Finite Element Heterogeneous Multiscale Method for the Helmholtz Equation

Patrick Ciarlet Jr., Christian Stohrer

Laboratoire POEMS, UMA, ENSTA ParisTech, 828, Boulevard des Maréchaux, 91762 Palaiseau Cedex, France

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Abstract

We show that standard Finite Element Heterogeneous Multiscale Method (FE-HMM) can be used to approximate the effective behavior of solutions of the classical Helmholtz equation in highly oscillatory media. Using a novel combination of well known results about FE-HMM and the notion of T-coercivity we derive an a priori error bound. Numerical experiments corroborate the analytical findings.

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Résumé

Nous montrons que la méthode multi-échelle hétérogène d'éléments finis (FE-HMM) peut être utilisée pour approcher le comportement effectif des solutions de l'équation de Helmholtz classique dans des milieux rapidement oscillants. A l'aide de cette méthode et de la notion de T-coercivité nous établissons une borne a priori de l'erreur. Des expériences numériques corroborent les résultats théoriques.

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Les méthodes multi-échelles hétérogènes introduites en 2003 [6] permettent de résoudre des problèmes faisant intervenir des coefficients rapidement oscillants. L'idée est d'utiliser un solveur macro – résolvant l'équation effective globale – couplé à un solveur micro qui permet de reconstruire des phénomènes petite échelle, absents de l'équation effective globale. Cette méthode permet de résoudre une large gamme de problèmes, voir [2]. Cependant, l'analyse de l'équation de Helmholtz n'a pas été réalisée jusqu'à présent.

Email addresses: `patrick.ciarlet@ensta-paristech.fr` (Patrick Ciarlet Jr.), `christian.stohrer@ensta-paristech.fr` (Christian Stohrer).

Pour rappel, l'équation est (1), avec a^ε et ρ^ε des coefficients rapidement oscillants (ε une longueur caractéristique de variation de a^ε et ρ^ε), k le nombre d'onde et f la donnée. Le phénomène est dit multi-échelle lorsque $\varepsilon \ll \operatorname{diam}\Omega$, où Ω est le domaine de calcul dans $\mathbb{R}^d$; nous faisons cette hypothèse par la suite. En outre, nous supposons que a^ε est un tenseur symétrique, respectivement ρ^ε un scalaire, qui vérifient (2). Lorsque les constantes $\alpha, \beta > 0$ sont indépendantes de la position, on parle de modèle classique. On réécrit l'équation (1) sous forme variationnelle, cf. (3), avec $(\cdot,\cdot)$ le produit scalaire usuel dans $L^2(\Omega)$, $(v,w)_\varepsilon = (\rho^\varepsilon v, w)$ et $B^\varepsilon(v,w) = (a^\varepsilon(x)\nabla v, \nabla w)$. Lorsque ε tend vers 0, la solution u^ε tend-elle vers u^0, la solution de l'équation de Helmholtz homogénéisée (4) ? Ici $B^0(v,w) = (a^0(x)\nabla v, \nabla w)$, où a^0 est le tenseur obtenu par G-convergence [7] et $(v,w)_0 = (\rho^0 v, w)$. Sous les hypothèses (2), on peut démontrer [7] que les valeurs propres du modèle exact tendent vers celles du modèle homogénéisé. Ainsi, si k^2 n'est pas valeur propre de (4), k^2 n'est pas valeur propre (3) pour ε suffisamment petit. En outre, on peut prouver rigoureusement que u^ε tend vers u^0 à l'aide de la convergence double échelle (voir [3]) si $a^\varepsilon(x) = a(x, x/\varepsilon)$. L'approximation directe du modèle homogénéisé est trop coûteuse, à cause de l'évaluation du tenseur homogénéisé $a^0(x)$ en tout x. La FE-HMM est une alternative. Supposons que le domaine Ω est polyédrique, et introduisons des triangulations régulières $(\mathcal{T}_H)_H$ de Ω en simplexes ou hexaèdres de diamètre maximal H, ainsi que les espaces d'éléments finis conformes $S^\ell(\Omega, \mathcal{T}_H)$ et $S_0^\ell(\Omega, \mathcal{T}_H)$, cf. (5). Sur l'élément de référence $\hat{K}$ on choisit une formule de quadrature compatible avec l'ordre des éléments finis, avec des nœuds et poids $(\hat{x}_j, \hat{\omega}_j)_j$ que l'on transporte sur la triangulation : pour chaque $K \in \mathcal{T}_H$ et tout j, soit $x_{K,j} = F_K(\hat{x}_j)$, où F_K est la transformation affine telle que $K = F_K(\hat{K})$. On note enfin $I_\delta = x_{K,j} + \delta\,[-1/2, 1/2]^d$ le domaine d'échantillonnage autour de chaque nœud de quadrature, avec $S^q(I_\delta, \mathcal{T}_h)$ l'espace d'éléments finis défini sur une (micro-)triangulation $\mathcal{T}_h$. La FE-HMM, de solution $u_H \in S_0^\ell(\Omega, \mathcal{T}_H)$, s'écrit sous la forme (6). Si on introduit les formes discrètes B_H^0 et $(\cdot,\cdot)_{0,H}$ (7), on peut établir les résultats de stabilité (8)-(9), où e_{HMM} est appelée l'erreur HMM. On en déduit ensuite le résultat de convergence principal

Théorème 0.1 *Supposons a^0 régulier et $\lim_{H\to 0} e_{\mathrm{HMM}} = \lim_{H\to 0} e_\rho = 0$. Alors, pour H suffisamment petit la norme $\left\|u^0 - u_H\right\|_{H^1}$ de l'erreur entre la solution u^0 du problème homogénéisé et la solution u_H de la FE-HMM est bornée par* (10).

Nous proposons enfin des illustrations numériques de la FE-HMM. En 1D, avec la solution u^0 gouvernée par (11) : $\left\|u^0 - u_H\right\|_{H^1} \leq C(H^\ell + (h/\varepsilon)^{2q})$ puisque u^0 est régulière. Ces résultats sont confirmés au tableau 1. En 2D, on constate sur la figure 1 que, même si la triangulation macro ne permet pas de résoudre les petites échelles, la solution u_H est proche de la solution du problème homogénéisé u^0.

1. Introduction

Heterogeneous Multiscale Methods (HMM), a general concept for the design of multiscale algorithms, were introduced in 2003 [6]. The main ingredients of an HMM scheme are a macro solver for the solution of an overall effective equation and a micro solver for the solution of micro problems used for the computation of missing data in the effective model. Since the size of the micro problem scales with the characteristic microscopic length scale of the medium, the computational load of an HMM does not depend on the fineness of the original multiscale problem. Implementations of FE-HMM, where standard FE are used for the macro and the micro solver, have been successfully applied to solve various homogenization problems in the elliptic, parabolic, and hyperbolic cases as well as more complicated problems like advection-diffusion and elasticity; see the review article [2].

Surprisingly, there seems to be a gap on the theory for the homogenization of the Helmholtz equation

$$-\nabla \cdot (a^\varepsilon(x)\nabla u^\varepsilon(x)) - k^2 \rho^\varepsilon(x) u^\varepsilon(x) = f(x), \qquad x \in \Omega \tag{1}$$

where $a^\varepsilon\colon \Omega \to \mathbb{R}^{d\times d}$ and $\rho^\varepsilon\colon \Omega \to \mathbb{R}$ describing the composite material, the wave number k, and the source term $f: \Omega \to \mathbb{R}$ are given. Note that ε is a characteristic length scale on which a^ε and ρ^ε vary, e.g.

for a periodic medium ε is the period. Equation (1) is a multiscale problem, when $\varepsilon \ll \operatorname{diam}\Omega$, where Ω is the computational domain. Our goal is to recover the macroscopic behavior of the solution neglecting microscopic oscillations.

We review the homogenization results in Section 2 and describe our method in Section 3. But because (1) is not elliptic, the a priori error analysis is more involved, see Section 4. We conclude by some numerical experiments.

2. Homogenization of the Helmholtz equation

We assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. Furthermore, let $a^\varepsilon \in (L^\infty(\Omega))^{d\times d}$, $\rho^\varepsilon \in L^2(\Omega)$ and assume that there are $\alpha, \beta > 0$ independent of ε, such that

$$\begin{aligned} a^\varepsilon(x) \text{ is symmetric,} \quad & \alpha\,|\xi|^2 \le a^\varepsilon(x)\xi\cdot\xi \le \beta\,|\xi|^2\,, \quad \forall\xi\in\mathbb{R}^d, \text{ for almost every } x\in\Omega, \\ & \alpha \le \rho^\varepsilon(x) \le \beta, \qquad\qquad\qquad \text{for almost every } x\in\Omega. \end{aligned} \tag{2}$$

We consider from now on the Helmholtz equation with homogeneous Dirichlet boundary condition in its variational form.

$$\text{Find } u^\varepsilon \in H_0^1(\Omega), \text{ such that} \quad B^\varepsilon(u^\varepsilon, v) - k^2(u^\varepsilon, v)_\varepsilon = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{3}$$

where $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$-inner product, $(v,w)_\varepsilon = (\rho^\varepsilon v, w)$, and $B^\varepsilon(v,w) = (a^\varepsilon(x)\nabla v, \nabla w)$ for $v, w \in H_0^1(\Omega)$.

For a fixed ε it is well known, that (3) is uniquely solvable, if and only if k^2 does not lie in the spectrum of B^ε with $(\cdot,\cdot)_\varepsilon$ as the $L^2(\Omega)$-inner product. However, since we consider a series of $\varepsilon > 0$ converging to 0, k^2 should not be an eigenvalue of any B^ε for ε small enough. This is in fact possible, since the i-th eigenvalue λ_i^ε of $B^\varepsilon(u_i^\varepsilon, v) = \lambda_i^\varepsilon(u_i^\varepsilon, v)_\varepsilon$ converges to the i-th eigenvalue λ_i^0 of $B^0(u_i^0, v) = \lambda_i^0(u_i^0, v)_0$, where $B^0(v,w) = (a^0(x)\nabla v, \nabla w)$ and $(v,w)_0 = (\rho^0 v, w)$ for $v, w \in H_0^1(\Omega)$, with a^0 the G-limit and ρ^0 the weak L^2-limit of the sequences $\{a^\varepsilon\}$, resp. $\{\rho^\varepsilon\}$, see [7].

The limiting sequence $\{\lambda_i^0\}_{i\ge1}$ for $\varepsilon \to 0$ constitutes the entire spectrum of B^0 with $(\cdot,\cdot)_0$ as $L^2(\Omega)$-inner product. Hence, if k^2 is no eigenvalue of B^0 there is a threshold value $\tilde\varepsilon > 0$ such that k^2 is not an eigenvalue of B^ε for all $0 < \varepsilon < \tilde\varepsilon$.

Given such a k^2, we consider now the series of $\{u^\varepsilon\}_{\varepsilon>0}$ of solutions of (3) for $\varepsilon \to 0$. Homogenization theory suggests that u^ε converges to u^0, where u^0 is the unique solution of the homogenized Helmholtz equation

$$\text{Find } u^0 \in H_0^1(\Omega), \text{ such that} \quad B^0(u^0, v) - k^2(u^0, v)_0 = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{4}$$

For a material with explicit scale separation into a slow varying macro scale and a periodic micro scale, this result can be proven using the notion of two-scale convergence [3].

3. FE Heterogeneous Multiscale Method

The final FE-HMM for the Helmholtz equation is closely related with standard FE-HMM for elliptic problems. Thus, we give only a brief description, more details can e.g. be found in [1]. For simplicity let Ω be a polytope with $d \le 3$. We consider shape regular triangulations $(\mathcal{T}_H)_H$ of Ω into simplicial or quadrilateral elements K of maximal diameter H. On $\mathcal{T}_H$ we define the conforming FE spaces

$$S^\ell(\Omega, \mathcal{T}_H) = \left\{ v_H \in H^1(\Omega); v_H|_K \in \mathcal{R}^\ell(K),\ \forall K \in \mathcal{T}_H \right\} \text{ and } S_0^\ell(\Omega, \mathcal{T}_H) = S^\ell(\Omega, \mathcal{T}_H) \cap H_0^1(\Omega), \tag{5}$$

where $\mathcal{R}^\ell(K)$ is the space $\mathcal{P}^\ell(K)$ of polynomials on K of total degree at most ℓ if K is a simplicial element, or $\mathcal{R}^\ell(K)$ is the space $\mathcal{Q}^\ell(K)$ of polynomials on K of degree at most ℓ in each variable if K is a quadrilateral element. Every element K in $\mathcal{T}_H$ is assumed affine equivalent to a reference element $\hat{K}$, either the unit simplex or the unit square. The associated affine mapping is denoted by $F_K\colon \hat{K} \to K$.

On the reference element $\hat{K}$ we choose a quadrature formula given by its nodes and weights, $(\hat{x}_j, \hat{\omega}_j)$ for $j = 1, \dots, J$. We assume that the quadrature formula satisfies standard assumptions to not degrade the expected order of convergence H^ℓ, see e.g. [5, Section 4.1]. Using the map F_K, a quadrature formula on $K \in \mathcal{T}_H$ is given by $x_{K,j} = F_K(\hat{x}_j)$ and $\omega_{K,j} = \det(DF_K)\hat{\omega}_j$ for $j = 1, \dots, J$. Let $I = [-1/2, 1/2]^d$ and denote by $I_\delta = I_\delta(x_{K,j}) = x_{K,j} + \delta I$ the sampling domain around the quadrature node with diameter $\delta \geq \varepsilon$. Every sampling domain is partitioned into a micro triangulation $\mathcal{T}_h$ and we consider the micro FE space $S^q(I_\delta, \mathcal{T}_h)$. We can now formulate the FE-HMM for the Helmholtz equation

$$\text{Find } u_H \in S_0^\ell(\Omega, \mathcal{T}_H), \text{ such that} \quad B_H(u_H, v_H) - k^2(u_H, v_H)_H = (f, v_H), \quad \forall v_H \in S_0^\ell(\Omega, \mathcal{T}_H)$$

$$\begin{aligned} \text{where} \quad B_H(v_H, w_H) &= \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K,j}}{|I_\delta|} \int_{I_\delta} a^\varepsilon(x) \nabla v_h(x) \cdot \nabla w_h(x)\, dx \\ \text{and} \quad (u_H, v_H)_H &= \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{K,j}}{|I_\delta|} \left(\int_{I_\delta} \rho^\varepsilon(x)\, dx \right) v_H(x_{K,j})\, w_H(x_{K,j}); \end{aligned} \tag{6}$$

above, v_h (resp. w_h) solves the micro problem with $v_{H,\text{lin}}(x) = v_H(x_{K,j}) + \nabla v_H(x_{K,j}) \cdot (x - x_{K,j}),\ x \in I_\delta$

$$\text{Find } v_h \in v_{H,\text{lin}} + S^q_{\text{per}}(I_\delta, \mathcal{T}_h), \text{such that} \quad \int_{I_\delta} a^\varepsilon(x) \nabla v_h \cdot \nabla z_h\, dx = 0, \quad \forall z_h \in S^q_{\text{per}}(I_\delta, \mathcal{T}_h).$$

Using the micro FE space $S^q_{\text{per}}(I_\delta, \mathcal{T}_h) = \{v_h \in S^q(I_\delta, \mathcal{T}_h) \cap H^1_{\text{per}}(I_\delta) : \int_{I_\delta} v_h\, dx = 0\}$, we apply a periodic coupling between macro and micro solvers. Other coupling conditions, e.g. Dirichlet coupling by replacing $S^q_{\text{per}}(I_\delta, \mathcal{T}_h)$ with $S^q_0(I_\delta, \mathcal{T}_h)$, could be used as well.

4. A priori error analysis

For the error analysis we combine the results from FE-HMM theory with the notion of T-coercivity, to realize the inf-sup condition numerically for the classical Helmholtz equation [4]. The error of an HMM scheme is usually decomposed into an error from the macro solver and the HMM error. Therefore, we introduce the discretized homogenized bilinear form and scalar product

$$\begin{aligned} B_H^0(v_H, w_H) &= \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \omega_{K,j} a^0(x_{K,j}) \nabla v_H(x_{K,j}) \cdot \nabla w_H(x_{K,j}), \\ (v_H, w_H)_{0,H} &= \sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \omega_{K,j} \rho^0(x_{K,j}) v_H(x_{K,j}) w_H(x_{K,j}), \end{aligned} \qquad \text{for } v_H, w_H \in S^\ell(\Omega, \mathcal{T}_H), \tag{7}$$

Following standard FEM error estimate, e.g. [5], we have for $a^0, \rho^0 \in W^{\ell,\infty}(\Omega)$,

$$\begin{aligned} \left| B^0(v_H, w_H) - B^0_H(v_H, w_H) \right| &\leq C H^\ell \max_{i,j} \left\| a^0_{ij} \right\|_{W^{\ell,\infty}} \|v_H\|_{H^1} \|w_H\|_{H^1}, \\ |(v_H, w_H)_0 - (v_H, w_H)_{0,H}| &\leq C H^\ell \left\| \rho^0 \right\|_{W^{\ell,\infty}} \|v_H\|_{L^2} \|w_H\|_{L^2}, \end{aligned} \qquad \text{for } v_H, w_H \in S^\ell(\Omega, \mathcal{T}_H), \tag{8}$$

where the constant $C > 0$ is independent of H, v_H, and w_H. On the other hand, the difference between B^0_H and the FE-HMM bilinear B_H, respectively between $(\cdot,\cdot)_{0,H}$ and $(\cdot,\cdot)_H$ can be controlled as

$$\begin{aligned} \left| B^0_H(v_H, w_H) - B_H(v_H, w_H) \right| &\leq C e_{\text{HMM}} \|v_H\|_{H^1} \|w_H\|_{H^1} \\ |(v_H, w_H)_{0,H} - (v_H, w_H)_H| &\leq C e_\rho \|v_H\|_{L^2} \|w_H\|_{L^2} \end{aligned} \qquad \forall v_H, w_H \in S^\ell(\Omega, \mathcal{T}_H), \tag{9}$$

where C is independent of H, v_H, and w_H, e_{HMM} is independent of v_H and w_H, and

$$e_\rho := \sup_{K,j} \Big(\frac{1}{|I_\delta|} \int_{I_\delta} \rho^\varepsilon(x)\, dx - \rho^0(x_{K,j}) \Big).$$

In the following we assume that $\lim_{H\to 0} e_{\text{HMM}} = \lim_{H\to 0} e_\rho = 0$, for what it is sufficient, that $a^\varepsilon(x)$ and $\rho^\varepsilon(x)$ are εI-periodic, $\delta \in \varepsilon\mathbb{N}$, and the macro and micro mesh are refined simultaneously.

In the subsequent analysis we follow [4], where T-coercivity was applied to classical Helmholtz-like problems. To this end we introduce the bilinear forms associated with the Helmholtz equation.

$$\mathfrak{B}^0(v, w) = B^0(v, w) - k^2(v, w)_0, \text{ and } \mathfrak{B}_H(v_H, w_H) = B_H(v_H, w_H) - k^2(v_H, w_H)_H,$$

for $v, w \in H_0^1(\Omega)$ and $v_H, w_H \in S_0^\ell(\Omega, \mathcal{T}_H)$, which are no longer elliptic. To compensate this drawback note that $\mathfrak{B}^0$ is T^0-coercive, i.e. there is a linear map T^0 and $\alpha^0 > 0$ such that

$$\left|\mathfrak{B}^0(v, T^0 v)\right| \ge \alpha^0 \|v\|_{H^1} \qquad \forall v \in H_0^1(\Omega).$$

Here one chooses $T^0 = Id - 2P^-$, where P^- is the orthogonal projection from $H_0^1(\Omega)$ to its subspace spanned by the eigenvectors of B^0 with eigenvalues $\lambda_i^0 < k^2$. Furthermore we consider for $H > 0$ its approximation $T_H = Id - 2P_H^-$, where P_H^- is the orthogonal projection from $S_0^\ell(\Omega, \mathcal{T}_H)$ to the subspace spanned by the approximations of the same eigenvalues in $S_0^\ell(\Omega, \mathcal{T}_H)$; see [4] for details. Let $|||\cdot|||$ denote the operator norm over $H_0^1(\Omega)$.

Lemma 4.1 *Let a^0 be smooth enough and assume that $\lim_{H\to 0} e_{\text{HMM}} = \lim_{H\to 0} e_\rho = 0$. For H sufficiently small the family of bilinear forms $\mathfrak{B}_H$ is uniformly T_H-coercive. This means, that there exist $\alpha^*, \beta^*, \tilde{H} > 0$ and for all $0 < H < \tilde{H}$ a linear operator $T_H \colon S^\ell(\Omega, \mathcal{T}_H) \to S^\ell(\Omega, \mathcal{T}_H)$, such that*

$$|\mathfrak{B}_H(v_H, T_H v_H)| \ge \alpha^* \|v_H\|_{H^1}^2 \text{ and } |||T_H||| \le \beta^*.$$

PROOF. The boundedness of T_H is proven in [4]. Hence, it remains to show the uniform coercivity. Due to the linearity of the bilinear form and the triangular inequality we have

$$\begin{aligned} |\mathfrak{B}_H(v_H, T_H v_H)| &\ge \left|\mathfrak{B}^0(v_H, T^0 v_H)\right| - \left|\mathfrak{B}^0(v_H, (T^0 - T_H) v_H)\right| - \left|(\mathfrak{B}_H - \mathfrak{B}^0)(v_H, T_H v_H)\right| \\ &\ge \alpha^0 \|v_H\|_{H^1}^2 - (C + k^2) \left|\left|\left|T^0 - T_H\right|\right|\right| \|v_H\|_{H^1}^2 - C\beta^*((1 + k^2)H^\ell + e_{\text{HMM}} + e_\rho) \|v_H\|_{H^1}^2, \end{aligned}$$

where we used the T^0-coercivity of $\mathfrak{B}^0$, the boundedness of B^0 and T_H, and the error bounds (8) and (9) in the second inequality. Since $\left|\left|\left|T^0 - T_H\right|\right|\right|, e_{\text{HMM}}, e_\rho \to 0$ as $H \to 0$ the lemma follows. □

Now our main theorem follows directly from [4, Theorem 2]

Theorem 4.2 *Under the same assumptions as above, we have for sufficiently small H*

$$\begin{aligned} \left\|u^0 - u_H\right\|_{H^1} \le C \inf_{v_H \in S^\ell(\Omega, \mathcal{T}_H)} \Bigg(&\left\|u^0 - v_H\right\|_{H^1} \\ &+ \sup_{\substack{w_H \in S^\ell(\Omega, \mathcal{T}_H) \\ w_H \ne 0}} \frac{\left|B^0(v_H, w_H) - B_H(v_H, w_H)\right|}{\|w_H\|_{H^1}} + \sup_{\substack{w_H \in S^\ell(\Omega, \mathcal{T}_H) \\ w_H \ne 0}} \frac{|(v_H, w_H)_0 - (v_H, w_H)_{0,H}|}{\|w_H\|_{H^1}} \Bigg). \end{aligned} \tag{10}$$

To further analyze the error one can treat the two parts individually. We perform such an analysis for the numerical illustrations given in the following section.

Table 1
Estimated order of convergence (EOC) for the FE-HMM applied to (11).

	Linear FEM		Quadratic FEM		Cubic FEM	
H	H^1-error	EOC	H^1-error	EOC	H^1-error	EOC
$3.13 \cdot 10^{-2}$	$1.73 \cdot 10^{-1}$	—	$4.52 \cdot 10^{-3}$	—	$9.64 \cdot 10^{-5}$	—
$1.56 \cdot 10^{-2}$	$8.66 \cdot 10^{-2}$	1.00	$1.13 \cdot 10^{-3}$	2.00	$1.20 \cdot 10^{-5}$	3.00
$7.81 \cdot 10^{-3}$	$4.33 \cdot 10^{-2}$	1.00	$2.82 \cdot 10^{-4}$	2.03	$1.51 \cdot 10^{-6}$	3.03
$3.91 \cdot 10^{-3}$	$2.16 \cdot 10^{-2}$	1.06	$7.04 \cdot 10^{-5}$	2.29	$1.88 \cdot 10^{-7}$	3.25

5. Numerical illustrations

First, we consider the following one dimensional example

$$-\nabla \cdot (a^\varepsilon(x)\nabla u^\varepsilon(x)) - k^2\rho^\varepsilon(x)u^\varepsilon(x) = f(x) \text{ in } \Omega = (0,1), \qquad u^\varepsilon(0) = u^\varepsilon(1) = 0, \tag{11}$$

where $\varepsilon = 0.001$, $k = 1$, $a^\varepsilon(x) = \sqrt{2} + \sin(2\pi x/\varepsilon)$, and $\rho^\varepsilon(x) = 1 + 0.5\sin(2\pi x/\varepsilon)$. Here we set $f(x) = (4\pi^2 - 1)x\sin(2\pi x) - 4\pi\cos(2\pi x)$, such that the homogenized solution is given by $u^0(x) = x\sin(2\pi x)$.

Since u^0 is smooth, its nodal interpolant is bounded uniformly and the difference between them is of order $\mathcal{O}(H^\ell)$ in the H^1-norm. Due to the periodicity of ρ^ε its weak limit is given by the average over one period and thus $e_\rho = 0$. Moreover, from the smoothness of a^ε and a^0 we get, following e.g. [2], that the consistency terms (second line of (10)) is bounded by $C(H^\ell + e_{\mathrm{HMM}})$, where $e_{\mathrm{HMM}} = (h/\varepsilon)^{2q}$. Note that in e_{HMM} the modelling error vanishes because of the periodicity of a^ε. Thus, the overall error is bounded by $\|u^0 - u_H\|_{H^1} \le C(H^\ell + (h/\varepsilon)^{2q})$.

We recover this convergence rate experimentally by computing the FE-HMM solution for a series of uniform meshes with diminishing meshsize H. To not degrade the convergence order the micromesh is refined simultaneously. On the other hand, to avoid degrading the convergence order by using suboptimal quadrature formulas, we use linear FE-HMM with a midpoint quadrature formula, quadratic FE-HMM with a Simpson quadrature formula, and cubic FE-HMM with a Gauss-Lobatto quadrature formula with four quadrature nodes. The results are shown in Table 1.

The same method can be used in higher dimension. Let $\Omega = (0,1)^2$ be the unit square and consider the stratified material given by $a^\varepsilon(x) = \mathrm{diag}(\sqrt{2} + \sin(2\pi x_1/\varepsilon), 1 + 1/2\sin(2\pi x_1/\varepsilon))$. Due to its special structure (dependence in the first coordinate only) it is still possible to compute the corresponding homogenized tensor a^0. Here a^0 is the identity. We set $\varepsilon = 0.0005$, $k = 1$, and $f(x) = 2(x_1 - x_1^2) + 2(x_2 - x_2^2) - x_1x_2(1 - x_1)(1 - x_2)$. To compute the FE-HMM solution we use bilinear finite element for the macro and the micro solver. To discretize the computational domain, we use a regular triangulation into squares of size $H = 0.01$ in each direction. The micro problems with size $\delta = \varepsilon$ and a periodic coupling condition are discretized with with 25 DOF in each direction. Although the macro mesh does not resolve the microscopic details of the underlying material the FE-HMM solution captures well the behavior of u^0, see Figure 1.

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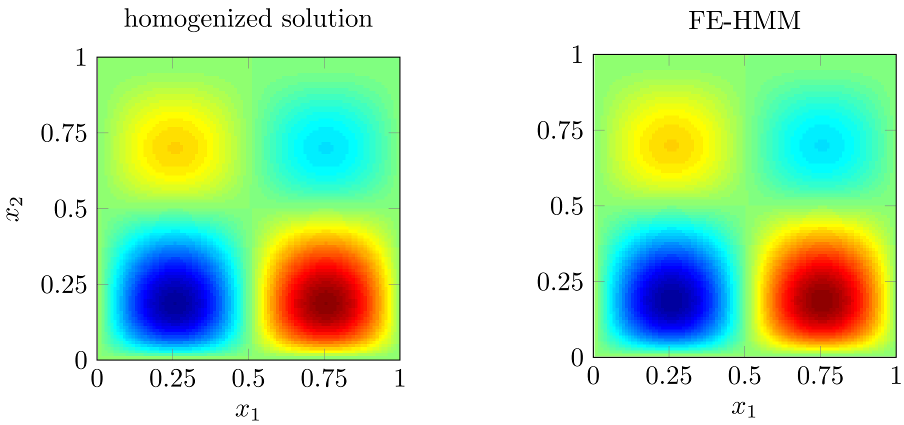

Figure 1. Analytically computed homogenized solution (left) and its FE-HMM approximation (right).

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