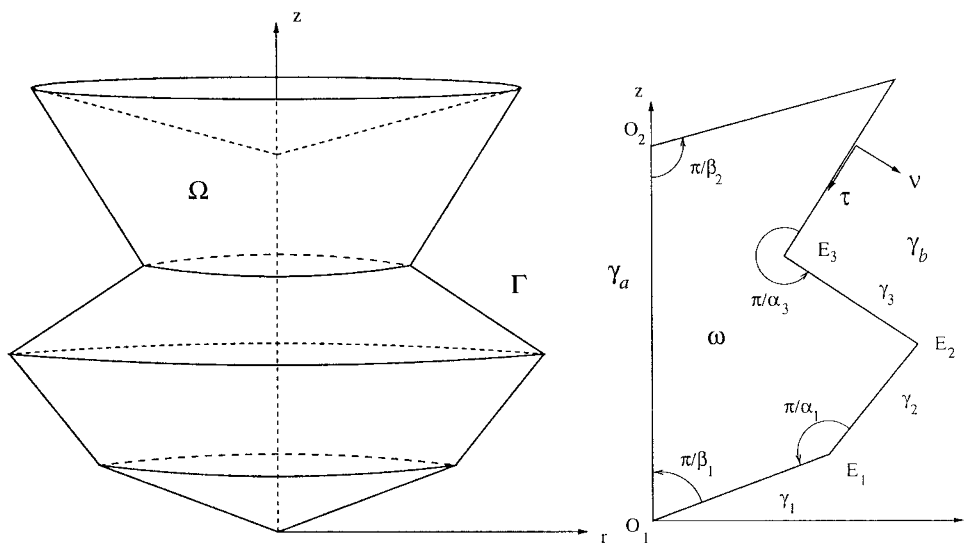

Figure 1. The domains Ω and ω.

of their traces (or the traces of their cylindrical components) in a meridian half-plane, or equivalently that their derivative (or the derivatives of their cylindrical components) with respect to θ vanishes: $T(x, y, z) = T(r, z)$ or $\partial_\theta T = 0$. We denote by the sign $\breve{}$ the subspaces of axisymmetric fields, e.g. $\breve{L}^2(\Omega)$, $\breve{\mathbf{H}}^s(\Omega), \breve{\mathscr{X}}^d \ldots$.

2.2. *Some results of functional analysis*

Weighted Sobolev spaces and traces. The characterization of the subspaces of axisymmetric fields *via* their traces in a meridian half-plane can be found in Reference [2, Section II.2; 1, Section 3]. Let us recall some notations: $L^2_\alpha(\omega)$ is the weighted Lebesgue space

$$L^2_\alpha(\omega) = \left\{ f\colon\ f \text{ is measurable on } \omega, \int_\omega |f|^2 r^\alpha \, \mathrm{d}r \, \mathrm{d}z < +\infty \right\}, \quad \alpha \in \mathbb{R}$$

with its canonical norm $\|\cdot\|_{0,\alpha,\omega}$, and $H^s_\alpha(\omega)$ is the related scale of Sobolev spaces, with the canonical norms $\|\cdot\|_{s,\alpha,\omega}$. We also use the following scales (cf. [2, Theorems II.2.1, II.2.6]):

- $H^s_+(\omega)$ is the trace space of scalar functions in $\breve{H}^s(\Omega)$, as well as of z components of vector fields in $\breve{\mathbf{H}}^s(\Omega)$.
- $H^s_-(\omega)$ is the trace space of r and θ components of vector fields in $\breve{\mathbf{H}}^s(\Omega)$.

Now we state some properties of the $H^s_\alpha(\omega)$ and $H^s_\pm(\omega)$ spaces, and the consequences on axisymmetric Sobolev spaces $\breve{\mathbf{H}}^s(\Omega)$; proofs are found in Reference [6].

Proposition 2.1
Let R be the isometry $L^2_\alpha(\omega) \to L^2_{\alpha-2}(\omega)$, $f \mapsto rf$. One has

$$\mathsf{R}[H^s_-(\omega)] = H^s_{-1}(\omega) \text{ for } 0 \leqslant s \leqslant 1, \quad \mathsf{R}[H^s_-(\omega)] \subset H^s_{-1}(\omega) \text{ for } 1 < s < 2$$

The canonical norm in $\mathsf{R}[H^s_-(\omega)]$ is $\|w\|_{\mathsf{R}[H^s_-(\omega)]}=\|w/r\|_{s,-,\omega}$. Hence, for any *axisymmetric* and *azimuthal* vector field $\mathbf{u}=u_\theta\,\mathbf{e}_\theta$, the following properties are equivalent:

$$\mathbf{u}\in\breve{\mathbf{H}}(\mathbf{curl};\Omega)\;\Leftrightarrow\;\mathbf{u}\in\breve{\mathbf{H}}(\mathbf{curl},\mathrm{div};\Omega)\;\Leftrightarrow\;\mathbf{u}\in\breve{\mathbf{H}}^1(\Omega)$$

and the canonical norms of these spaces are equivalent. Moreover, for $0\leqslant s<1$, $H^s_+(\omega)=H^s_-(\omega)=H^s_1(\omega)$, hence $\mathsf{R}[H^s_+(\omega)]=H^s_{-1}(\omega)$.

Proposition 2.2
Let A be a point in the meridian half-plane; (ρ,ϕ) local polar co-ordinates centred at A, and ω_A the bounded angular sector $\{(\rho,\phi)\colon\ 0<\rho<\rho_0,\ 0<\phi<\phi_0\}$. Let f be a function whose expression in ω_A is $f(\rho,\phi)=\rho^\alpha g(\phi)$, with $g(\phi)\in\mathscr{C}^\infty([0,\phi_0])$.

1. If A is in the *open* half-plane ($r(A)>0$), and ω_A is small enough to ensure $\bar{\omega}_A\cap\gamma_a=\emptyset$, then: $\forall s\in\mathbb{R}^+,\ H^s(\omega_A)=H^s_{\pm1}(\omega_A)$ and $\forall s\in\mathbb{R},\ f\in H^s(\omega_A)\Leftrightarrow s<\alpha+1$.
2. If A stands on the axis ($r(A)=0$), and the axis is taken as the origin of ϕ, then

$$\forall s\in\mathbb{R}^+,\quad\begin{cases} f\in H^s_1(\omega_A)\Leftrightarrow s<\alpha+3/2\\ f\in H^s_{-1}(\omega_A)\Leftrightarrow s<\alpha+1/2 \text{ and } g^{(j)}(0)=0\quad\text{for all } j\in\mathbb{N},\ j\leqslant s\end{cases}$$

The modified Laplacians and their variational spaces. The *modified* Laplacians Δ^+ and Δ^- are defined as

$$\Delta^\pm f=\frac{\partial^2 f}{\partial r^2}\pm\frac{1}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial z^2}$$

Δ^+ is the trace of the 3D scalar Laplacian for an axisymmetric function. And for an axisymmetric, azimuthal vector field $\mathbf{u}$, there holds

$$\mathbf{u}=\frac{\varphi}{r}\,\mathbf{e}_\theta\;\Rightarrow\;\Delta\mathbf{u}=\frac{1}{r}\,\Delta^-\varphi\;\mathbf{e}_\theta \tag{1}$$

The variational theory for Δ^+ can be found in [2, Section II.4.a]. The variational spaces for the Dirichlet, resp. Neumann boundary conditions are: $\mathsf{V}^{d+}=\overset{\diamond}{H}{}^1_1(\omega)=\{v\in H^1_1(\omega)\colon\ v=0 \text{ on } \gamma_b\}$ resp. $\mathsf{V}^{n+}=H^1_1(\omega)$. Similarly, the variational spaces for Dirichlet and Neumann problems with Δ^- are

$$\mathsf{V}^{d-}=\overset{\circ}{H}{}^1_{-1}(\omega)=\{v\in H^1_{-1}(\omega)\colon\ v=0 \text{ on } \gamma\}\quad\text{and}\quad \mathsf{V}^{n-}=H^1_{-1}(\omega)$$

We shall denote either space by $\mathsf{V}^{d/n-}$ or simply V^-. Besides the difficulty of the description of the space $[\mathsf{V}^-]'$ in 2D, another technical point is that the product by r or $1/r$ of a distribution is not defined in general. Yet these operations are underlying in the use of Δ^-, cf. (1). Fortunately, all we need in the sequel are the three particular cases dealt with in the following Proposition.

Proposition 2.3
Let $f\in[\mathsf{V}^-]'$ and $u\in\mathsf{V}^-$ as well as $\lambda\geqslant0$. The equality $-\Delta^-u+\lambda u=f$ in the sense of distributions, supplemented in the Neumann case with the boundary condition $\partial_\nu u|_{\gamma_b}=0\in H^{-1/2}_{-1}(\gamma_b)$, is equivalent to the variational formulation:

$$\forall v\in\mathsf{V}^-,\quad\iint_\omega[\mathbf{grad}\,u\cdot\mathbf{grad}\,v+\lambda uv]\,\frac{\mathrm{d}\omega}{r}=\langle f,v\rangle_{[\mathsf{V}^-]',\mathsf{V}^-} \tag{2}$$

in the three following cases:

1. f is an element of $L^2_{-1}(\omega)$, i.e. $\langle f, v\rangle_{[\mathsf{V}^-]',\mathsf{V}^-} = \iint_\omega f v \frac{d\omega}{r}$;
2. more generally, $f \in L^p_{1-p}(\omega) = \mathsf{R}[L^p_1(\omega)] \simeq \mathsf{R}[\breve{L}^p(\Omega)]$, for $p \geqslant 6/5$;
3. f is an element of the dual of $H^1(\omega)$ or $H^1_0(\omega)$ whose support is away from the axis.

Remark 2.1
The limiting value 6/5 for the Lebesgue exponent p stems from the Sobolev imbedding in the 3D domain Ω: $H^1(\Omega) \subset L^6(\Omega)$, and by duality $L^{6/5}(\Omega) \subset H^1(\Omega)'$.

As a straightforward consequence, we have the following Green's formulae for Δ^-:

Proposition 2.4
Let $\Phi^{n-} = \{u \in \mathsf{V}^{n-}\colon \Delta^- u \in L^2_{-1}(\omega) \text{ and } \partial_\nu u|_{\gamma_b} = 0\}$ and $\Phi^{d-} = \{u \in \mathsf{V}^{d-}\colon \Delta^- u \in L^2_{-1}(\omega)\}$. There holds

$$\forall u \in \mathsf{V}^{d/n-},\ \forall v \in \Phi^{d/n-},\ \iint_\omega \{\Delta^- u\, v + \mathbf{grad}\, u \cdot \mathbf{grad}\, v\} \frac{d\omega}{r} = 0 \tag{3}$$

$$\forall u, v \in \Phi^{d/n-},\ \iint_\omega \{\Delta^- u\, v - u \Delta^- v\} \frac{d\omega}{r} = 0 \tag{4}$$

3. ANALYSIS OF SINGULARITIES OF THE MODIFIED LAPLACIANS

In this section, we study the non-variational solutions in $L^2_{\pm 1}(\omega)$ to 'very weak' Dirichlet problems for $\Delta^\pm$ in ω. Let us define N^{d+} as the space of $p \in L^2_1(\omega)$ satisfying

$$\Delta^+ p = 0 \quad \text{in } \omega \tag{5}$$

$$p = 0 \quad \text{on } \gamma_i, \quad 1 \leqslant i \leqslant n+1 \tag{6}$$

$$p \in \mathscr{C}^\infty(\bar{\omega} \setminus \mathscr{V}_b) \quad \text{for any neighbourhood } \mathscr{V}_b \text{ of } \gamma_b \tag{7}$$

The trace on γ_i is understood in the suitable trace space, cf. [1, Section 5]. Since p is smooth up to any segment included in γ_a, one infers that $\partial_r p|_{\gamma_a} = 0$.

Similarly, let N^{d-} be the set of solutions in $L^2_{-1}(\omega)$ to

$$\Delta^- p = 0 \quad \text{in } \omega \tag{8}$$

$$p = 0 \quad \text{on } \gamma_i, \quad 1 \leqslant i \leqslant n+1 \tag{9}$$

$$\frac{p}{r} \in \mathscr{C}^\infty(\bar{\omega} \setminus \mathscr{V}_b) \quad \text{for any neighbourhood } \mathscr{V}_b \text{ of } \gamma_b \tag{10}$$

The smoothness of p/r up to any segment included in γ_a yields $p|_{\gamma_a} = 0$. We remark that $p \in N^{d-} \Leftrightarrow \mathbf{P} = (p/r)\mathbf{e}_\theta$ is a solution in $\breve{\mathbf{L}}^2(\Omega)$ to

$$\Delta \mathbf{P} = 0 \text{ in } \Omega, \quad P_\theta = 0 \text{ on } \Gamma_i, \quad 1 \leqslant i \leqslant n+1 \tag{11}$$

Obviously, there holds: $N^{d+} \cap H^1_1(\omega) = \{0\}$ and $N^{d-} \cap H^1_{-1}(\omega) = \{0\}$.

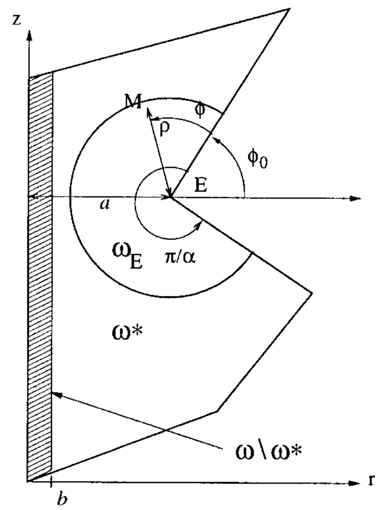

Figure 2. Edge singularity.

Lemma 3.1
Any $p\in N^{d+}$, resp. $p\in N^{d-}$, belongs to $\mathscr{C}^\infty(\bar{\omega}\backslash V)$, where V is any neighbourhood of the corners $E_1,\dots,E_n$, O_1, O_2. Any $\mathbf{P}$ solution to (11) belongs to $\check{\mathbf{C}}^\infty(\bar{\Omega}\backslash\mathbf{V})$, where $\mathbf{V}$ is any neighbourhood of the edges and conical vertices.

Proof
By (7), resp. (10), we only have to prove that p is $\mathscr{C}^\infty$ up to a neighbourhood of any segment included in γ_j. This is away from the axis, hence $\Delta^\pm$ has smooth coefficients and one concludes by a standard bootstrap argument [6]. □

3.1. Local study of singularities near the edges

We look for a local analytical expression of the solution p to (5)–(7), resp. (8)–(10) in a neighbourhood of an edge, i.e. and off-axis corner $E=E_j$ with opening π/α. [We drop the corner subscript j.] In a meridian half-plane, we use the local polar co-ordinates (ρ,ϕ) (see Figure 2). The expression of the modified Laplacians in these co-ordinates reads, with $\phi'=\phi+\phi_0$:

$$\Delta^\pm p=\frac{\partial^2 p}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial p}{\partial\rho}\pm\frac{1}{a+\rho\cos\phi'}\left(\cos\phi'\frac{\partial p}{\partial\rho}-\frac{\sin\phi'}{\rho}\frac{\partial p}{\partial\phi}\right)+\frac{1}{\rho^2}\frac{\partial^2 p}{\partial\phi^2} \tag{12}$$

We settle in a neighbourhood ω_E of E such that $\bar{\omega}_E$ is away from all corners except E and all sides except the ones which meet at E. $\eta\in\mathscr{C}^\infty(\bar{\omega})$ is a cutoff function with the following properties: (i) $\eta\equiv 1$ in ω_E; (ii) $\eta\equiv 0$ outside some neighbourhood $\omega'_E\supset\bar{\omega}_E$ which satisfies the same conditions as ω_E and (iii) η depends on ρ only.

In ω_E, there is no difference between functions of $L^2_1(\omega), L^2_{-1}(\omega)$ or $L^2(\omega)$; and the modified Laplacians are perturbations of the standard Laplacian by less singular terms. Thus, the singularities of $\Delta^\pm$ in $L^2_{\pm 1}(\omega)$ are locally 'close' to the singularities of Δ in $L^2(\omega)$.

Lemma 3.2
Let $p \in N^{d+}$, resp. $p \in N^{d-}$. There exist $c \in \mathbb{R}$ and $\ell \in \mathbb{Z}$, $\ell\alpha > -1$, such that $\eta(p - c\rho^{\ell\alpha}\sin(\ell\alpha\phi)) \in H^1(\omega_E)$.

Proof
Let ω^* be the polygonal domain $\{\mathbf{x} \in \omega\colon r(\mathbf{x}) > b\}$, and γ^* its boundary (see Figure 2). One has

$$\Delta(\eta p) = \mp \frac{\eta}{r}\frac{\partial p}{\partial r} + 2\,\mathbf{grad}\,\eta \cdot \mathbf{grad}\,p + p\Delta\eta \in H^{-1}(\omega^*)$$

Now let $v \in H^1(\omega^*)$ be the variational solution to the Dirichlet problem

$$\Delta v = \Delta(\eta p) \quad \text{in } \omega^*, \qquad v = 0 \quad \text{on } \gamma^*$$

and $p^* \stackrel{\text{def}}{=} \eta p - v$, p^* belongs to $L^2(\omega^*)$, has a vanishing Laplacian in ω^* and a vanishing trace on γ^*. Grisvard [8, pp. 45–56] has established that such a singularity satisfies

$$\eta(p^* - c\rho^{\ell\alpha}\sin(\ell\alpha\phi)) \in H^1(\omega^*)$$

for some c, ℓ such that $\ell\alpha > -1$. Hence $\eta(p - c\rho^{\ell\alpha}\sin(\ell\alpha\phi)) \in H^1(\omega_E)$. □

By Proposition 2.2, the condition $\ell\alpha > -1$ is needed for the term $\rho^{\ell\alpha}\sin(\ell\alpha\phi)$ to be in $L^2(\omega_E)$. If the corner is outgoing ($\alpha > 1$), this implies $\ell \geqslant 0$ and the latter expression is indeed in $H^1(\omega_E)$. Hence any element of $N^{d\pm}$ is locally H^1. For a reentrant corner ($1/2 < \alpha < 1$), however, the term $\rho^{\ell\alpha}\sin(\ell\alpha\phi)$ is locally L^2 but not H^1 for $\ell = -1$, and locally H^1 for $\ell \geqslant 0$. As a consequence, there exists a unique (up to a multiplication by a constant) *local* singular function, as shown by the following lemma.

Lemma 3.3
If the corner E is reentrant, there exists $\sigma^\pm \in N^{d\pm}$ such that

$$\sigma^\pm(\rho, \phi) - \eta\rho^{-\alpha}\sin\alpha\phi \in H^1_{\pm 1}(\omega)$$

Proof
Let $u(\rho, \phi) = \eta\rho^{-\alpha}\sin\alpha\phi$; this function vanishes on γ_b and γ_a, and so does its normal derivative on γ_a. In ω_E, $\eta \equiv 1$ and by (12),

$$f_\pm = \Delta^\pm u = \mp \frac{\alpha\rho^{-\alpha-1}}{a + \rho\cos\phi'}(\cos\phi'\sin\alpha\phi + \sin\phi'\cos\alpha\phi)$$

As $-\alpha - 1 > -2$, $f_\pm \in H^{-1}(\omega_E)$; elsewhere it is $\mathscr{C}^\infty$ and vanishes near the axis. Hence, by [2, Proposition II.4.1] and our Proposition 2.3, one can solve variationally the Dirichlet problems

$$\Delta^\pm w^\pm = f_\pm \quad \text{in } \omega, \qquad w^\pm = 0 \quad \text{on } \gamma_b$$

in $\mathring{H}^1_{-1}(\omega)$ or $\overset{\diamond}{H}{}^1_1(\omega)$. As $f_\pm$ vanishes near the axis, w^+ is smooth there and $\partial_r w|_{\gamma_a} = 0$. And since $w^- \in H^1_{-1}(\omega) \subset H^1_-(\omega)$, it satisfies $w|_{\gamma_a} = 0$ [1, Proposition 3.18]. Finally, the difference $\sigma^\pm = u - w^\pm \in L^2_{\pm 1}(\omega)$ has a vanishing modified Laplacian $\Delta^\pm$ and a vanishing trace on γ_b; on γ_a, $\sigma^\pm$ satisfies the same boundary condition as $w^\pm$. So, $\sigma^\pm \in N^{d\pm}$. □

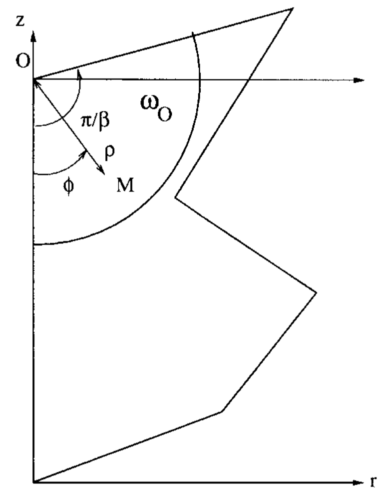

Figure 3. Conical singularity.

3.2. Local study of singularities near the conical vertices

Similarly to the previous subsection, we look for an analytical expression of p near a conical vertex O. [Here too, we drop the vertex subscript.] We use once more local polar co-ordinates (ρ, ϕ) in a meridian half-plane (see Figure 3). The expression of the modified Laplacians in these variables is

$$\Delta^+ p = \frac{\partial^2 p}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial p}{\partial \rho} + \frac{\cot\phi}{\rho^2}\frac{\partial p}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 p}{\partial \phi^2} \tag{13}$$

$$\Delta^- p = \frac{\partial^2 p}{\partial \rho^2} - \frac{\cot\phi}{\rho^2}\frac{\partial p}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 p}{\partial \phi^2} \tag{14}$$

Unlike the previous situation, the complementary terms (with respect to the standard Laplacian) are not less singular. However, the whole of $\Delta^\pm$ enjoys another nice feature: variable separation. To take advantage of it, let us introduce the angular parts

$$(\Lambda^\pm u)(\phi) = -u''(\phi) \mp \cot\phi \; u'(\phi)$$

and the Hilbert spaces

$$\mathscr{H}_+ = L^2\left(\left]0, \frac{\pi}{\beta}\right[, \sin\phi \,\mathrm{d}\phi\right), \quad \mathscr{H}_- = L^2\left(\left]0, \frac{\pi}{\beta}\right[, \frac{\mathrm{d}\phi}{\sin\phi}\right)$$

The boundary conditions in the definition of $N^{d\pm}$ suggest to consider the domains

$$D(\Lambda^+) = \{u \in \mathscr{H}_+ : \Lambda^+ u \in \mathscr{H}_+ \text{ and } u'(0) = u(\pi/\beta) = 0\} \tag{15}$$

$$D(\Lambda^-) = \{u \in \mathscr{H}_- : \Lambda^- u \in \mathscr{H}_- \text{ and } u(0) = u(\pi/\beta) = 0\} \tag{16}$$

We notice that $\Lambda^{\pm}$ defines an unbounded operator in $\mathscr{H}_{\pm}$, which is self-adjoint in the above domain, strictly positive and has a compact inverse; hence,

Theorem 3.4
There exists a Hilbert basis $(u_\ell^{\pm})_{\ell\in\mathbb{N}^*}$ of $\mathscr{H}_{\pm}$ made of eigenfunctions of $\Lambda^{\pm}$. The corresponding eigenvalues $(\lambda_\ell^{\pm})_{\ell\in\mathbb{N}^*}$ are strictly positive and go to infinity. Moreover, $((u_\ell^{\pm})'/\sqrt{\lambda_\ell^{\pm}})_{\ell\in\mathbb{N}^*}$ is an orthonormal family in $\mathscr{H}_{\pm}$.

These eigenfunctions and -values can be determined by simple calculations [6]. Denoting $P_\nu^\mu(x)$ the Legendre function, we define $(\nu_\ell^+)_{\ell\in\mathbb{N}^*}$, resp. $(\nu_\ell^-)_{\ell\in\mathbb{N}^*}$ as the increasing sequences of $\nu>0$, resp. $\nu>1$, satisfying the conditions

$$P_\nu^0\left(\cos\frac{\pi}{\beta}\right)=0 \quad \text{resp. } P_{\nu-1}^1\left(\cos\frac{\pi}{\beta}\right)=0 \tag{17}$$

The eigenpairs $(\lambda^{\pm},u^{\pm})$ of $\Lambda^{\pm}$ are

$$\lambda_\ell^+=\nu_\ell^+(\nu_\ell^++1),\quad u_\ell^+(\phi)=C_\ell^+P_{\nu_\ell^+}^0(\cos\phi) \tag{18}$$

$$\lambda_\ell^-=\nu_\ell^-(\nu_\ell^--1),\quad u_\ell^-(\phi)=C_\ell^-\sin\phi\, P_{\nu_\ell^--1}^1(\cos\phi) \tag{19}$$

where the $C_\ell^{\pm}$ are normalization coefficients in $\mathscr{H}_{\pm}$. The three following facts are worth noting: (i) all eigenvalues of $\Lambda^{\pm}$ are simple; (ii) as shown by tables of Legendre functions (see e.g. Reference [9]), there are no $x\in]-1,1[$ nor $\nu\in]1,2]$ such that $P_{\nu-1}^1(x)=0$; hence, $\nu_1^->2$; (iii) the $\nu_\ell^{\pm}$ have an asymptotic linear behaviour, as shown by

Lemma 3.5
One has $\nu_\ell^{\pm}\sim\beta\ell$ when $\ell\to+\infty$.

Proof
It relies on the asymptotic expansion [9, Equation (8.6.6)] of $P_\nu^\mu(\cos\phi)$ when $\nu\to+\infty$ with ϕ and $\mu\geqslant 0$ fixed, taking into account the following equivalence: $\Gamma(n+\alpha)/\Gamma(n)\sim n^\alpha$ when $n\to+\infty$ and α is fixed [9, Equation (6.1.45)]. □

Let us return to the singularities of the modified Laplacians. As $p(\rho,\phi)\in N^{d\pm}$ belongs to $L_{\pm1}^2(\omega)\cap\mathscr{C}^\infty(\bar{\omega}\backslash V)$, for any neighbourhood V of O, it can be viewed as a $\mathscr{C}^\infty$ function of $\rho\in]0,R[$ (for some $R>0$) with values in $\mathscr{H}_{\pm}$. Then (5), resp. (8), becomes

$$\frac{\partial^2 p}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial p}{\partial\rho}-\frac{1}{\rho^2}\Lambda^+p=0 \tag{20}$$

resp.

$$\frac{\partial^2 p}{\partial\rho^2}-\frac{1}{\rho^2}\Lambda^-p=0 \tag{21}$$

Hence p takes its values in $D(\Lambda^{\pm})$ defined by (15) and (16). We denote $D_R=\omega_O\cap\{0<\rho<R\}$.

Lemma 3.6
Let $p\in\mathscr{C}^\infty(]0,R[;D(\Lambda^-))$ be a solution to (21), and assume $p\in L^2_{-1}(D_R)=L^2(D_R,(\rho\sin\phi)^{-1}\rho\,\mathrm{d}\rho\,\mathrm{d}\phi)$. There exists a sequence $(c_\ell)_{\ell\in\mathbb{N}^*}$ satisfying

$$p(\rho,\phi)=\sum_{\ell=1}^{+\infty}c_\ell\rho^{v_\ell^-}u_\ell^-(\phi) \tag{22}$$

$$|c_\ell|\leqslant KR^{-v_\ell^-}\sqrt{v_\ell^-} \tag{23}$$

where the constant K depends only on p.

Proof
For a fixed ρ, one expands $\phi\mapsto p(\rho,\phi)$ on the Hilbert basis $(u_\ell^-)_{\ell\in\mathbb{N}^*}$:

$$p(\rho,\phi)=\sum_{\ell=1}^{+\infty}p_\ell(\rho)u_\ell^-(\phi),\quad \text{where } p_\ell(\rho)=\int_0^{\pi/\beta}p(\rho,\phi)u_\ell^-(\phi)\,\frac{\mathrm{d}\phi}{\sin\phi} \tag{24}$$

Then (21) yields

$$p_\ell''(\rho)-v_\ell^-(v_\ell^--1)\frac{p_\ell(\rho)}{\rho^2}=0,\quad \text{hence: } p_\ell(\rho)=c_\ell\rho^{v_\ell^-}+d_\ell\rho^{1-v_\ell^-}$$

By Proposition 2.2, $\rho^\sigma u_\ell^-(\phi)\in L^2_{-1}(D_R)$ iff $\sigma>-1/2$. As $v_\ell^->2$, $d_\ell\rho^{1-v_\ell^-}u_\ell^-$ cannot be in $L^2_{-1}(D_R)$ unless $d_\ell=0$; on the other hand, $c_\ell\rho^{v_\ell^-}u_\ell^-\in L^2_{-1}(D_R)$ for any ℓ.

To obtain the estimate (23), we apply the Schwarz inequality to (24):

$$p_\ell(\rho)^2\leqslant\int_0^{\pi/\beta}p(\rho,\phi)^2\,\frac{\mathrm{d}\phi}{\sin\phi}$$

since the functions u_ℓ^- are normalized. Hence,

$$\int_0^R p_\ell(\rho)^2\,\mathrm{d}\rho\leqslant\int_0^R\int_0^{\pi/\beta}p(\rho,\phi)^2\,\frac{\mathrm{d}\rho\,\mathrm{d}\phi}{\sin\phi}=\|p\|^2_{L^2_{-1}(D_R)}=\mathrm{cst}$$

The integral on the left-hand side is $c_\ell^2R^{2v_\ell^-+1}/(2v_\ell^-+1)$, and (23) follows. □

Lemma 3.7
Let $p\in\mathscr{C}^\infty(]0,R[;D(\Lambda^+))$ be a solution to (20), and assume $p\in L^2_1(D_R)=L^2(D_R,(\rho\sin\phi)\rho\mathrm{d}\rho\,\mathrm{d}\phi)$. There exists a sequence $(c_\ell)_{\ell\in\mathbb{N}^*}$ satisfying

$$p(\rho,\phi)=d_1\rho^{-1-v_1^+}u_1^+(\phi)+\sum_{\ell=1}^{+\infty}c_\ell\rho^{v_\ell^+}u_\ell^+(\phi) \tag{25}$$

$$|c_\ell|\leqslant KR^{-v_\ell^+}\sqrt{v_\ell^+} \tag{26}$$

for some constants K, d_1 depending only on p. Moreover, if β is greater than $\beta_\star$, the exceptional value for which $\check{\mathscr{X}}_R^d$ is not closed, then necessarily $d_1=0$. $\beta_\star$ is defined by $P^0_{1/2}(\cos\pi/\beta_\star)=0$; its value is $\beta_\star\simeq1,3771$, or $\pi/\beta_\star\simeq130°43'$.

Proof
Similarly to the previous proof, one writes: $p(\rho,\phi)=\sum_{\ell=1}^{+\infty} p_\ell(\rho)u_\ell^+(\phi)$. One finds that: $p_\ell(\rho)=c_\ell\rho^{v_\ell^+}+d_\ell\rho^{-1-v_\ell^+}$. By Proposition 2.2, $\rho^\sigma u_\ell^+(\phi)\in L_1^2(D_R)$ iff $\sigma>-3/2$; $d_\ell\rho^{-1-v_\ell^+}u_\ell^+(\phi)$ is $L_1^2(D_R)$, with $d_\ell\neq 0$, iff $v_\ell^+<1/2$. Tables [9] show that

- if $\beta<\beta_\star$, $v_1^+<1/2$ and $v_\ell^+>1/2$, hence $d_\ell=0$, for $\ell>1$;
- if $\beta>\beta_\star$, $v_\ell^+>1/2$, hence $d_\ell=0$, for $\ell\geqslant 1$;
- if $\beta=\beta_\star$, $v_1^+=1/2$, which does correspond to the eigenvalue $\lambda_1^+=3/4$ for Λ^+.

Moreover, $c_\ell\rho^{v_\ell^+}u_\ell^+\in L_1^2(D_R)$ for any ℓ. The Schwarz estimate for $p_\ell(\rho)$ yields

$$\int_0^R p_\ell(\rho)^2\rho^2\,\mathrm{d}\rho\leqslant\int_0^R\int_0^{\pi/\beta} p(\rho,\phi)^2\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi=\|p\|^2_{L_1^2(D_R)}=\mathrm{cst}$$

For $\ell>1$, the left-hand side is equal to $c_\ell^2R^{2v_\ell^++3}/(2v_\ell^++3)$, and (26) follows. □

We will call O a *sharp* vertex if its aperture angle is strictly greater than $\pi/\beta_\star$. The consequence of the previous results is that Δ^- has no singularity near O, while Δ^+ has *locally* one singular function if O is sharp, and no singularity if it is not.

Theorem 3.8
Let $p\in\mathscr{C}^\infty(]0,R[;D(\Lambda^-))$ be a solution to (21). If $p\in L^2_{-1}(D_R)$, then $p\in H^1_{-1}(D_{R'})$ for any $R'<R$.

Proof
Let us differentiate formally the expansion (22):

$$\frac{\partial p}{\partial\rho}=\sum_{\ell=1}^{+\infty}c_\ell v_\ell^-\rho^{v_\ell^--1}u_\ell^-(\phi),\qquad \frac{1}{\rho}\frac{\partial p}{\partial\phi}=\sum_{\ell=1}^{+\infty}c_\ell\rho^{v_\ell^--1}\sqrt{v_\ell^-(v_\ell^--1)}\,\frac{(u_\ell^-(\phi))'}{\sqrt{v_\ell^-(v_\ell^--1)}}$$

For a fixed $\rho\leqslant R'<R$, these expansions are performed on orthonormal families in $\mathscr{H}_-$, see Theorem 3.4. Thus, establishing their convergence amounts to checking that their coefficients are in $l^2(\mathbb{N}^*)$. This follows from the estimate (23), since by Lemma 3.5, $v_\ell^-\geqslant\beta\ell/2$ for ℓ large enough. Hence for ρ fixed, $\mathbf{grad}\,p\in[\mathscr{H}_-]^2$ and its norm is

$$\int_0^{\pi/\beta}|\mathbf{grad}\,p(\rho,\phi)|^2\,\frac{\mathrm{d}\phi}{\sin\phi}\leqslant 2KR\sum_{\ell=1}^{+\infty}(v_\ell^-)^3\left(\frac{\rho}{R}\right)^{2v_\ell^--2}\leqslant K'\,\frac{\rho^\beta}{R^\beta-\rho^\beta}$$

So, by Fubini's theorem,

$$\iint_{D_{R'}}|\mathbf{grad}\,p(\rho,\phi)|^2\,\frac{\mathrm{d}\rho\,\mathrm{d}\phi}{\sin\phi}\leqslant\int_0^{R'}K'\,\frac{\rho^\beta}{R^\beta-\rho^\beta}\,\mathrm{d}\rho<+\infty$$

and $p\in H^1_{-1}(D_{R'})$. □

Theorem 3.9
Let $p\in\mathscr{C}^\infty(]0,R[;D(\Lambda^+))$ be a solution to (20). If $p\in L_1^2(D_R)$, then

- if $\beta>\beta_\star$, $p\in H_1^1(D_{R'})$ for $R'<R$,
- if $\beta<\beta_\star$, $p-d_1\rho^{-1-v_1^+}u_1^+(\phi)\in H_1^1(D_{R'})$ for $R'<R$.

Proof
The fact that a function $f\in L^2_1(D_R)$ belongs to $H^1_1(D_{R'})$ is equivalent to the convergence of the integral $\iint_{D_{R'}} |\mathbf{grad}\, f|^2\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi$. According to the value of β we set $f=p$ or $f=p-d_1\rho^{-1-\nu_1^+}u_1^+(\phi)$; and applying to f a reasoning similar to the above one proves its $H^1_1(D'_R)$ regularity. □

Lemma 3.10
If $\beta<\beta_\star$, there exists $\sigma^+\in N^{d+}$ such that

$$\sigma^+(\rho,\phi)-\eta(\rho)\rho^{-1-\nu_1^+}u_1^+(\phi)\in H^1_1(\omega)$$

Proof
Let $u(\rho,\phi)=\eta\rho^{-1-\nu_1^+}u_1^+(\phi)$ and $f=\Delta^+u$. f vanishes everywhere except in a shell which stands away from all corners, and is smooth there. Define w as the variational solution in $\mathring{H}^1_1(\omega)$ to $\Delta^+w=f$. As $f\in L^2_1(\omega)$, w is locally H^2_+ away from γ_b [2, p. 44], hence it satisfies $\partial_r w=0$ on γ_a by [2, Theorem II.2.1]. As, moreover, $\partial_r u|_{\gamma_a}=0$, the difference $\sigma^+=u-w$ satisfies $\Delta^+\sigma^+=0$ and the boundary conditions; hence $\sigma^+\in N^{d+}$. □

3.3. Dimensions of the singular spaces

Definition 3.11
Let the sets of geometrical singularities be

$$\mathscr{K}_E=\{\text{edges}\},\quad \mathscr{K}_O=\{\text{vertices}\}$$

$$\mathscr{K}_{ES}=\{j\colon E_j \text{ is a reentrant edge}\}$$

$$\mathscr{K}_{OS}=\{j\colon O_j \text{ is a sharp vertex}\}$$

$$\mathscr{K}^b_S=\mathscr{K}_{ES},\quad \mathscr{K}^e_S=\mathscr{K}_{ES}\cup\mathscr{K}_{OS}$$

Theorem 3.12
The spaces N^{d+} and N^{d-} are finite-dimensional, with

$$\dim N^{d+}=\#\mathscr{K}^e_S,\quad \dim N^{d-}=\#\mathscr{K}^b_S$$

Proof
Let us introduce a cutoff function η_j for each singularity A_j, $j\in\mathscr{K}_E\cup\mathscr{K}_O$. (We take care that their supports are all disjoint.) For any $p\in N^{d\pm}$, there holds by Lemma 3.1:

$$T_\eta p\stackrel{\text{def}}{=}p\left(1-\sum_{j\in\mathscr{K}_E\cup\mathscr{K}_O}\eta_j\right)\in\mathscr{C}^\infty(\bar{\omega})$$

If $p\in N^{d-}$ and $\mathbf{P}=(p/r)\mathbf{e}_\theta$, then $T_\eta p=rT_\eta P_\theta$, where $T_\eta\mathbf{P}\in\check{\mathbf{C}}^\infty(\bar{\Omega})\subset\check{\mathbf{H}}^1(\Omega)$ (Lemma 3.1), so $T_\eta P_\theta\in H^1_-(\omega)$ and $T_\eta p\in H^1_{-1}(\omega)$ by Proposition 2.1.

If $j\in\mathscr{K}_O$, $\eta_j p\in H^1_{-1}(\omega)$ by Theorem 3.8. Near an outgoing edge E_j, p is locally H^1, hence $\eta_j p\in H^1_{-1}(\omega)$. If $j\in\mathscr{K}_{ES}$, there exists $\sigma_j^-\in N^{d-}$ such that $w_j=\eta_j p-c_j\sigma_j^-\in H^1_{-1}(\omega)$

by Lemma 3.3. Summarizing, we have

$$p = T_\eta p + \sum_{j\in\mathscr{K}_O} \eta_j p + \sum_{j\in\mathscr{K}_E\setminus\mathscr{K}_{ES}} \eta_j p + \sum_{j\in\mathscr{K}_{ES}} (c_j\sigma_j^- + w_j) = w + \sum_{j\in\mathscr{K}_{ES}} c_j\sigma_j^-$$

with w in $H^1_{-1}(\omega)$, as the sum of functions in $H^1_{-1}(\omega)$, and in N^{d+}, as the difference $p - \sum c_j\sigma_j^-$ of elements of N^{d-}. Thus $w=0$.

So, the $(\sigma_j^-)_{j\in\mathscr{K}_{ES}}$ are a generating family in N^{d-}: on the other hand, they are obviously linearly independent. Hence the dimension of N^{d-}.

The demonstration is very similar for $p\in N^{d+}$. Here $T_\eta p \in \mathscr{C}^\infty(\bar{\omega}) \subset H^1_1(\omega)$, and

$$p = T_\eta p + \sum_{j\in\mathscr{K}_{OS}} (c_j\sigma_j^+ + w_j) + \sum_{j\in\mathscr{K}_O\setminus\mathscr{K}_{OS}} \eta_j p + \sum_{j\in\mathscr{K}_E\setminus\mathscr{K}_{ES}} \eta_j p + \sum_{j\in\mathscr{K}_{ES}} (c_j\sigma_j^+ + w_j)$$
$$= w + \sum_{j\in\mathscr{K}_{OS}} c_j\sigma_j^+ + \sum_{j\in\mathscr{K}_{ES}} c_j\sigma_j^+$$

where w is both in $H^1_1(\omega)$ and N^{d+}, hence $w=0$. The conclusion follows once more from the obvious linear independence of $(\sigma_j^+)_{j\in\mathscr{K}_{ES}\cup\mathscr{K}_{OS}}$. □

4. ANALYSIS OF THE STATIC MAXWELL EQUATIONS

4.1. The two div–curl problems

The *electrostatic* or Dirichlet problem is, for $\mathbf{F}^n \in \mathscr{J}^n = \mathbf{H}_0(\operatorname{div} 0; \Omega)$ and $G^d \in \mathscr{L}^d = L^2(\Omega)$: *Find* $\mathscr{U}^d \in \mathbf{L}^2(\Omega)$ *such that*

$$\mathbf{curl}\, \mathscr{U}^d = \mathbf{F}^n \quad \text{in } \Omega \tag{27}$$

$$\operatorname{div} \mathscr{U}^d = G^d \quad \text{in } \Omega \tag{28}$$

$$\mathscr{U}^d \times \mathbf{n} = 0 \quad \text{on } \Gamma \tag{29}$$

The *magnetostatic* or Neumann problem is, given $\mathbf{F}^d \in \mathscr{J}^d = \mathbf{H}(\operatorname{div} 0, \Omega)$ and $G^n \in \mathscr{L}^n = \{u \in L^2(\Omega)\colon \iiint_\Omega u \, d\Omega = 0\}$: *Find* $\mathscr{U}^n \in \mathbf{L}^2(\Omega)$ *such that*

$$\mathbf{curl}\, \mathscr{U}^n = \mathbf{F}^d \quad \text{in } \Omega \tag{30}$$

$$\operatorname{div} \mathscr{U}^n = G^n \quad \text{in } \Omega \tag{31}$$

$$\mathscr{U}^n \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \tag{32}$$

The fact that G^n has a mean zero value stems from (32). This condition is satisfied by the actual magnetic field, which is divergence free (cf. (87) below).

We shall also need the *scalar* and *vector* potentials, which are associated to the Hodge decomposition

$$\mathscr{U}^{d/n} = -\mathbf{grad}\, V^{d/n} + \mathbf{curl}\, \mathscr{A}^{n/d} \overset{\text{def}}{=} \mathbf{Hodge}(V^{d/n}, \mathscr{A}^{n/d}) \tag{33}$$

The vector potential $\mathscr{A}$ is the solution in $\mathbf{L}^2(\Omega)$ to the Neumann resp. Dirichlet problem

$$-\Delta \mathscr{A}^{n/d} = \mathbf{F}^{n/d} \quad \text{in } \Omega \tag{34}$$

$$\operatorname{div} \mathscr{A}^{n/d} = 0 \quad \text{in } \Omega \tag{35}$$

$$\mathscr{A}^{n} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \tag{36}$$

$$(\mathbf{curl}\, \mathscr{A}^{n}) \times \mathbf{n} = 0 \quad \text{on } \Gamma \tag{37}$$

$$\text{resp. } \mathscr{A}^{d} \times \mathbf{n} = 0 \quad \text{on } \Gamma \tag{38}$$

The existence and uniqueness of the solutions to problems (27)–(29), (30)–(32) and (34)–(38) can be proven by a saddle-point approach (see [10] for details). *Notice* the swap of the boundary conditions, hence of the superscripts n/d, caused by the **curl** operator.

As for the scalar potential V, it is defined as the variational solution in $\mathscr{V}^d = H_0^1(\Omega)$, resp. $\mathscr{V}^n = \{u \in H^1(\Omega)\colon \iiint_\Omega u \,\mathrm{d}\Omega = 0\}$, to the Dirichlet, resp. Neumann problem

$$-\Delta V^{d/n} = G^{d/n} \quad \text{in } \Omega, \qquad V^d = 0, \text{ resp. } \frac{\partial V^n}{\partial n} = 0 \quad \text{on } \Gamma \tag{39}$$

Defining the spaces of potentials by

$$\Phi^{d/n} = \{\varphi \in \mathscr{V}^{d/n}\colon \Delta\varphi \in \mathscr{L}^{d/n} \text{ and, in the Neumann case, } \partial_n \varphi = 0\}$$
$$\mathscr{M}^{d/n} = \{\mathbf{M} \in \mathscr{X}^{d/n}\colon \mathbf{curl}\,\mathbf{M} \in \mathscr{X}^{n/d} \text{ and } \operatorname{div}\mathbf{M} = 0\}$$

and, as usual, by $\breve{\Phi}^{d/n}, \breve{\mathscr{L}}^{d/n}, \ldots$, the axisymmetric subspaces, the existence and uniqueness results are summarized in

Theorem 4.1
The following mappings are isomorphisms of vector spaces:

$$\breve{\Phi}^{d/n} \otimes \breve{\mathscr{M}}^{n/d} \xrightarrow{\mathbf{Hodge}} \breve{\mathscr{X}}^{d/n} \xrightarrow{\operatorname{div} \otimes \mathbf{curl}} \breve{\mathscr{L}}^{d/n} \otimes \breve{\mathscr{J}}^{n/d} \tag{40}$$

and so are $\Delta_{\text{scalar}}\colon \breve{\Phi}^{d/n} \to \breve{\mathscr{L}}^{d/n}$ and $\Delta_{\text{vector}}\colon \breve{\mathscr{M}}^{n/d} \to \breve{\mathscr{J}}^{n/d}$.

We shall also pay special attention to the *divergence-free* problem (i.e. $G = 0$ in (28) or (31)), which is of particular interest in the magnetostatic case:

Theorem 4.2
The following mappings are isomorphisms of vector spaces:

$$\breve{\mathscr{M}}^{n/d} \overset{\mathbf{curl}}{\rightarrow} \breve{\mathscr{X}}^{0d/n} \overset{\mathbf{curl}}{\rightarrow} \breve{\mathscr{J}}^{n/d} \tag{41}$$

with $\breve{\mathscr{X}}^{0d/n} = \{\mathbf{u} \in \breve{\mathscr{X}}^{d/n}\colon \operatorname{div}\mathbf{u} = 0\}$.

In the remainder of this section, we examine the simplification of the static problems induced by the axial symmetry. As noted in Reference [1, Proposition 2.2] there is a decoupling of meridian and azimuthal components in the divergence, curl and vector Laplacian operators; as a consequence, each of the static problems is decoupled into two problems, concerning the meridian and azimuthal components of the field $\mathscr{U}$, and set in the meridian cut ω. (For any vector field $\mathbf{w}$, its *meridian* and *azimuthal* components are defined as $\varpi_m(\mathbf{w}) = \mathbf{w}_m = w_r\mathbf{e}_r + w_z\mathbf{e}_z$ and $\varpi_\theta(\mathbf{w}) = \mathbf{w}_\theta = w_\theta\mathbf{e}_\theta$.) To study these 2D problems, we introduce the following operators in the meridian half-plane (r,z):

$$\operatorname{div}\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}, \quad \operatorname{div}_+\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$$

$$\operatorname{curl}\mathbf{u} = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \quad \operatorname{curl}_-\mathbf{u} = \frac{\partial u_r}{\partial z} + \frac{u_z}{r} - \frac{\partial u_z}{\partial r}, \quad \mathbf{curl}\, f = -\frac{\partial f}{\partial z}\mathbf{e}_r + \frac{\partial f}{\partial r}\mathbf{e}_z$$

Note that, in a 2D context, we denote by div the divergence in the cartesian plane (r,z), *not* the trace of the 3D divergence operator, which is denoted div_+. And the 3D **curl** operator is expressed in terms of the 2D operators **curl**, curl and curl_- as

$$\mathbf{curl}\,\mathbf{u} = (\operatorname{curl}\mathbf{u}_m)\mathbf{e}_\theta + r^{-1}\mathbf{curl}(ru_\theta) \quad \text{with } \operatorname{curl}\mathbf{u}_m = r^{-1}\operatorname{curl}_-(r\mathbf{u}_m) \tag{42}$$

4.2. The general meridian field problem

The meridian component $\mathbf{U}_m = U_r\mathbf{e}_r + U_z\mathbf{e}_z$ of $\mathscr{U}$ satisfies $\operatorname{div}_+\mathbf{U}_m = G$ and $\mathbf{curl}\,\mathbf{U}_m = \mathbf{F}_\theta$ (see above). Introducing $\mathbf{u}_m = r\mathbf{U}_m$, $g = rG$ and $f_\theta = rF_\theta$, one has

$$\operatorname{div}\mathbf{u}_m = g \text{ in } \omega, \quad \operatorname{curl}_-\mathbf{u}_m = f_\theta \text{ in } \omega, \quad \mathbf{u}_m^n\cdot\mathbf{v}, \text{ resp. } \mathbf{u}_m^d\cdot\boldsymbol{\tau} = 0 \text{ on } \gamma_b \tag{43}$$

Any vector field in $\check{\mathbf{H}}(\mathbf{curl},\operatorname{div};\Omega)$ is $\check{\mathbf{H}}^1$ away from the boundary Γ [11]; hence its r-component vanishes on the axis: $\mathbf{U}\cdot\mathbf{v}|_{\gamma_a} = U_r|_{\gamma_a} = 0$; the same is true for $\mathbf{u}_m$.

We shall treat in some detail the electrostatic case only. For the magnetostatic case, the superscripts d and n and the boundary conditions *on* γ_b *only* are to be swapped: $\mathbf{u}\cdot\boldsymbol{\tau} = 0$ is to be replaced with $\mathbf{u}\cdot\mathbf{v} = 0$, $V = 0$ becomes $\partial_v V = 0$, and so on. The boundary conditions on the axis, which stem from the axial symmetry, are the same.

Defining $W^n = rA_\theta^n$, V^d and W^n are related to $\mathbf{u}_m^d$ by

$$\mathbf{u}_m^d = \mathbf{curl}\,W^n - r\,\mathbf{grad}\,V^d \stackrel{\text{def}}{=} \mathbf{hodge}(V^d, W^n) \tag{44}$$

As a function of the source f_θ, W^n is obtained as the solution to

$$-\Delta^- W^n = f_\theta^n \text{ in } \omega, \quad W^n = 0 \text{ on } \gamma_a, \quad \frac{\partial W^n}{\partial v} = 0 \text{ on } \gamma_b \tag{45}$$

The boundary condition on γ_a stems from Proposition 2.1 and [1, Proposition 3.18]; on γ_b, it is the trace of (37); the condition (36) only concerns the meridian components of $\mathscr{A}^n$.

Similarly, V^d is the solution to

$$-\Delta^+ V^d = G^d \text{ in } \omega, \quad \frac{\partial V^d}{\partial v} = 0 \text{ on } \gamma_a, \quad V^d = 0 \text{ on } \gamma_b \tag{46}$$

The boundary condition on γ_a is justified as in the proof of Lemma 3.10.

f_0 belongs to $L^2_{-1}(\omega)$, and so does g. As for $\mathbf{U}^d_m$, it belongs to

$$\breve{\mathscr{X}}^d_m = \{\mathbf{U}^d_m \in L^2_1(\omega)^2\colon\ \mathrm{div}_+\,\mathbf{U}^d_m \in L^2_1(\omega) \text{ and } \mathrm{curl}\,\mathbf{U}^d_m \in L^2_1(\omega) \text{ on } \mathbf{U}^d_m\cdot\boldsymbol{\tau}=0 \text{ and } \gamma_b\}$$

So, by Proposition 2.1, the variable $\mathbf{u}^d_m$ belongs to the space

$$\mathsf{U}^d = \{\mathbf{u}^d \in L^2_{-1}(\omega)^2\colon\ \mathrm{div}\,\mathbf{u}^d \in L^2_{-1}(\omega) \text{ and } \mathrm{curl}_-\,\mathbf{u}^d \in L^2_{-1}(\omega) \text{ and } \mathbf{u}^d\cdot\boldsymbol{\tau}=0 \text{ on } \gamma_b\}$$

Finally, by [1, Propositions 3.13 and 3.19], the potentials W^n and V^d belong, respectively, to Φ^{n-} (defined in Proposition 2.3) and to

$$\Phi^{d+} = \{V^d \in H^1_1(\omega)\colon\ \Delta^+ V^d \in L^2_1(\omega) \text{ and } \partial_\nu V^d = 0 \text{ on } \gamma_a \text{ and } V^d = 0 \text{ on } \gamma_b\}$$

Theorem 4.3

For $(f_0, g) \in L^2_{-1}(\omega)\times L^2_{-1}(\omega)$, i.e. $G\in L^2_1(\omega)$, Eqs. (43), (45) and (46) have unique solutions in U^d, Φ^{n-} and Φ^{d+}. As a consequence, Eq. (44) has a unique solution in $\Phi^{d+}\times\Phi^{n-}$. Hence, Δ^+ and Δ^- are isomorphisms, respectively, between Φ^{d+} and $L^2_1(\omega)$, and Φ^{n-} and $L^2_{-1}(\omega)$, and the mappings defined by (43) and (44) are isomorphisms between the relevant spaces. These isomorphisms are linked by: $\Delta^+ = \mathrm{div}_+\,\mathbf{grad}$ and $\Delta^- = -\,\mathrm{curl}_-\,\mathbf{curl}$.

Proof

Consider the following diagram:

$$\begin{array}{ccccc}
\breve{\Phi}^d\otimes\breve{\mathscr{M}}^n & \xrightarrow{\mathbf{Hodge}} & \breve{\mathscr{X}}^d & \xrightarrow{\mathrm{div}\,\otimes\,\mathbf{curl}} & \breve{\mathscr{L}}^d\otimes\breve{\mathscr{J}}^n \\
{\scriptstyle 1\otimes\varpi_\theta}\big\downarrow & & {\scriptstyle \varpi_m}\big\downarrow & & {\scriptstyle 1\otimes\varpi_\theta}\big\downarrow \\
\breve{\Phi}^d\otimes\breve{\mathscr{M}}^n_\theta & \xrightarrow{\mathbf{Hodge}_m} & \breve{\mathscr{X}}^d_m & \xrightarrow{\mathrm{div}_+\,\otimes\,\mathrm{curl}} & L^2_1(\omega)\otimes L^2_1(\omega) \\
{\scriptstyle 1\otimes\mathsf{R}}\big\downarrow & & {\scriptstyle \mathsf{R}}\big\downarrow & & {\scriptstyle 1\otimes\mathsf{R}}\big\downarrow \\
\Phi^{d+}\otimes\Phi^{n-} & \xrightarrow{\mathbf{Hodge}} & \mathsf{U}^d & \xrightarrow{(1/r)\,\mathbf{div}\,\otimes\,\mathrm{curl}_-} & L^2_1(\omega)\otimes L^2_1(\omega)
\end{array} \qquad (47)$$

where ϖ_m and ϖ_θ are the meridian and azimuthal projections: they are surjective (onto) morphisms by [1, Propositions 3.17, 3.19]. R is as in Proposition 2.1; I is the identity mapping, or the trace operator in a meridian half-plane, which we (more or less) merge. $\mathbf{Hodge}_m$ is the meridian component of the $\mathbf{Hodge}$ operator. Straightforward calculations (see p. 875) show that the diagram (47) is commutative. As (by Theorem 4.1) the first row is made of isomorphisms, the same is true for the second and third rows. The last assertion, too, is straightforward. □

4.3. The divergence-free meridian field problem

To study it—in the magnetostatic case—let us introduce the divergence-face spaces:

$$\breve{\mathscr{X}}^{0n}_m = \{\mathbf{B}_m \in L^2_1(\omega)^2\colon\ \mathrm{div}_+\,\mathbf{B}_m = 0 \text{ in } \omega \text{ and } \mathrm{curl}\,\mathbf{B}_m \in L^2_1(\omega) \text{ and } \mathbf{B}_m\cdot\boldsymbol{\nu}=0 \text{ on } \gamma_b\}$$

$$\mathsf{U}^{0n} = \{\mathbf{v} \in L^2_{-1}(\omega)^2\colon\ \mathrm{div}\,\mathbf{v} = 0 \text{ in } \omega \text{ and } \mathrm{curl}_-\mathbf{v} \in L^2_{-1}(\omega) \text{ and } \mathbf{v}\cdot\boldsymbol{\nu}=0 \text{ on } \gamma_b\}$$

and the space of potentials Φ^{n-} as in Proposition 2.3. There holds the simplified result:

Lemma 4.4
The following mappings are isomorphisms:

$$\mathbf{curl}:\Phi^{d-}\to \mathsf{U}^{0n},\quad \mathrm{curl}_-:\mathsf{U}^{0n}\to L^2_{-1}(\omega),\quad \Delta^-:\Phi^{d-}\to L^2_{-1}(\omega)$$

and are linked by $-\Delta^- = \mathrm{curl}_-\,\mathbf{curl}$.

Proof
Consider this time the diagram

$$\begin{array}{ccccc}
\check{\mathscr{M}}^d & \xrightarrow{\mathbf{curl}} & \check{\mathscr{X}}^{0n} & \xrightarrow{\mathbf{curl}} & \check{\mathscr{J}}^d \\
\varpi_\theta\big\downarrow & & \varpi_m\big\downarrow & & \varpi_\theta\big\downarrow \\
\check{\mathscr{M}}^d_\theta & \xrightarrow{1/r\,\mathbf{curl}(r\cdot)} & \check{\mathscr{X}}^{0n}_m & \xrightarrow{\mathrm{curl}} & L^2_1(\omega) \\
\mathsf{R}\big\downarrow & & \mathsf{R}\big\downarrow & & \mathsf{R}\big\downarrow \\
\Phi^{d-} & \xrightarrow{\mathbf{curl}} & \mathsf{U}^{0n} & \xrightarrow{\mathrm{curl}_-} & L^2_{-1}(\omega)
\end{array} \tag{48}$$

and show that it is commutative. □

If we were interested in divergence-free electro'static' problems, similar results would hold, with boundary conditions on γ_b swapped.

4.4. Space decomposition results for the meridian problems

Let us summarize the main results of Reference [1, Section 5] in the following:

Theorem 4.5
The regular subspaces $\check{\mathscr{X}}^d_R = \check{\mathscr{X}}_R \cap \check{\mathbf{H}}^1(\Omega)$, $\check{\mathscr{X}}^{0n}_R = \check{\mathscr{X}}^{0n} \cap \check{\mathbf{H}}^1(\Omega)$. $\check{\Phi}^{d/n}_R = \check{\Phi}^{d/n} \cap \check{H}^2(\Omega)$, and $\check{\mathscr{M}}^{n/d}_R = \{\mathbf{M} \in \check{\mathscr{X}}^{n/d}\colon \operatorname{div}\mathbf{M} = 0 \text{ and } \mathbf{curl}\,\mathbf{M} \in \check{\mathscr{X}}^{d/n}_R\}$ are closed within $\check{\mathscr{X}}^d$, $\check{\mathscr{X}}^{0n}$, $\check{\Phi}^{d/n}$ and $\check{\mathscr{M}}^{n/d}$. $\check{\mathscr{X}}^d_R$ and $\check{\Phi}^d_R$ admit complementary subspaces isomorphic to N^{d+}. $\check{\mathscr{X}}^{0n}_R$ and $\check{\mathscr{M}}^d_R$ admit complementary subspaces isomorphic to N^{d-}. And the complements of $\check{\mathscr{X}}^d_R$ and $\check{\mathscr{X}}^{0n}_R$ are made of *meridian* fields.

Hence by Theorem 3.12, $\operatorname{codim}\check{\mathscr{X}}^d_R = \operatorname{codim}\check{\Phi}^d_R = \#\mathscr{K}^e_S$ and $\operatorname{codim}\check{\mathscr{X}}^{0n}_R = \operatorname{codim}\check{\mathscr{M}}^d_R = \#\mathscr{K}^b_S$. The following technical result will be useful in order to analyse the space and space–time regularity of the electromagnetic fields. Let $\check{\mathscr{M}}^{n/d}_{0R} = \{\mathbf{M} \in \check{\mathscr{M}}^{n/d}_R\colon\ \mathbf{M} \parallel \mathbf{e}_\theta\}$; for all $\mathbf{M} \in \check{\mathscr{M}}^{n/d}_R$, both $\mathbf{M}_m$ and $\mathbf{M}_\theta$ belong to $\check{\mathscr{M}}^{n/d}_R$. Then

Proposition 4.6
$\check{\mathscr{M}}^{d/n}_{0R}$ is algebraically and topologically equal to $\mathbf{V}^{2,d/n}_\theta$ defined as

$$\mathbf{V}^{2,d}_\theta \stackrel{\text{def}}{=} \{\mathbf{v} \in \check{\mathbf{H}}^2(\Omega) \cap \check{\mathbf{H}}^1_0(\Omega)\colon\ \mathbf{v} \parallel \mathbf{e}_\theta\}$$

$$\mathbf{V}^{2,n}_\theta \stackrel{\text{def}}{=} \{\mathbf{v} \in \check{\mathbf{H}}^2(\Omega)\colon\ \ \mathbf{v} \parallel \mathbf{e}_\theta \text{ and } \partial_\nu(rv_\theta)|_\Gamma = 0\}$$

Proof
Obviously, $\mathbf{V}_\theta^{2,d/n} \subset \check{\mathscr{M}}_{\theta R}^{d/n}$. Moreover, it is easily proven that both $\Delta\check{\mathscr{M}}_{\theta R}^{d/n}$ and $\Delta\mathbf{V}_\theta^{2,d/n}$ are closed subspaces of $\check{\mathbf{L}}^2(\Omega)$. By using 'very weak' integration by parts formulae of [1, Lemma 5.12; 6, Lemma 4.16] one proves that $[\Delta\mathbf{V}_\theta^{2,d/n}]^\perp \subset [\Delta\check{\mathscr{M}}_{\theta R}^{d/n}]^\perp$. The conclusion follows. □

We now study the trace of these results in a meridian half-plane. For both div–curl problems, there holds the following result:

Lemma 4.7
In $\mathsf{U}^{d/n}$, the semi-norm $(\|\mathrm{curl}_-\,\mathbf{u}\|_{0,-1,\omega}^2 + \|\mathrm{div}\,\mathbf{u}\|_{0,-1,\omega}^2)^{1/2}$ defines a norm equivalent to the canonical norm; and so do the norms $\|\mathrm{curl}_-\,\mathbf{u}\|_{0,-1,\omega}$ in $\mathsf{U}^{0d/n}$, $\|\Delta^+ V\|_{0,1,\omega}$ in $\Phi^{d/n+}$ and $\|\Delta^- W\|_{0,-1,\omega}$ in $\Phi^{n/d-}$. With these norms, the isomorphisms of Lemmas 4.3 and 4.4 are isometries.

The traces of the spaces $\check{\mathscr{X}}_R^d$, $\check{\Phi}_R^d$, $\check{\mathscr{M}}_R^d$, etc. are closed within U^d, Φ^{d+}, Φ^{n-}, and so on. One finds that the trace of $\check{\mathscr{X}}_R^d$ is

$$\mathsf{U}_R^d = \{\mathbf{u} \in L_{-1}^2(\omega)^2\colon\ (\partial_r u_r, \partial_z u_r, u_r/r, \partial_z u_z, \partial_r u_z - u_z/r) \in L_{-1}^2(\omega)^5 \text{ and } \mathbf{u}\cdot\boldsymbol{\tau} = 0 \text{ on } \gamma_b\}$$

For the vector potential, the trace of $\mathbf{curl}\,\mathscr{A} \in \check{\mathbf{H}}^1(\Omega)$ is $\mathbf{curl}\,W \in \mathsf{U}_R$, i.e.

$$\frac{\partial^2 W}{\partial r^2} - \frac{1}{r}\frac{\partial W}{\partial r} \in L_{-1}^2(\omega), \quad \frac{\partial^2 W}{\partial r \partial z} \in L_{-1}^2(\omega), \quad \frac{\partial^2 W}{\partial z^2} \in L_{-1}^2(\omega) \tag{49}$$

hence, the range of $\check{\mathscr{M}}_R^n$ is

$$\Phi_R^{n-} = \{W \in H_{-1}^1(\omega)\colon\ W \text{ satisfies (49) and } W = 0 \text{ on } \gamma_a \text{ and } \partial_\nu W = 0 \text{ on } \gamma_b\}$$

The range of $\check{\Phi}_R^d$ is $\Phi_R^{d+} = \Phi^{d+} \cap H_+^2(\omega)$. Similarly, one defines U_R^n, Φ_R^{d-}, Φ_R^{n+} by swapping the boundary conditions on γ_b; and U_R^{0n} by imposing the extra condition $\mathrm{div}\,\mathbf{u} = 0$.

By Proposition 4.6, the spaces Φ_R^{d-} and Φ_R^{n-} enjoy the following property:

Proposition 4.8
Φ_R^{d-} and Φ_R^{n-} are the range by R of closed subspaces in $H_-^2(\omega)$; and the norms $\|\Delta^- W\|_{0,-1,\omega}$ and $\|W/r\|_{2,-,\omega}$ are equivalent on these spaces.

Explicit singular potentials. As we already know the codimensions of the regular spaces, we just need to determine explicit complements. Let

$$\Phi^{d+} = \Phi_R^{d+} \oplus \Phi_S^{d+}, \quad \Phi^{d-} = \Phi_R^{d-} \oplus \Phi_S^{d-}, \quad \Phi^{n-} = \Phi_R^{n-} \oplus \Phi_S^{n-}$$

be direct-sum decompositions of the various potential spaces. Thanks, respectively, to [1, Theorem 5.3] and the isomorphisms of Lemma 4.4, there holds

$$\mathsf{U}^d = \mathsf{U}_R^d \oplus r\,\mathbf{grad}\,\Phi_S^{d+}, \quad \mathsf{U}^{0n} = \mathsf{U}_R^{0n} \oplus \mathbf{curl}\,\Phi_S^{d-}$$

Moreover, $\mathsf{U}_R^d \cap \mathbf{curl}\,\Phi_S^{n-} = \{0\}$; so, if we exhibit a sufficient number of linearly independent functions in Φ^{n-} that do not belong to Φ_R^{n-}, we will have additionally obtained

$$\dim \Phi_S^{n-} = \#\mathscr{K}_S^e \quad \text{and} \quad \mathsf{U}^d = \mathsf{U}_R^d \oplus \mathbf{curl}\,\Phi_S^{n-}$$

a decomposition that will be useful in the analysis of time-dependent problems. These different splittings are the analogues, in axisymmetric geometry, of those exhibited by Costabel–Dauge [4] in the 2D Cartesian case.

Using the notations of Section 3, we define for each reentrant edge E_j the functions

$$S_j^{d-}(\rho_j,\phi_j)=\eta_j(\rho_j)\rho_j^{\alpha_j}\Upsilon_j^{d-}(\phi_j), \quad \text{where } \Upsilon_j^{d-}(\phi_j)=\sqrt{\frac{2\alpha_j}{\pi}}\sin(\alpha_j\phi_j) \tag{50}$$

$$S_j^{d+}(\rho_j,\phi_j)=\eta_j(\rho_j)\rho_j^{\alpha_j}\Upsilon_j^{d+}(\phi_j), \quad \text{where } \Upsilon_j^{d+}(\phi_j)=\Upsilon_j^{d-}(\phi_j) \tag{51}$$

$$S_j^{n-}(\rho_j,\phi_j)=\eta_j(\rho_j)\rho_j^{\alpha_j}\Upsilon_j^{n-}(\phi_j), \quad \text{where } \Upsilon_j^{n-}(\phi_j)=\sqrt{\frac{2\alpha_j}{\pi}}\cos(\alpha_j\phi_j) \tag{52}$$

It is easy to check that these functions satisfy all the criteria that define respectively Φ^{d-}, Φ^{d+} and Φ^{n-}. On the other hand, they are not in $H^2(\omega_{E_j})$ because of the exponent $\alpha_j<1$; hence they fail to belong to the regularized subspaces.

For a sharp vertex O_j, let us introduce

$$S_j^{d+}(\rho_j,\phi_j)=\eta_j(\rho_j)\rho_j^{\nu_j}\Upsilon_j^{d+}(\phi_j), \quad \text{where } \Upsilon_j^{d+}(\phi_j)=C_j^{d+}P_{\nu_j}^0(\cos\phi_j) \tag{53}$$

$$S_j^{n-}(\rho_j,\phi_j)=\eta_j(\rho_j)\rho_j^{1+\nu_j}\Upsilon_j^{n-}(\phi_j), \quad \text{where } \Upsilon_j^{n-}(\phi_j)=C_j^{n-}\sin\phi_j\, P_{\nu_j}^1(\cos\phi_j) \tag{54}$$

We recall that $\nu_j=\nu_{j,1}^+$ is the unique root of $P_\nu^0(\cos\pi/\beta_j)=0$ in the interval $]0,1/2[$. The C_j are normalization factors in $\mathscr{H}_\pm$. Thanks to the expression [8, Eq. (8.5.2)] of the derivative of the Legendre function, one shows that $S_j^{n-}(\rho_j,\phi_j)$ satisfies the homogeneous Neumann condition on γ_b, and that Υ_j^{n-} is an eigenfunction of the operator Λ^- with mixed boundary conditions, with eigenvalue $\nu_j\ (\nu_j+1)$. Then, using Equations (13) and (14) one shows that $\Delta^+S_j^{d+}=\Delta^-S_j^{n-}\equiv 0$ near O_j. Elsewhere, these modified Laplacians vanish except in a shell where they are $\mathscr{C}^\infty$ up to the boundary. In addition, thanks to the factor $\sin\phi_j$ in $\Upsilon_j^{n-}(\phi_j)$, there holds $\Delta^-S_j^{n-}(\rho_j,0)=0$, so $\Delta^-S_j^{n-}\in L^2_{-1}(\omega)$.

By Proposition 2.2, $S_j^{d+}\in H^1_1(\omega)$, but $S_j^{d+}\notin H^2_+(\omega)$; $S_j^{n-}\in H^1_{-1}(\omega)$, but $S_j^{n-}\notin \mathsf{R}[H^2_-(\omega)]$. So we have $S_j^{d+}\in\Phi^{d+}$ and $S_j^{d+}\notin\Phi_R^{d+}$; $S_j^{n-}\in\Phi^{n-}$ and $S_j^{n-}\notin\Phi_R^{n-}$.

Finally, the obvious linear independence of the S_j associated with different geometrical singularities, and the dimensions of the singular spaces, imply that $(S_j^{d+})_{j\in\mathscr{K}_S^e}$, $(S_j^{n-})_{j\in\mathscr{K}_S^e}$, $(S_j^{d-})_{j\in\mathscr{K}_S^b}$, generate complements of Φ_R^{d+}, Φ_R^{n-} and Φ_R^{d-} respectively.

Explicit singular electric fields. Near a reentrant edge E_j, one has

$$\mathbf{grad}\, S_j^{d+}(\rho_j,\phi_j)=\sqrt{2\pi\alpha_j}\,\eta_j(\rho_j)\,\rho_j^{\alpha_j-1}[\sin(\alpha_j\phi_j)\mathbf{e}_{\rho_j}+\cos(\alpha_j\phi_j)\mathbf{e}_{\phi_j}]$$
$$+\sqrt{\frac{2\pi}{\alpha_j}}\eta_j'(\rho_j)\,\rho_j^{\alpha_j}\sin(\alpha_j\phi_j)\mathbf{e}_{\rho_j}$$

$$\mathbf{curl}\, S_j^{n-}(\rho_j,\phi_j) = \sqrt{2\pi\alpha_j}\,\eta_j(\rho_j)\rho_j^{\alpha_j-1}[\sin(\alpha_j\phi_j)\mathbf{e}_{\rho_j} + \cos(\alpha_j\phi_j)\mathbf{e}_{\phi_j}]$$
$$+\sqrt{\frac{2\pi}{\alpha_j}}\eta_j'(\rho_j)\rho_j^{\alpha_j}\cos(\alpha_j\phi_j)\mathbf{e}_{\phi_j}$$

These two types of singularities have the same 'principal part', i.e. they only differ by an $\mathbf{H}^1$ term. Besides, $(r/a_j)\,\mathbf{grad}\, S_j^{d+}$ still has the same principal part, since

$$\left(\frac{r}{a_j}-1\right)\mathbf{grad}\, S_j^{d+}(\rho_j,\phi_j) = \rho_j\cos\phi_j'\,\mathbf{grad}\, S_j^{d+}(\rho_j,\phi_j) \approx \rho_j^{\alpha_j} \in \mathbf{H}^1(\omega_{E_j})$$

In the absence of divergence constraint, we shall use a 'practical' singular field

$$\mathbf{S}_j^d(\rho_j,\phi_j) = \sqrt{2\pi\alpha_j}\,\eta_j(\rho_j)\rho_j^{\alpha_j-1}[\sin(\alpha_j\phi_j)\mathbf{e}_{\rho_j} + \cos(\alpha_j\phi_j)\mathbf{e}_{\phi_j}]$$
$$= \frac{1}{2}\left\{\frac{r}{a_j}\,\mathbf{grad}\, S_j^{d+} + \mathbf{curl}\, S_j^{n-}\right\} + \mathbf{w}_j \quad \text{with } \mathbf{w}_j \in \mathsf{U}_R^d \tag{55}$$

Let us now construct a singular field near a sharp conical vertex O_j. Using the differentiation formulae for the Legendre functions [9, Equations (8.5.2) and (8.6.6)], we find

$$r\,\mathbf{grad}\, S_j^{d+}(\rho_j,\phi_j) = C_j^{d+}\eta_j(\rho_j)\rho_j^{v_j}\sin\phi_j[v_j P_{v_j}^0(\cos\phi_j)\mathbf{e}_{\rho_j} + P_{v_j}^1(\cos\phi_j)\mathbf{e}_{\phi_j}]$$
$$+ C_j^{d+}\eta_j'(\rho_j)\rho_j^{v_j+1}\sin\phi_j P_{v_j}^0(\cos\phi_j)\mathbf{e}_{\rho_j}$$
$$\mathbf{curl}\, S_j^{n-}(\rho_j,\phi_j) = (v_j+1)C_j^{n-}\eta_j(\rho_j)\rho_j^{v_j}\sin\phi_j[v_j P_{v_j}^0(\cos\phi_j)\mathbf{e}_{\rho_j} + P_{v_j}^1(\cos\phi_j)\mathbf{e}_{\phi_j}]$$
$$+ C_j^{n-}\eta_j'(\rho_j)\rho_j^{v_j+1}\sin\phi_j\sin\phi_j\, P_{v_j}^1(\cos\phi_j)\mathbf{e}_{\phi_j}$$

Here too, the two types of singularities have proportional principal parts, since

$$(v_j+1)C_j^{n-}r\,\mathbf{grad}\, S_j^{d+} - C_j^{d+}\,\mathbf{curl}\, S_j^{n-} \approx \rho_j^{v_j+1}$$

so the r and z components of this field are in $H^1_{-1}(\omega)$, and the field belongs to U_R^d. In the absence of divergence constraint, we shall use the singular field

$$\mathbf{S}_j^d(\rho_j,\phi_j) = \eta_j(\rho_j)\rho_j^{v_j}\sin\phi_j[v_j P_{v_j}^0(\cos\phi_j)\mathbf{e}_{\rho_j} + P_{v_j}^1(\cos\phi_j)\mathbf{e}_{\phi_j}]$$
$$= \frac{1}{2}\left\{\frac{r\,\mathbf{grad}\, S_j^{d+}}{C_j^{d+}} + \frac{\mathbf{curl}\, S_j^{n-}}{(v_j+1)C_j^{n-}}\right\} + \mathbf{w}_j \quad \text{with } \mathbf{w}_j \in \mathsf{U}_R^d \tag{56}$$

By a dimension argument, we conclude that the $(\mathbf{S}_j^d)_{j\in\mathscr{K}_S^e}$ make up a basis of U_S^d, a complement of U_R^d within U^d. Since the azimuthal component of any field in $\breve{\mathbf{H}}(\mathbf{curl};\Omega)$ is in $\breve{\mathbf{H}}^1(\Omega)$ (Proposition 2.1), we infer that the fields $\mathbf{S}_j^d(\rho_j,\phi_j)/r$ span a complement of $\breve{\mathscr{X}}_R^d$ within $\breve{\mathscr{X}}^d$.

Explicit singular magnetic fields. The space span $\{\mathbf{S}_j^{0n}\}_{j\in\mathscr{K}_S^b}$, where $\mathbf{S}_j^{0n} \stackrel{\text{def}}{=} \mathbf{curl}\, S_j^{d-}$, obviously complements U_R^{0n} within U^{0n}. One has explicitly

$$\begin{aligned}\mathbf{S}_j^{0n}(\rho_j,\phi_j) &= \sqrt{2\pi\alpha_j}\,\eta_j(\rho_j)\rho_j^{\alpha_j-1}[-\cos(\alpha_j\phi_j)\mathbf{e}_{\rho_j}+\sin(\alpha_j\phi_j)\mathbf{e}_{\phi_j}]\\ &\quad+\sqrt{\frac{2\pi}{\alpha_j}}\eta_j'(\rho_j)\rho_j^{\alpha_j}\sin(\alpha_j\phi_j)\mathbf{e}_{\phi_j}\end{aligned} \tag{57}$$

The last term vanishes near E_j, and is of global H^1 regularity. It is necessary to preserve the divergence constraint. If, however, we work within U^n, i.e., without divergence constraint, or if we just need local expressions, we can use the singular fields

$$\begin{aligned}\mathbf{S}_j^{n}(\rho_j,\phi_j) &= \sqrt{2\pi\alpha_j}\,\eta_j(\rho_j)\rho_j^{\alpha_j-1}[-\cos(\alpha_j\phi_j)\mathbf{e}_{\rho_j}+\sin(\alpha_j\phi_j)\mathbf{e}_{\phi_j}]\\ &= \mathbf{curl}\, S_j^{d-}+\mathbf{w}_j \quad \text{with } \mathbf{w}_j\in\mathsf{U}_R^n\end{aligned} \tag{58}$$

We remark that $\mathbf{S}_j^n$ is $\mathbf{S}_j^d$ rotated of $\pi/2$: this is similar to the Cartesian geometry.

Like in the electric case, we infer that the fields $\mathbf{S}_j^{0n}(\rho_j,\phi_j)/r$ span a complement of $\breve{\mathscr{X}}_R^{0n}$ within $\breve{\mathscr{X}}^{0n}$. If we work within $\breve{\mathscr{X}}^n$, i.e. without divergence constraint, we can use the fields $\mathbf{S}_j^n$: because of their support, they belong to $\breve{\mathbf{H}}(\mathbf{curl},\mathrm{div};\Omega)$ but not to $\breve{\mathbf{H}}^1(\Omega)$, and satisfy the magnetic boundary condition.

4.5. *The azimuthal field problem*

The azimuthal component $\mathbf{U}_\theta = U_\theta\mathbf{e}_\theta$ satisfies $\mathbf{curl}\,\mathbf{U}_\theta=\mathbf{F}_m$, and because of the axial symmetry, $\mathrm{div}\,\mathbf{U}_\theta=0$. In the electrostatic case, $\mathscr{U}\times\mathbf{n}=0$ implies $\mathbf{U}_\theta=0$ on Γ.

Introducing $u_\theta = rU_\theta$ and $\mathbf{f}_m = r\mathbf{F}_m$, Equation (27) resp. (30) becomes

$$\mathbf{curl}\, u_\theta = \mathbf{f}_m \text{ in } \omega \tag{59}$$

In the electrostatic case, on has $\mathbf{f}_m\in\mathscr{F}^n=\{\mathbf{f}\in L^2_{-1}(\omega)^2:\ \mathrm{div}\,\mathbf{f}=0 \text{ and } \omega \text{ and } \mathbf{f}\cdot\mathbf{v}=0 \text{ on } \gamma_b\}$, and by [1, Proposition 3.19], $u_\theta\in\mathsf{V}^{d-}=\mathring{H}^1_{-1}(\omega)$. In the magnetostatic case, $\mathbf{f}_m\in\mathscr{F}^d=\{\mathbf{f}\in L^2_{-1}(\omega)^2:\ \mathrm{div}\,\mathbf{f}=0\}$, and $u_\theta\in\mathsf{V}^{n-}=H^1_{-1}(\omega)$. The following diagram is commutative and isometric (see p. 875):

$$\begin{array}{ccc} \breve{\mathscr{X}} & \xrightarrow{\text{curl}} & \breve{\mathscr{J}} \\ \varpi_\theta\big\downarrow & & \big\downarrow\varpi_m \\ \breve{\mathscr{X}}_\theta & \xrightarrow{1/r\ \text{curl}\ (r.)} & \breve{\mathscr{J}}_m \\ \mathsf{R}\big\downarrow & & \big\downarrow\mathsf{R} \\ \mathsf{V}^- & \xrightarrow{\text{curl}} & \mathscr{F} \end{array} \tag{60}$$

We recall the absence of azimuthal singularities; the azimuthal component of any field in $\check{\mathbf{H}}(\mathbf{curl}, \mathrm{div}; \Omega)$ is automatically in $\check{\mathbf{H}}^1(\Omega)$, i.e. $u_\theta \in H^1_{-1}(\omega)$.

4.6. Space regularity of the electric and magnetic fields

Definition 4.9

The *maximal* and *minimal exponents of singularity* are:

$$\sigma^e_m = \min \mathbb{S}^e, \quad \sigma^e_M = \max \mathbb{S}^e, \quad \text{where: } \mathbb{S}^e = \{\alpha_j,\ j \in \mathscr{K}_{ES}\text{: } v_j + 1/2,\ j \in \mathscr{K}_{OS}\}$$

$$\sigma^b_m = \min \mathbb{S}^b, \quad \sigma^b_M = \max \mathbb{S}^b, \quad \text{where: } \mathbb{S}^e = \{\alpha_j,\ j \in \mathscr{K}_{ES}\}$$

Theorem 4.10

There holds: $\check{\mathscr{X}}^d_m \subset H^s_1(\omega)^2$, i.e. $\mathsf{U}^d \subset H^s_{-1}(\omega)^2$, or $\check{\mathscr{X}}^d \in \check{\mathbf{H}}^s(\Omega)$, iff $s < \sigma^e_m$; similarly, $\check{\mathscr{X}}^{0n}_m \subset H^s_1(\omega)^2$, i.e. $\mathsf{U}^{0n} \subset H^s_{-1}(\omega)^2$, or $\check{\mathscr{X}}^{0n} \subset \check{\mathbf{H}}^s(\Omega)$, iff $s < \sigma^b_m$.

Proof

For any $\mathscr{U}^d \in \check{\mathscr{X}}^d$, its azimuthal component is in $\check{\mathbf{H}}^1(\Omega)$; so the global space regularity will be determined by the singular part of its meridian component. We have: $\check{\mathscr{X}}^d = \check{\mathscr{X}}^d_R \oplus \mathrm{span}\{\mathbf{S}^d_j/r,\ j \in \mathscr{K}^e_S\}$. By Proposition 2.2, the formula (55) proves that the $\mathbf{S}^d_j$ associated to reentrant edges are in $H^s(\omega_{E_j})^2$ iff $s < \alpha_j$. Then, because of their support, $\mathbf{S}^d_j \in H^s_{-1}(\omega)^2$, and $\mathbf{S}^d_j/r \in H^s_1(\omega)^2 = H^s_+(\omega) \times H^s_-(\omega)$. Similarly, for conical vertices, (56) proves that $\mathbf{S}^d_j \in H^s_{-1}(\omega)^2$ and $\mathbf{S}^d_j/r \in H^s_1(\omega)^2 = H^s_-(\omega) \times H^s_+(\omega)$, iff $s < v_j + 1/2$.

The conclusions follow. The proof in the magnetic case is similar (and simpler). □

5. ANALYSIS OF THE MODIFIED WAVE EQUATION

Setting $q = \omega \times]0, T[$, $\sigma_{a/b} = \gamma_{a/b} \times]0, T[$, we consider the evolution problem

$$\begin{cases} u''(t) - \Delta^- u(t) = f(t) \quad \text{in } q \\ u = 0 \text{ on } \sigma_a, \quad u = 0, \text{ resp. } \partial_v u = 0 \text{ on } \sigma_b \\ u(0) = u_0, \quad u'(0) = u_1 \text{ in } \omega \end{cases} \tag{61}$$

We are mainly interested in the decomposition into regular and singular parts, and in the space–time regularity, of the solution to (61). This study, carried out in Subsections 5.1 and 5.2, closely parallels and relies upon the similar work on the standard Laplacian and the standard wave operator by Grisvard [8, Paragraphs 2.5.2 and 5.3]; so many proofs will be sketched or even omitted.

Let $H = L^2_{-1}(\omega)$, $\mathsf{V}^{-1} = \mathring{H}^1_{-1}(\omega)$ or $H^1_{-1}(\omega)$, A the unbounded operator $-\Delta^-$ on H, and $D_A = \Phi^{d/n-}$ its domain. As A is a strictly positive self-adjoint operator with compact inverse, we define the power A^ϑ, $0 \leqslant \vartheta \leqslant 1$, by interpolation in the usual way; in particular, $\mathsf{V}^- = D_{A^{1/2}}$. Then by adapting [8, Theorem 5.1.3], we have the basic result of

Theorem 5.1
Assume $f \in L^1(0,T;\mathsf{V}^-)$, $u_0 \in D_A$ and $u_1 \in \mathsf{V}^-$. The problem (61) admits a unique solution $u \in \mathscr{C}^0(0,T;D_A) \cap \mathscr{C}^1(0,T;\mathsf{V}^-)$, depending continuously on the data f, u_0, u_1 in their respective spaces. If, moreover, $f \in \mathscr{C}^0(0,T;H)$, then $u \in \mathscr{C}^2(0,T;H)$.

The spaces $\mathscr{C}^m(0,T;X)$, $\mathscr{C}^{0,\alpha}(0,T;X)$, $L^p(0,T;X), W^{s,p}(0,T;X)$, where X is some Banach space of functions over ω or Ω, are the usual ones; the time regularity defining the $\mathscr{C}$-spaces is assumed to extend up to $t=0$ and T.

5.1. *Estimates with parameter for* Δ^-

Given $\xi \geqslant 0$, we study the variational solution in $\mathsf{V}^{d-} = \overset{\circ}{H}{}^1_{-1}(\omega)$ resp. $\mathsf{V}^{n-} = H^1_{-1}(\omega)$, in the sense of Proposition 2.3, to the Dirichlet, resp. Neumann problem

$$-\Delta^- u + \xi^2 u = f \ \text{ in } \omega, \quad u = 0 \ \text{ on } \gamma_a, \quad u, \text{ resp. } \partial_\nu u = 0 \ \text{ on } \gamma_b \tag{62}$$

If $f \in L^2_{-1}(\omega)$, u belongs to $\Phi^- = \Phi^{d-}$, resp. Φ^{n-}; in this section, we shall usually omit the index d or n: and we write $S_j = S_j^{d-}$ or S_j^{n-}, $\Upsilon_j = \Upsilon_j^{d-}$ or $\Upsilon_j^{n-}, \nu_j = \nu_{j,1}^+ \ldots$.

The relevant sets of geometrical singularities are $\mathscr{K}_S^b$ in the Dirichlet case and $\mathscr{K}_S^e$ in the Neumann case; in this section, we write $\mathscr{K}_S$ to cover both cases. u is split onto $\Phi^- = \Phi_R^- \oplus \Phi_S^-$:

$$u = u_R^*(\xi) + \sum_{j \in \mathscr{K}_S} c_j(\xi) S_j \quad \text{with } u_R^* \in \Phi_R^- \tag{63}$$

It is indeed more convenient to use the following decomposition:

$$u = u_R(\xi) + \sum_{j \in \mathscr{K}_S} c_j(\xi) \mathrm{e}^{-\xi \rho_j} S_j \quad \text{with } u_R \in \Phi_R^-, \tag{64}$$

which holds with *the same* $c_j(\xi)$, since one easily checks that $(1 - \mathrm{e}^{-\xi\rho_j}) S_j \in \Phi_R^-$. The main goal of this subsection is to obtain estimates of the various terms in (64) as $\xi \to \infty$.

For a given ξ, the mapping $f \mapsto u$ is linear and continuous, and so is the projection on the closed subspace spanned by $\mathrm{e}^{-\xi\rho_j} S_j$: so the mapping $f \mapsto c_j(\xi)$ is a linear continuous form on $L^2_{-1}(\omega)$, and there exists $g_j(\xi) \in L^2_{-1}(\omega)$ such that

$$c_j(\xi) = \iint_\omega f g_j(\xi) \frac{\mathrm{d}\omega}{r} \tag{65}$$

Obviously, $g_j(\xi) \in N_\xi^-$, the orthogonal of $(\Delta^- - \xi^2 \mathsf{I}) \Phi_R^-$ within $L^2_{-1}(\omega)$.

We shall now give a characterization of N_ξ^- (Proposition 5.2) and local expressions for a basis of this space (Lemma 5.3). Then, we shall give a representation formula for $c_j(\xi)$ in Proposition 5.4, which will allow us to give the desired bounds in Theorem 5.7.

As a consequence of References [1, Lemma 5.12; 6, Lemma 4.16], we have the following 'very weak' integration by parts formulae for Δ^-. If $p \in L^2_{-1}(\omega)$, $\Delta^- p \in L^2_{-1}(\omega)$, and $w \in \Phi_R^{d-}$, resp. $w \in \Phi_R^{n-}$ and $w|_{\gamma_i}$ belongs to the suitable trace space, there holds

$$\iint_\omega \{ p \Delta^- w - w \Delta^- p \} \frac{\mathrm{d}\omega}{r} = \sum_i \left\langle p, \frac{\partial w}{\partial \nu} \right\rangle_{\text{on } \gamma_i} \tag{66}$$

resp.

$$\iint_\omega \{p\Delta^- w - w\Delta^- p\}\frac{d\omega}{r} = -\sum_i \left\langle \frac{\partial p}{\partial v}, w \right\rangle_{\text{on }\gamma_i} \tag{67}$$

the duality brackets on the right-hand sides being taken between the suitable spaces. In particular, the hypotheses of (67) are automatically satisfied when the trace of w vanishes on all sides γ_i except one. As a consequence of these formulae, one has

Proposition 5.2
Let $v \in L^2_{-1}(\omega)$; v belongs to N^-_ξ iff

$$\Delta^- v - \xi^2 v = 0 \text{ in } \omega, \quad v = 0 \text{ on } \gamma_a, \quad v, \text{ resp. } \partial_v v = 0 \text{ on } \gamma_b \tag{68}$$

The boundary condition is understood in the suitable space (see above) on γ_b, and in the strong sense on γ_a.

Lemma 5.3
For the reentrant edge E_j, resp., in the Neumann case, a sharp vertex O_j, let $u_j \in L^2_{-1}(\omega)$ be the function

$$u_j(\rho_j, \phi_j) = \eta_j(\rho_j) e^{-\xi\rho_j} \rho_j^{-\alpha_j} \Upsilon_j(\phi_j) \tag{69}$$

resp.

$$u_j(\rho_j, \phi_j) = \eta_j(\rho_j) e^{-\xi\rho_j} \rho_j^{-v_j} \Upsilon_j(\phi_j) \tag{70}$$

Then there exists $v_j \in N^-_\xi$ such that $-w_j = v_j - u_j \in \mathsf{V}^-$.

Proof
For the edges, it is similar to that of Lemma 3.3. For a sharp vertex, checking that $f_j = \Delta^- u_j - \xi^2 u_j \in L^p_{1-p}(\omega)$ for $p < p_\star = 3/(2+v_j)$ is straightforward. Since $v_j < 1/2$, $p_\star > 6/5$ and one can define w_j as the variational solution in V^- of $\Delta^- w_j - \xi^2 w_j = f_j$ by Proposition 2.3. Then, $v_j = u_j - w_j$ is in $L^2_{-1}(\omega)$ and satisfies (68). □

Remark 5.1
By a dimension argument, the $(v_j)_{j\in\mathscr{K}_S}$ make up a basis of N^-_ξ.

Proposition 5.4
The coefficient $c_j(\xi)$ in (64) is explicitly given by

$$c_j(\xi) = -\frac{1}{2\lambda_j(\xi)} \iint_\omega f v_j \frac{d\omega}{r} \tag{71}$$

where $\lambda_j(\xi)$ satisfies, for ξ large enough,

$$0 < \lambda_{\min} \leqslant \lambda_j(\xi) \leqslant \lambda_{\max} \tag{72}$$

and the constants $\lambda_{\min}$ and $\lambda_{\max}$ depend only on the geometry, not on ξ.

Proof
Using decomposition (63), we calculate

$$I = \iint_\omega f v_j \frac{\mathrm{d}\omega}{r} = \iint_\omega v_j(\Delta^- - \xi^2 \mathsf{I})u \frac{\mathrm{d}\omega}{r}$$

$$= \iint_\omega v_j(\Delta^{-1} - \xi^2 \mathsf{I}) \left\{ u_R^*(\xi) + \sum_k c_k(\xi) S_k \right\} \frac{\mathrm{d}\omega}{r}$$

But $u_R^*(\xi) \in \Phi_R^-$ and $v_j \in N_\xi^-$, so $(\Delta^- - \xi^2 \mathsf{I})u_R^*(\xi)$ and v_j are orthogonal in $L^2_{-1}(\omega)$. Then, the orthogonality of v_j and $(\Delta^- - \xi^2\mathsf{I})S_k$ for $j \neq k$ (see next lemma) implies $I = -2\lambda_j(\xi)c_j(\xi)$. □

Lemma 5.5
There holds

$$I \stackrel{\text{def}}{=} \iint_\omega v_j(\Delta^- - \xi^2 \mathsf{I}) S_k \frac{\mathrm{d}\omega}{r} = -2\lambda_j(\xi)\delta_{jk} \tag{73}$$

and $\lambda_j(\xi)$ satisfies the bound (72).

Proof
Orthogonality. The integral

$$I_1 = \iint_\omega u_j(\Delta^- - \xi^2 \mathsf{I}) S_k \frac{\mathrm{d}\omega}{r}$$

vanishes for $j \neq k$ since u_j and S_k have disjoint supports. Moreover, using Proposition 2.4 and the variational definition of w_j, we calculate

$$I_2 = \iint_\omega w_j(\Delta^- - \xi^2 \mathsf{I}) S_k \frac{\mathrm{d}\omega}{r} = \iint_\omega S_k(\Delta^- - \xi^2 \mathsf{I}) w_j \frac{\mathrm{d}\omega}{r}$$

$$= \iint_\omega S_k(\Delta^- - \xi^2 \mathsf{I}) u_j \frac{\mathrm{d}\omega}{r} \tag{74}$$

which vanishes again for $j \neq k$. So we are left with the case $j = k$, and we drop the subscript j. It stems from (74) that

$$I = I_1 - I_2 = \iint_\omega \{u(\Delta^- - \xi^2 \mathsf{I})S - S(\Delta^- - \xi^2 \mathsf{I})u\} \frac{\mathrm{d}\omega}{r}$$

$$= \iint_\omega \{u\Delta^- S - S\Delta^- u\} \frac{\mathrm{d}\omega}{r} \tag{75}$$

We shall examine the cases of a reentrant edge and a sharp vertex.

Edge singularity. We write (75) as

$$I = \iint_\omega \{u\Delta S - S\Delta u\} \frac{\mathrm{d}\omega}{r} - \iint_\omega \left\{ u \frac{\partial S}{\partial r} - S \frac{\partial u}{\partial r} \right\} \frac{\mathrm{d}\omega}{r^2} \stackrel{\text{def}}{=} I_3 - I_4$$

In the region where u and S are non-zero, there holds $0 < R_{\min} \leqslant r \leqslant R_{\max}$, hence it is enough to estimate the integrals

$$I_3' \stackrel{\text{def}}{=} \iint_\omega \{u\Delta S - S\Delta u\}\,\mathrm{d}\omega \quad \text{and} \quad I_4' \stackrel{\text{def}}{=} \iint_\omega \left\{u\,\frac{\partial S}{\partial r} - S\,\frac{\partial u}{\partial r}\right\}\mathrm{d}\omega$$

Grisvard [8, p. 66] has calculated $I_3' = -2\alpha$; hence, $-2\alpha/R_{\min} \leqslant I_3 \leqslant -2\alpha/R_{\max}$.

Now let us estimate I_4'. In the Dirichlet case, a straightforward calculation yields

$$u\,\frac{\partial S}{\partial r} - S\,\frac{\partial u}{\partial r} = \frac{2\alpha}{\pi}\,\mathrm{e}^{-\xi\rho}\eta(\rho)^2\rho^{-1}\cos\phi'\sin^2(\alpha\phi)(2\alpha + \xi\rho)$$

and thus

$$I_4' = \frac{2\alpha}{\pi}\int_{\phi=0}^{\pi/\alpha}\cos(\phi+\phi_0)\sin^2(\alpha\phi)d\phi\int_{\rho=0}^{+\infty}\mathrm{e}^{-\xi\rho}\eta(\rho)^2(2\alpha+\xi\rho)\,\mathrm{d}\rho \stackrel{\text{def}}{=} k_1 J(\xi)$$

where k_1 depends only on the geometry. Moreover,

$$0 \leqslant J(\xi) = \int_{s=0}^{+\infty}\mathrm{e}^{-s}\eta\left(\frac{s}{\xi}\right)^2(2\alpha+s)\frac{\mathrm{d}s}{\xi} \leqslant \frac{1}{\xi}\int_{s=0}^{+\infty}\mathrm{e}^{-s}(2\alpha+s)\,\mathrm{d}s \stackrel{\text{def}}{=} k_2/\xi$$

where k_2 is independent of ξ. So, $I_4 = O(\xi^{-1})$. This is still true in the Neumann case: the only change concerns the angular part of the integrand, which is once more bounded and independent of ξ. The estimate (72) follows.

In fact, one can prove that I is independent of ξ: $I = -2\alpha/a$. This calculation uses a generalized version of (4) and an approximation of ω which avoids the singularity.

Conical singularity. Using (14) and $\Lambda^-\Upsilon = \nu(\nu+1)\Upsilon$, we calculate

$$u\Delta^- S - S\Delta^- u = \left\{\mathrm{e}^{-\xi\rho}\rho^{-\nu}\eta(\rho)\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}[\rho^{1+\nu}\eta(\rho)] - \rho^{1+\nu}\eta(\rho)\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}[\mathrm{e}^{-\xi\rho}\rho^{-\nu}\eta(\rho)]\right\}\Upsilon(\phi)^2$$

Using the normalization of $\Upsilon(\phi)$ and an integration by parts formula in ρ, Equation (75) yields

$$I = \left[\mathrm{e}^{-\xi\rho}\rho^{-\nu}\eta(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}[\rho^{1+\nu}\eta(\rho)]\right]_0^{+\infty} - \left[\rho^{1+\nu}\eta(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}[\mathrm{e}^{-\xi\rho}\rho^{-\nu}\eta(\rho)]\right]_0^{+\infty} = -2\nu$$

This value is constant, so it satisfies (72). □

In close analogy with Lemma 2.5.7 in [8], and using techniques similar to that of the proof of Lemma 5.5, we obtain

Lemma 5.6

The function w_j defined in Lemma 5.3, associated with a reentrant edge E_j, resp. a sharp vertex O_j, satisfies for ξ large enough

$$\|w_j\|_{0,-1,\omega} \leqslant K\xi^{\alpha_j - 1}, \quad \text{resp. } \|w_j\|_{0,-1,\omega} \leqslant K\xi^{\nu_j - 1/2}$$

Theorem 5.7
There exists a constant K such that the different terms in (64) satisfy, for ξ large enough

$$\|u_R(\xi)\|_{\Phi_R^-} + \xi\|u_R(\xi)\|_{1,-1,\omega} + \xi^2\|u_R(\xi)\|_{0,-1,\omega} \leqslant K\|f\|_{0,-1,\omega} \tag{76}$$

$$\text{(for a reentrant edge)} \quad |c_j(\xi)| \leqslant K(1+\xi)^{\alpha_j-1}\|f\|_{0,-1,\omega} \tag{77}$$

$$\text{(for a sharp vertex)} \quad |c_j(\xi)| \leqslant K(1+\xi)^{\nu_j-1/2}\|f\|_{0,-1,\omega} \tag{78}$$

Proof
Step 1. Let us first prove the estimates for $c_j(\xi)$. One has

$$c_j(\xi) = -\frac{1}{2\lambda_j(\xi)}\iint_\omega f v_j \frac{\mathrm{d}\omega}{r} = -\frac{1}{2\lambda_j(\xi)}\iint_\omega f(u_j - w_j)\frac{\mathrm{d}\omega}{r}$$

Hence by the Schwarz and triangle inequalities

$$|c_j(\xi)| \leqslant \frac{1}{2\lambda_j(\xi)}\|f\|_{0,-1,\omega} \times \{\|u_j\|_{0,-1,\omega} + \|w_j\|_{0,-1,\omega}\} \tag{79}$$

The estimate for $\|w_j\|_{0,-1,\omega}$ was obtained in the previous lemma, and simple calculations show that $\|u_j\|_{0,-1,\omega} = O(\xi^{\alpha_j-1})$ or $O(\xi^{\nu_j-1/2})$. Then (77), resp. (78) follows from (79) and estimate (72) for $\lambda_j(\xi)$.

Step 2. To obtain (76), we shall consider: $f_R = -(\Delta^- - \xi^2\mathsf{I})u_R$ and prove the bound

$$\|f_R\|_{0,-1,\omega} \leqslant K\|f\|_{0,-1,\omega} \tag{80}$$

One has

$$f_R = f + (\Delta^- - \xi^2\mathsf{I})\sum_{j\in\mathscr{K}_S} c_j(\xi)\mathrm{e}^{-\xi\rho_j}S_j$$

so it is enough to bound each $c_j(\xi)(\Delta^- - \xi^2\mathsf{I})\{\mathrm{e}^{-\xi\rho_j}S_j\}$. By adapting the proof of [8, Lemma 2.5.9], one can show that

$$\|(\Delta^- - \xi^2\mathsf{I})\{\mathrm{e}^{-\xi\rho_j}S_j\}\|^2_{0,-1,\omega} = O(\xi^{2-2\alpha_j}) \text{ or } (\xi^{1-2\nu_j})$$

Then we use (77) or (78) to estimate

$$\|c_j(\xi)(\Delta^- - \xi^2\mathsf{I})\{\mathrm{e}^{-\xi\rho_j}S_j\}\|^2_{0,-1,\omega} = O(1)\|f\|_{0,-1,\omega}$$

Estimate (80) follows.

Step 3. On the other hand, there holds

$$\|(\Delta^- - \xi^2\mathsf{I})u_R\|^2_{0,-1,\omega} = \|\Delta^- u_R\|^2_{0,-1,\omega} - 2\xi^2(\Delta^- u_R \mid u_R)_{0,-1,\omega} + \xi^4\|u_R\|^2_{0,-1,\omega}$$

According to (3), the scalar product in the second term, with its minus sign, is equal to $\|\mathbf{grad}\, u_R\|^2_{0,-1,\omega}$, which is a norm equivalent to $\|u_R\|^2_{1,-1,\omega}$. As for the first term, it is a norm equivalent to the canonical norm of Φ_R^- (Lemma 4.7). Hence, by (80)

$$\|u_R(\xi)\|^2_{\Phi_R^-} + 2\xi^2\|u_R(\xi)\|_{1,-1,\omega} + \xi^4\|u_R(\xi)\|_{0,-1,\omega} \leqslant K_1\|f_R\|^2_{0,-1,\omega} \leqslant K_2\|f\|^2_{0,-1,\omega}$$

which is equivalent to (76). □

5.2. Space–time regularity of the solution to the wave-like problem

The solution to (61) belongs to Φ^- at any time, hence we have the decomposition

$$u(t) = u_R^*(t) + \sum_{j \in \mathscr{K}_S} c_j(t) S_j \quad \text{with } u_R^*(t) \in \Phi_R^- \tag{81}$$

It stems from the continuity of projections onto closed subspaces that

$$u_R^* \in \mathscr{C}^0(0, T; \Phi_R^-), \quad c_j \in \mathscr{C}^0(0, T; \mathbb{R})$$

Moreover, we have the more precise result of

Theorem 5.8
Define σ_M and σ_m as σ_M^b and σ_m^b in the Dirichlet case, and σ_M^e and σ_m^e in the Neumann case. Then, the different terms in (81) satisfy

$$u_R^* \in \mathscr{C}^{0,1-\sigma_M-\varepsilon}(0, T; \mathsf{R}[H_-^{1+\sigma_M+\delta}(\omega)]), \quad c_j \in \mathscr{C}^{0,1-\sigma_M-\varepsilon}(0, T; \mathbb{R}) \quad \text{for } \varepsilon > \delta > 0$$

$$u \in \mathscr{C}^{0,1-\sigma_M-\varepsilon}(0, T; \mathsf{R}[H_-^{1+\sigma_m-\varepsilon'}(\omega)]) \quad \text{for } \varepsilon, \varepsilon' > 0$$

Proof
By Theorem 5.1 $u \in \mathscr{C}^0(0, T; D_A) \cap \mathscr{C}^1(0, T; D_{A^{1/2}})$. By convexity, it follows that $u \in \mathscr{C}^{0,\sigma}(0, T; D_{A^{1-\sigma/2}})$ for $0 \leqslant \sigma \leqslant 1$. Now, let $\sigma < 1 - \sigma_M$ and $1 + \sigma_M < s < 2 - \sigma$. Applying Lemma 5.9 (see below), we split u on the direct sum $E = \mathsf{R}[H_-^s(\omega)] + \operatorname{span}\{S_j,\ j \in \mathscr{K}_S\}$.

As $\Phi_R^- \subset \mathsf{R}[H_-^2(\omega)] \subset \mathsf{R}[H_-^s(\omega)]$, the components of u in the two direct sums $\Phi_R^- \oplus \operatorname{span}\{S_j\}$ and $\mathsf{R}[H_-^s(\omega)] \oplus \operatorname{span}\{S_j\}$ are the same. This shows $u_R^* \in \mathscr{C}^{0,\sigma}(0, T; \mathsf{R}[H_-^s(\omega)])$ and $c_j \in \mathscr{C}^{0,\sigma}(0, T; \mathbb{R})$.

Finally, we note that u_R^* and all the S_j belong to $\mathsf{R}[H_-^{1+\sigma_m-\varepsilon'}(\omega)]$, hence $u \in \mathscr{C}^{0,1-\sigma_M-\varepsilon}(0, T; \mathsf{R}[H_-^{1+\sigma_m-\varepsilon'}(\omega)])$. □

Here is the announced technical lemma:

Lemma 5.9
Let $\vartheta > (1 + \sigma_M)/2$; then for any $s < 2\vartheta$:

$$D_{A^\vartheta} \subset \mathsf{E} \equiv \mathsf{R}[H_-^s(\omega)] + \operatorname{span}\{S_j,\ j \in \mathscr{K}_S\} \tag{82}$$

Proof
The regularity of the S_j in the scale $\mathsf{R}[H_-^s(\omega)]$ is given by Proposition 2.2. Equations (50) and (52) show that the edge functions belong to $H^s(\omega)$—or $\mathsf{R}[H_-^s(\omega)]$ by a support argument—iff $s < 1 + \alpha_j$. Similarly, the vertex functions S_j belong to $\mathsf{R}[H_-^s(\omega)]$, $1 < s < 2$, iff $S_j/r \in H_-^s(\omega) = \{w \in H_1^s(\Omega) \colon w|_{\gamma_a} = 0\}$. Since

$$\frac{S_j}{r} = \frac{\rho_j^{1+\nu_j} \Upsilon_j(\phi_j)}{\rho_j \sin \phi_j} = \rho_j^{\nu_j} P_{\nu_j}^1(\cos \phi_j) \tag{83}$$

this holds iff $s < \nu_j + 3/2$: the condition $S_j/r|_{\gamma_a} = 0$ is ensured by the Legendre function.

Hence for $1 + \sigma_M < s < 2\vartheta$, the space E is a direct sum, which we equip with the product topology; and it is enough to prove (82) in this case. We follow Lemmas 5.3.2 and 5.3.3 of [8]. First, we show that

$$\|(A + t\mathsf{I})^{-1}\|_{H \to \mathsf{E}} = O(t^{s/2-1}) \tag{84}$$

as $t \to +\infty$. Let $f \in H$ and $u = (A + t\mathsf{I})^{-1} f$: the decomposition (64) and the estimates (76) and (77) hold, with $\xi = \sqrt{t}$. One also has the decomposition (63)

$$u = \left[u_R + \sum_{j \in \mathscr{K}_S} c_j (\mathrm{e}^{-\rho_j \sqrt{t}} - 1) S_j \right] + c_j S_j \tag{85}$$

As the bracket belongs to $\Phi_R^- \subset \mathsf{R}[H_-^2(\omega)] \subset \mathsf{R}[H_-^s(\omega)]$ (Proposition 4.8). Equation (85) coincides with the decomposition of u in the direct sum E. Then we use Sobolev injections and interpolation arguments to show that the $\mathsf{R}[H_-^s(\omega)]$ norm of the bracket is $O(t^{s/2-1})$; as $c_j S_j$ decays faster, (84) is proven.

Finally, consider $u \in D_{A^\vartheta}$ and write $u = A^{-\vartheta} A^\vartheta u$, i.e.

$$u = \frac{\sin \pi\vartheta}{\pi} \int_0^{+\infty} t^{-\vartheta} (A + t\mathsf{I})^{-1} A^\vartheta u \, \mathrm{d}t$$

By (84), the norm of the integrand in E is $O(t^{s/2-1-\vartheta}) \|A^\vartheta u\|_H$, and the integral converges for $s < 2\vartheta$. □

6. ANALYSIS OF THE TIME-DEPENDENT MAXWELL EQUATIONS

6.1. The Maxwell equations

Given $T > 0$, let $Q = \Omega \times]0, T[$ and $\Sigma = \Gamma \times]0, T[$; let c and ε_0 be the speed of light and the dielectric permittivity; ϱ and $\mathscr{J}$ the *source terms* (charge and current densities). First, there are the evolution equations

$$\frac{\partial \mathscr{E}}{\partial t} - c^2 \,\mathbf{curl}\, \mathscr{B} = -\frac{1}{\varepsilon_0} \mathscr{J}, \quad \frac{\partial \mathscr{B}}{\partial t} + \mathbf{curl}\, \mathscr{E} = 0 \text{ in } Q \tag{86}$$

Then, the constraint equations, viz. divergence and boundary conditions:

$$\operatorname{div} \mathscr{E} = \frac{\varrho}{\varepsilon_0}, \quad \operatorname{div} \mathscr{B} = 0 \text{ in } Q \tag{87}$$

$$\mathscr{E} \times \mathbf{n} = 0, \quad \mathscr{B} \cdot \mathbf{n} = 0 \text{ on } \Sigma \tag{88}$$

The charge conservation equation,

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} \mathscr{J} = 0 \text{ in } Q \tag{89}$$

appears as a compatibility condition for the first equation in (86), given the first in (87). Last, initial conditions are provided to close the system of equations,

$$\mathscr{E}(0) = \mathscr{E}_0, \quad \mathscr{B}(0) = \mathscr{B}_0 \text{ in } \Omega \tag{90}$$

The existence and uniqueness of the electromagnetic field can be proven under suitable assumptions on the data and the initial conditions, cf. [1, Theorem 6.1]. In the axisymmetric case, we have:

Proposition 6.1
If ϱ, $\mathscr{J}$ and $(\mathscr{E}_0, \mathscr{B}_0)$ are axisymmetric, so is the solution to (86)–(90). And, provided that $\mathscr{J} \in \mathscr{C}^1(0,T;\breve{\mathbf{L}}^2(\Omega))$ and that $\varrho \in \mathscr{C}^0(0,T;\breve{L}^2(\Omega))$, there holds

$$\mathscr{E} \in \mathscr{C}^0(0,T;\breve{\mathscr{X}}^d) \cap \mathscr{C}^1(0,T;\breve{\mathbf{L}}^2(\Omega)), \quad \mathscr{B} \in \mathscr{C}^0(0,T;\breve{\mathscr{X}}^{0n}) \cap \mathscr{C}^1(0,T;\breve{\mathbf{L}}^2(\Omega)) \tag{91}$$

6.2. Reduction to two-dimensional problems and basic regularity results

We now examine the simplification of the time-dependent Maxwell problem (86)–(90) induced by the axial symmetry. Similarly to the static problems (see Section 4), there is a decoupling of meridian and azimuthal components: namely, the problem (86)–(90) is decoupled into two sub-systems:

- the 'first system' links the meridian electric field and the azimuthal magnetic field;
- the 'second system' links the azimuthal electric and meridian magnetic fields.

Like in Section 4, it is convenient to introduce the product by r of the 'natural' variables

$$\mathbf{u} = r\mathscr{E}, \quad \mathbf{v} = r\mathscr{B}, \quad \mathbf{f} = (r/\varepsilon_0)\mathscr{J}, \quad g = r\varrho/\varepsilon_0$$

The following forms for the two systems are obtained through simple calculations.
The first system. The evolution and constraint equations are:

$$\frac{\partial \mathbf{u}_m}{\partial t} - c^2 \,\mathbf{curl}\, v_\theta = -\mathbf{f}_m \quad \text{in } q \tag{92}$$

$$\frac{\partial v_\theta}{\partial t} + \mathrm{curl}_-\, \mathbf{u}_m = 0 \quad \text{in } q \tag{93}$$

$$\mathrm{div}\, \mathbf{u}_m = g \quad \text{in } q \tag{94}$$

$$\mathbf{u}_m \cdot \mathbf{v} = 0 \text{ on } \sigma_a, \quad \mathbf{u}_m \cdot \boldsymbol{\tau} = 0 \text{ on } \sigma_b \tag{95}$$

The compatibility condition between $\mathbf{f}_m$ and g reads

$$\mathrm{div}\, \mathbf{f}_m + \frac{\partial g}{\partial t} = 0 \text{ in } q \tag{96}$$

As for the initial data, they are

$$\mathbf{u}_m(0) = \mathbf{u}_{m0} = r\mathbf{E}_{0m}, \quad v_\theta(0) = v_{\theta 0} = rB_{0\theta} \text{ in } \omega \tag{97}$$

they satisfy

$$\mathrm{div}\, \mathbf{u}_{m0} = g(0) \text{ in } \omega, \quad \mathbf{u}_{m0} \cdot \mathbf{v} = 0 \text{ on } \gamma_a, \quad \mathbf{u}_{m0} \cdot \boldsymbol{\tau} = 0 \text{ on } \gamma_b \tag{98}$$

The second system. The evolution and constraint equations read:

$$\frac{\partial u_\theta}{\partial t} - c^2 \operatorname{curl}_- \mathbf{v}_m = -f_\theta \quad \text{in } q \tag{99}$$

$$\frac{\partial \mathbf{v}_m}{\partial t} + \mathbf{curl}\, u_\theta = 0 \quad \text{in } q \tag{100}$$

$$\operatorname{div} \mathbf{v}_m = 0 \quad \text{in } q \tag{101}$$

$$u_\theta = 0 \text{ and } \mathbf{v}_m \cdot \mathbf{v} = 0 \quad \text{on } \sigma \tag{102}$$

There is no compatibility condition for this problem. The initial data are:

$$\mathbf{v}_m(0) = \mathbf{v}_{m0} = r\mathbf{B}_{0m}, \quad u_\theta(0) = u_{\theta 0} = rE_{0\theta} \text{ in } \omega \tag{103}$$

they satisfy

$$\operatorname{div} \mathbf{v}_{m0} = 0 \text{ in } \omega, \quad \mathbf{v}_{m0} \cdot \mathbf{v} = 0 \text{ on } \gamma, \quad u_{\theta 0} = 0 \text{ on } \gamma \tag{104}$$

Basic regularity results. Combining Proposition 6.1 with the results of Sections 2.2 and 4, we obtain the following regularity result:

$$(\mathbf{u}_m, v_\theta) \in \mathscr{C}^0(0, T; \mathsf{U}^d \times H^1_{-1}(\omega)) \cap \mathscr{C}^1(0, T; L^2_{-1}(\omega)^2 \times L^2_{-1}(\omega)) \tag{105}$$

$$(\mathbf{v}_m, u_\theta) \in \mathscr{C}^0(0, T; \mathsf{U}^{0n} \times \overset{\circ}{H}{}^1_{-1}(\omega)) \cap \mathscr{C}^1(0, T; L^2_{-1}(\omega)^2 \times L^2_{-1}(\omega)) \tag{106}$$

The rest of this section is devoted to the improvement of this result, as announced in Reference [12]. For the azimuthal components, there holds by interpolation:

$$(u_\theta, v_\theta) \in \mathscr{C}^{0,1-\sigma}(0, T; \overset{\circ}{H}{}^\sigma_{-1}(\omega) \times H^\sigma_{-1}(\omega)) \quad \text{for } 0 < \sigma < 1 \tag{107}$$

We now focus on the meridian components and their splitting, valid at any time, into a regular and a singular part, the latter being chosen within the explicit singular spaces of Section 4.4:

$$\mathbf{u}_m(t) = \mathbf{u}_R(t) + \sum_{i \in \mathscr{K}_S^e} \kappa_i(t) \mathbf{S}_i^d \tag{108}$$

$$\mathbf{v}_m(t) = \mathbf{v}_R(t) + \sum_{j \in \mathscr{K}_S^b} \delta_j(t) \mathbf{S}_j^{0n} \tag{109}$$

By the continuity of projections, there holds: $(\mathbf{u}_R, \mathbf{v}_R) \in \mathscr{C}^0(0, T; \mathsf{U}_R^d \times \mathsf{U}_R^{0n})$ and, for $(i, j) \in \mathscr{K}_S^e \times \mathscr{K}_S^b$, $(\kappa_i, \delta_j) \in \mathscr{C}^0(0, T; \mathbb{R} \times \mathbb{R})$.

6.3. Regularity in time of the singular coefficients and global space–time regularity of the fields

The first system. We consider the Hodge decomposition, valid at any time:

$$\mathbf{u}_m(t) = \mathbf{curl}\, W(t) - r\, \mathbf{grad}\, V(t) = \mathbf{hodge}(V(t), W(t)) \tag{110}$$

If we assume: $\forall t$, $V(t) \in \Phi^{d+}$ and $W(t) \in \Phi^{n-}$, the results of Subsection 4.2 allow us to state that they are uniquely defined and satisfy

$$(V, W) \in \mathscr{C}^0(0, T; \Phi^{d+} \times \Phi^{n-}) \cap \mathscr{C}^1(0, T; \mathsf{V}^{d+} \times \mathsf{V}^{n-}) \tag{111}$$

Combining (110) with (94) and (95) shows that, for all t, $V(t)$ is the variational solution to

$$-\Delta^+ V(t) = \frac{\varrho(t)}{\varepsilon_0} \text{ in } \omega, \quad V(t) = 0 \text{ on } \gamma_b, \quad \frac{\partial V(t)}{\partial \nu} = 0 \text{ on } \gamma_a \tag{112}$$

Obviously, the time regularity of ϱ and V are related:

Proposition 6.2
If $\varrho \in \mathscr{C}^m(0, T; [\mathsf{V}^{d+}]')$ resp. $W^{s,p}(0, T; [\mathsf{V}^{d+}]')$ or $\mathscr{C}^{0,\sigma}(0, T; L^2_1(\omega))$, then $V \in \mathscr{C}^m(0, T; \mathsf{V}^{d+})$ resp. $W^{s,p}(0, T; \mathsf{V}^{d+})$ or $\mathscr{C}^{0,\sigma}(0, T; \Phi^{d+})$.

Let us now look for the equation satisfied by $W(t)$. Plugging (110) into (92) and (93) yields

$$\frac{\partial v_\theta}{\partial t} - \Delta^- W = 0 \tag{113}$$

$$\mathbf{curl}\left(\frac{\partial W}{\partial t} - c^2 v_\theta\right) = -\mathbf{f}_m + \frac{\partial}{\partial t}(r\, \mathbf{grad}\, V) \tag{114}$$

The right-hand side of (114) is divergence free, thanks to (96) and (112). Hence, there exists a function χ such that $\mathbf{curl}\, \chi = -\mathbf{f}_m + \partial_t(r\, \mathbf{grad}\, V)$. But the left-hand side of (114), by (111) and (105), is at any time in $L^2_{-1}(\omega)^2$. Hence, it has a unique potential in $H^1_{-1}(\omega)$, and we can choose $\chi = \partial_t W - c^2 v_\theta$. Combining this equation with (113) yields

$$\frac{\partial^2 W}{\partial t^2} - c^2 \Delta^- W = \frac{\partial \chi}{\partial t} \stackrel{\text{def}}{=} \psi \tag{115}$$

Proposition 6.3
Assume $\mathbf{j}_m \in W^{1,1}(0, T; \check{\mathbf{L}}^2(\Omega))$, hence $\mathbf{f}_m \in W^{1,1}(0, T; L^2_{-1}(\omega)^2)$. Then there exists a strong solution $W(t) \in \mathscr{C}^1(0, T; \mathsf{V}^{n-}) \cap \mathscr{C}^0(0, T; \Phi^{n-})$ to the evolution equation (115), supplemented with the boundary and initial conditions

$$\begin{cases} W = 0 \text{ on } \sigma_a, \quad \partial_\nu W = 0 \text{ on } \sigma_b \\ W(0) = W_0, \quad W'(0) = W_1 \text{ in } \omega \end{cases} \tag{116}$$

where the initial conditions satisfy

$$W_0 \in \Phi^{n-}, \quad W_1 \in H^1_{-1}(\omega), \quad \mathbf{curl}\, W_0 - r\, \mathbf{grad}\, V(0) = \mathbf{u}_{m0}, \quad W_1 = c^2 v_{\theta 0} + \chi(0) \tag{117}$$

Proof
$\mathbf{j}_m \in W^{1,1}(0,T;\check{\mathbf{L}}^2(\Omega))$ implies $\partial_t \varrho \in W^{1,1}(0,T;\check{H}^{-1}(\Omega))$ by (89) and $\partial_t V \in W^{1,1}(0,T;\mathsf{V}^{d+})$ by Proposition 6.2. Hence $\mathbf{curl}\,\chi = -\mathbf{f}_m + \partial_t(r\,\mathbf{grad}\,V) \in W^{1,1}(0,T;L^2_{-1}(\omega)^2)$, $\chi \in W^{1,1}(0,T;H^1_{-1}(\omega))$, and finally $\psi \in L^1(0,T;H^1_{-1}(\omega))$. Hence the existence of W by Theorem 5.1. Now (117) is clear, since χ and $\partial_t(r\,\mathbf{grad}\,V)$, as $W^{1,1}$ functions of time, are continuous up to $t=0$, like $\mathbf{u}_m$ and v_θ. □

Conversely, checking that V and W defined, respectively, by (112) and (115)–(116) satisfy (110), provided (117) holds, is straightforward. Thus, we can apply the results of Section 5. We set

$$V(t) = V_R(t) + \sum_{i \in \mathscr{K}_S^e} \kappa_i^d(t)\{\lambda_i^d S_i^{d+}\} \tag{118}$$

$$W(t) = W_R(t) + \sum_{i \in \mathscr{K}_S^e} \kappa_i^n(t)\{\lambda_i^n S_i^{n-}\} \tag{119}$$

As stated in Subsection 4.4, we can choose the constants λ_i^d and λ_i^n so as to have

$$-r\,\mathbf{grad}\{\lambda_i^d S_i^{d+}\} + \mathbf{curl}\{\lambda_i^n S_i^{n-}\} = 2\mathbf{S}_i^d + 2\mathbf{w}_i^+, \quad \mathbf{w}_i^+ \in \mathsf{U}_R^d$$

$$-r\,\mathbf{grad}\{\lambda_i^d S_i^{d+}\} - \mathbf{curl}\{\lambda_i^n S_i^{n-}\} = 2\mathbf{w}_i^- \in \mathsf{U}_R^d$$

Combining (110) with (118) and (119), we find

$$\mathbf{u}_m(t) = \mathbf{curl}\,W_R(t) - r\,\mathbf{grad}\,V_R(t) + \sum_{i \in \mathscr{K}_S^e} \{(\kappa_i^d(t) - \kappa_i^n(t))\mathbf{w}_i^- + (\kappa_i^d(t) + \kappa_i^n(t))\mathbf{w}_i^+\}$$
$$+ \sum_{i \in \mathscr{K}_S^e} (\kappa_i^d(t) + \kappa_i^n(t))\mathbf{S}_i^d$$

Comparing this equation to (108), we infer

$$\mathbf{u}_R(t) = \mathbf{curl}\,W_R(t) - r\,\mathbf{grad}\,V_R(t) + \sum_{i \in \mathscr{K}_S^e} \{(\kappa_i^d(t) - \kappa_i^n(t))\mathbf{w}_i^- + \kappa_i(t)\mathbf{w}_i^+\} \tag{120}$$

$$\kappa_i(t) = \kappa_i^d(t) + \kappa_i^n(t) \quad \forall i \in \mathscr{K}_S^e \tag{121}$$

Theorem 6.4
Assume that the sources enjoy the following space–time regularity:

$$\varrho \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;L^2_1(\omega)) \quad \forall \varepsilon > 0 \quad \text{and} \quad \mathbf{f}_m \in W^{1,1}(0,T;L^2_{-1}(\omega)^2)$$

Then the following regularity results hold:

$$\kappa_i \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;\mathbb{R}) \quad \forall \varepsilon > 0 \tag{122}$$

$$\mathbf{u}_m \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;H_{-1}^{\sigma_m^e-\varepsilon'}(\omega)^2) \quad \forall \varepsilon, \varepsilon' > 0 \tag{123}$$

Proof
The projection onto closed subspaces, such as Φ_R^{d+}, span S_i^{n-}, etc., is smooth. Hence, the assumed regularity of ϱ, together with Proposition 6.2, yields

$$V_R \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;\Phi_R^{d+}), \quad \kappa_i^d \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;\mathbb{R})$$

and Theorem 5.8 implies

$$W_R \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;\mathsf{R}[H_-^{1+\sigma_M^e+\delta}(\omega)]), \quad \kappa_i^n \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;\mathbb{R})$$

So, under the above hypotheses, $\kappa_i \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;\mathbb{R})$.

Moreover, we know from Theorem 4.10 that for all t, $\mathbf{u}_m(t) \in H_{-1}^{\sigma_m^e-\varepsilon'}(\omega)$; and this space regularity is optimal. Since $\Phi_R^{d+} \subset H_+^2(\omega) \subset H_+^{1+\sigma_m^e-\varepsilon'}(\omega)$, one has

$$-r\,\mathbf{grad}\,V_R \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;\mathsf{R}[H_1^{\sigma_m^e-\varepsilon'}(\omega)^2]) = \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;H_{-1}^{\sigma_m^e-\varepsilon'}(\omega)^2)$$

by Proposition 2.1. The same proposition yields

$$W_R \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;\mathsf{R}[H_-^{1+\sigma_m^e-\varepsilon'}(\omega)]) \subset \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;H_{-1}^{1+\sigma_m^e-\varepsilon'}(\omega))$$

thus $\mathbf{curl}\,W_R \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;H_{-1}^{\sigma_m^e-\varepsilon'}(\omega)^2)$. Then (123) follows from (120). □

Corollary 6.5
Under the hypotheses of the above theorem, there holds

$$\mathscr{E} \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;\breve{\mathbf{H}}^{\sigma_m^e-\varepsilon'}(\Omega)) \tag{124}$$

Proof
It stems from (123) that $\mathbf{E}_m \in \mathscr{C}^{0,1-\sigma_M^e-\varepsilon}(0,T;H_1^{\sigma_m^e-\varepsilon'}(\omega)^2)$, where $H_1^{\sigma_m^e-\varepsilon'}(\omega)^2 = H_-^{\sigma_m^e-\varepsilon'}(\omega)\times H_+^{\sigma_m^e-\varepsilon'}(\omega)$ is the meridian component of $\breve{\mathbf{H}}^{\sigma_m^e-\varepsilon'}(\Omega)$. On the other hand, u_θ satisfies (107); taking $\sigma = \sigma_m^e - \varepsilon'$ yields $u_\theta \in \mathscr{C}^{0,1-\sigma_m^e+\varepsilon'}(0,T;H_{-1}^{\sigma_m^e-\varepsilon'}(\omega))$ or $E_\theta \in \mathscr{C}^{0,1-\sigma_m^e+\varepsilon'}(0,T;H_-^{\sigma_m^e-\varepsilon'}(\omega))$. The conclusion follows. □

The second system. According to Section 4.3, we set at any time

$$\mathbf{v}_m(t) = \mathbf{curl}\,\varphi(t) \tag{125}$$

with $\varphi(t) \in \Phi^{d-}$. It stems from the regularity (106) of $\mathbf{v}_m$, and Section 4.3 that

$$\varphi \in \mathscr{C}^0(0,T;\Phi^{d-}) \cap \mathscr{C}^1(0,T;\mathsf{V}^{d-}) \tag{126}$$

Let us look for the equation satisfied by φ. It follows from (100) that $\mathbf{curl}(\partial_t\varphi + u_\theta) = 0$. Since $u_\theta \in \mathsf{V}^{d-}$, we infer $u_\theta = -\partial_t\varphi$, and (99) becomes

$$\frac{\partial^2\varphi}{\partial t^2} - c^2\Delta^-\varphi = f_\theta \tag{127}$$

Given (126), Theorem 5.1 implies:

Proposition 6.6
If $j_\theta \in L^1(0,T;H^1_-(\omega))$ and $j_\theta|_\sigma = 0$—hence $f_\theta \in L^1(0,T;\overset{\circ}{H}{}^1_{-1}(\omega))$—$\varphi$ is the strong solution in $\mathscr{C}^0(0,T;\Phi^{d-}) \cap \mathscr{C}^1(0,T;\mathsf{V}^{d-})$ to the evolution equation (127), supplemented with the initial and boundary conditions

$$\begin{cases} \varphi = 0 \quad \text{on } \sigma \\ \varphi(0) = \varphi_0, \quad \varphi'(0) = \varphi_1 \ \text{ in } \omega \end{cases} \tag{128}$$

where the initial conditions satisfy

$$\varphi_0 \in \Phi^{d-}, \quad \varphi_1 \in \mathsf{V}^{d-}, \quad \mathbf{curl}\, \varphi_0 = \mathbf{v}_{m0}, \quad \varphi_1 = -u_{\theta 0} \tag{129}$$

The proof is similar to Proposition 6.3, and simpler. Conversely, checking that φ solution to (127) and (128) satisfies (125), provided (129) holds, is straightforward. To apply the results of Section 5, we set

$$\varphi(t) = \varphi_R(t) + \sum_{j \in \mathscr{K}_S^b} \delta_j(t) S_j^{d-} \tag{130}$$

We recall that $\mathbf{curl}\, S_j^{d-} = \mathbf{S}_j^{0n}$. Comparing (125) and (130), one sees that the singular coefficients $\delta_j(t)$ are indeed the same as in (109), and that $\mathbf{v}_R(t) = \mathbf{curl}\, \varphi_R(t)$.

Theorem 6.7
Assume that the current f_θ belongs to $W^{1,1}(0,T;\mathsf{V}^{d-})$. Then the following regularity results holds:

$$\delta_j \in \mathscr{C}^{0,1-\sigma_M^b-\varepsilon}(0,T;\mathbb{R}) \quad \forall \varepsilon > 0 \tag{131}$$

$$\mathbf{v}_m \in \mathscr{C}^{0,1-\sigma_M^b-\varepsilon}(0,T;H_{-1}^{\sigma_m^b-\varepsilon'}(\omega)^2) \quad \forall \varepsilon, \varepsilon' > 0 \tag{132}$$

Proof
By Theorem 5.8

$$\varphi_R \in \mathscr{C}^{0,1-\sigma_M^b-\varepsilon}(0,T;\mathsf{R}[H_-^{1+\sigma_M^b+\delta}(\omega)]), \quad \delta_j \in \mathscr{C}^{0,1-\sigma_M^b-\varepsilon}(0,T;\mathbb{R})$$

Moreover, we know from Theorem 4.10 that for all t, $\mathbf{v}_m(t) \in H_{-1}^{\sigma_m^b-\varepsilon'}(\omega)$; again, this space regularity is optimal. But

$$\varphi_R \in \mathscr{C}^{0,1-\sigma_M^b-\varepsilon}(0,T;\mathsf{R}[H_-^{1+\sigma_m^b-\varepsilon'}(\omega)]) \subset \mathscr{C}^{0,1-\sigma_M^b-\varepsilon}(0,T;H_{-1}^{1+\sigma_m^b-\varepsilon'}(\omega))$$

by Proposition 2.1, hence, $\mathbf{curl}\, \varphi_R \in \mathscr{C}^{0,1-\sigma_M^b-\varepsilon}(0,T;H_{-1}^{\sigma_m^b-\varepsilon'}(\omega)^2)$. As δ_j is an element of $\mathscr{C}^{0,1-\sigma_M^b-\varepsilon}(0,T;\mathbb{R})$, this implies (132). □

Corollary 6.8
Under the hypotheses of the above theorem, there holds

$$\mathscr{B} \in \mathscr{C}^{0,1-\sigma_M^b-\varepsilon}(0,T;\check{\mathbf{H}}^{\sigma_m^b-\varepsilon'}(\Omega)) \tag{133}$$

Proof
Similar to Corollary 6.5. □

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