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MULTISCALED ASYMPTOTIC EXPANSIONS FOR THE ELECTRIC POTENTIAL: SURFACE CHARGE DENSITIES AND ELECTRIC FIELDS AT ROUNDED CORNERS

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We are interested in computing the charge density and the electric field at the rounded tip of an electrode of small curvature. As a model, we focus on solving the electrostatic problem for the electric potential. For this problem, Peek's empirical formulas describe the relation between the electric field at the surface of the electrode and its curvature radius. However, it can be used only for electrodes with either a purely cylindrical, or a purely spherical, geometrical shape. Our aim is to justify rigorously these formulas, and to extend it to more general, two-dimensional, or three-dimensional axisymmetric, geometries. With the help of multiscaled asymptotic expansions, we establish an explicit formula for the electric potential in geometries that coincide with a cone at infinity. We also prove a formula for the surface charge density, which is very simple to compute with the Finite Element Method. In particular, the meshsize can be chosen independently of the curvature radius. We illustrate our mathematical results by numerical experiments.

0. Introduction

Electrostatic phenomena are of great interest in many domestic and industrial applications. For instance, processing the smokes in thermal power stations: the principle is to charge the particles in suspension with an ionizing device, which then aggregate on the grounded inner walls so that they can first be gathered, and later be disposed of. Let us also mention electrostatic spray painting. Below, we describe in detail this second application, which motivates our mathematical studies.

When an object (a "target") is painted with the help of a gun (electrode hereafter), the paint is made of microscopic droplets, a fraction of which actually reaches the target, while the rest dissolves into the atmosphere. In order to improve the method (greater fraction reaching the object, less pollution), one can apply a high voltage between the electrode and the target. Thanks to the applied voltage, the paint droplets are charged electrically. The difference between the electric potentials (the voltage) produces a stationary electric field $\mathbf{E} = -\nabla\phi$. Then, the charged droplets (or charges hereafter) follow the lines of force, from the electrode to the target. An electric corona discharge is produced at the onset, and the phenomena are then considered to be stationary. The electric field solves Coulomb's law

$$\mathrm{div}(\varepsilon\mathbf{E}) = \varrho. \tag{0.1}$$

In the above, ε is the electric permittivity of the medium, and ϱ is the charge density. Assuming that the medium is homogeneous, Eq. (0.1) can be rewritten

$$-\Delta\phi = \frac{\varrho}{\varepsilon}. \tag{0.2}$$

In order to "close" the model with the potential ϕ as the main unknown, one has to state precisely what the (computational) domain is, and also to add suitable boundary conditions.

On the surface of the electrode and of the target, it is natural to prescribe a constant value of ϕ (ϕ_1 and respectively ϕ_0, with for instance $\phi_0 = 0$), i.e. a Dirichlet boundary condition.

Then, in a first approach, one can close the domain by putting artificial boundaries, on which a zero-flux condition is imposed (a Neumann boundary condition): the resulting computational domain is bounded. This amounts to saying that there is no loss of charges through these artificial boundaries, which is reasonable. Some numerical experiments have been carried out on this model by Timouyas *et al.*,[23,17] where the quantity of interest was the maximum (absolute value of the) surface charge density on the electrode.

On the other hand, at the charges level, one can safely assume that the domain of interest is unbounded. In order to define a well-posed problem, one has to prescribe the behavior of the potential at the infinity: this defines a second approach, set in an unbounded domain.

The stationary conservation equation on charges is

$$\mathrm{div}\,\mathbf{J} = 0, \tag{0.3}$$

where $\mathbf{J}$ denotes the current density. Under some suitable assumptions,[5] it can be linked to the electric field as

$$\mathbf{J} = K\varrho\mathbf{E}, \tag{0.4}$$

where K is the constant *charge mobility*. This leads finally to

$$\nabla\phi \cdot \nabla\varrho = \frac{\varrho^2}{\varepsilon}. \tag{0.5}$$

This conservation equation for the charge density has also be supplemented with a boundary condition around the tip of the electrode, where (all) charges are emitted: it can be modelled as an injection law.[2,5] In this setting the boundary value of ϱ is a function of the norm of the electric field, denoted by E. This is a more delicate topic. Accepting Kaptzov's hypothesis, which states that the electric field remains constant at the surface of the electrode after the discharge occurs, allows to simplify the model. One can simply choose near the tip of the electrode a boundary condition $\varrho = f(E - E_c)$, with E_c the threshold field, and f such that $f(x) = 0$ if $x \leq 0$, and $f(x) > 0$ if $x > 0$. See Refs. 2 and 5 for examples. In the air, the value of the threshold E_c is described by Peek's heuristic formulas[2]:

- In a cylindrical geometry,

$$E_c = 3.1 \times 10^4 \mathrm{d} \left(1 + \frac{0.308}{\sqrt{\mathrm{d} r_c}}\right) \mathrm{kV} \cdot \mathrm{cm}^{-1}. \tag{0.6}$$

- In a spherical geometry,

$$E_c = 3.1 \times 10^4 \mathrm{d} \left(1 + \frac{0.308\sqrt{2}}{\sqrt{\mathrm{d} r_c}}\right) \mathrm{kV} \cdot \mathrm{cm}^{-1}. \tag{0.7}$$

In the above, r_c denotes the curvature radius of the electrode (expressed in centimeters), and d the ratio $\frac{(P/P_0)}{(T/T_0)}$, with T_0 and P_0 respectively the reference temperature and pressure, and T and P the actual temperature and pressure. It is crucial to note that Peek's heuristic formulas are valid only around specific electrodes, i.e. thin, either cylindrical or spherical, electrodes.

Then, the full model — coupled in ϕ, ϱ — consists of Eqs. (0.2) and (0.5), together with the boundary conditions presented before.

Solving numerically the problem in ϱ is standard, for example with the help of the method of characteristics.[2,5]

In this paper, we focus instead on the theoretical and numerical solutions of the problem on the electrostatic potential, that is

$$-\Delta u = f \tag{0.8}$$

with a homogeneous Dirichlet boundary condition on the boundary and a prescribed behavior at the infinity. Among others, we propose to justify mathematically Peek's formulas and to provide a rigorous and easily computable formula for the charge density, at the tip of the electrode. We extend those results to other, less specific geometries (still, under the assumption that at infinity the geometry coincides with that of a cone).

The outline is as follows. First, we describe in some details the functional setting. To that aim, we introduce a series of weighted Sobolev spaces around the unbounded, sharp (at the vertex), cone, and around the unbounded, rounded (at the tip), cones. Second, we solve the problem around the unbounded sharp cone: we establish existence and uniqueness, together with some *a priori* regularity estimates. The third and fourth parts are devoted to a detailed solution of the problem around rounded cones, either in Cartesian, or in axisymmetric coordinates. Using multiscale expansions, we establish rigorously the asymptotic — when the curvature radius goes to zero — behavior of the solution. In particular, we address the problem of mathematically justifying Peek's formulas. Moreover, we can characterize the behavior of the trace of the normal derivative at the tip, which is exactly the maximum surface charge density on the electrode. Among others, the multiscale expansion formula can be computed accurately with a Finite Element approximation, on meshes with a meshsize which is independent of the curvature radius. Finally, we conclude by a series of numerical experiments, and we compare our numerical method to a "standard" discretized integral representation.

The results of this paper have been announced in Ref. 10.

1. Notations and Functional Spaces

1.1. *Preliminary notations*

$\mathbb{R}^n$ is the Euclidean space of dimension n, with $n = 2, 3$.

For $\varepsilon > 0$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$, let us introduce

$$\rho(x) := \sqrt{1 + |x|^2},$$

$$\rho_\varepsilon(x) := \varepsilon\rho\left(\frac{x}{\varepsilon}\right) = \sqrt{\varepsilon^2 + |x|^2}, \quad \text{where } |x| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2},$$

$$\partial^\lambda u = \frac{\partial^{|\lambda|} u}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}}, \quad \text{where } |\lambda| = \lambda_1 + \cdots + \lambda_n.$$

For $\varepsilon = 0$, we introduce $\rho_0 = \rho$ (note that $\rho_0(x) \neq \lim_{\varepsilon\to 0} \rho_\varepsilon(x)$).

Given an open subset A of $\mathbb{R}^n$ and $\sigma > 0$, σA is equal to: $\sigma A := \{\sigma m\,;\, m \in A\}$.

$B_\sigma(x)$ is the open ball centered at x with radius σ. It is called B_σ when $x = O$.

Given A, B two topological spaces with $A \subset B$ topologically and algebraically, $\overline{A}^B$ is the closure of A in B.

If the open subset under study is axisymmetric ($n = 3$), it is denoted by $\breve{A}$. Then, A stands for its trace in a meridian half-plane, the location of any point of A being given by the coordinates (r, z), with $r = \sqrt{x_1^2 + x_2^2}$, $z = x_3$.

The set of compactly supported, $\mathcal{C}^\infty$-class, functions defined in A is denoted by $\mathcal{C}_0^\infty(A)$. Then, $\mathcal{C}_0^\infty(\overline{A})$ is the set of restrictions (to $\overline{A}$) of elements of $\mathcal{C}_0^\infty(\mathbb{R}^n)$. Finally, $\mathcal{D}'(A)$ is the set of distributions defined on A, and $L^2_{\rm loc}(A)$ is the space of measurable and locally square-integrable functions on A.

1.2. *Classes of domains*

Let us first introduce the classes of domains of interest, for $\varepsilon > 0$ given. We consider either Cartesian geometries, set in $\mathbb{R}^2$, or axisymmetric geometries, set in $\mathbb{R}^3$.

- Cartesian case: for $1/2 < \alpha < 1$, let Ω denote the *unbounded, sharp* (*at its vertex*), *cone* of $\mathbb{R}^2$ with vertex O and angle π/α (see Fig. 1), and let Γ be its boundary.

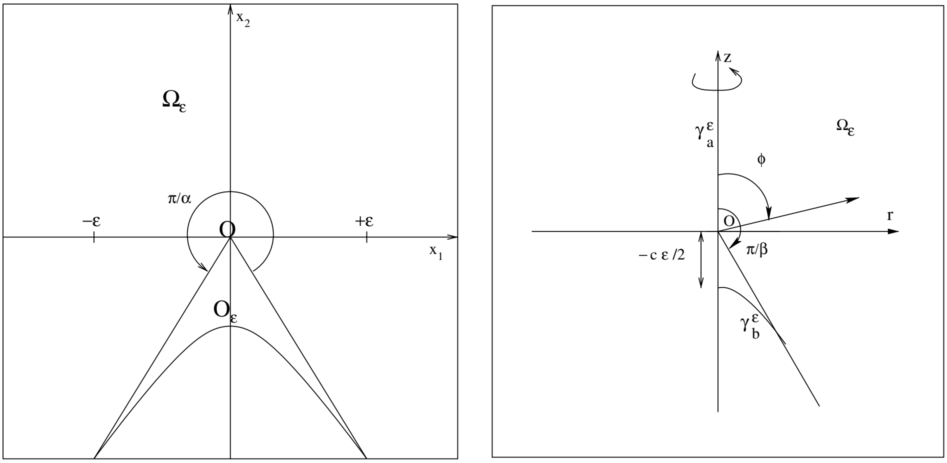

Fig. 1. Domains Ω and Ω_ε: Cartesian ($n = 2$) and axisymmetric ($n = 3$) geometries.

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be the mapping

$$x \longmapsto \begin{cases} cx, & x < -1, \\ -\dfrac{c}{2}(x^2+1), & |x| \le 1, \\ -cx, & x > 1. \end{cases} \tag{1.1}$$

We set the constant c to the value $c = -\left(\tan[\pi/(2\alpha)]\right)^{-1}$. In this way, the mapping φ globally belongs to $\mathcal{C}^1$. In addition, it is $\mathcal{C}^2$ everywhere, except at the coupling points $x = \pm 1$. Then, let ω be the open subset of $\mathbb{R}^2$

$$\omega := \left\{(x_1, x_2) \in \mathbb{R}^2 | x_2 > \varphi(x_1)\right\}.$$

Next, we introduce the *unbounded*, *rounded* (*at the tip*), *cone*

$$\Omega_\varepsilon := \varepsilon\omega = \left\{(x_1, x_2) \in \mathbb{R}^2 \middle| \frac{x_2}{\varepsilon} > \varphi\left(\frac{x_1}{\varepsilon}\right)\right\}.$$

Note that $\Omega_1 = \omega$.

The boundary and vertex of Ω_ε are respectively called Γ_ε and O_ε, with $O_\varepsilon = (0, -(c\varepsilon)/2)$.

- Axisymmetric case: for $1 < \beta < 2$, let Ω denote the 2D, unbounded, sharp, half-cone with vertex O and angle π/β, located in the half plane $\mathbb{R}^2_+ := \left\{(r, z) \in \mathbb{R}^2, r > 0\right\}$ (see Fig. 1). Then, let $\breve{\Omega}$ be the 3D *unbounded sharp cone*, generated by the rotation of Ω around the axis (Oz).

In the axisymmetric case, the mapping φ is still defined by (1.1), with the constant c set to $c = -\left(\tan[\pi/\beta]\right)^{-1}$.

The open subset ω of $\mathbb{R}^2_+$ is equal to

$$\omega := \left\{(r, z) \in \mathbb{R}^2_+ | z > \varphi(r)\right\}.$$

By rotation around the (Oz) axis, one generates the open subset of $\mathbb{R}^3$ called $\breve{\omega}$. As above, one defines the 2D open set Ω_ε and then the 3D *unbounded rounded cone* $\breve{\Omega}_\varepsilon$ by rotation. Let Γ_a (resp. Γ_a^ε) be the part of the boundary of Ω (resp. Ω_ε) located on the axis $r = 0$ (see Fig. 1).

For $\varepsilon = 0$, we introduce $\Omega_0 = \Omega$ in 2D, and $\breve{\Omega}_0 = \breve{\Omega}$ in 3D.

Remark 1.1. For practical reasons, we choose a parabolic shape. However, one can check that any shape defined with a smooth function φ such that $\varphi(0) < 0$ and $\varphi'(0) = 0$ can be used.

1.3. *Sobolev spaces in unbounded domains*

We define in this subsection the many functional spaces required to carry out the subsequent analysis. Note that we have to deal both with the behavior of solutions near the vertex of the domain, and with the behavior of the same solution at the infinity. As in the case of the Dirichlet problem set in an exterior domain, the weighted Sobolev spaces are an appropriate setting. But, in addition, we have to take into account the unbounded character of its boundary. One has in particular to be careful in the definition of the trace spaces. Finally, we have to deal with either sharp or rounded (at the tip) cones.

Let us consider the first family of Sobolev spaces below.

Definition 1.1. Given $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, introduce

- around rounded cones, given $\varepsilon > 0$, the functional space:

$$W_\beta^m(\Omega_\varepsilon) := \{u \in L^2_{\mathrm{loc}}(\Omega_\varepsilon) \mid w_\lambda^\varepsilon \partial^\lambda u \in L^2(\Omega_\varepsilon) \text{ for } |\lambda| \le m\},$$

- around the sharp cone ($\varepsilon = 0$), the functional space:

$$W_\beta^m(\Omega) := \{u \in L^2_{\mathrm{loc}}(\Omega) \mid w_\lambda \partial^\lambda u \in L^2(\Omega) \text{ for } |\lambda| \le m\}.$$

In the above, the weights w_λ^ε and w_λ are respectively defined by

$$w_\lambda^\varepsilon(x) := \rho_\varepsilon(x)^{\beta+|\lambda|-m}, \quad w_\lambda(x) := \rho(x)^{\beta+|\lambda|-m}.$$

The weights are introduced for three reasons. First, they allow to control the behavior at the infinity (as previoulsy mentioned). Second, they are chosen in such a way that compact embeddings and coercivity inequalities (Poincaré-like) hold. Third, using them in the definition of the W_β^m spaces does not modify the behavior near the origin O: in this way, local properties are similar to those of elements of Sobolev spaces set in bounded domains. More precisely, one has:

$$H^m(\Omega_\varepsilon) \subset W_\beta^m(\Omega_\varepsilon) \subset H^m_{\mathrm{loc}}(\Omega_\varepsilon).$$

In the case of a bounded domain Ω_ε, for a given $\varepsilon > 0$, one finds $W_\beta^m(\Omega_\varepsilon) = H^m(\Omega_\varepsilon)$.

Remark 1.2. The relevance of the use of ε-dependent weights will appear when we derive error bounds for the asymptotic expansions.

Proposition 1.1. *The functional space $W^m_\beta(\Omega_\varepsilon)$ is a Hilbert space, endowed with the scalar product*

$$(u,v)_{W^m_\beta(\Omega_\varepsilon)} := \sum_{\lambda\in\mathbb{N}^2, 0\le|\lambda|\le m} \int_{\Omega_\varepsilon} w^\varepsilon_\lambda(x)^2\, \partial^\lambda u \partial^\lambda v \,\mathrm{d}x.$$

By truncation,[15] one can recover properties of the spaces $H^m(\Omega_\varepsilon)$.

Proposition 1.2. *$\mathcal{C}^\infty_0(\overline{\Omega}_\varepsilon)$ is dense in $W^m_\beta(\Omega_\varepsilon)$.*

One can then consider the closure of $\mathcal{C}^\infty_0(\Omega_\varepsilon)$ in $W^m_\beta(\Omega_\varepsilon)$.

Definition 1.2. Introduce $\overset{\circ}{W}{}^m_\beta(\Omega_\varepsilon) := \overline{\mathcal{C}^\infty_0(\Omega_\varepsilon)}^{W^m_\beta(\Omega_\varepsilon)}$.

Classically, it follows that, if $v \in \overset{\circ}{W}{}^1_\beta(\Omega_\varepsilon)$, then $\Delta v \in (\overset{\circ}{W}{}^1_\beta(\Omega_\varepsilon))'$.

The space $W^m_\beta(\Omega_\varepsilon)$ coincides locally with the "usual" space $H^m(\Omega_\varepsilon)$: traces can therefore be defined locally. Since the space of choice for the Laplace problem is $W^1_0(\Omega_\varepsilon)$, let us focus on the properties of traces of its elements. From Ref. 15, it is known that the behavior at infinity is comparable to those of traces in a half-plane. So, let us introduce

Definition 1.3.

$$W^{\frac12}_0(\Gamma_\varepsilon) := \left\{ u \in L^2_{\mathrm{loc}}(\Gamma_\varepsilon) \,\middle|\, \frac{u}{\sqrt{\rho_\varepsilon}} \in L^2(\Gamma_\varepsilon), \int_{\Gamma_\varepsilon\times\Gamma_\varepsilon} \frac{|u(\sigma)-u(\sigma')|^2}{|\sigma-\sigma'|^2}\,\mathrm{d}\sigma\,\mathrm{d}\sigma' < \infty \right\}.$$

The dual space of $W^{1/2}_0(\Gamma_\varepsilon)$ is denoted by $W^{-1/2}_0(\Gamma_\varepsilon)$. The next proposition provides a characterization of traces of elements of $W^1_0(\Omega_\varepsilon)$.

Proposition 1.3. *Given $\varepsilon \ge 0$, the trace mapping*

$$\gamma : u \in \mathcal{C}^\infty_0(\overline{\Omega}_\varepsilon) \to u_{|\Gamma_\varepsilon} \in \mathcal{C}^0(\Gamma_\varepsilon)$$

can be extended by continuity to a linear, continuous and surjective mapping, from $W^1_0(\Omega_\varepsilon)$ to $W^{1/2}_0(\Gamma_\varepsilon)$. Its kernel is $\gamma^{-1}(\{0\}) = \overset{\circ}{W}{}^1_0(\Omega_\varepsilon)$.

To build suitable Green formulas, we need some other functional spaces.

Definition 1.4. Consider $W(\mathrm{div}\,,\Omega_\varepsilon) := \{\mathbf{u}\in L^2(\Omega_\varepsilon)^2, \rho_\varepsilon \mathrm{div}\,\mathbf{u} \in L^2(\Omega_\varepsilon)\}$.

$W(\mathrm{div}\,,\Omega_\varepsilon)$ is a Hilbert space, endowed with the scalar product $(\mathbf{u},\mathbf{v})_{\mathrm{div}\,,\Omega_\varepsilon} := (\mathbf{u},\mathbf{v})_{L^2(\Omega_\varepsilon)^2} + (\rho_\varepsilon \mathrm{div}\,\mathbf{u}, \rho_\varepsilon \mathrm{div}\,\mathbf{v})_{L^2(\Omega_\varepsilon)}$.

Proposition 1.4. *The space $\mathcal{C}^\infty_0(\overline{\Omega}_\varepsilon)^2$ is dense in $W(\mathrm{div}\,,\Omega_\varepsilon)$.*

Proof. Cf. Ref. 16. □

The normal trace of elements of $W(\mathrm{div}\,,\Omega_\varepsilon)$ on Γ_ε can then be introduced.

Proposition 1.5. *Given $\varepsilon \geq 0$, the mapping $\gamma_n : \mathbf{u} \in \mathcal{C}_0^\infty(\overline{\Omega}_\varepsilon)^2 \to \mathbf{u}.\mathbf{n}|_{\Gamma_\varepsilon} \in \mathcal{C}^0(\Gamma_\varepsilon)$ can be extended by continuity to a linear and continuous mapping, from $W(\mathrm{div}\,,\Omega)$ to $W_0^{-1/2}(\Gamma_\varepsilon)$. In addition, the following integration by parts formula holds*:
$\forall\, \mathbf{u} \in W(\mathrm{div}\,,\Omega_\varepsilon),\ v \in W_0^1(\Omega_\varepsilon),$

$$(\mathbf{u}, \nabla v)_{L^2(\Omega_\varepsilon)^2} + (\mathrm{div}\,\mathbf{u}, v)_{L^2(\Omega_\varepsilon)} = {}_{W_0^{-1/2}(\Gamma_\varepsilon)}\langle \mathbf{u}\cdot\mathbf{n}|_{\Gamma_\varepsilon}, v|_{\Gamma_\varepsilon}\rangle_{W_0^{1/2}(\Gamma_\varepsilon)}. \tag{1.2}$$

Proof. Cf. Ref. 16. □

1.4. *Axisymmetric setting*

Given $\varepsilon \geq 0$, $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, we can define similarly the spaces $W_\beta^m(\breve{\Omega}_\varepsilon)$ in an axisymmetric domain $\breve{\Omega}_\varepsilon$. The trace space of $W_0^1(\breve{\Omega}_\varepsilon)$ is now characterized by the

Definition 1.5. Introduce

$$W_0^{1/2}(\breve{\Gamma}_\varepsilon) := \left\{ u \in L^2_{\mathrm{loc}}(\breve{\Gamma}_\varepsilon) \,\middle|\, \frac{u}{\sqrt{\rho_\varepsilon}} \in L^2(\breve{\Gamma}_\varepsilon), \int_{\breve{\Gamma}_\varepsilon \times \breve{\Gamma}_\varepsilon} \frac{|u(\sigma) - u(\sigma')|^2}{|\sigma - \sigma'|^3}\, \mathrm{d}\sigma\, \mathrm{d}\sigma' < \infty \right\}.$$

Let us consider the dimension reduction, from $n = 3$ to $n = 2$. For $\mu \in \mathbb{R}$, let $L^2_\mu(\Omega_\varepsilon)$ be the space of mesurable and square-integrable functions over Ω_ε, with respect to the measure $r^\mu drdz$. In particular, there holds the

Lemma 1.1. *Let $u \in W_\beta^2(\breve{\Omega}_\varepsilon)$ be invariant by rotation around the axis $r = 0$. If D^2u denotes its Hessian, then there holds*

$$\int_{\breve{\Omega}_\varepsilon} \rho_\varepsilon(x)^{2\beta} |D^2 u(x)|^2 \,\mathrm{d}\breve{\Omega}_\varepsilon$$
$$= 2\pi \int_{\Omega_\varepsilon} \rho_\varepsilon(r,z)^{2\beta} \left\{ \left|\frac{\partial^2 u}{\partial r^2}\right|^2 + \left|\frac{1}{r}\frac{\partial u}{\partial r}\right|^2 + \left|\frac{\partial^2 u}{\partial r \partial z}\right|^2 + \left|\frac{\partial^2 u}{\partial z^2}\right|^2 \right\} r\,\mathrm{d}r\,\mathrm{d}z.$$

This helps to define the *ad hoc* functional spaces in the meridian half-plane.

Definition 1.6. Consider, for $\varepsilon \geq 0$, $m \in \mathbb{N}$, $\beta \in \mathbb{R}$,

$$W^m_{\beta,a}(\Omega_\varepsilon) := \{u \in L^2_{1,\mathrm{loc}}(\Omega_\varepsilon) | \rho_\varepsilon^{\beta+|\lambda|-m} \partial^\lambda_{r,z} u \in L^2_1(\Omega_\varepsilon), |\lambda| \leq m\}.$$

With the help of Lemma 1.1, and according to Refs. 6 and 9, one finds that the dimension reduction works in the following manner.

Proposition 1.6. *For $\varepsilon \geq 0$, $\beta \in \mathbb{R}$ and $m \in \{0, 1\}$, the space $W^m_{\beta,a}(\Omega_\varepsilon)$ is exactly the space of traces in the meridian half-plane of invariant by rotation elements of $W_\beta^m(\breve{\Omega}_\varepsilon)$. For $m = 2$, the relevant trace space is*

$$W^2_{\beta,a,+}(\Omega_\varepsilon) := \{u \in W^2_{\beta,a}(\Omega_\varepsilon) | \rho_\varepsilon^\beta \partial_r u \in L^2_{-1}(\Omega_\varepsilon)\}.$$

1.5. *Sobolev spaces on the unbounded sharp cone*

For a fixed $x \in \Omega$, when $\varepsilon \to 0$, the value of the weight $w_\lambda^\varepsilon(x)$ goes to $|x|^{\beta+|\lambda|-m}$. Based on this simple observation, and from Refs. 19 and 22 it follows that the Sobolev spaces thus weighted are the "natural" ones to describe the solutions to elliptic problems in a sharp cone. The corresponding definitions are

Definition 1.7. Given $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, introduce

$$V_\beta^m(\Omega) := \left\{ u \in L^2_{\mathrm{loc}}(\Omega) \,\middle|\, \sum_{\mu \in \mathbb{N}^2, |\mu| \le m} \int_\Omega |x|^{2(\beta+|\mu|-m)} |\partial^\mu u(x)|^2 \,\mathrm{d}x < \infty \right\}.$$

Note that there holds

$$u \in V_\beta^m(\Omega) \Longleftrightarrow u \in L^2_{\mathrm{loc}}(\Omega), \quad \forall\, \mu \in \mathbb{N}^2,\ |\mu| \le m, \qquad |x|^{|\mu|-m} \partial^\mu (|x|^\beta u) \in L^2(\Omega).$$

1.6. *The problems to be solved*

In the next sections, for $\varepsilon \ge 0$, we shall solve the Laplace equation with Dirichlet boundary condition in two-dimensional or three-dimensional domains.

$$\text{In 2D: } \begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = g & \text{on } \Gamma_\varepsilon. \end{cases} \qquad \text{In 3D: } \begin{cases} -\Delta \breve{u}_\varepsilon = f & \text{in } \breve{\Omega}_\varepsilon, \\ \breve{u}_\varepsilon = g & \text{on } \breve{\Gamma}_\varepsilon. \end{cases} \tag{1.3}$$

We make the following simplifying assumptions: first, that $g = 0$, and second that f vanishes in a (fixed) neighborhood of the origin O. See Ref. 19 (Chap. 7, Sec. 7.1), or Remark 3.3, for a more general assumption on f. In 3D, we further assume that f is invariant by rotation.

We call (1.3), with $g = 0$, the *Dirichlet problem* hereafter.

2. Dirichlet Problem in the Sharp Cone

2.1. *Setting of the problem*

For the Cartesian case, existence and uniqueness results[19,22] in the spaces $V_\beta^m(\Omega)$ come from the study of the Dirichlet problem in the strip $\mathbb{R}\times]0, \frac{\pi}{\alpha}[$. As a matter of fact, one easily checks that a one-to-one mapping from Ω to this strip is defined by the change of variable $t = \log(r)$. In particular, one finds that the Laplace operator maps $V_\beta^{m+2}(\Omega)$ onto $V_\beta^m(\Omega)$, except for specific values of the index β. More precisely, one has

Theorem 2.1. *Let* $\beta \in \mathbb{R}$, $m \ge 2$, $f \in V_\beta^{m-2}(\Omega)$.

If $\beta - m + 1 \notin \mathbb{Z}^* \alpha$, *then the Dirichlet problem* (1.3) *set in the sharp cone has one, and only one solution* $u_0 \in V_\beta^m(\Omega)$, *which satisfies the a priori estimate*

$$\|u_0\|_{V_\beta^m} \le c\, \|f\|_{V_\beta^{m-2}},$$

with a constant c independent of the right-hand side f.

Remark 2.1. For our purpose, it is important to note that, in the case when $m = 2$ and if β is such that $|\beta - 1| < \alpha$, then the solution to (1.3) belongs to $V^2_\beta(\Omega)$ for any $f \in V^0_\beta(\Omega)$ (see Refs. 19 and 22 for details).

For the axisymmetric case, let us introduce G, the intersection between $\breve{\Omega}$ and the unit sphere $\mathbb{S}^2$. In other words, $\breve{\Omega} = \{\rho\sigma, \rho > 0, \sigma \in G\}$. Let us call (ρ, θ, ϕ) the spherical coordinates. The Laplace–Beltrami operator Δ_G on G is defined as follows:

$$\Delta_G := \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2}{\partial\theta^2}.$$

On G, we consider the surface gradient

$$\nabla_G u := \begin{pmatrix} \dfrac{1}{\sin\phi}\dfrac{\partial u}{\partial\theta} \\ \dfrac{\partial u}{\partial\phi} \end{pmatrix}.$$

Then, we define the *ad hoc* Hilbert space (and accompanying scalar product)

$$H^1(G) := \left\{u \in L^2(G) | \nabla_G u \in L^2(G)^2\right\},$$

$$(u, v)_{H^1(G)} := \int_G \left(\frac{1}{4}uv + \nabla_G u \cdot \nabla_G v\right) \mathrm{d}\sigma$$

(with $\mathrm{d}\sigma = \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$).

Then, we introduce the closure of smooth surface fields $H^1_0(G) := \overline{\mathcal{C}^\infty_0(G)}^{H^1(G)}$. It is standard knowledge that the bilinear form on $H^1_0(G) \times H^1_0(G)$, defined by

$$A(u, v) := \int_G \nabla_G u \cdot \nabla_G v \,\mathrm{d}\sigma$$

is symmetric and coercive. As a consequence, one has

Theorem 2.2. *There exist a countable sequence of positive real numbers $(\Lambda_\ell)_{\ell\geq 1}$ (sorted by increasing values), which goes to $+\infty$, and a countable set $(\Psi_\ell)_{\ell\geq 1}$ of elements of $H^1_0(G)$ such that*

- $(\Psi_\ell)_{\ell\geq 1}$ *is a Hilbert basis of* $L^2(G)$;
- $(\Psi_\ell/\sqrt{\Lambda_\ell})_{\ell\geq 1}$ *is a Hilbert basis of* $H^1_0(G)$;
- $-\Delta_G \Psi_\ell = \Lambda_\ell \Psi_\ell$, *for all* $\ell \geq 1$.

In addition, the first eigenvalue Λ_1 is simple, and $\Lambda_1 > 0$.

As in the Cartesian case, the change of variables $t = \log(r)$ transforms problem (1.3), set in $\breve{\Omega}$, in another one, set in $\mathbb{R} \times G$.

Theorem 2.3. *Let $\beta \in \mathbb{R}$ such that $|\beta - 1| < \sqrt{\Lambda_1 + \frac{1}{4}}$, $f \in V^0_\beta(\breve{\Omega})$. The Dirichlet problem (1.3) set in the 3D sharp cone has one, and only one solution $u_0 \in V^2_\beta(\breve{\Omega})$, which satisfies the a priori estimate*

$$\|u_0\|_{V^2_\beta} \leq c\|f\|_{V^0_\beta},$$

with a constant c independent of the right-hand side f.

Proof. See Refs. 13, 19 and 16. □

2.2. *Cartesian case*

In this subsection, we propose a variational setting for the Dirichlet problem, set in the unbounded, sharp cone Ω. Recall that the space $W_0^1(\Omega)$ combines the local regularity of the space $H^1_{\rm loc}$, with a prescribed behavior at infinity, which allows to recover Poincaré inequalities in unbounded domains. More precisely, there holds the

Proposition 2.1. *There exists a constant* $C > 0$, *such that one has*

$$\forall\, u \in \overset{\circ}{W}{}_0^1\,(\Omega), \quad \int_\Omega \frac{|u(x)|^2}{1+|x|^2}\,\mathrm{d}x \le C \int_\Omega |\nabla u|^2\,\mathrm{d}x. \tag{2.1}$$

The semi-norm $|u|_{W_0^1} = (\int_\Omega |\nabla u|^2\,\mathrm{d}x)^{1/2}$ *is a norm on* $\overset{\circ}{W}{}_0^1\,(\Omega)$, *which is equivalent to the full norm.*

As a consequence, the Lax–Milgram Theorem can be applied to the variational formulation of the Dirichet problem, set in $\overset{\circ}{W}{}_0^1\,(\Omega)$.

Proposition 2.2. *Given* $f \in L^2_{\rm loc}(\Omega)$ *such that* $\rho f \in L^2(\Omega)$, *there exists one, and only one solution* u_0 *to the Dirichlet problem* (1.3) *which belongs to* $\overset{\circ}{W}{}_0^1\,(\Omega)$, *and satisfies* (*with a constant* C *independent of* f)

$$\|u_0\|_{W_0^1} \le C\|f\|_{W_1^0}.$$

Remark 2.2. The Dirichlet problem is also well-posed with $f \in (\overset{\circ}{W}{}_0^1\,(\Omega))'$. Thanks to the property that f vanishes in a neighborhood of O, f also belongs to the space $V_1^0(\Omega)$, and so $u_0 \in V_1^2(\Omega)$.

When f vanishes in a neighborhood of O, the asymptotic behavior of u_0, that is when $r \to 0$, is governed by its local regularity. We follow here Grisvard[13,14]: consider the polar coordinates (r,θ) with respect to the vertex O, such that Γ_+ (resp. Γ_-) is a subset of $\{\theta = 0\}$ (resp. $\{\theta = \frac{\pi}{\alpha}\}$). Let $(\varphi_\ell)_{\ell\ge1}$, $\varphi_\ell(\theta) = \sqrt{2\alpha/\pi}\sin(\ell\alpha\theta)$ be the orthonormal basis of $L^2(]0,\frac{\pi}{\alpha}[)$, which is associated to the 1D operator $\Delta_\theta := -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}$ with homogeneous Dirichlet boundary conditions at $\theta = 0$ and $\theta = \pi/\alpha$. For $\ell \in \mathbb{N}$, we introduce the dual functions $s_D^\ell(r,\theta) = r^{-\ell\alpha}\varphi_\ell(\theta)$, as well as the primal ones $\phi_\ell(r,\theta) = r^{\ell\alpha}\varphi_\ell(\theta)$. One has

Proposition 2.3. *There exists coefficients* $(\lambda_\ell)_{\ell\ge1}$ *such that one can write, for any* $N \in \mathbb{N}$, *the expansion* (*in increasing powers of* r):

$$u_0(r,\theta) = \sum_{\ell=1}^{N} \lambda_\ell \phi_\ell(r,\theta) + O(r^{(N+1)\alpha}), \quad r \to 0. \tag{2.2}$$

In addition, each coefficient λ_ℓ *depends linearly on the right-hand side* f:

$$\lambda_\ell = \frac{1}{2\ell\alpha}\int_\Omega f s_D^\ell\,\mathrm{d}x.$$

Remark 2.3. For each $\ell \in \{1, \ldots, m\}$ such that $\ell\alpha \in \mathbb{N}$, a log term appears in the expansion[13]: more precisely, $\phi_\ell(r, \theta) = d_\ell r^{\ell\alpha}(\log r \sin(\ell\alpha\theta) + \theta\cos(\ell\alpha\theta))$, with d_ℓ a normalizing factor. Anyway, all computations can still be carried out in this case.

We note that since α belongs to $]1/2, 1[$, this cannot happen for $\ell \leq 2$.

2.3. *Axisymmetric case*

Since we consider here the axisymmetric case, it is assumed that the right-hand side f is invariant by rotation, in addition to vanishing in a neighborhood of the origin O. The solution to the Dirichlet problem (1.3) set in the meridian half-plane is invariant by rotation and satisfies

$$\begin{cases} -\Delta^+ u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma_b, \\ \dfrac{\partial u_0}{\partial n} = 0 & \text{on } \Gamma_a. \end{cases} \tag{2.3}$$

In the above, the 2D operator Δ^+ is defined as

$$\Delta^+ := \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad \text{or}$$
$$\Delta^+ := \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho} + \frac{\cot(\phi)}{\rho^2}\frac{\partial}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} \quad \text{in spherical coordinates } (\rho, \theta, \phi).$$

Propositions 2.1 and 2.2 are still valid, so that u_0 exists, and is unique.

Let us consider for a moment that f is not invariant by rotation. The domain $\breve{\Omega}$ can include a conical *geometrical singularity*.[a] In addition, when the aperture angle π/β is greater than a limit value, the behavior of the solution to the Dirichlet problem becomes singular in the sense that $\breve{u}_0$ does not belong to $H^2_{\text{loc}}(\breve{\Omega})$ (see for instance Refs. 6, 4 and 9). This value is related to the zero of the Legendre function $P^0_{1/2}$, where $t \mapsto P^0_\nu(t)$ is the Legendre polynomial of order 0 and index $\nu > 0$, which is bounded at $t = 0$. More precisely, let us define $\beta_c \in]1, 2[$, such that $P^0_{1/2}(\cos(\pi/\beta_c)) = 0$, and assume that the aperture angle π/β is larger than π/β_c: in this case, the solution can be singular. Moreover, if one performs a Fourier expansion in the azimuthal coordinate θ, it can be proven[9] that the singular behavior is contained in the mode $k = 0$. So, albeit this mode is the only one when the solution is invariant by rotation, it is nonetheless exactly the one which behaves singularly.

As we shall see, the regularity of the solution is related to the value of the index $\nu > 0$, and to the zero(s) of the Legendre polynomials P^0_ν. As a matter of fact, the

[a]By definition,[3,4] for a bounded axisymmetric domain, a geometrical singularity is located on the boundary of the domain, and is either a re-entrant circular edge, or a conical re-entrant corner (with a circular basis), both of which yield a non-convex domain.

solution u_0 in a neighborhood of O can be expressed as an infinite sum, using the eigenfunctions of the 1D operator derived from (2.3): consider

$$\Delta_\phi^+ := -\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} - \cot(\phi)\frac{\mathrm{d}}{\mathrm{d}\phi}.$$

Its domain (taking into account the boundary conditions on $\phi = 0$ and $\phi = \pi/\beta$) is described below. Following Ref. 3, we introduce the functional space

$$\mathcal{H} := \left\{u \in L^2_{\mathrm{loc}}\left(\left]0, \frac{\pi}{\beta}\right[\right), (\sin\phi)^{1/2}u \in L^2\left(\left]0, \frac{\pi}{\beta}\right[\right)\right\}. \tag{2.4}$$

The domain of Δ_ϕ^+ is

$$D(\Delta_\phi^+) := \left\{u \in \mathcal{H} \middle| \Delta_\phi^+ u \in \mathcal{H}, u'(0) = u\left(\frac{\pi}{\beta}\right) = 0\right\}. \tag{2.5}$$

Theorem 2.4. *There exists a Hilbert basis* $(u_\ell)_{\ell\geq 1}$ *of* $\mathcal{H}$, *made of eigenfunctions of* Δ_ϕ^+, *which are associated to the eigenvalues (sorted by increasing values)* $(\lambda_\ell)_{\ell\geq 1}$, *with* $\lambda_\ell \in \mathbb{R}_*^+$ *for all* $\ell \geq 1$ *and* $\lambda_\ell \to +\infty$ *when* $\ell \to +\infty$. *All eigenfunctions* $(u_\ell)_{\ell\geq 1}$ *belong to*

$$\mathcal{H}^1 := \left\{u \in \mathcal{H}, u' \in \mathcal{H}, u\left(\frac{\pi}{\beta}\right) = 0\right\},$$

and, in addition, $(u_\ell/\sqrt{\lambda_\ell})_{\ell\geq 1}$ *is a Hilbert basis of* $\mathcal{H}^1$.

Now, let $(\nu_\ell)_{\ell\geq 1}$ be the sequence of non-negative real numbers, sorted by increasing values, such that $P_\nu^0[\cos(\pi/\beta)] = 0$. The eigenpairs of the operator Δ_ϕ^+ are

$$\lambda_\ell = \nu_\ell(\nu_\ell + 1),\ u_\ell(\phi) = \frac{P_{\nu_\ell}^0(\cos\phi)}{\|P_{\nu_\ell}^0(\cos\cdot)\|_{\mathcal{H}}}, \quad \ell \geq 1.$$

When $\pi/\beta > \pi/\beta_c$, according to the tables of solutions to

find $\nu > 0$ *such that* $P_\nu^0[\cos(\pi/\beta)] = 0$,

which can be found in Ref. 1, there holds $\nu_1 < 1/2$ and $\nu_\ell > 1/2$ for all $\ell > 1$. The fact that ν_1 is strictly smaller than $1/2$ implies that the first term in the expansion of the solution u_0 to (2.3) generates a term in $\breve{u}_0$, which does not belong to $H^2_{\mathrm{loc}}(\breve{\Omega})$ (except if it vanishes...).

For $\ell \in \mathbb{N}$, define the primal and dual functions, resp. $s_D^\ell(\rho, \phi) = \rho^{-\nu_\ell}u_\ell(\phi)$ and $\phi_\ell^c(\rho, \phi) = \rho^{\nu_\ell}u_\ell(\phi)$. Than, the expansion of u_0 when $\rho \to 0$ is as follows.

Proposition 2.4. *There exist coefficients* $(\lambda_\ell)_{\ell\geq 1}$ *such that one can write, for any* $N \in \mathbb{N}$, *the expansion (in increasing powers of* ρ):

$$u_0(\rho, \phi) = \sum_{\ell=1}^{N} \lambda_\ell \phi_\ell^c(\rho, \phi) + O(\rho^{\nu_{N+1}}), \quad \rho \to 0. \tag{2.6}$$

Each coefficient λ_ℓ *depends linearly on the right-hand side* f:

$$\lambda_\ell = \frac{1}{1 + 2\nu_\ell}\int_\Omega f(\rho, \phi)s_D^\ell(\rho, \phi)\rho^2 \sin\phi\,\mathrm{d}\phi.$$

3. Asymptotic Expansions Around Rounded Cones

In the previous section, we examined the behavior of the solution u_0 to the Dirichlet problem in the sharp cone (for $n = 2, 3$). We shall now derive the asymptotic expansions (in powers of ε) of the solution u_ε to the Dirichlet problem, around the rounded cones (for $n = 2, 3$). We assume that the right-hand side f is *independent of* ε and, as before, that it vanishes in a neighborhood of O.

Let us introduce a cut-off function $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^+)$, such that $0 \leq \xi \leq 1$, and moreover $\xi \equiv 1$ in $[0, a]$ and $\xi \equiv 0$ in $[2a, +\infty[$ ($a > 0$ is given). Then, for $\gamma > 0$, let us consider the scaled function χ_γ defined by $x \mapsto \xi(|x|/\gamma)$; for $\gamma = 1$, the function will be denoted simply by χ.

3.1. *Rounded corners in two-dimensional domains*

As for the sharp cone, the inequality (2.1) and the integration by parts formula (1.2) remain valid in Ω_ε. There follows

Proposition 3.1. *Given* $f \in W_1^0(\Omega_\varepsilon)$, *there exists one, and only one solution* u_ε, *which belongs to* $W_1^2(\Omega_\varepsilon)$, *to the problem* (1.3). *Moreover, it satisfies*

$$\|u_\varepsilon\|_{W_1^2} \leq C_\varepsilon \|f\|_{W_1^0}.$$

Existence and uniqueness of the solution are still valid, provided that f belongs to $W_\beta^0(\Omega_\varepsilon)$, under the condition $|\beta - 1| < \alpha$.

A priori, C_ε depends on the domain Ω_ε, so the index $_\varepsilon$. As we shall see, it is not the case *a posteriori*.

Theorem 3.1. (Stability) *The constant* C_ε *of Proposition* 3.1 *can be chosen independently of* ε.

Proof. Let us consider the operator

$$A_\varepsilon^{-1} : W_1^0(\Omega_\varepsilon) \to W_1^2(\Omega_\varepsilon)$$

which to $f \in W_1^0(\Omega_\varepsilon)$ associates the solution $u \in W_1^2(\Omega_\varepsilon)$ of the Dirichlet problem (1.3). The idea of the proof is to show that its operator norm is bounded, independently of ε.

Let us write

$$f(x) = f_1(x) + \varepsilon^{-2} f_2\left(\frac{x}{\varepsilon}\right), \quad \text{with } f_1 := \left(1 - \chi_{\sqrt{\varepsilon}}\right) f, \quad f_2\left(\frac{\cdot}{\varepsilon}\right) := \varepsilon^2 \chi_{\sqrt{\varepsilon}} f. \tag{3.1}$$

Consider next the two problems:

$$-\Delta u_1 = f_1 \quad \text{in} \quad \Omega, \qquad u_1 = 0 \quad \text{on } \Gamma, \tag{3.2}$$

$$-\Delta u_2 = f_2 \quad \text{in} \quad \omega, \qquad u_2 = 0 \quad \text{on } \partial\omega. \tag{3.3}$$

One can easily find the bounds:

$$\|f_1\|_{V_1^0(\Omega)} + \|f_2\|_{W_1^0(\omega)} \leq c\|f\|_{W_1^0(\Omega_\varepsilon)},$$

with c independent of ε. In particular, u_1 and u_2 exist, and are unique, respectively in $V_1^2(\Omega)$ and $W_1^2(\omega)$ respectively. Moreover, one has

$$\|u_1\|_{V_1^2(\Omega)} + \|u_2\|_{W_1^2(\omega)} \le c\|f\|_{W_1^0(\Omega_\varepsilon)}.$$

Define (with a continuation of u_1 by zero to $\Omega_\varepsilon \setminus \Omega$ in the first term, still called u_1)

$$U_\varepsilon = (1-\chi_\varepsilon)u_1 + \chi u_2\left(\frac{\cdot}{\varepsilon}\right).$$

Let R_ε denote the operator which to $f \in W_1^0(\Omega_\varepsilon)$ associates $U_\varepsilon \in W_1^2(\Omega_\varepsilon)$. We prove first that the operator norm of R_ε is bounded independently of ε. As a matter of fact

$$\|U_\varepsilon\|_{W_1^2(\Omega_\varepsilon)} \le \|(1-\chi_\varepsilon)u_1\|_{W_1^2(\Omega_\varepsilon)} + \left\|\chi u_2\left(\frac{\cdot}{\varepsilon}\right)\right\|_{W_1^2(\Omega_\varepsilon)}.$$

Let us bound the first term of the right-hand side. Since $1-\chi_\varepsilon$ vanishes on $\{x \in \Omega_\varepsilon \mid |x| \le a\varepsilon\}$, one finds

$$\|(1-\chi_\varepsilon)u_1\|^2_{W_1^2(\Omega_\varepsilon)} = \sum_{|\lambda|\le 2}\int_{|x|\ge a\varepsilon}\left(\varepsilon^2+|x|^2\right)^{|\lambda|-1}\left|\partial^\lambda(1-\chi_\varepsilon)u_1)(x)\right|^2\,\mathrm{d}x.$$

Terms which do not include a derivative of $(1-\chi_\varepsilon)$ are bounded by $\|u_1\|_{V_1^2(\Omega)}$. For the other terms ($|\lambda| \ge 1$), the integral is actually carried out over the region $\mathcal{C}_\varepsilon = \{x \in \Omega_\varepsilon \mid a\varepsilon < |x| < 2a\varepsilon\}$. In this region, one has $|x| \simeq \varepsilon$, so $(\varepsilon^2+|x|^2)^{|\lambda|-1} \simeq |x|^{2(|\lambda|-1)}$. Therefore

$$\|(1-\chi_\varepsilon)u_1\|_{W_1^2(\Omega_\varepsilon)} \le C_1\|u_1\|_{V_1^2(\Omega)},$$

with C_1 independent of ε.

For the second term, we perform the change of variables $y = x/\varepsilon$ to recover

$$\left\|\chi u_2\left(\frac{\cdot}{\varepsilon}\right)\right\|_{W_1^2(\Omega_\varepsilon)} \le C_2\|u_2\|_{W_1^2(\omega)}.$$

As expected, the above allows to prove that $\|R_\varepsilon\|_{W_1^0\to W_1^2} \le C$, with C independent of ε.

Next, let $T_\varepsilon = (A_\varepsilon^{-1})^{-1}R_\varepsilon - I$. In this way, the norm $A_\varepsilon^{-1} = R_\varepsilon(I+T_\varepsilon)^{-1}$ will be bounded independently of ε, provided that the norm of T_ε is small. So, let $f \in W_1^0(\Omega_\varepsilon)$ and compute:

$$\begin{aligned}
T_\varepsilon f &= -\Delta U_\varepsilon - f,\\
&= -\chi_{\sqrt{\varepsilon}}f + \Delta\chi_\varepsilon u_1 + \chi_\varepsilon\Delta u_1 + 2\nabla\chi_\varepsilon\cdot\nabla u_1\\
&\quad - \Delta\chi u_2\left(\frac{\cdot}{\varepsilon}\right) - \chi\Delta\left[u_2\left(\frac{\cdot}{\varepsilon}\right)\right] - 2\nabla\left[u_2\left(\frac{\cdot}{\varepsilon}\right)\right]\cdot\nabla\chi,\\
&= \Delta\chi_\varepsilon u_1 + 2\nabla\chi_\varepsilon\cdot\nabla u_1 - \Delta\chi u_2\left(\frac{\cdot}{\varepsilon}\right) - 2\nabla\left[u_2\left(\frac{\cdot}{\varepsilon}\right)\right]\cdot\nabla\chi.
\end{aligned}$$

Note that the last equality comes from the fact that the respective supports of χ_ε and of $\left(1-\chi_{\sqrt{\varepsilon}}\right)$ are disjoint for small enough ε.

We are now in a position to evaluate $\|T_\varepsilon f\|_{W_1^0}$. Let E be the term which includes the contributions from u_1: $E = \Delta\chi_\varepsilon u_1 + 2\nabla\chi_\varepsilon \cdot \nabla u_1$. There holds $\mathrm{supp}(E) \subset \mathcal{C}_\varepsilon$ and also

$$\begin{aligned}\int_{\Omega_\varepsilon} \left(\varepsilon^2 + |x|^2\right) |E(x)|^2 \,\mathrm{d}x \le C \biggl(&\int_{\Omega_\varepsilon} \left(\varepsilon^2 + |x|^2\right) |\Delta\chi_\varepsilon(x) u_1(x)|^2 \,\mathrm{d}x \\ &+ \int_{\Omega_\varepsilon} \left(\varepsilon^2 + |x|^2\right) |\nabla\chi_\varepsilon(x) \cdot \nabla u_1(x)|^2 \,\mathrm{d}x \biggr) \\ &\le C\,(①+②)\,.\end{aligned}$$

Let $d \in]0, \alpha[$ be given. For the first term, ①, one can write:

$$\begin{aligned}\|\Delta\chi_\varepsilon u_1\|^2_{W_1^0(\Omega_\varepsilon)} &= \int_{\mathcal{C}_\varepsilon} \frac{\left(\varepsilon^2 + |x|^2\right)}{\varepsilon^4} \left| \left\{ \xi''\left(\frac{|x|}{\varepsilon}\right) + \frac{\varepsilon}{|x|} \xi'\left(\frac{|x|}{\varepsilon}\right) \right\} u_1(x) \right|^2 \mathrm{d}x, \\ &\le 2\left(\|\xi''\|^2_\infty + \frac{1}{a^2}\|\xi'\|^2_\infty \right) \int_{\mathcal{C}_\varepsilon} \frac{\left(\varepsilon^2 + |x|^2\right)}{\varepsilon^4} |u_1(x)|^2 \,\mathrm{d}x, \\ &\le C_\xi \sup_{x \in \mathcal{C}_\varepsilon} \left(\frac{(\varepsilon^2 + |x|^2)|x|^{2+2d}}{\varepsilon^4} \right) \int_{\mathcal{C}_\varepsilon} |x|^{2((1-d)-2)} |u_1(x)|^2 \,\mathrm{d}x.\end{aligned}$$

Since $|x| = O(\varepsilon)$ for $x \in \mathcal{C}_\varepsilon$, the term $\varepsilon^{-4}(\varepsilon^2 + |x|^2)|x|^{2+2d}$ is $O(\varepsilon^{2d})$. Therefore, *under the condition that* u_1 *belongs to* $V^2_{1-d}(\Omega)$, one finds:

$$\|\Delta\chi_\varepsilon u_1\|_{W_1^0(\Omega_\varepsilon)} \le C\varepsilon^d \|u_1\|_{V^2_{1-d}(\Omega)}.$$

The same kind of estimate can be obtained for the second term ②.

One has to *check that* f_1 *belongs to* $V^0_{1-d}(\Omega)$ to conclude, thanks to Remark 2.1, that $u_1 \in V^2_{1-d}(\Omega)$ and $\|u_1\|_{V^2_{1-d}(\Omega)} \le C\|f_1\|_{V^0_{1-d}(\Omega)}$. Now,

$$\begin{aligned}\|f_1\|^2_{V^0_{1-d}(\Omega)} &= \int_{\Omega \cap \{x : |x| \ge a\sqrt{\varepsilon}\}} |x|^{2(1-d)} \left| 1 - \chi\left(\frac{|x|}{\sqrt{\varepsilon}}\right) \right|^2 |f(x)|^2 \,\mathrm{d}x, \\ &= \int_{\Omega \cap \{x : |x| \ge a\sqrt{\varepsilon}\}} |x|^2 \left| 1 - \chi\left(\frac{|x|}{\sqrt{\varepsilon}}\right) \right|^2 |f(x)|^2 \frac{1}{|x|^{2d}} \,\mathrm{d}x, \\ &\le C\varepsilon^{-d} \|f\|^2_{W_1^0(\Omega_\varepsilon)}.\end{aligned}$$

Therefore, u_1 belongs to $V^2_{1-d}(\Omega)$ and $\|u_1\|_{V^2_{1-d}(\Omega)} \le C\varepsilon^{-d/2}\|f\|^2_{W_1^0(\Omega_\varepsilon)}$, so

$$\|\Delta\chi_\varepsilon u_1 + \nabla\chi_\varepsilon \cdot \nabla u_1\|_{W_1^0(\Omega_\varepsilon)} \le C\varepsilon^{d/2}\|f\|_{W_1^0(\Omega_\varepsilon)}.$$

For the term which includes u_2 in $T_\varepsilon f$, one finds in a similar manner:

$$\|f_2\|_{W^0_{1+d}(\omega)} \le C\varepsilon^{-d}\|f\|^2_{W_1^0(\Omega_\varepsilon)}.$$

So, $u_2 \in W^2_{1+d}(\omega)$ and

$$\left\| \Delta\chi u_2\left(\frac{\cdot}{\varepsilon}\right) + 2\nabla\left[u_2\left(\frac{\cdot}{\varepsilon}\right)\right] \cdot \nabla\chi \right\|_{W_1^0(\Omega_\varepsilon)} \le C\varepsilon^d \|u_2\|_{W^2_{1+d}(\omega)}.$$

To conclude, we proved the bound

$$\forall\, f \in W_1^0(\Omega_\varepsilon), \quad \|T_\varepsilon f\|_{W_1^0(\Omega_\varepsilon)} \leq C\varepsilon^{d/2}\|f\|_{W_1^0(\Omega_\varepsilon)}.$$

Recall that $A_\varepsilon^{-1} = R_\varepsilon(I+T_\varepsilon)^{-1}$, so there holds finally

$$\|A_\varepsilon^{-1}\| = \|R_\varepsilon(I+T_\varepsilon)^{-1}\| \leq \frac{C_1}{1-C_2\varepsilon^{d/2}}. \qquad \square$$

Remark 3.1. The fact that f vanishes in a neighborhood of the origin O has not been used in the previous proof. So, this method provides an approximation U_ε of the solution u_ε to the Dirichlet problem (1.3) for any f in the *ad hoc* space $W_1^0(\Omega_\varepsilon)$. However, in the case when f does not vanish near O, this approximation is not usable to find an expansion at the order one or more (see Remark 3.3 below).

In the sequel, we establish an asymptotic, multi-scale, expansion of the solution u_ε to (1.3), when $\varepsilon \to 0$. The underlying principle of proof follows the scheme we used for Theorem 3.1. We choose a term, which is not too different from the exact solution, and then we estimate the error, by studying the PDE it solves.

Note that since the perturbation of the boundary (Γ versus Γ_ε) is *local*, we can (safely!) assume that, away from the origin, u_ε is similar to u_0. Based on this observation, we put u_0 as the first term in our expansion. Since it is not defined in $\Omega_\varepsilon \setminus \Omega$, we choose instead its continuation by zero (still called u_0) and introduce

$$U_\varepsilon^0 = (1-\chi_\varepsilon)\, u_0.$$

Define $R_\varepsilon^0 = u_\varepsilon - U_\varepsilon^0$. The error reads as follows, in the PDE,

$$\begin{aligned} -\Delta R_\varepsilon^0 &= -\mathbf{1}_{[a,2a]}\left(\frac{\cdot}{\varepsilon}\right)(\Delta\chi_\varepsilon u_0 + 2\nabla\chi_\varepsilon\cdot\nabla u_0 - \chi_\varepsilon f), \\ &= -\mathbf{1}_{[a,2a]}\left(\frac{\cdot}{\varepsilon}\right)(\Delta\chi_\varepsilon u_0 + 2\nabla\chi_\varepsilon\cdot\nabla u_0), \end{aligned}$$

for small enough ε. In the above, we used the fact that f vanishes in a neighborhood of O. Thanks to the behavior of u_0 near O, we find (C depends only on f)

$$\|\Delta R_\varepsilon^0\|_{W_1^0(\Omega_\varepsilon)} \leq C\varepsilon^\alpha.$$

According to Proposition 3.1 and Theorem 3.1, we infer

Proposition 3.2. *Assume that f vanishes in a neighborhood of the origin O, then*

$$\|u_\varepsilon - U_\varepsilon^0\|_{W_1^2(\Omega_\varepsilon)} \leq C_0\varepsilon^\alpha,$$

with a constant C_0 that depends only on the data f.

This first approximation is not enough to provide information concerning the trace of the normal derivative at the tip, so we look at the next term in the expansion.

After studying the behavior of u_0 at the origin (see (2.2)), we introduce the function Q_1 defined by

$$\varepsilon^{\alpha-2}Q_1\left(\frac{\cdot}{\varepsilon}\right) = \varepsilon^\alpha\left\{\Delta\chi_\varepsilon\phi_1\left(\frac{\cdot}{\varepsilon}\right) + 2\nabla\chi_\varepsilon\cdot\nabla\left[\phi_1\left(\frac{\cdot}{\varepsilon}\right)\right]\right\}.$$

Then, in ω, we consider the problem

$$\begin{cases} -\Delta q_1 = Q_1 & \text{in } \omega \\ q_1 = 0 & \text{in } \partial\omega. \end{cases} \tag{3.4}$$

According to Proposition 3.1, there exists one and only one solution q_1 in $W_1^2(\omega)$. Next, define

$$U_\varepsilon^1 = (1 - \chi_\varepsilon)\, u_0 - \lambda_1 \varepsilon^\alpha q_1 \left(\frac{\cdot}{\varepsilon}\right),$$

with λ_1 the coefficient which appears in (2.2). Define $R_\varepsilon^1 = u_\varepsilon - U_\varepsilon^1$. For small enough ε, the error reads, in the PDE,

$$\begin{aligned} -\Delta R_\varepsilon^1 &= -\mathbf{1}_{[a,2a]}\left(\frac{\cdot}{\varepsilon}\right)\left(\Delta\chi_\varepsilon(u_0 - \lambda_1\phi_1) + 2\nabla\chi_\varepsilon \cdot \nabla(u_0 - \lambda_1\phi_1) - \chi_\varepsilon f\right), \\ &= -\mathbf{1}_{[a,2a]}\left(\frac{\cdot}{\varepsilon}\right)\left(\Delta\chi_\varepsilon(u_0 - \lambda_1\phi_1) + 2\nabla\chi_\varepsilon \cdot \nabla(u_0 - \lambda_1\phi_1)\right). \end{aligned}$$

Let $y_1(\zeta) = (1 - \chi(\zeta))\, \phi_1(\zeta) - q_1(\zeta)$. It satisfies

$$\begin{cases} -\Delta y_1 = 0 \quad \text{in } \omega, \\ y_1 = 0 \quad \text{on } \partial\omega, \\ y_1(\zeta) = \phi_1(\zeta) + c_1 s_D^1(\zeta) + O(|\zeta|^{-2\alpha}), \quad |\zeta| \to +\infty. \end{cases} \tag{3.5}$$

By construction, it is independent of ε. See Remark 3.4 for some further comments.

The term U_ε^1 can finally be rewritten as

$$U_\varepsilon^1 = (1 - \chi_\varepsilon)\,(u_0 - \lambda_1\phi_1) + \lambda_1 \varepsilon^\alpha y_1 \left(\frac{\cdot}{\varepsilon}\right).$$

Now, the error between u_ε and U_ε^1 behaves as indicated below.

Proposition 3.3. *Assume that f vanishes in a neighborhood of the origin O.*

$$\|u_\varepsilon - U_\varepsilon^1\|_{W_1^2(\Omega_\varepsilon)} \le C_1 \varepsilon^{2\alpha},$$

with a constant C_1 that depends only on the data f.

More generally, one can prove

Theorem 3.2. (Consistency)

1. *By induction, one can build a sequence of functions $(y_\ell)_{\ell \ge 1}$ such that*

$$\begin{cases} -\Delta y_\ell = 0 \quad \text{in } \omega, \\ y_\ell = 0 \quad \text{in } \partial\omega, \\ y_\ell(\zeta) = \phi_\ell(\zeta) + O(|\zeta|^{-\alpha}), \; |\zeta| \to +\infty. \end{cases}$$

2. *Assume that f vanishes in a neighborhood of the origin O.*
Then, for all $m \in \mathbb{N}$, if U_ε^m is defined by

$$U_\varepsilon^m = (1-\chi_\varepsilon)\left(u_0 - \sum_{\ell=1}^{m} \lambda_\ell \phi_\ell\right) + \sum_{\ell=1}^{m} \lambda_\ell \varepsilon^{\ell\alpha} y_\ell\left(\frac{\cdot}{\varepsilon}\right), \tag{3.6}$$

then the error of order m is bounded by

$$\|u_\varepsilon - U_\varepsilon^m\|_{W_1^2(\Omega_\varepsilon)} \le C_m \varepsilon^{(m+1)\alpha}. \tag{3.7}$$

Remark 3.2. Let us note that the sequence $(y_\ell)_{\ell\ge1}$ is independent of f, and that the coefficients $(\lambda_\ell)_{\ell\ge1}$ are the ones that appear in (2.2).

If $m\alpha \in \mathbb{N}$, (3.7) holds with the right-hand side $C_m\varepsilon^{(m+1)\alpha}|\log\varepsilon|$.

Moreover, if Ω and Ω_ε are bounded, then by restricting y_ℓ to Ω_ε, one finds

$$\begin{aligned} \|u_\varepsilon\|_{H^2(\Omega_\varepsilon)} &\le C\varepsilon^{\alpha-1}, \\ \|u_\varepsilon - U_\varepsilon^m\|_{H^1(\Omega_\varepsilon)} &\le C\varepsilon^{(m+1)\alpha}, \\ \|u_\varepsilon - U_\varepsilon^m\|_{H^2(\Omega_\varepsilon)} &\le C\varepsilon^{(m+1)\alpha-1}. \end{aligned}$$

Corollary 3.1. *For $\varepsilon > 0$, one can write in any given ε-neighborhood of O_ε,*

$$u_\varepsilon(x) = \sum_{\ell=1}^{\ell=m} \lambda_\ell \varepsilon^{\ell\alpha} y_\ell\left(\frac{x}{\varepsilon}\right) + O(\varepsilon^{(m+1)\alpha-1}). \tag{3.8}$$

Moreover, when $\varepsilon \to 0^+$, the trace of the normal derivative at the tip behaves as

$$\frac{\partial u_\varepsilon}{\partial n}(O_\varepsilon) = \lambda_1 \frac{\partial y_1}{\partial n}(O_1)\varepsilon^{\alpha-1} + O(\varepsilon^{2\alpha-1}). \tag{3.9}$$

Outside any given neighborhood of the origin O, one can write, for $\varepsilon > 0$ sufficiently small,

$$u_\varepsilon(x) = u_0(x) + c_1\lambda_1\varepsilon^{2\alpha} s_D^1(x) + O(\varepsilon^{3\alpha}), \tag{3.10}$$

where c_1 appears in (3.5).

We conclude this subsection by two important remarks. The first one provides some pointers as to the relevance of the assumption of f that it vanishes near O. The second one gives some insight as to how one can relate the behavior near the origin to the behavior at the infinity.

Remark 3.3. What happens if f does not vanish in a neighborhood of the origin O? In the proof of Theorem 3.1, one can still split f as the sum of a term that vanishes near O (f_1), and a term with compact support (f_2). Then, the first term in the expansion of u_ε is as before, but, following the proof of Theorem 3.1, one has only $\|u_\varepsilon - U_\varepsilon^0\|_{W_1^2(\Omega_\varepsilon)} \le C_\delta \varepsilon^{\alpha/2-\delta}$, for $\delta \in]0, \alpha/2[$, with a constant C_δ that depends on the data f and on δ.

However, in the case when f is *smooth and rapidly decreasing near O* (see Chap. 7 of Ref. 19), that is

$$\forall\, N \in \mathbb{N},\ \ \forall\, \beta \in \mathbb{N}^2,\ \ \exists\, C_\beta > 0, \quad \left|\partial_\theta^{\beta_1}(r\partial_r)^{\beta_2} f\right| \le C_\beta r^N,$$

the expansion (2.2) of u_0 still holds, and so does the asymptotic expansion of u_ε (3.6) and the error (3.7).

Remark 3.4. What is the value of the constant c_1 in the expression of y_1 in (3.5)? After some elementary computations, one finds

$$c_1 = \frac{1}{2\alpha} \lim_{R\to+\infty} \int_{\Omega\cap\partial B_R} \left(y_1 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial y_1}{\partial n} \right) \mathrm{d}\sigma_R = \frac{1}{2\alpha} \int_{\{\sigma\in\Gamma, |\sigma|<a\}} \frac{\partial \phi_1}{\partial n}(\sigma) y_1(\sigma)\, \mathrm{d}\sigma.$$

Actually, c_1 can be seen as the multiplicative coefficient that enables one to go from the local behavior of u_ε (3.8) to the behavior at infinity of $u_\varepsilon - u_0$ (3.10):

$$\begin{aligned} &\text{For } x \text{ such that } |x| \le \varepsilon, && u_\varepsilon(x) = \lambda_1 \varepsilon^\alpha y_1\left(\frac{x}{\varepsilon}\right) + O(\varepsilon^{2\alpha-1}),\\ &\text{For } x \text{ such that } |x| \ge r_0 > 0, && u_\varepsilon(x) - u_0(x) = c_1 \lambda_1 \varepsilon^{2\alpha} s_D^1(x) + O(\varepsilon^{3\alpha}). \end{aligned}$$

3.2. *Rounded corners in three-dimensional axisymmetric domains*

Let us consider the axisymmetric case. We assume that the solution to the Dirichlet problem can exhibit a singularity, i.e. $\pi/\beta > \pi/\beta_c$ (see Sec. 2.3). The techniques are very similar to the 2D case.

Theorem 3.3. (Stability)

(1) *Given $f \in W_1^0(\breve{\Omega}_\varepsilon)$, there exists one, and only one solution $\breve{u}_\varepsilon$, which belongs to $W_1^2(\breve{\Omega}_\varepsilon)$, to the problem* (1.3)*. Moreover, it satisfies*

$$\|\breve{u}_\varepsilon\|_{W_1^2} \le C_\varepsilon \|f\|_{W_1^0},$$

with a constant C_ε independent of ε.

(2) *If β is such that $|\beta - 1| < \sqrt{\Lambda_1 + \frac{1}{4}}$ and if f belongs to $W_\beta^0(\breve{\Omega}_\varepsilon)$, then the same results apply with $\breve{u}_\varepsilon$ in $W_\beta^2(\breve{\Omega}_\varepsilon)$ and the estimate $\|\breve{u}_\varepsilon\|_{W_\beta^2} \le C_\varepsilon \|f\|_{W_\beta^0}$.*

As in the Cartesian case, we define in the meridian half-plane $U_\varepsilon^0 = (1-\chi_\varepsilon)\, u_0$. One has

Proposition 3.4. *Assume that f vanishes in a neighborhood of the origin O.*

$$\|u_\varepsilon - U_\varepsilon^0\|_{W_{1,a,+}^2} \le C_0 \varepsilon^{\nu_1 + \frac{1}{2}},$$

with a constant C_0 that depends only on the data f.

Let us define Q_1^c such that

$$\varepsilon^{\nu_1 - 2} Q_1^c\left(\frac{\cdot}{\varepsilon}\right) = \varepsilon^{\nu_1} \left\{ \Delta^+ \chi_\varepsilon \phi_1^c\left(\frac{\cdot}{\varepsilon}\right) + 2\nabla\chi_\varepsilon \cdot \nabla \left[\phi_1^c\left(\frac{\cdot}{\varepsilon}\right)\right] \right\}$$

and $q_1^c \in W^2_{1,a,+}(\omega)$ the solution to

$$\begin{cases} -\Delta^+ q_1^c = Q_1^c & \text{in } \omega, \\ \dfrac{\partial q_1^c}{\partial n} = 0 & \text{on } \partial\omega_a, \\ q_1^c = 0 & \text{on } \partial\omega_b. \end{cases} \tag{3.11}$$

Next, we define

$$U_\varepsilon^1 = (1-\chi_\varepsilon)\, u_0 - \lambda_1 \varepsilon^{\nu_1} q_1^c\left(\frac{\cdot}{\varepsilon}\right) = (1-\chi_\varepsilon)\,(u_0 - \lambda_1 \phi_1^c) + \lambda_1 \varepsilon^{\nu_1} y_1^c\left(\frac{\cdot}{\varepsilon}\right),$$

with $y_1^c(\zeta) = (1-\chi(\zeta))\phi_1^c(\zeta) - q_1^c(\zeta)$. It is possible to prove

Proposition 3.5. *Assume that f vanishes in a neighborhood of the origin O, then*

$$\|u_\varepsilon - U_\varepsilon^1\|_{W^2_{1,a,+}} \leq C_1 \varepsilon^{\nu_2 + \frac{1}{2}},$$

with a constant C_1 that depends only on the data f.

More generally, one can prove the general result below.

Theorem 3.4. (Consistency) 1. *By induction, one can build a sequence of functions $(y_\ell^c)_{\ell\geq 1}$ in $H^1_{1,\mathrm{loc}}(\omega)$ such that*

$$\begin{cases} -\Delta^+ y_\ell^c = 0 & \text{in } \omega, \\ \dfrac{\partial y_\ell^c}{\partial n} = 0 & \text{on } \partial\omega_a, \\ y_\ell^c = 0 & \text{on } \partial\omega_b, \\ y_\ell^c(\zeta) = \phi_\ell^c(\zeta) + O(|\zeta|^{-\nu_1 - 1}), & |\zeta| \to +\infty. \end{cases}$$

2. *Assume that f vanishes in a neighborhood of the origin O.*

Then, for all $m \in \mathbb{N}$, if U_ε^m is defined by

$$U_\varepsilon^m = (1-\chi_\varepsilon)\left(u_0 - \sum_{\ell=1}^{m} \lambda_\ell \phi_\ell^c\right) + \sum_{\ell=1}^{m} \lambda_\ell \varepsilon^{\nu_\ell} y_\ell^c\left(\frac{\cdot}{\varepsilon}\right), \tag{3.12}$$

then the error of order m is bounded by

$$\|u_\varepsilon - U_\varepsilon^m\|_{W^2_{1,a,+}} \leq C_m \varepsilon^{\nu_{m+1} + \frac{1}{2}}. \tag{3.13}$$

Corollary 3.2. *When $\varepsilon \to 0^+$, there holds*

$$\frac{\partial u_\varepsilon}{\partial n}(O_\varepsilon) = \lambda_1 \frac{\partial y_1^c}{\partial n}(O_1)\varepsilon^{\nu_1 - 1} + O(\varepsilon^{\nu_2 - 1}). \tag{3.14}$$

4. Integral Representation of the Normal Derivative

In this section, we establish computable integral boundary representation formulas for the value of the normal derivative at the tip of rounded corners. In this way, it will be possible to compare the results obtained here, with those derived from the asymptotic expansion techniques. Among others, this involves a splitting of the Green's function, where the most singular terms are separated from the smooth ones. Since the singular terms are known explicitly, they can be computed. As far as the smooth terms are concerned, they are numerically approximated, via the discretization of "standard" variational formulations. We begin by some well-known results to illustrate our purpose, and then focus on the results we are interested in.

4.1. *Sharp corners in two-dimensional domains*

One can compute the solution u_0 to the Dirichlet problem (1.3) set in Ω, with the help of the Green's function G_0 of the domain Ω. As a matter of fact, there holds

$$u_0(x) = \int_\Omega f(y) G_0(x,y)\,\mathrm{d}y. \tag{4.1}$$

Below, we recall some results concerning this Green's function, so that we can first derive a computable approximation of the above formula. For $y \in \Omega$, let δ_y be the Dirac mass at y: by definition, G_0 is the solution to

$$\begin{cases} -\Delta_x G_0(\cdot,y) = \delta_y & \text{in } \mathcal{D}'(\Omega), \\ G_0(\cdot,y) = 0 & \text{on } \Gamma, \end{cases} \tag{4.2}$$

with the additional condition $G_0(x,y) = O(1)$ when $|x| \to +\infty$.

Consider $E_2(x,y) := -\frac{1}{2\pi}\log(|x-y|)$, which satisfies the identity below, in the sense of distributions:

$$-\Delta_x E_2 = \delta_y.$$

We would like to compare G_0 to E_2, and be able to compute the remainder. We note that $x \mapsto E_2(x,y)$ behaves like $\log(|x|)$ when $|x| \to +\infty$: its trace on Γ does not belong to $W_0^{1/2}(\Gamma)$. So, we introduce a smooth truncation function $\eta \in \mathcal{C}^\infty(\mathbb{R}^+)$, built in such a way that

$$\eta \equiv \begin{cases} 0 & \text{for } t \in \left[0, \dfrac{1}{2}\right] \cup [2, +\infty[\\ 1 & \text{for } t \in \left[\dfrac{3}{4}, \dfrac{5}{4}\right] \end{cases}$$

and split Green's function $G_0(\cdot,y)$ as

$$G_0(x,y) = \eta\left(\frac{|x|}{|y|}\right) E_2(x,y) + H(x,y), \quad x,y \in \Omega.$$

Since E_2 is known, one only has to characterize $H(\cdot, y)$. After (4.2), it solves

$$\begin{cases} -\Delta_x H(\cdot, y) = -\Delta_x \left(\left(1 - \eta\left(\frac{|\cdot|}{|y|}\right)\right) E_2(\cdot, y)\right) & \text{in } \Omega, \\ H(\cdot, y) = -\eta\left(\frac{|\cdot|}{|y|}\right) E_2(\cdot, y) & \text{on } \Gamma. \end{cases} \tag{4.3}$$

Interestingly, this problem can now be solved in $W_0^1(\Omega)$, since one has

$$\operatorname{supp}\left(\eta\left(\frac{|\cdot|}{|y|}\right)\right) \subset \left\{x \in \Omega, \frac{|y|}{2} < |x| < 2|y|\right\}$$

and

$$\operatorname{supp}\left(\Delta_x \left(\left(1 - \eta\left(\frac{|\cdot|}{|y|}\right)\right) E_2(\cdot, y)\right)\right) \subset \left\{x \in \Omega, \frac{|y|}{2} < |x| < 2|y|\right\}.$$

This splitting of the Green's function, where the singular part E_2 is extracted, is now computable. As a matter of fact, the expression of E_2 is *explicit*. Moreover, one can discretize the variational formulation of problem (4.3), set in $W_0^1(\Omega)$, to recover a numerical approximation of H, and therefore of G_0 and of u_0, via (4.1).

Remark 4.1. There exist other ways of computing the Green's function. For the 2D unbounded, sharp cone, G_0 can be computed explicitly, using the separation of variables in polar coordinates (r, θ). After some computations,[16] for a given (r_0, θ_0) (in Ω), G_0 can be explicitly written as the expansion:

$$\begin{aligned} G_0((r,\theta),(r_0,\theta_0)) = &\left(\sum_{m\geq 1} \frac{1}{2m\alpha}\left(\frac{r}{r_0}\right)^{m\alpha} \varphi_m(\theta)\varphi_m(\theta_0)\right) \mathbf{1}_{]0,1[}\left(\frac{r}{r_0}\right) \\ &+ \left(\sum_{m\geq 1} \frac{1}{2m\alpha}\left(\frac{r_0}{r}\right)^{m\alpha} \varphi_m(\theta)\varphi_m(\theta_0)\right) \mathbf{1}_{]0,1[}\left(\frac{r_0}{r}\right). \end{aligned} \tag{4.4}$$

4.2. *Rounded corners in two-dimensional domains*

As for the sharp cone, the Green's function G_ε of Ω_ε can be split as: $G_\varepsilon(x, y) = \eta(|x|/|y|)E_2(x, y) + H_\varepsilon(x, y)$, where η has been introduced in Sec. 4.1, and $H_\varepsilon(\cdot, y)$ is the solution to a problem analogous to (4.3) set in Ω_ε. As we already noticed:

$$u_\varepsilon(x) = \int_{\Omega_\varepsilon} f(y) G_\varepsilon(x, y)\, \mathrm{d}y.$$

Since the quantity of interest is the trace of the normal derivative $\partial_n u_\varepsilon(x)$, one has to know $\partial_n G_\varepsilon$ on Γ_ε, which requires an *ad hoc* splitting of this object. We proceed as before.

Lemma 4.1. *The Green's function G_ε satisfies*:

(1) $\nabla_x G_\varepsilon(x,y) = -\frac{x-y}{2\pi|x-y|^2} + O(1)$, *when* $y \to x$;
(2) *for any* $y \in \Omega_\varepsilon$ *and* $\psi \in \mathcal{C}_0^\infty\left(\overline{\Omega}_\varepsilon\right)$, *there holds*

$$\int_{\Omega_\varepsilon} G_\varepsilon(x,y)\Delta\psi(x)\,\mathrm{d}x = -\psi(y) + \int_{\Gamma_\varepsilon} \psi(x)\frac{\partial G_\varepsilon}{\partial n_x}(x,y)\,\mathrm{d}\sigma(x).$$

Proof. The first item is straightforward.

As far as the second item is concerned, let us introduce another truncation function, $\chi \in \mathcal{C}_0^\infty(\Omega_\varepsilon)$, the support of which is included in a ball $B_\gamma(y)$, and which is equal to 1 in $B_{\frac{\gamma}{2}}(y)$.

$$\begin{aligned}
\int_{\Omega_\varepsilon} G_\varepsilon(x,y)\Delta\psi(x)\,\mathrm{d}x &= \int_{\Omega_\varepsilon} G_\varepsilon(x,y)\Delta\left(\chi\psi\right)(x)\,\mathrm{d}x + \int_{\Omega_\varepsilon} G_\varepsilon(x,y)\Delta\left((1-\chi)\psi\right)(x)\,\mathrm{d}x,\\
&= -\psi(y) + \int_{\Omega_\varepsilon - B_{\gamma/2}(y)} G_\varepsilon(x,y)\Delta\left((1-\chi)\psi\right)(x)\,\mathrm{d}x,\\
&= -\psi(y) + \int_{\Omega_\varepsilon - B_{\gamma/2}(y)} \Delta_x\left(G_\varepsilon(x,y)\right)\left((1-\chi)\psi\right)(x)\,\mathrm{d}x\\
&\quad - \int_{\Gamma_\varepsilon} \frac{\partial G_\varepsilon}{\partial n_x}(x,y)(1-\chi)(x)\psi(x)\,\mathrm{d}\sigma(x),\\
&= -\psi(y) - \int_{\Gamma_\varepsilon} \frac{\partial G_\varepsilon}{\partial n_x}(x,y)\psi(x)\,\mathrm{d}\sigma(x).
\end{aligned}$$

□

One can then prove

Proposition 4.1. *Consider $\sigma \in \Gamma_\varepsilon$. The integral representation formula holds*

$$\frac{\partial u_\varepsilon}{\partial n}(\sigma) = \int_{\Omega_\varepsilon} f(y)\frac{\partial G_\varepsilon}{\partial n_\sigma}(\sigma,y)\,\mathrm{d}y. \tag{4.5}$$

Proof. On the one hand, consider $\psi \in \mathcal{C}_0^\infty\left(\overline{\Omega}_\varepsilon\right)$ and compute

$$\left\langle \frac{\partial u_\varepsilon}{\partial n}, \psi \right\rangle = \int_{\Omega_\varepsilon} \left(\Delta u_\varepsilon \psi - u_\varepsilon \Delta\psi\right)\,\mathrm{d}x = \int_{\Omega_\varepsilon} \left(-f\psi - u_\varepsilon\Delta\psi\right)\,\mathrm{d}x.$$

On the other hand, the singularity $\eta(|x|/|y|)\log(|x-y|)$ is locally integrable over the domain $\Omega_\varepsilon \times \Omega_\varepsilon$ (see Ref. 18). Therefore, according to Fubini's Theorem:

$$\begin{aligned}
\int_{\Omega_\varepsilon} u_\varepsilon \Delta\psi\,\mathrm{d}x &= \int_{\Omega_\varepsilon\times\Omega_\varepsilon} f(y)G_\varepsilon(x,y)\Delta\psi(x)\,\mathrm{d}x dy,\\
&= \int_{\Omega_\varepsilon} f(y)\left(\int_{\Omega_\varepsilon} G_\varepsilon(x,y)\Delta\psi(x)\,\mathrm{d}x\right)\,\mathrm{d}y,\\
&= \int_{\Omega_\varepsilon} f(y)\left(-\psi(y) - \int_{\Gamma_\varepsilon}\psi(x)\frac{\partial G_\varepsilon}{\partial n_x}(x,y)\,\mathrm{d}\sigma(x)\right)\,\mathrm{d}y.
\end{aligned}$$

The last equality is a consequence of Lemma 4.1. To conclude, since the singularity $|x-y|^{-1}$ is locally integrable over the domain $\Gamma_\varepsilon \times \Omega_\varepsilon$, one finds, according once more to Fubini's Theorem:

$$\begin{aligned}\left\langle \frac{\partial u_\varepsilon}{\partial n}, \psi \right\rangle &= \int_{\Omega_\varepsilon} f(y) \left(\int_{\Gamma_\varepsilon} \psi(x) \frac{\partial G_\varepsilon}{\partial n_x}(x,y)\,\mathrm{d}\sigma(x) \right) \mathrm{d}y, \\ &= \int_{\Gamma_\varepsilon} \psi(x) \left(\int_{\Omega_\varepsilon} f(y) \frac{\partial G_\varepsilon}{\partial n_x}(x,y)\,\mathrm{d}y \right) \mathrm{d}\sigma(x).\end{aligned}$$

This yields the desired formula. □

To characterize the function $G^\sigma_{\varepsilon,n} = \partial_{n_\sigma} G_\varepsilon(\sigma,\cdot)$ for a given $\sigma \in \Gamma_\varepsilon$, let us split it as the sum of a singular part (when $y \to \sigma$) and of a part, which belongs to $H^1_{\mathrm{loc}}(\Omega_\varepsilon)$:

$$G^\sigma_{\varepsilon,n}(y) = (n_x(\sigma)\cdot\nabla_x)\, G_\varepsilon(\sigma,y) = (n_x(\sigma)\cdot\nabla_x)\, E_2(\sigma,y) + H^\sigma_{\varepsilon,n}(y).$$

If one notices that one can write

$$G^\sigma_{\varepsilon,n}(y) = \lim_{h\to 0^+} \frac{G_\varepsilon(\sigma - h n_\sigma, y) - G_\varepsilon(\sigma,y)}{h},$$

the boundary condition follows and the function $G^\sigma_{\varepsilon,n}$ is a solution to:

$$\begin{cases} \Delta G^\sigma_{\varepsilon,n} = 0 & \text{in } \Omega_\varepsilon, \\ G^\sigma_{\varepsilon,n} = 0 & \text{on } \Gamma_\varepsilon - \{\sigma\}. \end{cases} \tag{4.6}$$

Assume that, in a neighborhood of $\sigma = (x_0, \varphi(x_0))$, Γ_ε is of $\mathcal{C}^2$-class (this assumption is satisfied everywhere, with the exception of the two coupling points). In this way, $\partial_{n_\sigma} E_2(\sigma,\cdot)$ belongs to $L^1_{\mathrm{loc}}(\Gamma_\varepsilon)$ (it is bounded when $y \longrightarrow \sigma$). Consider next the problem

$$\begin{cases} -\Delta H^\sigma_{\varepsilon,n} = 0 & \text{in } \Omega_\varepsilon, \\ H^\sigma_{\varepsilon,n} = -\dfrac{\partial E_2}{\partial n_\sigma}(\sigma,\cdot) & \text{on } \Gamma_\varepsilon. \end{cases} \tag{4.7}$$

It is clear that, under the condition $\partial_{n_\sigma} E_2(\sigma,\cdot) \in W^{1/2}_0(\Gamma_\varepsilon)$, problem (4.7) is well-posed in $W^1_0(\Omega_\varepsilon)$, so $H^\sigma_{\varepsilon,n}$ exists in this space.

For the specific point $\sigma = O_\varepsilon$ (which is the one with smallest curvature radius) one finds by direct computations:

$$\frac{\partial E_2}{\partial n_\sigma}(O_\varepsilon,(x,\varphi(x))) = \begin{cases} -\dfrac{1}{2\pi}\dfrac{cx + \frac{c\varepsilon}{2}}{x^2 + (cx + \frac{c\varepsilon}{2})^2}, & x < -\varepsilon \\ \dfrac{1}{2\pi}\dfrac{\frac{c}{2\varepsilon}}{1 + \left(\frac{c}{2\varepsilon}\right)^2 x^2}, & x \in]-\varepsilon,\varepsilon[, \\ -\dfrac{1}{2\pi}\dfrac{-cx + \frac{c\varepsilon}{2}}{x^2 + (cx - \frac{c\varepsilon}{2})^2}, & x > \varepsilon. \end{cases}$$

One then infers by a direct estimate that $\partial_{n_\sigma} E_2(O_\varepsilon,\cdot)$ is such that

$$\left\| \frac{\partial E_2}{\partial n_\sigma}(O_\varepsilon,\cdot) \right\|_{W^{1/2}_0(\Gamma^\varepsilon)} = O(\varepsilon^{-1}),$$

which allows to conclude that $H^{O_\varepsilon}_{\varepsilon,n}$ indeed exists and is in $W^1_0(\Omega_\varepsilon)$. Moreover, an approximation is numerically computable, see the end of Sec. 4.1.

Remark 4.2. For $\sigma \neq O_\varepsilon$, one reaches a similar conclusion. More generally, for a different shape of Γ_ε, i.e. a different mapping φ, one can find explicit formulas by direct computations (see details in Ref. 16).

4.3. *Rounded corners in three-dimensional axisymmetric domains*

The Green's function in the domain $\breve{\Omega}_\varepsilon$ is such that

$$\begin{cases} -\Delta_x G_\varepsilon(\cdot, y) = \delta_y & \text{in } \mathcal{D}'(\breve{\Omega}_\varepsilon), \\ G_\varepsilon(\cdot, y) = 0 & \text{on } \breve{\Gamma}_\varepsilon. \end{cases} \tag{4.8}$$

Whereas in a 2D domain the logarithmic behavior requires the use of a truncation function to resolve the Green's function, in 3D the restriction of the function $x \mapsto |x|^{-1}$ to $\Omega_\varepsilon \cap \{x, |x| > 1\}$ now belongs to $W^1_0(\Omega_\varepsilon \cap \{x, |x| > 1\})$, so that the trace of $x \mapsto |x-y|^{-1}$ belongs to $W^{1/2}_0(\Gamma_\varepsilon)$ (for $y \notin \Gamma_\varepsilon$). Thus, G_ε can be split directly as $G_\varepsilon(x, y) = \frac{1}{4\pi}|x-y|^{-1} + H_\varepsilon(x, y)$. According to the above, one has $H_\varepsilon(x, y) = O(1)$ when $|x| \to +\infty$, for fixed y. The gradient of G_ε (with respect to x) behaves in the following way, when $x \to y$:

$$-\frac{x-y}{4\pi|x-y|^3} + O(1).$$

By performing computations, which are very similar to those carried out in 2D, one can prove (provided the boundary is of $\mathcal{C}^2$-class) that the normal trace derivative at $\sigma \in \breve{\Gamma}_\varepsilon$ is given by the integral representation formula below.

$$\frac{\partial u_\varepsilon}{\partial n}(\sigma) = \int_{\breve{\Omega}_\varepsilon} f(y) \frac{\partial G_\varepsilon}{\partial n_\sigma}(\sigma, y)\,\mathrm{d}y. \tag{4.9}$$

Remark 4.3. Notice that, for a point y which does not belong to the axis (Oz), the Green's function is not invariant by rotation.

From now on, we carry out the computations at the point with the smallest curvature radius, that is $\sigma = O_\varepsilon$. To emphasize this, we shall use the superscrit * for the related functionals. Due to the nature of $\breve{\Omega}_\varepsilon$, (4.9) can be rewritten as:

$$\frac{\partial u_\varepsilon}{\partial n}(O_\varepsilon) = 2\pi \int_{\Omega_\varepsilon} f(r, z) \frac{\partial G_\varepsilon}{\partial n_\sigma}(O_\varepsilon, (r, z)) r\,\mathrm{d}r\,\mathrm{d}z. \tag{4.10}$$

As in 2D, introduce $G^\star_{\varepsilon,n} = \partial_{n_\sigma} G_\varepsilon(O_\varepsilon, \cdot)$. It is the solution to a homogeneous problem which reads

$$\begin{cases} -\Delta^+ G^\star_{\varepsilon,n} = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial G^\star_{\varepsilon,n}}{\partial n} = 0 & \text{on } \Gamma^\varepsilon_a, \\ G^\star_{\varepsilon,n} = 0 & \text{on } \Gamma^\varepsilon_b. \end{cases} \tag{4.11}$$

Consider $E_3(x,y) = \frac{1}{4\pi}|x-y|^{-1}$. At $\sigma = O_\varepsilon$, the outward unit normal to $\breve{\Gamma}_\varepsilon$ is equal to $-\mathbf{e}_z$. Therefore, for $y = (r,z) \in \Omega_\varepsilon$, there holds

$$-\frac{\partial G_\varepsilon}{\partial n_\sigma}(O_\varepsilon, y) = \frac{\partial G_\varepsilon}{\partial z_\sigma}(O_\varepsilon, y) = \frac{\partial E_3}{\partial z_\sigma}(O_\varepsilon, y) - H^\star_{\varepsilon,n}(y),$$
$$= -\frac{1}{4\pi}\frac{z - z_{O_\varepsilon}}{(r^2 + (z - z_{O_\varepsilon}))^{3/2}} - H^\star_{\varepsilon,n}(y),$$

with a function $H^\star_{\varepsilon,n}$ such that $-\Delta^+ H^\star_{\varepsilon,n} = 0$. Moreover, on Γ^ε_a, it satisfies the boundary condition: $\partial_n H^\star_{\varepsilon,n} = 0$. On the other part of the boundary, Γ^ε_b, its trace $H^\star_{\varepsilon,n} = \partial_{z_\sigma} E_3(O_\varepsilon, \cdot)$ can be computed explicitly at the location $\sigma' = (r, \varphi(r))$:

$$H^\star_{\varepsilon,n}(r,\varphi(r)) = \begin{cases} \dfrac{c}{8\pi\varepsilon}\dfrac{1}{r\left(1 + \left(\frac{c}{2\varepsilon}\right)^2 r^2\right)^{\frac{3}{2}}} & \text{for } r < \varepsilon, \\ \dfrac{c}{4\pi}\dfrac{r - \frac{\varepsilon}{2}}{\left(r^2 + c^2\left(r - \frac{\varepsilon}{2}\right)^2\right)^{\frac{3}{2}}} & \text{for } r \geq \varepsilon. \end{cases} \tag{4.12}$$

Unfortunately, the trace of $H^\star_{\varepsilon,n}$ on Γ^ε_b is still singular at O_ε, so that $H^\star_{\varepsilon,n} \notin W^{1/2}_0(\Gamma^\varepsilon_b)$: the variational approach set in $W^1_{0,a}(\Omega_\varepsilon)$ fails. So, for practical purposes, i.e. to be able to compute a variational approximation of $\partial_{z_\sigma} G_\varepsilon(O_\varepsilon, y)$, we introduce

$$F^\star(y) = \frac{\partial E_3}{\partial z_\sigma}(O_\varepsilon, y) - \frac{c}{2\varepsilon}E_3(O_\varepsilon, y).$$

With the help of $F^\star$, which is computable, it is possible to remove the singularity: $\partial_{z_\sigma} G_\varepsilon(O_\varepsilon, y)$ is now split as

$$\frac{\partial G_\varepsilon}{\partial z_\sigma}(O_\varepsilon, y) = F^\star(y) - I^\star_{\varepsilon,n}(y) = \frac{\partial E_3}{\partial z_\sigma}(O_\varepsilon, y) - \frac{c}{2\varepsilon}E_3(O_\varepsilon, y) - I^\star_{\varepsilon,n}(y).$$

$I^\star_{\varepsilon,n}$ is such that

$$\begin{cases} -\Delta^+ I^\star_{\varepsilon,n} = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial I^\star_{\varepsilon,n}}{\partial n} = 0 & \text{on } \Gamma^\varepsilon_a, \\ I^\star_{\varepsilon,n} = F^\star & \text{on } \Gamma^\varepsilon_b. \end{cases} \tag{4.13}$$

Then, one can compute

$$F^\star|_{\Gamma^\varepsilon_b}(r,\varphi(r)) = \begin{cases} \dfrac{1}{\varepsilon^3}\dfrac{c^3}{32\pi}\dfrac{r}{\left(1 + \left(\frac{c}{2\varepsilon}\right)^2 r^2\right)^{\frac{3}{2}}} & \text{for } r < \varepsilon, \\ \dfrac{c}{4\pi}\dfrac{r - \frac{\varepsilon}{2}}{\left(r^2 + c^2\left(r - \frac{\varepsilon}{2}\right)^2\right)^{\frac{3}{2}}} - \dfrac{c}{8\pi\varepsilon}\dfrac{1}{r\sqrt{1 + \left(\frac{c}{2\varepsilon}\right)^2 r^2}} & \text{for } r \geq \varepsilon. \end{cases}$$

The behavior of $F^\star$ when $r \to +\infty$ is in r^{-2}, so that one gets $F^\star \in W_0^{1/2}(\Gamma_b^\varepsilon)$, with the estimate

$$\|F^\star\|_{W_0^{1/2}(\Gamma_b^\varepsilon)} = O\left(\varepsilon^{-1}\right).$$

These computations allow one to conclude that there exists one, and only one solution to (4.13) in $W^1_{0,a}(\Omega_\varepsilon)$. It can be discretized, so that one recovers a numerical approximation of (4.9).

5. Numerical Experiments

In this section, we compute the surface charge density at the tip, by using a numerical approximation of the asymptotic expansion formulas, that is (3.9) in the Cartesian case, and (3.14) in the axisymmetric case. Interestingly, we can compare the results to those obtained by approximating the integral representation formulas, respectively (4.5) and (4.9). For simplicity, we shall consider numerical experiments, set in a bounded domain: an angular sector of radius 5 (rounded at the tip). The results we report below have been obtained using Matlab.

Before we proceed, let us begin by some brief comments. We focus on the Cartesian case, albeit they are very similar in the axisymmetric case.

If we evaluate the asymptotic expansion formulas, we note first that the approximation is *independent of* ε. As a matter of fact, $\partial_n y_1(O_1)$ and α depend solely on the geometry of the domain ω, whereas λ_1 is equal to

$$\lambda_1 = \frac{1}{2\alpha}\int_\Omega f s_D^1 \,\mathrm{d}x.$$

So, assuming $\partial_n y_1(O_1)$ is known (note that the value of $\partial_n y_1(O_1)$ is directly computable via a numerical approximation of either (3.4) or (3.5)), one only has to evaluate λ_1. This amounts to computing an approximation of s_D^1, for which one can use the Singular Complement Method.[8] It can be implemented by using the Lagrange P_1 finite element on a series of regular triangulations $(\mathcal{T}_h)_h$, where h denotes the meshsize. From the above, it is clear that h can be chosen *independently of* ε. The convergence result is as follows:

$$|\lambda_1 - \lambda_1^h| = O(h^{2\alpha}).$$

In the above, we took into account the classical result of Ref. 20 on the approximation of integrals, when the exact and discretized domains differ: the error is in the order of $O(h^2)$.

Remark 5.1. According to Merlet,[21] one can refine the Singular Complement Method to reach an error estimate $O(h^2)$. One could also use graded meshes (near the re-entrant corner) for a similar result. Neither techniques require the knowledge of ε.

On the other hand, one can choose to approximate the integral representation formulas, using the same finite element for simplicity. Then, to resolve the part of the boundary near the (rounded) tip, one has to have a meshsize in the order of ε, i.e. $h_\varepsilon \simeq \varepsilon$. Now, if this requirement is met, one has to evaluate $\partial_n u_\varepsilon(O_\varepsilon)$ as given by (4.5). With the splitting $G^\star_{\varepsilon,n} = -\partial_y E^\star_2 + H^\star_{\varepsilon,n}$ one only has to approximate the variational formulation $H^\star_{\varepsilon,n}$ solves, that is (4.7). The convergence result is as follows:

$$\left|\frac{\partial u_\varepsilon}{\partial n}(O_\varepsilon) - \int_{\Omega_{\varepsilon,h}} f G^{\star,h}_{\varepsilon,n} dx\right| = O(h^2_\varepsilon).$$

Notice that the integrals are not evaluated over Ω_ε, but over $\Omega^{\text{ext}} = \Omega_\varepsilon \cap \overset{\circ}{\text{supp}}(f)$. The estimate on the $L^2(\Omega^{\text{ext}})$-norm of the difference $(G^\star_{\varepsilon,n} - G^{\star,h}_{\varepsilon,n}) = (H^\star_{\varepsilon,n} - H^{\star,h}_{\varepsilon,n})$ is standard: $O(h^2_\varepsilon \,\|H^\star_{\varepsilon,n}\|_{H^2(\Omega^{\text{ext}})}) = O(h^2_\varepsilon)$, since $\|H^\star_{\varepsilon,n}\|_{H^2(\Omega^{\text{ext}})} = O(1)$.

Then, if we want to compare the two approaches, we introduce the ratio

$$R_h(\alpha) = \frac{1}{\lambda^h_1 \varepsilon^{\alpha-1}} \int_{\Omega_{\varepsilon,h}} f G^{\star,h}_{\varepsilon,n} dx.$$

As far as the numerical results are concerned, let us begin with the Cartesian case, with the opening $3\pi/2$ ($\alpha = 2/3$). We choose the data:

$$f(x) = \begin{cases} 0, & |x| < 2, \\ \dfrac{1}{1+|x|^2}, & |x| \geq 2. \end{cases}$$

In other words, f vanishes in a neighborhood of the origin O. We consider the results obtained via the discretization resulting from the asymptotic expansion (3.9) and from the integral representation formula (4.5). So in Table 1, we give the value of the coefficient λ^h_1, the approximation of λ_1, the only term of (3.9) that depends on the data. Then, in Table 2, we provide the approximation of (4.5), scaled by the factor $\varepsilon^{\alpha-1}$ so that the result is independent of ε. Then, the ratio between the two computed values is reported in Table 3: it corresponds to the value of the coefficient $\partial_n y_1(O_1)$, which depends only on the geometry of the domain. Here, we give the

Table 1. Values of λ^h_1 for $\alpha = \frac{2}{3}$ and f vanishing near O.

	h			
α	0.2	0.1	0.05	0.025
2/3	1.077e−2	1.075e−2	1.076e−2	1.075e−2

Table 2. Values of $\varepsilon^{1-\alpha} \int_{\Omega_{\varepsilon,h}} f G^{\star,h}_{\varepsilon,n} dx$ for $\alpha = \frac{2}{3}$ and f vanishing near O.

	ε			
α	0.5	0.25	0.125	0.062
2/3	9.34e−2	9.29e−2	9.28e−2	9.28e−2

Table 3. Ratio $R_h(\alpha)$ for $\alpha = \frac{2}{3}, \frac{3}{4}$ and f vanishing near O.

	h			
α	0.5	0.25	0.125	0.062
2/3	−0.856	−0.868	−0.862	−0.862
3/4	−0.918	−0.883	−0.887	−0.890

Table 4. Ratio $R_h(\alpha)$ for $\alpha = \frac{2}{3}, \frac{3}{4}$ and f nonvanishing near O.

	h			
α	0.5	0.25	0.125	0.062
2/3	−0.500	−0.436	−0.394	−0.376
3/4	−0.411	−0.360	−0.336	−0.305

Table 5. Ratio $R_h(\nu_1)$ for $\beta = \frac{4}{3}$ and f rapidly decreasing at O.

	h			
β	0.5	0.25	0.125	0.062
4/3	−0.106	−0.106	−0.105	−0.105

results for two different values of the opening, namely $3\pi/2$ and $4\pi/3$. It appears that the numerical convergence is good (for a fixed α).

What happens numerically, if one considers some data, which does not vanish anymore near the origin O, and which is not rapidly decreasing? See the results in Table 4. We choose here

$$f(x) = \frac{1}{1 + |x|^2}.$$

Finally, we consider the axisymmetric case, with an opening corresponding to $\beta = 4/3$. We choose the data

$$f(x) = e^{-\frac{1}{|x|^2}}.$$

In other words, f is rapidly decreasing in a neighborhood of the origin O. See Table 5 for the results.

6. Conclusion

Peek's empirical formulas (0.6) and (0.7) are valid only around specific electrodes. Under the assumption that the data (physically, the charge density) vanishes in a neighborhood of the tip, we established a mathematical justification of those formulas for geometries that coincide at infinity with a cone. The case of a parabolic shape at the tip has been thoroughly studied, and extensions to other shapes are straightforward. In particular, we proved that the value of the normal derivative of the electric potential at the tip behaves like (cf. (3.9)):

$$\frac{\partial u_\varepsilon}{\partial n}(O_\varepsilon) = \lambda_1 \frac{\partial y_1}{\partial n}(O_1)\varepsilon^{\alpha-1} + O(\varepsilon^{2\alpha-1}).$$

In other words, it depends only of the curvature radius at the tip like $\varepsilon^{\alpha-1}$, the data (through λ_1), and the geometry of the domain (the coefficient $\partial_n y_1(O_1)$).

Numerical experiments corroborate the theoretical results.

In addition, we provided some information about the link between the behavior of the potential "close to the tip", and the behavior of the potential "far from the tip". They are related by a constant, which we call c_1, that depends only on the geometry of the domain (see Remark 3.4).

As far as other applications are concerned, we note that the case of a (bounded) wire could be handled similarly. Also, it is possible to extend our results to $2\frac{1}{2}$D geometries[11]: an axisymmetric tip with data that is not invariant by rotation, or a prismatic domain with rounded edge.

Finally, we mention that one derives easily from our framework that a phenomenon similar to a boundary layer occurs, but only near the tip (namely, for $|x| \lesssim \varepsilon$): a *corona layer.* Therefore, one can relate our work to the study of more classical boundary layers. We refer to Refs. 12 and 7 for results on this problem.

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