Mathematical Models and Methods in Applied Sciences

LAGRANGE MULTIPLIERS IN INTRINSIC ELASTICITY

PHILIPPE G. CIARLET*

Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong
mapgc@cityu.edu.hk

PATRICK CIARLET, Jr.

Laboratoire POEMS, Ecole Nationale Supérieure de Techniques Avancées, 32, Boulevard Victor, 75739 Paris cedex 15, France
ciarlet@ensta.fr

OANA IOSIFESCU

Département de Mathématiques, Université de Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex 5, France
iosifescu@math.univ-montp2.fr

STEFAN SAUTER

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland
stas@math.uzh.ch

JUN ZOU

Department of Mathematics, Lady Shaw Building, The Chinese University of Hong Kong, Shatin, Hong Kong
zou@math.cuhk.edu.hk

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In an intrinsic approach to three-dimensional linearized elasticity, the unknown is the linearized strain tensor field (or equivalently the stress tensor field by means of the constitutive equation), instead of the displacement vector field in the classical approach. We consider here the pure traction problem and the pure displacement problem and we show that, in each case, the intrinsic approach leads to a quadratic minimization problem constrained by Donati-like relations (the form of which depends on the type of boundary conditions considered). Using the Babuška-Brezzi inf-sup condition, we then show that, in each case, the minimizer of the constrained minimization problem found in an intrinsic approach is the first argument of the saddle-point of an *ad hoc* Lagrangian, so that the second argument of this saddle-point is the Lagrange multiplier associated with the corresponding constraints. Such results have potential applications to the nu-

*Corresponding author.

merical analysis and simulation of the intrinsic approach to three-dimensional linearized elasticity.

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1. Introduction

All notions, notations, etc., not defined here are defined in the next section.

Classically, the *displacement vector field* $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$ of the reference configuration $\overline{\Omega}$ of an elastic body is the unknown of choice appearing in the mathematical models of linearized and nonlinear three-dimensional elasticity. The minimization of the associated energy over an ad hoc function space (or the solution of the associated variational equations in the linear case), then provides a weak solution. When the data are smooth enough and the boundary condition is of the same type along the entire boundary of the reference configuration, the weak solution is also a classical solution of the associated boundary value problem.

In linearized elasticity, mixed models, where both the *displacement vector field and the stress tensor field* are unknowns, are also often used. One major reason is that their finite element discretizations provide approximations of the stress tensor field, the knowledge of which is of primary importance in computational mechanics.

Be that as it may, the displacement vector field is always the unknown, or one of the unknowns, in the *classical approach*.

Yet another approach is slowly coming of age, where the *strain tensor field*, or equivalently the *stress tensor field* by means of the constitutive equation, is the *only unknown*. The *idea* of such an approach, which bears the name of *intrinsic approach*, is not new: in nonlinear three-dimensional elasticity, it was first suggested, albeit only briefly, by Antman [3] in 1977; a similar idea for shells goes back even earlier to Synge & Chien [25] (see also Chien [7]), who already in 1941 advocated using the change of metric and change of curvature tensors as the primary unknowns. But it is only recently that the *mathematical analysis* and *numerical analysis* of the intrinsic approach to *three-dimensional elasticity* were undertaken, by Ciarlet & Ciarlet, Jr. [8,9], Ciarlet, Ciarlet, Jr. & Vicard [12] and Amrouche, Ciarlet, Gratie & Kesavan [2] in the linear case, and by Ciarlet & Mardare [13,14] in the nonlinear case. For *elastic shells*, see Opoka & Pietraszkiewicz [23].

Let us recall some terminology: In a *pure displacement problem*, the unknown displacement field $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$ is subjected to the *boundary condition of place* $\boldsymbol{u} = \boldsymbol{0}$ along the entire boundary $\Gamma = \partial\Omega$ (we only consider here homogeneous boundary conditions), whereas in a *pure traction problem*, no boundary condition of place is imposed on the field $\boldsymbol{u}$.

As shown in Section 3, one intrinsic approach to the *pure traction problem* of linearized elasticity takes the following form. Let

$$\mathbb{M} = \{\boldsymbol{\mu} \in \mathbb{L}^2_s(\Omega);\ \mathbf{div}\,\boldsymbol{\mu} = \boldsymbol{0} \text{ in } \boldsymbol{H}^{-1}(\Omega),\ \boldsymbol{\gamma}\boldsymbol{\mu} = 0 \text{ in } \boldsymbol{H}^{-1/2}(\Gamma)\},$$

where $\mathbb{L}^2_s(\Omega)$ denotes the space of all 3×3 symmetric tensor fields with components in $L^2(\Omega)$. Then one seeks a linearized strain tensor field $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ that satisfies

$$\boldsymbol{\varepsilon} \in \mathbb{E} = \{\boldsymbol{e} = (e_{ij}) \in \mathbb{L}^2_s(\Omega);\ \int_\Omega e_{ij}\mu_{ij}\,\mathrm{d}x = 0 \text{ for all } \boldsymbol{\mu} = (\mu_{ij}) \in \mathbb{M}\},$$
$$j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e}\in\mathbb{E}} j(\boldsymbol{e}), \quad \text{with } j(\boldsymbol{e}) = \frac{1}{2}\int_\Omega A_{ijk\ell}e_{k\ell}e_{ij}\,\mathrm{d}x - \ell(\boldsymbol{e}),$$

where $(A_{ijk\ell})$ is the uniformly positive-definite elasticity tensor of the elastic body, and ℓ is a continuous linear form over the space $\mathbb{E}$ that takes into account the applied body and surface forces.

As shown in Ref. 2, the *pure displacement problem* of linearized elasticity is likewise amenable to an *intrinsic approach.* This approach now takes the following form: Let

$$\mathbb{M}_0 = \{\boldsymbol{\mu} \in \mathbb{L}^2_s(\Omega);\ \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega)\}.$$

Then one now seeks a linearized strain tensor $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ that satisfies

$$\boldsymbol{\varepsilon} \in \mathbb{E}_0 = \{\boldsymbol{e} = (e_{ij}) \in \mathbb{L}^2_s(\Omega);\ \int_\Omega e_{ij}\mu_{ij}\,\mathrm{d}x = 0 \text{ for all } \boldsymbol{\mu} = (\mu_{ij}) \in \mathbb{M}_0\},$$
$$j_0(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e}\in\mathbb{E}_0} j_0(\boldsymbol{e}), \quad \text{with } j_0(\boldsymbol{e}) = \frac{1}{2}\int_\Omega A_{ijk\ell}e_{k\ell}e_{ij}\,\mathrm{d}x - \ell_0(\boldsymbol{e}),$$

where ℓ_0 is a continuous linear form over the space $\mathbb{E}_0$ that takes into account the applied body forces.

Each one of the above problems thus provides an example of a *quadratic minimization problem* ("quadratic" because the functional to be minimized is quadratic), subjected to *linear constraints* (in the sense of optimization theory), which take the form $\int_\Omega e_{ij}\mu_{ij}\,\mathrm{d}x = 0$ for all $\boldsymbol{\mu} = (\mu_{ij})$ in an *ad hoc* subspace of the space $\mathbb{L}^2_s(\Omega)$.

Note that an important feature of an *intrinsic approach* is the following: since the unknown *stresses* σ_{ij} inside the elastic body are given by means of the *constitutive equation*

$$\sigma_{ij} = A_{ijk\ell}\varepsilon_{k\ell}, \quad 1 \le i,\, j \le 3,$$

each one of the above minimization problems provides a *direct way of computing the stresses.*

The objective of this paper consists in identifying the *Lagrange multiplier* $\boldsymbol{\lambda}$ (again in the sense of optimization theory) that is in each case associated with the minimizer $\boldsymbol{\varepsilon}$ of the associated constrained minization problem. To this end, an essential usage is made of the *Babuška-Brezzi inf-sup condition*, so named after Babuška [4] and Brezzi [5], which allows to identify $(\boldsymbol{\varepsilon}, \boldsymbol{\lambda})$ as the saddle-point of an *ad hoc Lagrangian* (these results are reviewed in Section 2).

As indicated in Section 7, such results have potential applications for the numerical analysis and simulation of the intrinsic approach.

The results of this paper were announced in Ref. 11.

2. Notations and Preliminaries

Throughout this article, Latin indices vary in the set $\{1, 2, 3\}$, save when they are used for indexing sequences, and the summation convention with respect to repeated indices is systematically used in conjunction with this rule.

All the vector spaces, matrices, etc., considered in this article are real. Let V denote a normed vector space with norm $\|\cdot\|_V$. Given a closed subspace Z of V, the equivalence class of $v \in V$ in the quotient space $\dot{V} := V/Z$ is denoted $\dot{v}$ and its norm is defined by $\|\dot{v}\|_{\dot{V}} := \inf_{z \in Z} \|v + z\|_V$. The notation V^* designates the dual space of V and ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denotes the duality bracket between V^* and V.

Let U and V denote two vector spaces and let $A : U \to V$ be a linear operator. Then $\mathbf{Ker}\, A \subset U$ and $\mathrm{Im}\, A \subset V$ respectively designate the kernel and the image of A.

Let Ω be an open subset of $\mathbb{R}^3$ and let $x = (x_i)$ designate a generic point in Ω. Partial derivative operators of the first, second, and third order are then denoted $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$. The same symbols also designate partial derivatives in the sense of distributions.

Spaces of functions, vector fields in $\mathbb{R}^3$, and 3×3 matrix fields, defined over Ω are respectively denoted by italic capitals, boldface Roman capitals, and special Roman capitals. The space of all symmetric matrices of order 3 is denoted $\mathbb{S}^3$. The subscript s appended to a special Roman capital denotes a space of symmetric matrix fields.

Notations such as $C^m(\Omega)$, $m \geq 0$, and $C^\infty(\overline{\Omega})$ denote the usual spaces of continuously differentiable functions. The notation $D(\Omega)$ denotes the space of functions that are infinitely differentiable in Ω and have compact supports in Ω. The notation $D'(\Omega)$ denotes the space of distributions defined over Ω. The notations $H^m(\Omega)$, $m \in \mathbb{Z}$, with $H^0(\Omega) = L^2(\Omega)$, and $H_0^m(\Omega)$, $m \geq 1$, designate the usual Sobolev spaces.

Combining the above rules, we shall thus encounter spaces such as $D'(\Omega)$, $\boldsymbol{D}'(\Omega)$, $\mathbb{D}'(\Omega)$, $\mathbb{L}_s^2(\Omega)$, $\mathbb{H}_{0,s}^1(\Omega)$, etc.

The notation $(\boldsymbol{v})_i$ designates the i-th component of a vector $\boldsymbol{v} \in \mathbb{R}^3$ and the notation $\boldsymbol{v} = (v_i)$ means that $v_i = (\boldsymbol{v})_i$. The notation $(\boldsymbol{A})_{ij}$ designates the element at the i-th row and j-th column of a square matrix $\boldsymbol{A}$ of order three and the notation $\boldsymbol{A} = (a_{ij})$ means that $a_{ij} = (\boldsymbol{A})_{ij}$. The inner-product and vector product of $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$ are denoted $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$. The notation $\boldsymbol{s} : \boldsymbol{t} := s_{ij} t_{ij}$ designates the matrix inner-product of two matrices $\boldsymbol{s} := (s_{ij})$ and $\boldsymbol{t} := (t_{ij})$ of order three.

The *divergence operator* $\mathbf{div} : \boldsymbol{D}'(\Omega) \to D'(\Omega)$ is defined by

$$\mathrm{div}\, \boldsymbol{v} := \partial_i v_i \quad \text{for any } \boldsymbol{v} = (v_i) \in \boldsymbol{D}'(\Omega).$$

The *matrix gradient operator* $\boldsymbol{\nabla} : \boldsymbol{D}'(\Omega) \to \mathbb{D}'(\Omega)$ is defined by

$$(\boldsymbol{\nabla} \boldsymbol{v})_{ij} := \partial_j v_i \quad \text{for any } \boldsymbol{v} = (v_i) \in \boldsymbol{D}'(\Omega).$$

For any vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{D}'(\Omega)$, the associated *linearized strain tensor* is the symmetric matrix field $\boldsymbol{\nabla}_s \boldsymbol{v} \in \mathbb{D}'_s(\Omega)$ defined by

$$\boldsymbol{\nabla}_s \boldsymbol{v} := \frac{1}{2}(\boldsymbol{\nabla} \boldsymbol{v}^T + \boldsymbol{\nabla} \boldsymbol{v}),$$

or equivalently, by

$$(\boldsymbol{\nabla}_s \boldsymbol{v})_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i).$$

When Ω is connected, the kernel of the operator $\boldsymbol{\nabla}_s$ has the well-known characterization (see, e.g., Theorem 6.3-4 in Ref. 8), viz.,

$$\mathbf{Ker}\, \boldsymbol{\nabla}_s = \{\boldsymbol{v} \in \boldsymbol{D}'(\Omega);\ \boldsymbol{\nabla}_s \boldsymbol{v} = \mathbf{0} \text{ in } \mathbb{D}'(\Omega)\} = \{\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b} \wedge \mathbf{id}_\Omega;\ \boldsymbol{a} \in \mathbb{R}^3,\ \boldsymbol{b} \in \mathbb{R}^3\},$$

where $\mathbf{id}_\Omega$ denotes the identity mapping of the set Ω.

The *vector divergence operator* $\mathbf{div} : \mathbb{D}'(\Omega) \to \mathbf{D}'(\Omega)$ is defined by

$$(\mathbf{div}\, \boldsymbol{\mu})_i := \partial_j \mu_{ij} \quad \text{for any } \boldsymbol{\mu} = (\mu_{ij}) \in \mathbb{D}'(\Omega).$$

The *matrix Laplacian* $\Delta : \mathbb{D}'(\Omega) \to \mathbb{D}'(\Omega)$ is defined by

$$(\Delta \boldsymbol{e})_{ij} := \Delta e_{ij} \quad \text{for any } \boldsymbol{e} = (e_{ij}) \in \mathbb{D}'(\Omega).$$

Finally, a *domain in* $\mathbb{R}^3$ is a bounded, connected, open subset of $\mathbb{R}^3$ whose boundary is Lipschitz-continuous, the set Ω being locally on a single side of its boundary (see, e.g., Nečas [20]).

To conclude these preliminaries, we state the well-known *Babuška-Brezzi inf-sup theorem*, so named after Babuška [4] and Brezzi [5] (a proof is also found in Brezzi & Fortin [6]). Together with two corollaries (Theorems 2.2 and 2.3) this fundamental result will pervade the rest of this article. For convenience, we state these results in a form and with notations that are directly adapted to our purposes. The inequality involving the constant β constitutes the *Babuška-Brezzi inf-sup condition.*

Theorem 2.1. *Let $\mathbb{V}$ and $\mathbb{M}$ be two Hilbert spaces, let $\ell : \mathbb{V} \to \mathbb{R}$ be a continuous linear form, and let $a(\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and $b : \mathbb{V} \times \mathbb{M} \to \mathbb{R}$ be two continuous bilinear forms with the following properties: There exist constants $\alpha > 0$ and $\beta > 0$ such that*

$$a(\boldsymbol{e}, \boldsymbol{e}) \geq \alpha \,\|\boldsymbol{e}\|^2_{\mathbb{V}} \quad \text{for all } \boldsymbol{e} \in \mathbb{E} := \{\boldsymbol{e} \in \mathbb{V};\ b(\boldsymbol{e}, \boldsymbol{\mu}) = 0 \text{ for all } \boldsymbol{\mu} \in \mathbb{M}\},$$

$$\inf_{\begin{cases}\boldsymbol{\mu}\in\mathbb{M}\\ \boldsymbol{\mu}\neq\mathbf{0}\end{cases}} \sup_{\begin{cases}\boldsymbol{e}\in\mathbb{V}\\ \boldsymbol{e}\neq\mathbf{0}\end{cases}} \frac{|b(\boldsymbol{e}, \boldsymbol{\mu})|}{\|\boldsymbol{e}\|_{\mathbb{V}}\, \|\boldsymbol{\mu}\|_M} \geq \beta.$$

Then the variational problem: Find $(\boldsymbol{\varepsilon}, \boldsymbol{\lambda}) \in \mathbb{V} \times \mathbb{M}$ such that

$$\begin{aligned} a(\boldsymbol{\varepsilon}, \boldsymbol{e}) + b(\boldsymbol{e}, \boldsymbol{\lambda}) &= \ell(\boldsymbol{e}) \quad \text{for all } \boldsymbol{e} \in \mathbb{V}, \\ b(\boldsymbol{\varepsilon}, \boldsymbol{\mu}) &= 0 \qquad \text{for all } \boldsymbol{\mu} \in \mathbb{M}, \end{aligned}$$

has one and only one solution.

Under a mild additional assumption on the bilinear form $a(\cdot,\cdot)$, Theorem 2.1 provides a way to solve a specific class of *constrained quadratic minimization problems*, precisely of the form considered here (for a proof, see Brezzi & Fortin [6] or Girault & Raviart [18]).

Theorem 2.2. *Let the assumptions be as in* Theorem 2.1*, and assume in addition that the bilinear form $a(\cdot,\cdot)$ is symmetric, i.e., $a(\boldsymbol{\varepsilon},\boldsymbol{e}) = a(\boldsymbol{e},\boldsymbol{\varepsilon})$ for all $\boldsymbol{\varepsilon},\boldsymbol{e} \in \mathbb{V}$.*

Then $(\boldsymbol{\varepsilon},\boldsymbol{\lambda}) \in \mathbb{V}\times\mathbb{M}$ is the unique solution to the variational problem of Theorem 2.1 *if and only if $\boldsymbol{\varepsilon} \in \mathbb{E}$ and $\boldsymbol{\varepsilon}$ is the unique solution to the constrained quadratic minimization problem*

$$j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e}\in\mathbb{E}} J(\boldsymbol{e}), \quad \text{where } j(\boldsymbol{e}) := \frac{1}{2}a(\boldsymbol{e},\boldsymbol{e}) - \ell(\boldsymbol{e}) \text{ for all } \boldsymbol{e} \in \mathbb{V}.$$

Under a slightly stronger assumption on the bilinear form $a(\cdot,\cdot)$, the solution $(\boldsymbol{\varepsilon},\boldsymbol{\lambda}) \in \mathbb{V}\times\mathbb{M}$ to the variational problem of Theorem 2.1 can be also characterized as the *saddle-point* of an appropriate *Lagrangian* $\mathscr{L}$ (for a proof, see again Brezzi & Fortin [6] or Girault & Raviart [18]).

Theorem 2.3. *Let the assumptions be as in* Theorem 2.1*, and assume in addition that the bilinear form $a(\cdot,\cdot)$ is symmetric and nonnegative-definite on $\mathbb{V}$, i.e.,*

$$a(\boldsymbol{\varepsilon},\boldsymbol{e}) = a(\boldsymbol{e},\boldsymbol{\varepsilon}) \quad \text{and} \quad a(\boldsymbol{e},\boldsymbol{e}) \geq 0 \quad \text{for all } \boldsymbol{\varepsilon},\boldsymbol{e} \in \mathbb{V}.$$

Then the unique solution $(\boldsymbol{\varepsilon},\boldsymbol{\lambda}) \in \mathbb{V}\times\mathbb{M}$ to the variational problem of Theorem 2.1 *is the unique saddle-point of the Lagrangian $\mathscr{L} : \mathbb{V}\times\mathbb{M} \to \mathbb{R}$ defined by*

$$\mathscr{L}(\boldsymbol{e},\boldsymbol{\mu}) := \frac{1}{2}a(\boldsymbol{e},\boldsymbol{e}) - \ell(\boldsymbol{e}) + b(\boldsymbol{e},\boldsymbol{\mu}) \quad \text{for all } (\boldsymbol{e},\boldsymbol{\mu}) \in \mathbb{V}\times\mathbb{M},$$

i.e.,

$$\inf_{\boldsymbol{e}\in\mathbb{V}} \sup_{\boldsymbol{\mu}\in\mathbb{M}} \mathscr{L}(\boldsymbol{e},\boldsymbol{\mu}) = \sup_{\boldsymbol{\mu}\in\mathbb{M}} \mathscr{L}(\boldsymbol{\varepsilon},\boldsymbol{\mu}) = \mathscr{L}(\boldsymbol{\varepsilon},\boldsymbol{\lambda}) = \inf_{\boldsymbol{e}\in\mathbb{V}} \mathscr{L}(\boldsymbol{e},\boldsymbol{\lambda}) = \sup_{\boldsymbol{\mu}\in\mathbb{M}} \inf_{\boldsymbol{e}\in\mathbb{V}} \mathscr{L}(\boldsymbol{e},\boldsymbol{\mu}).$$

In the language of optimization theory, $\boldsymbol{\lambda} \in \mathbb{M}$ is the *Lagrange multiplier* associated with the *constrained quadratic minimization problem* of Theorem 2.2. The object of this paper is to identify such Lagrange multipliers, associated with constrained quadratic minimization problems that arise in intrinsic three-dimensional linearized elasticity.

3. An Intrinsic Approach to the Pure Traction Problem

Let a *domain* Ω in $\mathbb{R}^3$, with boundary Γ, be the reference configuration of a linearly elastic body, characterized by its elasticity tensor field $\mathbf{A} = (A_{ijk\ell})$ with components $A_{ijk\ell} \in L^\infty(\Omega)$. It is assumed as usual that these components satisfy the symmetry relations $A_{ijk\ell} = A_{jik\ell} = A_{k\ell ij}$, and that there exists a constant $\alpha > 0$ such that

$$\mathbf{A}(x)\boldsymbol{t} : \boldsymbol{t} \geq \alpha \boldsymbol{t} : \boldsymbol{t} \quad \text{for almost all } x \in \Omega \text{ and all matrices } \boldsymbol{t} = (t_{ij}) \in \mathbb{S}^3,$$

where $(\boldsymbol{A}(x)\boldsymbol{t})_{ij} := A_{ijk\ell}(x)t_{k\ell}$. The body is subjected to applied body forces with density $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$ in its interior and to applied surface forces of density $\boldsymbol{g} \in \boldsymbol{L}^{4/3}(\Gamma)$ on its boundary. Finally, it is assumed that the linear form $L \in \mathscr{L}(\boldsymbol{H}^1(\Omega); \mathbb{R})$ defined by

$$L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{v} \, \mathrm{d}\Gamma \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$$

vanishes for all $\boldsymbol{v} \in \mathbf{Ker}\, \boldsymbol{\nabla}_s$, where

$$\mathbf{Ker}\, \boldsymbol{\nabla}_s := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \boldsymbol{\nabla}_s \boldsymbol{v} = \mathbf{0} \text{ in } \mathbb{L}^2_s(\Omega)\}.$$

Then the corresponding *pure traction problem of three-dimensional linearized elasticity* classically consists in finding $\dot{\boldsymbol{u}} \in \dot{\boldsymbol{H}}^1(\Omega) := \boldsymbol{H}^1(\Omega)/\, \mathbf{Ker}\, \boldsymbol{\nabla}_s$ such that

$$J(\dot{\boldsymbol{u}}) = \inf_{\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)} J(\dot{\boldsymbol{v}}), \quad \text{where } J(\dot{\boldsymbol{v}}) := \frac{1}{2} \int_\Omega \mathbf{A} \boldsymbol{\nabla}_s \dot{\boldsymbol{v}} : \boldsymbol{\nabla}_s \dot{\boldsymbol{v}} \, \mathrm{d}x - L(\dot{\boldsymbol{v}}).$$

As is well known (see, e.g., Theorem 3.4 in Duvaut & Lions [15]), this minimization problem has one and only one solution.

An *intrinsic approach* to the above pure traction problem consists in considering the linearized strain tensor field $\boldsymbol{\varepsilon} := \boldsymbol{\nabla}_s \dot{\boldsymbol{u}} : \Omega \to \mathbb{S}^3$ as the primary unknown, instead of the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ itself ($\dot{\boldsymbol{u}} \in \dot{\boldsymbol{H}}^1(\Omega)$ is unique; hence $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ is unique up to the addition of any vector field in $\mathbf{Ker}\, \boldsymbol{\nabla}_s$) in the "classical" approach.

Accordingly, one first needs to characterize those 3×3 matrix fields $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$ that can be written as $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ for some vector fields $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. The characterization given in the next theorem is not the only possible one; others, which are briefly reviewed in Section 7, are equally possible, but, as explained in *ibid.*, they do not seem to be suitable for our purposes.

To begin with, we recall some functional analytic preliminaries due to Geymonat & Krasucki [16,17], which are the "matrix analogs" of results of Girault & Raviart [18] for spaces of vector fields (see Section 2.2 in Chapter 1 in *ibid.*). Given a domain Ω in $\mathbb{R}^3$, define the space

$$\mathbb{H}(\mathbf{div}; \Omega) := \{\boldsymbol{\mu} \in \mathbb{L}^2_s(\Omega);\ \mathbf{div}\, \boldsymbol{\mu} \in \boldsymbol{L}^2(\Omega)\},$$

which is thus the matrix analog of the familiar space $\boldsymbol{H}(\mathbf{div}; \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega); \mathbf{div}\, \boldsymbol{v} \in L^2(\Omega)\}$ (in this definition $\mathbf{div}\, \boldsymbol{\mu}$ is of course to be understood in the sense of distributions). Equipped with the norm defined by

$$\|\boldsymbol{\mu}\|_{\mathbb{H}(\mathbf{div};\Omega)} := \left(\|\boldsymbol{\mu}\|^2_{\mathbb{L}^2_s(\Omega)} + \|\, \mathbf{div}\, \boldsymbol{\mu}\|^2_{\boldsymbol{L}^2(\Omega)} \right)^{1/2} \quad \text{for all } \boldsymbol{\mu} \in \mathbb{H}(\mathbf{div}; \Omega),$$

the space $\mathbb{H}(\mathbf{div}; \Omega)$ is a Hilbert space. Let $\boldsymbol{\nu} : \Gamma \to \mathbb{R}^3$ denote the unit outer normal vector field along the boundary of Ω (such a field is thus defined dΓ-everywhere). The set Ω being a domain, the density of the space $\mathbb{C}^\infty_s(\overline{\Omega})$ in the space $\mathbb{H}(\mathbf{div}; \Omega)$

then implies that the mapping $\boldsymbol{\mu} \in \mathbb{C}_s^\infty(\overline{\Omega}) \to \boldsymbol{\mu\nu}|_\Gamma$ can be extended to a continuous linear mapping

$$\boldsymbol{\gamma} : \mathbb{H}(\mathbf{div};\Omega) \to \boldsymbol{H}^{-1/2}(\Gamma).$$

Note that the notations $\mathbb{M}$ and $\mathbb{E}$ used below are the same as those used in Theorem 2.1, but they now designate *specific* function spaces. Note also that the next result was already alluded to in Ref. 10.

Theorem 3.1. *Let Ω be a domain in $\mathbb{R}^3$ and let there be given a matrix field $\boldsymbol{e} \in \mathbb{L}_s^2(\Omega)$. Then there exists a vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ if and only if*

$$\int_\Omega \boldsymbol{e} : \boldsymbol{\mu} \, \mathrm{d}x = 0 \quad \text{for all } \boldsymbol{\mu} \in \mathbb{M},$$

where the space $\mathbb{M}$ is defined as

$$\mathbb{M} := \{\boldsymbol{\mu} \in \mathbb{L}_s^2(\Omega); \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega),\ \boldsymbol{\gamma\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1/2}(\Gamma)\}.$$

All other vector fields $\widetilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ that satisfy $\boldsymbol{e} = \boldsymbol{\nabla}_s \widetilde{\boldsymbol{v}}$ are of the form $\widetilde{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{a} + \boldsymbol{b} \wedge \mathbf{id}_\Omega$ for some vectors $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$.

Proof. Geymonat & Krasucki [16] have shown that the space

$$\boldsymbol{\mathcal{V}} := \{\boldsymbol{\mu} \in \mathbb{D}_s(\Omega); \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \Omega\}$$

is dense in the space $\mathbb{M}$ with respect to the norm of the space $\mathbb{L}_s^2(\Omega)$ (this result is the matrix analog of Theorem 2.8, Chapter 1, of Ref. 18). Hence the space

$$\boldsymbol{\mathcal{W}} := \{\boldsymbol{\mu} \in \mathbb{H}_{0,s}^1(\Omega); \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{L}^2(\Omega)\}$$

is likewise dense in the space $\mathbb{M}$ with respect to the norm of the space $\mathbb{L}_s^2(\Omega)$, since $\boldsymbol{\mathcal{V}} \subset \boldsymbol{\mathcal{W}} \subset \mathbb{M}$.

If a tensor field $\boldsymbol{e} \in \mathbb{L}_s^2(\Omega)$ satisfies $\int_\Omega \boldsymbol{e} : \boldsymbol{\mu} \, \mathrm{d}x = 0$ for all $\boldsymbol{\mu} \in \mathbb{M}$, it thus *a fortiori* satisfies $\int_\Omega \boldsymbol{e} : \boldsymbol{\mu} \, \mathrm{d}x = 0$ for all $\boldsymbol{\mu} \in \boldsymbol{\mathcal{W}}$. The existence of $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ then follows from Theorem 4.3 of Amrouche, Ciarlet, Gratie & Kesavan [2].

If conversely, a tensor field $\boldsymbol{e} \in \mathbb{L}_s^2(\Omega)$ is of the form $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, then the Green's formula (see Ref. 16)

$$\int_\Omega \boldsymbol{\mu} : \boldsymbol{\nabla}_s \boldsymbol{v} \, \mathrm{d}x + \int_\Omega (\mathbf{div}\,\boldsymbol{\mu}) \cdot \boldsymbol{v} \, \mathrm{d}x = {}_{\boldsymbol{H}^{-1/2}(\Gamma)} \langle \boldsymbol{\gamma\mu}, \boldsymbol{v} \rangle_{\boldsymbol{H}^{1/2}(\Gamma)},$$

which holds for all $\boldsymbol{\mu} \in \mathbb{H}(\mathbf{div};\Omega)$ and all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, shows that

$$\int_\Omega \boldsymbol{e} : \boldsymbol{\mu} \, \mathrm{d}x = \int_\Omega \boldsymbol{\nabla}_s \boldsymbol{v} : \mu \, \mathrm{d}x = 0 \quad \text{for all } \mu \in \mathbb{M}.$$

The characterization of $\mathbf{Ker}\,\boldsymbol{\nabla}_s$ (Section 2) shows that all the other solutions $\widetilde{\boldsymbol{v}} \in \boldsymbol{H}^1(\Omega)$ to the equation $\boldsymbol{e} = \boldsymbol{\nabla}_s \widetilde{\boldsymbol{v}}$ are indeed of the announced form. □

Thanks to Theorem 3.1, the pure traction problem of three-dimensional elasticity can be recast as *a constrained quadratic minimization problem*, with $\boldsymbol{\varepsilon} := \boldsymbol{\nabla}_s \dot{\boldsymbol{u}} \in \mathbb{L}^2_s(\Omega)$ as the primary unknown. Note that this minimization problem could be in turn immediately recast as yet another one, this time with the *linearized stress tensor* $\mathbf{A}\boldsymbol{\varepsilon}$ as the primary unknown, since the elasticity tensor field $\mathbf{A}$ is invertible almost everywhere in Ω.

Theorem 3.2. *Let Ω be a domain in $\mathbb{R}^3$, and let the space $\mathbb{M}$ be defined as in* Theorem 3.1. *Define the Hilbert space*

$$\mathbb{E} := \{\boldsymbol{e} \in \mathbb{L}^2_s(\Omega); \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\, \mathrm{d}x = 0 \text{ for all } \boldsymbol{\mu} \in \mathbb{M}\},$$

and, for each $\boldsymbol{e} \in \mathbb{E}$, let $\boldsymbol{\mathcal{F}}(\boldsymbol{e})$ denote the unique element in the quotient space $\dot{\boldsymbol{H}}^1(\Omega)$ that satisfies $\boldsymbol{\nabla}_s \boldsymbol{\mathcal{F}}(\boldsymbol{e}) = \boldsymbol{e}$ (Theorem 3.1)*. Then the mapping $\boldsymbol{\mathcal{F}} : \mathbb{E} \to \dot{\boldsymbol{H}}^1(\Omega)$ defined in this fashion is an isomorphism between the Hilbert spaces $\mathbb{E}$ and $\dot{\boldsymbol{H}}^1(\Omega)$.*

The minimization problem: Find $\boldsymbol{\varepsilon} \in \mathbb{E}$ such that

$$j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e} \in \mathbb{E}} j(\boldsymbol{e}), \quad \text{where } j(\boldsymbol{e}) := \frac{1}{2} \int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\, \mathrm{d}x - L \circ \boldsymbol{\mathcal{F}}(\boldsymbol{e}),$$

has one and only one solution $\boldsymbol{\varepsilon}$, and this solution satisfies $\boldsymbol{\varepsilon} = \boldsymbol{\nabla}_s \dot{\boldsymbol{u}}$, where $\dot{\mathbf{u}}$ is the unique minimizer of the functional J in the space $\dot{\boldsymbol{H}}^1(\Omega)$.

Proof. By Theorem 3.1, the mapping $\boldsymbol{\mathcal{F}}$ is a bijection between the Hilbert spaces $\mathbb{E}$ and $\dot{\boldsymbol{H}}^1(\Omega)$. Furthermore, its inverse is continuous since there evidently exists a constant c such that

$$\|\boldsymbol{\nabla}_s(\boldsymbol{v} + \boldsymbol{r})\|_{\mathbb{L}^2_s(\Omega)} \le c\, \|\boldsymbol{v} + \boldsymbol{r}\|_{\mathbf{H}^1(\Omega)}$$

for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and any $\boldsymbol{r} \in \mathbf{Ker}\, \boldsymbol{\nabla}_s$, so that

$$\|\boldsymbol{\nabla}_s(\dot{\boldsymbol{v}})\|_{\mathbb{L}^2_s(\Omega)} \le c \inf_{\boldsymbol{r} \in \mathbf{Ker}\, \boldsymbol{\nabla}_s} \|\boldsymbol{v} + \boldsymbol{r}\|_{\boldsymbol{H}^1(\Omega)} = c\, \|\dot{\boldsymbol{v}}\|_{\dot{\boldsymbol{H}}^1(\Omega)}$$

for all $\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega)$.

Hence $\boldsymbol{\mathcal{F}} : \mathbb{E} \to \mathbf{H}^1(\Omega)$ is an isomorphism by the Banach open mapping theorem.

The bilinear form $(\boldsymbol{e}, \widetilde{\boldsymbol{e}}) \in \mathbb{E} \times \mathbb{E} \to \int_\Omega \mathbf{A}\boldsymbol{e} : \widetilde{\boldsymbol{e}}\, \mathrm{d}x \in \mathbb{R}$ and the linear form $\Lambda := L \circ \boldsymbol{\mathcal{F}} : \mathbb{E} \to \mathbb{R}$ thus satisfy all the assumptions of the Lax-Milgram lemma (Λ is continuous since $\boldsymbol{\mathcal{F}}$ is an isomorphism). Consequently, there exists one, and only one, minimizer $\boldsymbol{\varepsilon}$ of the functional j over $\mathbb{E}(\Omega)$. That $\dot{\boldsymbol{u}}$ minimizes the functional J over $\dot{\boldsymbol{H}}^1(\Omega)$ implies that $\boldsymbol{\nabla}_s \dot{\boldsymbol{u}}$ minimizes the functional j over $\mathbb{E}$. Hence $\boldsymbol{\varepsilon} = \boldsymbol{e}(\dot{\boldsymbol{u}})$ since the minimizer is unique. □

Remark 3.1. A proof similar to that of the corollary to Theorem 4.1 in Ref. 8 shows that the *Korn inequality in the space* $\dot{\boldsymbol{H}}^1(\Omega)$ can then be recovered as a simple corollary to Theorem 3.2, which thus provides an entirely new proof of this classical inequality.

4. A Lagrangian Approach to the Pure Traction Problem

We now identify the *Lagrangian*, and consequently the *Lagrange multiplier* (as the second argument of the saddle-point of the Lagrangian), associated with the constrained quadratic minimization problem of Theorem 3.2. The spaces $\mathbb{M}$ and $\mathbb{E}$ defined in the next theorem are those defined in Theorems 3.1 and 3.2.

Theorem 4.1. *Define the spaces*

$$\mathbb{V} := \mathbb{L}^2_s(\Omega) \quad \text{and} \quad \mathbb{M} := \{\boldsymbol{\mu} \in \mathbb{L}^2_s(\Omega); \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega),\, \boldsymbol{\gamma\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1/2}(\Gamma)\},$$

and the Lagrangian

$$\mathcal{L}(\boldsymbol{e}, \boldsymbol{\mu}) := \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x - \ell(\boldsymbol{e}) + \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x \quad \text{for all } (\boldsymbol{e}, \boldsymbol{\mu}) \in \mathbb{V} \times \mathbb{M},$$

where $\ell : \mathbb{L}^2_s(\Omega) \to \mathbb{R}$ is any continuous linear extension of the continuous linear form $L \circ \mathcal{F} : \mathbb{E} \to \mathbb{R}$, where

$$\mathbb{E} := \{\boldsymbol{e} \in \mathbb{L}^2_s(\Omega); \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \text{ for all } \boldsymbol{\mu} \in \mathbb{M}\}.$$

Then the Lagrangian $\mathcal{L}$ has a unique saddle-point $(\boldsymbol{\varepsilon}, \boldsymbol{\lambda}) \in \mathbb{V} \times \mathbb{M}$ over the space $\mathbb{V} \times \mathbb{M}$. Besides, the first argument $\boldsymbol{\varepsilon}$ of the saddle-point is the unique solution of the minimization problem of Theorem 3.2*, i.e.,*

$$\boldsymbol{\varepsilon} \in \mathbb{E}(\Omega) \quad \text{and} \quad j(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e} \in \mathbb{E}} j(\boldsymbol{e}),$$

and the second argument $\boldsymbol{\lambda} \in \mathbb{M}$ of the saddle-point is a Lagrange multiplier associated with this minimization problem.

Proof. To start with, it is clear that both $\mathbb{V}$ and $\mathbb{M}$ are Hilbert spaces.

Define two bilinear forms $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and $b : \mathbb{V} \times \mathbb{M} \to \mathbb{R}$ by

$$a(\boldsymbol{\varepsilon}, \boldsymbol{e}) := \int_\Omega \mathbf{A}\boldsymbol{\varepsilon} : \boldsymbol{e}\,\mathrm{d}x \quad \text{for all } (\boldsymbol{\varepsilon}, \boldsymbol{e}) \in \mathbb{V} \times \mathbb{V},$$

$$b(\boldsymbol{e}, \boldsymbol{\mu}) := \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x \quad \text{for all } (\boldsymbol{e}, \boldsymbol{\mu}) \in \mathbb{V} \times \mathbb{M}.$$

Clearly, both bilinear forms are continuous. Besides, the bilinear form $a(\cdot,\cdot)$ is symmetric on $\mathbb{V}$, and $\mathbb{V}$-coercive since

$$a(\boldsymbol{e}, \boldsymbol{e}) = \int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x \geq \alpha\, \|\boldsymbol{e}\|^2_{\mathbb{V}} \quad \text{for all } \boldsymbol{e} \in \mathbb{V}$$

(the constant $\alpha > 0$ is that appearing in the uniform positive-definiteness of the elasticity tensor field $\mathbf{A}$).

Finally, the Babuška-Brezzi inf-sup condition follows from the inclusion $\mathbb{M} \subset \mathbb{V}$, which implies that, for each $\boldsymbol{\mu} \in \mathbb{M}$,

$$\sup_{\begin{cases}\boldsymbol{e} \in \mathbb{V} \\ \boldsymbol{e} \neq \mathbf{0}\end{cases}} \frac{\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x}{\|\boldsymbol{e}\|_{\mathbb{L}^2(\Omega)}\, \|\boldsymbol{\mu}\|_{\mathbb{L}^2(\Omega)}} \geq \frac{\int_\Omega \boldsymbol{\mu} : \boldsymbol{\mu}\,\mathrm{d}x}{\|\boldsymbol{\mu}\|^2_{\mathbb{L}^2(\Omega)}} = 1.$$

Hence all the announced assertions follows from Theorems 2.1 to 2.3. □

5. An Intrinsic Approach to the Pure Displacement Problem

Consider now the *pure displacement problem of three-dimensional linearized elasticity*, which classically consists in finding $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ such that

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)} J(\boldsymbol{v}), \quad \text{where } J(\boldsymbol{v}) := \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{\nabla}_s\boldsymbol{v} : \boldsymbol{\nabla}_s\boldsymbol{v}\,\mathrm{d}x - L(\boldsymbol{v}),$$

where

$$L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$$

for some given body force density $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$ (no extra condition needs to be imposed on the linear form L in this case, since $\mathbf{Ker}\,\boldsymbol{\nabla}_s = \{\mathbf{0}\}$ in $\boldsymbol{H}_0^1(\Omega)$).

An *intrinsic approach* to the above pure displacement problem consists again in considering the linearized strain tensor $\boldsymbol{\varepsilon} := \boldsymbol{\nabla}_s\boldsymbol{u} : \Omega \to \mathbb{S}^3$ as the primary unknown, instead of the displacement vector field $\boldsymbol{u} : \Omega \to \mathbb{R}^3$. Accordingly, we need to characterize those 3×3 matrix fields $\boldsymbol{e} \in \mathbb{L}_s^2(\Omega)$ that can be written as $\boldsymbol{e} = \boldsymbol{\nabla}_s\boldsymbol{v}$ for some vector field $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$. The following result, established in Theorem 4.2 of Ref. 2, constitutes such a characterization.

Theorem 5.1. *Let Ω be a domain in $\mathbb{R}^3$ and let there be given a matrix field $\boldsymbol{e} \in \mathbb{L}_s^2(\Omega)$. Then there exists a vector field $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ such that $\boldsymbol{e} = \boldsymbol{\nabla}_s\boldsymbol{v}$ if and only if*

$$\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \quad \text{for all } \boldsymbol{\mu} \in \mathbb{M}_0,$$

where the space $\mathbb{M}_0$ is defined as

$$\mathbb{M}_0 := \{\boldsymbol{\mu} \in \mathbb{L}_s^2(\Omega); \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega)\}.$$

If this is the case, the vector field $\boldsymbol{v}$ is unique.

Thanks to Theorem 5.1, this problem can be again recast as *a constrained quadratic minimization problem* with $\boldsymbol{\varepsilon} := \boldsymbol{\nabla}_s\boldsymbol{u} \in \mathbb{L}_s^2(\Omega)$ as the primary unknown:

Theorem 5.2. *Let Ω be a domain in $\mathbb{R}^3$, and let the space $\mathbb{M}_0$ be defined as in* Theorem 5.1. *Define the Hilbert space*

$$\mathbb{E}_0 := \{\boldsymbol{e} \in \mathbb{L}_s^2(\Omega); \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \text{ for all } \boldsymbol{\mu} \in \mathbb{M}_0\},$$

and, for each $\boldsymbol{e} \in \mathbb{E}_0$, let $\boldsymbol{\mathcal{F}}_0(\boldsymbol{e})$ denote the unique element in the space $\boldsymbol{H}_0^1(\Omega)$ that satisfies $\boldsymbol{\nabla}_s\boldsymbol{\mathcal{F}}_0(\boldsymbol{e}) = \boldsymbol{e}$ (Theorem 5.1). *Then the mapping $\boldsymbol{\mathcal{F}}_0 : \mathbb{E}_0 \to \boldsymbol{H}_0^1(\Omega)$ defined in this fashion is an isomorphism between the Hilbert spaces $\mathbb{E}_0$ and $\boldsymbol{H}_0^1(\Omega)$.*

The minimization problem: Find $\boldsymbol{\varepsilon} \in \mathbb{E}_0$ such that

$$j_0(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e} \in \mathbb{E}_0} j_0(\boldsymbol{e}), \quad \text{where } j_0(\boldsymbol{e}) := \frac{1}{2}\int_\Omega \boldsymbol{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x - L \circ \boldsymbol{\mathcal{F}}_0(\boldsymbol{e}),$$

has one and only one solution $\boldsymbol{\varepsilon}$, and this solution satisfies $\boldsymbol{\varepsilon} = \boldsymbol{\nabla}_s \boldsymbol{u}$, where $\boldsymbol{u}$ is the unique minimizer of the functional J in the space $\boldsymbol{H}_0^1(\Omega)$.

Proof. The proof is similar to that of Theorem 3.2 and, for this reason, omitted. $\square$

6. A Lagrangian Approach to the Pure Displacement Problem

We now identify the *Lagrangian* and *Lagrange multiplier* associated with the constrained quadratic minimization problem of Theorem 5.2. The spaces $\mathbb{M}_0$ and $\mathbb{E}_0$ defined in the next theorem are those defined in Theorems 5.1 and 5.2.

Theorem 6.1. *Define the spaces*

$$\mathbb{V} := \mathbb{L}_s^2(\Omega) \quad \text{and} \quad \mathbb{M}_0 := \{\boldsymbol{\mu} \in \mathbb{L}_s^2(\Omega); \mathbf{div}\, \boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega)\},$$

and the Lagrangian

$$\mathcal{L}_0(\boldsymbol{e}, \boldsymbol{\mu}) := \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\, \mathrm{d}x - \ell_0(\boldsymbol{e}) + \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\, \mathrm{d}x \quad \text{for all } (\boldsymbol{e}, \boldsymbol{\mu}) \in \mathbb{V} \times \mathbb{M}_0,$$

where $\ell_0 : \mathbb{L}_s^2(\Omega) \to \mathbb{R}$ is any continuous linear extension of the continuous linear form $L \circ \mathcal{F}_0 : \mathbb{E}_0 \to \mathbb{R}$, where

$$\mathbb{E}_0 := \{\boldsymbol{e} \in \mathbb{L}_s^2(\Omega); \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\, \mathrm{d}x = 0 \text{ for all } \boldsymbol{\mu} \in \mathbb{M}_0\}.$$

Then the Lagrangian $\mathcal{L}_0$ has a unique saddle-point $(\boldsymbol{\varepsilon}, \boldsymbol{\lambda}) \in \mathbb{V} \times \mathbb{M}_0$ over the space $\mathbb{V} \times \mathbb{M}_0$. Besides, the first argument $\boldsymbol{\varepsilon}$ of the saddle-point is the unique solution of the minimization problem of Theorem 5.2*, i.e.,*

$$\boldsymbol{\varepsilon} \in \mathbb{E}_0(\Omega) \quad \text{and} \quad j_0(\boldsymbol{\varepsilon}) = \inf_{\boldsymbol{e} \in \mathbb{E}_0} j_0(\boldsymbol{e}),$$

and the second argument $\boldsymbol{\lambda} \in \mathbb{M}_0$ of the saddle-point is a Lagrange multiplier associated with this minimization problem.

Proof. The proof is similar to that of Theorem 4.1 and for this reason, only sketched. Let again the spaces $\mathbb{V}$ and $\mathbb{M}_0$ be both equipped with the norm of the space $\mathbb{L}_s^2(\Omega)$. Hence both $\mathbb{V}$ and $\mathbb{M}_0$ are Hilbert spaces (it is clear that $\mathbb{M}_0$ is closed in $\mathbb{V}$).

Define two bilinear forms $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and $b : \mathbb{V} \times \mathbb{M}_0 \to \mathbb{R}$ by

$$a(\boldsymbol{\varepsilon}, \boldsymbol{e}) := \int_\Omega \mathbf{A}\boldsymbol{\varepsilon} : \boldsymbol{e}\, \mathrm{d}x \quad \text{for all } (\boldsymbol{\varepsilon}, \boldsymbol{e}) \in \mathbb{V} \times \mathbb{V},$$

$$b(\boldsymbol{e}, \boldsymbol{\mu}) := \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\, \mathrm{d}x \quad \text{for all } (\boldsymbol{e}, \boldsymbol{\mu}) \in \mathbb{V} \times \mathbb{M}_0.$$

Then both bilinear forms are continuous. Besides, the bilinear form $a(\cdot, \cdot)$ is symmetric on $\mathbb{V}$, and $\mathbb{V}$-coercive. Finally, the Babuška-Brezzi condition holds since $\mathbb{M}_0 \subset \mathbb{V}$. $\square$

7. Miscellaneous Remarks

(a) Let Ω be an open subset of $\mathbb{R}^3$. The question of characterizing those symmetric matrix fields $\boldsymbol{e} = (e_{ij})$ that can be written over Ω as $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ for some vector field $\boldsymbol{v}$, has been arousing considerable interest for quite a long time. Indeed A.J.C.B. de Saint Venant announced as early as 1864 what is since then known as *Saint Venant's theorem* (in fact, it was not until 1886 that E. Beltrami provided a rigorous proof of this result): *Assume that the open set Ω is simply-connected. Then there exists a vector field $\boldsymbol{v} \in \boldsymbol{C}^3(\Omega)$ such that $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ in Ω if* (and clearly only if, even if Ω is not simply-connected) *the functions e_{ij} are in the space $C^2(\Omega)$ and they satisfy*

$$R_{ijk\ell}(\boldsymbol{e}) := \partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0 \quad \text{in } \Omega \text{ for all } i, j, k, \ell \in \{1, 2, 3\}.$$

For any matrix field $\boldsymbol{e} = (e_{ij}) \in \mathbb{D}'(\Omega)$, the matrix field $\mathbf{CURLCURL}\, \boldsymbol{e} \in \mathbb{D}'(\Omega)$ is defined by

$$(\mathbf{CURLCURL}\, \boldsymbol{e})_{ij} := \varepsilon_{ik\ell} \varepsilon_{jmn} \partial_{\ell n} e_{km},$$

where (ε_{ijk}) denotes the orientation tensor.

It is then easily seen that the *Saint Venant compatibility conditions* $R_{ijk\ell}(\boldsymbol{e}) = 0$ in Ω are equivalent to the relations

$$\mathbf{CURLCURL}\, \boldsymbol{e} = \mathbf{0} \quad \text{in } \Omega;$$

besides the matrix field $\mathbf{CURLCURL}\, \boldsymbol{e}$ is always symmetric. Hence the eighty-one relations $R_{ijk\ell}(\boldsymbol{e}) = 0$ reduce in effect to six relations only.

The Saint Venant compatibility conditions have been recently shown to remain sufficient under substantially weaker regularity assumptions. More specifically, Ciarlet & Ciarlet, Jr. [8] have established the following *Saint Venant theorem in $\mathbb{L}^2_s(\Omega)$*: *Let Ω be a simply-connected domain in $\mathbb{R}^3$ and let $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$ be a matrix field that satisfies the Saint Venant compatibility conditions $\mathbf{CURLCURL}\, \boldsymbol{e} = \mathbf{0}$ in $\mathbb{H}^{-2}_s(\Omega)$. Then there exists a vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$* in $\mathbb{L}^2_s(\Omega)$. Further extensions, to Sobolev spaces of weaker regularity, have since then been given, in Refs. 1 and 2.

A natural question arises as to whether a Lagrange multiplier can be associated with the constraint $\mathbf{CURLCURL}\, \boldsymbol{e} = \mathbf{0}$ in $\mathbb{H}^{-2}_s(\Omega)$. In this direction, one can prove (cf. Theorem 2.3 in Ref. 10), that given $\boldsymbol{\mu} \in \mathbb{M}$, there exists a symmetric *tensor potential* $\boldsymbol{w} \in \mathbb{H}^2_{0,s}(\Omega)$, such that $\boldsymbol{\mu} = \mathbf{CURLCURL}\, \boldsymbol{w}$ in Ω. Therefore, in the case of the pure traction problem, a *Lagrange multiplier* can be associated with the constraint $\mathbf{CURLCURL}\, \boldsymbol{e} = \mathbf{0}$ in $\mathbb{H}^{-2}_s(\Omega)$. Let us denote by $\mathbb{K}$ the kernel of the operator $\mathbf{CURLCURL} : \mathbb{H}^2_{0,s}(\Omega) \to \mathbb{L}^2_s(\Omega)$.

With the same space $\mathbb{V} = \mathbb{L}^2_s(\Omega)$ and bilinear form $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ as in Theorem 4.1, natural candidates for the Lagrange multiplier space $\widetilde{\mathbb{M}}$ and bilinear form $b : \mathbb{V} \times \widetilde{\mathbb{M}} \to \mathbb{R}$ are then:

$$\widetilde{\mathbb{M}} = \mathbb{H}^2_{0,s}(\Omega)/\mathbb{K} \text{ and } b(\boldsymbol{e}, \boldsymbol{\mu}) = \int_\Omega \boldsymbol{e} : \mathbf{CURLCURL}\,\boldsymbol{\mu}\,\mathrm{d}x \text{ for all } (\boldsymbol{e}, \boldsymbol{\mu}) \in \mathbb{L}^2_s(\Omega) \times \widetilde{\mathbb{M}}.$$

Let $\mathbb{K}_0$ denote the kernel of the operator $\mathbf{CURLCURL} : \mathbb{L}^2_s(\Omega) \to \mathbb{H}^{-2}(\Omega)$ and let $\mathbb{K}_0^\perp$ denote its orthogonal complement in $\mathbb{L}^2_s(\Omega)$. Then we can define the operators $\boldsymbol{B}^* := \mathbf{CURLCURL} : \widetilde{\mathbb{M}} \to \operatorname{Im} \boldsymbol{B}^* \subset \mathbb{L}^2_s(\Omega)$, and $\boldsymbol{B} := \mathbf{CURLCURL} : \mathbb{K}_0^\perp \to \widetilde{\mathbb{M}}'$. Thanks to the Saint Venant theorem in $\mathbb{L}^2_s(\Omega)$ and to the existence of potentials, we obtain $\mathbb{K}_0 = \operatorname{Im} \boldsymbol{B}^*$. Hence the Babuška-Brezzi inf-sup condition is equivalent to the condition that $\boldsymbol{B}^*$ is an isomorphism between $\mathbb{M}$ and $\mathbb{K}_0^\perp$. But, according to the above, $\boldsymbol{B}^*$ is clearly a linear, one-to-one and surjective operator (by construction), which is continuous, so we conclude that $\boldsymbol{B}^*$ is an isomorphism.

(b) In 1890, L. Donati proved that, *if Ω is an open subset of $\mathbb{R}^3$ and the components e_{ij} of a symmetric matrix field $\boldsymbol{e} = (e_{ij})$ are in the space $C^2(\Omega)$ and they satisfy:*

$$\int_\Omega e_{ij}\mu_{ij}\,\mathrm{d}x = 0 \quad \text{for all } \boldsymbol{\mu} = (\mu_{ij}) \in \mathbb{D}_s(\Omega) \text{ such that } \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \Omega,$$

then $\mathbf{CURLCURL}\,\boldsymbol{e} = \mathbf{0}$ in Ω. This result, known as *Donati's theorem*, thus provides, once combined with Saint Venant's theorem, another characterization of symmetric matrix fields as linearized strain tensor fields for simply-connected open subsets Ω of $\mathbb{R}^3$.

A first extension of Donati's theorem was given in 1974 by Ting [26]: *Let Ω be a domain in $\mathbb{R}^3$. If a tensor field $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$ satisfies*

$$\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \quad \text{for all } \boldsymbol{\mu} \in \mathbb{D}_s(\Omega) \text{ such that } \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \Omega,$$

then there exists $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ *such that* $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ *in* $\mathbb{L}^2_s(\Omega)$.

Then Moreau [19] showed in 1979 that Donati's theorem holds even in the sense of distributions, according to the following theorem, where Ω is now an arbitrary open subset of $\mathbb{R}^3$: *If the components e_{ij} of a tensor field $\boldsymbol{e} \in \mathbb{D}'_s(\Omega)$ satisfy*

$$_{D'(\Omega)}\langle e_{ij}, s_{ij}\rangle_{D(\Omega)} = 0 \quad \text{for all } \boldsymbol{s} = (s_{ij}) \in \mathbb{D}_s(\Omega) \text{ such that } \mathbf{div}\,\boldsymbol{s} = \mathbf{0} \text{ in } \Omega,$$

then there exists a vector field $\boldsymbol{v} \in \boldsymbol{D}'(\Omega)$ such that $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ in the sense of distributions. Note that Ting's and Moreau's extensions, or more generally, the other extensions of Donati's theorem considered here, *do not* require that Ω be simply-connected.

A further extension of Donati's theorem was given in Theorem 4.3 in Ref. 2, where it was shown that, given $\boldsymbol{e} \in \mathbb{L}^2_s(\Omega)$, there exists $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{e} = \boldsymbol{\nabla}_s \boldsymbol{v}$ in $\mathbb{L}^2_s(\Omega)$ if and only if

$$\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \quad \text{for all } \boldsymbol{\mu} \in \mathbb{H}^1_{0,s}(\Omega) \text{ such that } \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{L}^2(\Omega).$$

By contrast with that of Theorem 6.1, this extension does not seem to be amenable to a Lagrangian approach, however.

Note that Theorems 3.1 and 5.1 likewise constitute yet other extensions of Donati's theorem, the latter producing a vector field $\boldsymbol{v}$ in the space $\boldsymbol{H}_0^1(\Omega)$.

(c) Regarding the *mechanical interpretation* of the Lagrange multipliers, we refer to Podio-Guidugli[24], where it is shown that they are in effect *"reactive stress" fields*. This means that the various types of constraints that have been imposed there on the tensor fields $\boldsymbol{e} \in \mathbb{L}_s^2(\Omega)$ can be regarded as "internal constraints" on the admissible strains, and that these constraints are precisely maintained by such reactive stress fields.

(d) A key issue consists in devising efficient *numerical approximation* to intrinsic elasticity problems. One approach, which was first analyzed in Ref. 9 and then successfully implemented in Ref. 12 for problems of *planar* elasticity, consists in constructing *"edge" finite elements* (in the sense of Nédélec [21,22]), in such a way that the Saint-Venant compatibility conditions are *exactly satisfied* in the resulting finite element space.

The present paper should thus pave the way for a different class of approaches, where the unknown to be approximated is the saddle-point found in either Theorem 4.1 or Theorem 6.1.

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