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A family of Crouzeix–Raviart finite elements in 3D

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In this paper, we will develop a family of non-conforming "Crouzeix–Raviart" type finite elements in three dimensions. They consist of local polynomials of maximal degree $p \in \mathbb{N}$ on simplicial finite element meshes while certain jump conditions are imposed across adjacent simplices. We will prove optimal *a priori* estimates for these finite elements. The characterization of this space via jump conditions is implicit and the derivation of a local basis requires some deeper theoretical tools from orthogonal polynomials on triangles and their representation. We will derive these tools for this purpose. These results allow us to give explicit representations of the local basis functions. Finally, we will analyze the linear independence of these sets of functions and discuss the question whether they span the whole non-conforming space.

Keywords: Finite element; non-conforming; Crouzeix–Raviart; orthogonal polynomials on triangles; symmetric orthogonal polynomials.

Mathematics Subject Classification 2010: 33C45, 33C50, 65N12, 65N30, 33C80

1. Introduction

For the numerical solution of partial differential equations, Galerkin finite element methods are among the most popular discretization methods. In the last decades, *non-conforming* Galerkin discretizations have become very attractive where the

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test and trial spaces are not subspaces of the natural energy spaces and/or the variational formulation is modified on the discrete level. These methods have nice properties, e.g., in different parts of the domain different discretizations can be easily used and glued together or, for certain classes of problems (Stokes problems, highly indefinite Helmholtz and Maxwell problems, problems with "locking", etc.), the non-conforming discretization enjoys a better stability behavior compared to the conforming one. One of the first non-conforming finite element space was the Crouzeix–Raviart element ([12], see [5] for a survey). It is piecewise affine with respect to a triangulation of the domain while interelement continuity is required only at the barycenters of the edges/facets (2D/3D).

Our paper can be considered as Part II of [9], where a family of high-order non-conforming (intrinsic) finite elements have been introduced which corresponds to a family of high-order Crouzeix–Raviart elements in *two* dimensions. For Poisson's equation in 2D, this family includes the non-conforming Crouzeix–Raviart element [12], the Fortin–Soulie element [15], the Crouzeix–Falk element [11], and the Gauss–Legendre elements [1, 21] as well as the standard conforming hp-finite elements. However, we focus here on the formulation in the primal variables instead its intrinsic version.

The definition of Crouzeix–Raviart type finite elements was given in the original paper [12] in d spatial dimensions and for general polynomial order p. However, this definition was fully implicit and global. An explicit representation of basis functions of order p for this space has been derived in [9] for $d = 2$ while the representation of general basis functions in three spatial dimensions is the topic of this paper. These new finite element spaces are non-conforming but the (broken version of the) continuous bilinear form can still be used. Thus, our results also give insights on how far one can go in the non-conforming direction while keeping the original forms.

One important application of Crouzeix–Raviart finite elements is the stable discretization of the Stokes equation. In our paper, the mathematical focus is on the explicit construction of a basis for these finite elements and this requires the development of some deeper theoretical tools in the field of orthogonal polynomials on triangles and their representations. A first step for the definition of these basis functions is to decompose the space of two variate orthogonal polynomials on the unit triangle into irreducible $\mathcal{S}_3$ modules. Then, a frame for each of these modules can be constructed by considering eigenvalue problems for combinations of the two generating reflections for an action of the symmetric group $\mathcal{S}_3$. Finally, the frame is reduced to a basis by deriving and employing appropriate transformations which allow to determine linear combinations of the frame functions which result in linear independent functions.

The investigation of the stability of these new Crouzeix–Raviart basis functions for Stokes equations would overload this paper and will be the topic of future research. Hence, we consider here Poisson's equation as a simple model problem for the derivation of p-explicit representations of Crouzeix–Raviart finite elements in three dimensions.

There is a vast literature on various conforming and non-conforming, primal, dual, mixed formulations of elliptic differential equations and conforming as well as non-conforming discretization. Our main focus is the characterization and construction of non-conforming Crouzeix–Raviart type finite elements from theoretical principles. For this reason, we do not provide an extensive list of references on the analysis of specific families of finite elements spaces but refer to the classical monographs [8, 20, 2] and the references therein.

The paper is organized as follows.

In Sec. 2, we introduce our model problem, Poisson's equation, the relevant function spaces and standard conditions on its well-posedness.

In Sec. 3, we briefly recall classical, conforming *hp*-finite element spaces and their Lagrange basis.

The new non-conforming finite element spaces are introduced in Sec. 4. We introduce an appropriate compatibility condition at the interfaces between elements of the mesh so that the non-conforming perturbation of the original bilinear form is consistent with the local error estimates. We will see that this compatibility condition can be inferred from the proof of the second Strang lemma applied to our setting. The weak compatibility condition allows to characterize the non-conforming family of high-order Crouzeix–Raviart type elements in an *implicit* way. In this section, we will also present explicit representations of non-conforming basis functions of general degree p while their derivation and analysis is the topic of the following sections.

Section 5 is devoted to the explicit construction of a basis for these new non-conforming finite elements. It requires deeper theoretical tools from orthogonal polynomials on triangles and their representation which we will derive for this purpose in this section.

It is by no means obvious whether the constructed set of functions is linearly independent and span the non-conforming space which was defined implicitly in Sec. 4. These questions will be treated in Sec. 6.

Finally, in Sec. 7, we summarize the main results and give some comparison with the two-dimensional case which was developed in [9].

2. Model Problem

As a model problem we consider the Poisson equation in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with boundary $\Gamma := \partial\Omega$. First, we introduce some spaces and sets of functions for the coefficient functions and solution spaces.

The Euclidean scalar product in $\mathbb{R}^d$ is denoted for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ by $\mathbf{a} \cdot \mathbf{b}$. For $s \geq 0$, $1 \leq p \leq \infty$, let $W^{s,p}(\Omega)$ denote the classical (real-valued) Sobolev spaces with norm $\|\cdot\|_{W^{s,p}(\Omega)}$. The space $W_0^{s,p}(\Omega)$ is the closure with respect to the $\|\cdot\|_{W^{s,p}(\Omega)}$ of all $C^\infty(\Omega)$ functions with compact support. As usual we write $L^p(\Omega)$ short for $W^{0,p}(\Omega)$. The scalar product and norm in $L^2(\Omega)$ are denoted by $(u,v) := \int_\Omega uv$ and $\|\cdot\| := (\cdot,\cdot)^{1/2}$. For $p = 2$, we use $H^s(\Omega)$, $H_0^s(\Omega)$ as shorthands for $W^{s,2}(\Omega)$,

$W_0^{s,2}(\Omega)$. The dual space of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$. We recall that, for positive integers s, the seminorm $|\cdot|_{H^s(\Omega)}$ in $H^s(\Omega)$ which contains only the derivatives of order s is a norm in $H_0^s(\Omega)$.

We consider the Poisson problem in weak form:

$$\text{Given } f \in L^2(\Omega) \text{ find } u \in H_0^1(\Omega) \quad a(u,v) := (\mathbb{A}\nabla u, \nabla v) = (f,v) \quad \forall\, v \in H_0^1(\Omega). \tag{1}$$

Throughout the paper we assume that the diffusion matrix $\mathbb{A} \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ is symmetric and satisfies

$$\begin{aligned} 0 < a_{\min} &:= \operatorname*{ess\,inf}_{\mathbf{x}\in\Omega} \inf_{\mathbf{v}\in\mathbb{R}^d\backslash\{0\}} \frac{(\mathbb{A}(\mathbf{x})\mathbf{v})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} \\ &\le \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} \sup_{\mathbf{v}\in\mathbb{R}^d\backslash\{0\}} \frac{(\mathbb{A}(\mathbf{x})\mathbf{v})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} =: a_{\max} < \infty \end{aligned} \tag{2}$$

and that there exists a partition $\mathcal{P} := (\Omega_j)_{j=1}^J$ of Ω into J (possibly curved) polygons (polyhedra for $d = 3$) such that, for some appropriate $r \in \mathbb{N}$, it holds

$$\|\mathbb{A}\|_{PW^{r,\infty}(\Omega)} := \max_{1\le j\le J} \|\mathbb{A}|_{\Omega_j}\|_{W^{r,\infty}(\Omega_j)} < \infty. \tag{3}$$

Assumption (2) implies the well-posedness of problem (1) via the Lax–Milgram lemma.

3. Conforming hp-Finite Element Galerkin Discretization

In this paper, we restrict our studies to bounded, polygonal ($d = 2$) or polyhedral ($d = 3$) Lipschitz domains $\Omega \subset \mathbb{R}^d$ and regular finite element meshes $\mathcal{G}$ (in the sense of [8]) consisting of (closed) simplices K, where hanging nodes are not allowed. The local and global mesh width is denoted by $h_K := \operatorname{diam} K$ and $h := \max_{K\in\mathcal{G}} h_K$. The boundary of a simplex K can be split into $(d-1)$-dimensional simplices (facets for $d = 3$ and triangle edges for $d = 2$) which are denoted by T. The set of all facets in $\mathcal{G}$ is called $\mathcal{F}$; the set of facets lying on $\partial\Omega$ is denoted by $\mathcal{F}_{\partial\Omega}$ and defines a triangulation of the surface $\partial\Omega$. The set of facets in Ω is denoted by $\mathcal{F}_\Omega$. As a convention we assume that simplices and facets are closed sets. The interior of a simplex K is denoted by $\overset{\circ}{K}$ and we write $\overset{\circ}{T}$ to denote the (relative) interior of a facet T. The set of all simplex vertices in the mesh $\mathcal{G}$ is denoted by $\mathcal{V}$, those lying on $\partial\Omega$ by $\mathcal{V}_{\partial\Omega}$, and those lying in Ω by $\mathcal{V}_\Omega$. Similar the set of simplex edges in $\mathcal{G}$ is denoted by $\mathcal{E}$, those lying on $\partial\Omega$ by $\mathcal{E}_{\partial\Omega}$, and those lying in Ω by $\mathcal{E}_\Omega$.

We recall the definition of conforming *hp*-finite element spaces (see, e.g., [20]). For $p \in \mathbb{N}_0 := \{0, 1, \ldots\}$, let $\mathbb{P}_p^d$ denote the space of d-variate polynomials of total degree $\le p$. For a connected subset $\omega \subset \Omega$, we write $\mathbb{P}_d^p(\omega)$ for polynomials of degree $\le p$ defined on ω. For a connected m-dimensional manifold $\omega \subset \mathbb{R}^d$, for which there exists a subset $\hat\omega \in \mathbb{R}^m$ along an affine bijection $\chi_\omega : \hat\omega \to \omega$, we set

$\mathbb{P}_p^m(\omega) := \{v \circ \chi_\omega^{-1} : v \in \mathbb{P}_p^m(\hat{\omega})\}$. If the dimension m is clear from the context, we write $\mathbb{P}_p(\omega)$ short for $\mathbb{P}_p^m(\omega)$.

The conforming hp-finite element space is given by

$$S_{\mathcal{G},\mathrm{c}}^p := \{u \in C^0(\overline{\Omega}) \,|\, \forall K \in \mathcal{G}\ u|_K \in \mathbb{P}_p(K)\} \cap H_0^1(\Omega). \tag{4}$$

A Lagrange basis for $S_{\mathcal{G},\mathrm{c}}^p$ can be defined as follows. Let

$$\widehat{\mathcal{N}}^p := \left\{ \frac{\mathbf{i}}{p} : \mathbf{i} \in \mathbb{N}_0^d \text{ with } i_1 + \cdots + i_d \leq p \right\} \tag{5}$$

denote the equispaced unisolvent set of nodal points on the d-dimensional unit simplex

$$\widehat{K} := \left\{ \mathbf{x} \in \mathbb{R}_{\geq 0}^d \,|\, x_1 + \cdots + x_d \leq 1 \right\}. \tag{6}$$

For a simplex $K \in \mathcal{G}$, let $\chi_K : \widehat{K} \to K$ denote an affine mapping. The set of nodal points is given by

$$\mathcal{N}^p := \left\{ \chi_K(\hat{\mathbf{N}}) \,|\, \hat{\mathbf{N}} \in \widehat{\mathcal{N}}^p, K \in \mathcal{G} \right\}, \quad \mathcal{N}_\Omega^p := \mathcal{N}^p \cap \Omega, \quad \mathcal{N}_{\partial\Omega}^p := \mathcal{N}^p \cap \partial\Omega. \tag{7}$$

The Lagrange basis for $S_{\mathcal{G},\mathrm{c}}^p$ can be indexed by the nodal points $\mathbf{N} \in \mathcal{N}_\Omega^p$ and is characterized by

$$B_{p,\mathbf{N}}^{\mathcal{G}} \in S_{\mathcal{G},\mathrm{c}}^p \quad \text{and} \quad \forall\, \mathbf{N}' \in \mathcal{N}_\Omega^p \quad B_{p,\mathbf{N}}^{\mathcal{G}}(\mathbf{N}') = \delta_{\mathbf{N},\mathbf{N}'}, \tag{8}$$

where $\delta_{\mathbf{N},\mathbf{N}'}$ is the Kronecker delta.

Definition 1. For all $K \in \mathcal{G}$, $T \in \mathcal{F}_\Omega$, $E \in \mathcal{E}_\Omega$, $\mathbf{V} \in \mathcal{V}_\Omega$, the conforming spaces $S_{K,\mathrm{c}}^p$, $S_{T,\mathrm{c}}^p$, $S_{E,\mathrm{c}}^p$, $S_{\mathbf{V},\mathrm{c}}^p$ are given as the spans of the following basis functions:

$$S_{K,\mathrm{c}}^p := \operatorname{span}\left\{ B_{p,\mathbf{N}}^{\mathcal{G}} \,|\, \mathbf{N} \in \overset{\circ}{K} \cap \mathcal{N}_\Omega^p \right\}, \quad S_{T,\mathrm{c}}^p := \operatorname{span}\left\{ B_{p,\mathbf{N}}^{\mathcal{G}} \,|\, \mathbf{N} \in \overset{\circ}{T} \cap \mathcal{N}_\Omega^p \right\},$$

$$S_{E,\mathrm{c}}^p := \operatorname{span}\left\{ B_{p,\mathbf{N}}^{\mathcal{G}} \,|\, \mathbf{N} \in \overset{\circ}{E} \cap \mathcal{N}_\Omega^p \right\}, \quad S_{\mathbf{V},\mathrm{c}}^p := \operatorname{span}\left\{ B_{p,\mathbf{V}}^{\mathcal{G}} \right\}.$$

The following proposition shows that these spaces give rise to a direct sum decomposition and that these spaces are locally defined. To be more specific we first have to introduce some notations.

For any facet $T \in \mathcal{F}_\Omega$, vertex $\mathbf{V} \in \mathcal{V}_\Omega$, and $E \in \mathcal{E}_\Omega$ we define the sets

$$\begin{aligned}
\mathcal{G}_T &:= \{K \in \mathcal{G} : T \subset \partial K\}, \quad \omega_T := \bigcup_{K \in \mathcal{G}_T} K, \\
\mathcal{G}_{\mathbf{V}} &:= \{K \in \mathcal{G} : \mathbf{V} \in \partial K\}, \quad \omega_{\mathbf{V}} := \bigcup_{K \in \mathcal{G}_{\mathbf{V}}} K, \\
\mathcal{G}_E &:= \{K \in \mathcal{G} : E \subset \partial K\}, \quad \omega_E := \bigcup_{K \in \mathcal{G}_E} K.
\end{aligned} \tag{9}$$

Proposition 2. *Let* $S_{K,\mathrm{c}}^p, S_{T,\mathrm{c}}^p, S_{E,\mathrm{c}}^p, S_{\mathbf{V},\mathrm{c}}^p$ *be as in Definition* 1. *Then the direct sum decomposition holds*

$$S_{\mathcal{G},\mathrm{c}}^p = \left(\bigoplus_{\mathbf{V} \in \mathcal{V}_\Omega} S_{\mathbf{V},\mathrm{c}}^p \right) \oplus \left(\bigoplus_{E \in \mathcal{E}_\Omega} S_{E,\mathrm{c}}^p \right) \oplus \left(\bigoplus_{T \in \mathcal{F}_\Omega} S_{T,\mathrm{c}}^p \right) \oplus \left(\bigoplus_{K \in \mathcal{G}} S_{K,\mathrm{c}}^p \right). \tag{10}$$

4. Galerkin Discretization with Non-Conforming Crouzeix–Raviart Finite Elements

4.1. *Non-conforming finite elements with weak compatibility conditions*

In this section, we will characterize a class of non-conforming finite element spaces implicitly by a weak compatibility condition across the facets. For each facet $T \in \mathcal{F}$, we fix a unit vector $\mathbf{n}_T$ which is orthogonal to T. The orientation for the inner facets is arbitrary but fixed while the orientation for the boundary facets is such that $\mathbf{n}_T$ points toward the exterior of Ω. Our non-conforming finite element spaces will be a subspace of

$$C^0_{\mathcal{G}}(\Omega) := \left\{ u \in L^\infty(\Omega) \mid \forall K \in \mathcal{G} \; u|_{\overset{\circ}{K}} \in C^0\left(\overset{\circ}{K}\right)\right\}$$

and we consider the skeleton $\bigcup_{T\in\mathcal{F}} T$ as a set of measure zero.

For $K \in \mathcal{G}$, we define the restriction operator $\gamma_K : C^0_{\mathcal{G}}(\Omega) \to C^0(K)$ by

$$(\gamma_K w)(\mathbf{x}) = w(\mathbf{x}) \quad \forall \mathbf{x} \in \overset{\circ}{K}$$

and on the boundary ∂K by continuous extension. For the inner facets $T \in \mathcal{F}$, let K^1_T, K^2_T be the two simplices which share T as a common facet with the convention that $\mathbf{n}_T$ points into K_2. We set $\omega_T := K^1_T \cup K^2_T$. The jump $[\,\cdot\,]_T : C^0_{\mathcal{G}}(\Omega) \to C^0(T)$ across T is defined by

$$[w]_T = (\gamma_{K_2} w)|_T - (\gamma_{K_1} w)|_T. \tag{11}$$

For vector-valued functions, the jump is defined component-wise. The definition of the non-conforming finite elements involves orthogonal polynomials on triangles which we introduce first.

Let $\widehat{T}$ denote the (closed) unit simplex in $\mathbb{R}^{d-1}$, with vertices $\mathbf{0}$, $(1,0,\ldots,0)^\intercal$, $(0,1,0,\ldots,0)^\intercal$, $(0,\ldots,0,1)^\intercal$. For $n \in \mathbb{N}_0$, the set of orthogonal polynomials on $\widehat{T}$ is given by

$$\mathbb{P}^\perp_{n,n-1}(\widehat{T}) := \begin{cases} \mathbb{P}_0(\widehat{T}) & n = 0, \\ \left\{ u \in \mathbb{P}_n(\widehat{T}) \;\middle|\; \int_{\widehat{T}} uv = 0 \;\forall\, v \in \mathbb{P}_{n-1}(\widehat{T}) \right\} & n \geq 1. \end{cases} \tag{12}$$

We lift this space to a facet $T \in \mathcal{F}$ by employing an affine transform $\chi_T : \widehat{T} \to T$

$$\mathbb{P}^\perp_{n,n-1}(T) := \left\{ v \circ \chi_T^{-1} : v \in \mathbb{P}^\perp_{n,n-1}(T) \right\}.$$

The orthogonal polynomials on triangles allow us to formulate the *weak compatibility condition* which is employed for the definition of non-conforming finite element spaces:

$$[u]_T \in \mathbb{P}^\perp_{p,p-1}(T),\ \forall T \in \mathcal{F}_\Omega \quad \text{and} \quad u|_T \in \mathbb{P}^\perp_{p,p-1}(T),\ \forall T \in \mathcal{F}_{\partial\Omega}. \tag{13}$$

We have collected all ingredients for the (implicit) characterization of the non-conforming Crouzeix–Raviart finite element space.

Definition 3. The non-conforming finite element space $S^p_{\mathcal{G}}$ with weak compatibility conditions across facets is given by

$$S^p_{\mathcal{G}} := \{u \in L^\infty(\Omega) \,|\, \forall\, K \in \mathcal{G}\ \gamma_K u \in \mathbb{P}_p(K) \text{ and } u \text{ satisfies (13)}\}. \tag{14}$$

The non-conforming Galerkin discretization of (1) for a given finite element space S which satisfies $S^p_{\mathcal{G},\mathrm{nc}} \subset S \subset S^p_{\mathcal{G}}$ reads:

$$\text{Given } f \in L^2(\Omega) \text{ find } u_S \in S \quad a_{\mathcal{G}}(u_S, v) := (\mathbb{A}\nabla_{\mathcal{G}} u_S, \nabla_{\mathcal{G}} v) = (f, v) \quad \forall\, v \in S \tag{15}$$

where

$$\nabla_{\mathcal{G}} u(\mathbf{x}) := \nabla u(\mathbf{x}) \quad \forall\, \mathbf{x} \in \Omega \Bigg\backslash \left(\bigcup_{T\in\mathcal{F}} \partial T\right).$$

4.2. *Non-conforming finite elements of Crouzeix–Raviart type in 3D*

The definition of the non-conforming space $S^p_{\mathcal{G}}$ in (14) is implicit via the weak compatibility condition. In this section, we will present explicit representations of non-conforming basis functions of Crouzeix–Raviart type for general polynomial order p. These functions together with the conforming basis functions span a space $S^p_{\mathcal{G},\mathrm{nc}}$ which satisfies the inclusions $S^p_{\mathcal{G},\mathrm{c}} \subsetneq S^p_{\mathcal{G},\mathrm{nc}} \subseteq S^p_{\mathcal{G}}$ (cf. Theorem 10). The derivation of the formula and their algebraic properties will be the topic of the following sections.

We will introduce two types of non-conforming basis functions: those whose support is one tetrahedron and those whose support consists of two adjacent tetrahedrons, that is tetrahedrons which have a common facet. For details and their derivation we refer to Sec. 5 while here we focus on the representation formulae.

4.2.1. *Non-conforming basis functions supported on one tetrahedron*

The construction starts by defining *symmetric orthogonal polynomials* $b^{\mathrm{sym}}_{p,k}$, $0 \le k \le d_{\mathrm{triv}}(p) - 1$ on the reference triangle $\widehat{T}$ with vertices $(0,0)^\intercal, (1,0)^\intercal, (0,1)^\intercal$, where

$$d_{\mathrm{triv}}(p) := \left\lfloor \frac{p}{2} \right\rfloor - \left\lfloor \frac{p-1}{3} \right\rfloor. \tag{16}$$

We define the coefficients

$$M^{(p)}_{i,j} = (-1)^p {}_4F_3 \left(\begin{matrix} -j, j+1, -i, i+1 \\ -p, p+2, 1 \end{matrix}; 1 \right) \frac{2i+1}{p+1} \quad 0 \le i, j \le p,$$

where ${}_pF_q$ denotes the generalized hypergeometric function (cf. [17, Chap. 16]). The ${}_4F_3$-sum is understood to terminate at i to avoid the $0/0$ ambiguities in the formal ${}_4F_3$-series. These coefficients allow to define the polynomials

$$r_{p,2k}(x_1, x_2) := 2 \sum_{0\le j\le p/2} M^{(n)}_{2j,2k} b_{p,2j} + b_{p,2k} \quad 0 \le k \le p/2,$$

where $b_{p,k}$, $0 \leq k \leq p$, are the basis for the orthogonal polynomials of degree p on $\widehat{T}$ as defined afterwards in (35). Then, a basis for the symmetric orthogonal polynomials is given by

$$b_{p,k}^{\mathrm{sym}} := \begin{cases} r_{p,p-2k} & \text{if } p \text{ is even,} \\ r_{p,p-1-2k} & \text{if } p \text{ is odd,} \end{cases} \quad k = 0, 1, \ldots, d_{\mathrm{triv}}(p) - 1. \tag{17}$$

The non-conforming Crouzeix–Raviart basis function $B_{p,k}^{\widehat{K},\mathrm{nc}} \in \mathbb{P}_p(\widehat{K})$ on the unit tetrahedron $\widehat{K}$ is characterized by its values at the nodal points in $\widehat{\mathcal{N}}^p$ (cf. (5)). For a facet $T \subset \partial\widehat{K}$, let $\chi_T : \widehat{T} \to T$ denote an affine pullback to the reference triangle. Then $B_{p,k}^{\widehat{K},\mathrm{nc}} \in \mathbb{P}_p(\widehat{K})$ is uniquely defined by

$$B_{p,k}^{\widehat{K},\mathrm{nc}}(\mathbf{N}) := \begin{cases} b_{p,k}^{\mathrm{sym}} \circ \chi_T^{-1}(\mathbf{N}) & \forall \mathbf{N} \in \widehat{\mathcal{N}}^p \text{ s.t. } \mathbf{N} \in T \\ & \quad \text{for some facet } T \subset \partial\widehat{K}, \\ 0 & \forall \mathbf{N} \in \widehat{\mathcal{N}}^p \backslash \partial\widehat{K}, \end{cases} \quad k = 0, 1, \ldots, d_{\mathrm{triv}}(p) - 1. \tag{18}$$

Remark 4. In Sec. 5.3, we will prove that the polynomials $b_{p,k}^{\mathrm{sym}}$ are totally symmetric, i.e. invariant under affine bijections $\chi : \widehat{K} \to \widehat{K}$. Thus, any of these functions can be lifted to the facets of a tetrahedron via affine pullbacks and the resulting function on the surface is continuous. As a consequence, the value $B_{p,k}^{\widehat{K},\mathrm{nc}}(\mathbf{N})$ in definition (18) is independent of the choice of T also for nodal points $\mathbf{N}$ which belong to different facets.

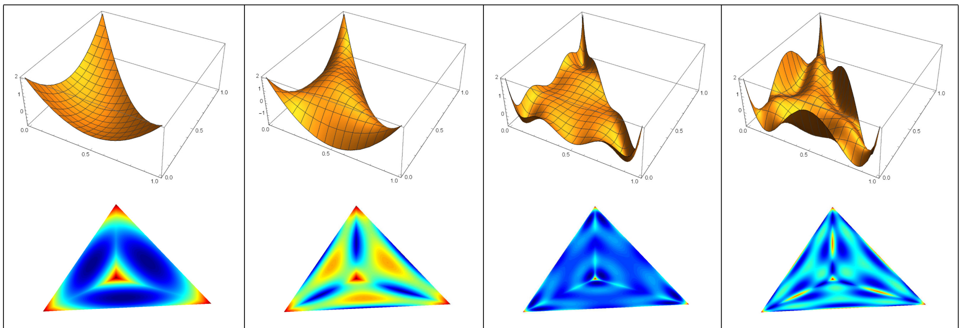
It will turn out that the value 0 at the inner nodes could be replaced by other values without changing the arising non-conforming space. Other choices could be preferable in the context of inverse inequalities and the condition number of the stiffness matrix. However, we recommend to choose these values such that the symmetries of $B_{p,k}^{\widehat{K},\mathrm{nc}}$ are preserved.

Definition 5. The non-conforming tetrahedron-supported basis functions on the reference element are given by

$$B_{p,k}^{\widehat{K},\mathrm{nc}} = \sum_{\mathbf{N} \in \widehat{\mathcal{N}}^p \cap \partial\widehat{K}} B_{p,k}^{\widehat{K},\mathrm{nc}}(\mathbf{N}) B_{p,\mathbf{N}}^{\mathcal{G}} \quad k = 0, 1, \ldots, d_{\mathrm{triv}}(p) - 1 \tag{19}$$

with values $B_{p,k}^{\widehat{K},\mathrm{nc}}(\mathbf{N})$ as in (18). For a simplex $K \in \mathcal{G}$ the corresponding non-conforming basis functions $B_{p,k}^{K,\mathrm{nc}}$ are given by lifting $B_{p,k}^{\widehat{K},\mathrm{nc}}$ via an affine pullback χ_K from $\widehat{K}$ to $K \in \mathcal{G}$:

$$B_{p,k}^{K,\mathrm{nc}}\Big|_{\overset{\circ}{K'}} := \begin{cases} B_{p,k}^{\widehat{K},\mathrm{nc}} \circ \chi_K^{-1} & K = K', \\ 0 & K \neq K'. \end{cases}$$



$b^{\text{sym}}_{2,0}, B^{K,\text{nc}}_{2,0}$ $b^{\text{sym}}_{3,0}, B^{K,\text{nc}}_{3,0}$ $b^{\text{sym}}_{6,0}, B^{K,\text{nc}}_{6,0}$ $b^{\text{sym}}_{6,1}, B^{K,\text{nc}}_{6,1}$

Fig. 1. (Color online) Symmetric orthogonal polynomials on the reference triangle and corresponding tetrahedron-supported non-conforming basis functions.

and span the space

$$S^p_{K,\text{nc}} := \text{span}\{B^{K,\text{nc}}_{p,k} : k = 0, 1, \ldots, d_{\text{triv}}(p) - 1\}. \tag{20}$$

Example 6. The lowest-order of p such that $d_{\text{triv}}(p) \geq 1$ is $p = 2$. In this case, we get $d_{\text{triv}}(p) = 1$. In Fig. 1 the function $b^{\text{sym}}_{p,k}$ and corresponding basis functions $B^{K,\text{nc}}_{p,k}$ are depicted for $(p, k) \in \{(2, 0), (3, 0), (6, 0), (6, 1)\}$.

4.2.2. *Non-conforming basis functions supported on two adjacent tetrahedrons*

The starting point is to define orthogonal polynomials $b^{\text{refl}}_{p,k}$ on the reference triangle $\widehat{T}$ which are mirror symmetric[a] with respect to the angular bisector in $\widehat{T}$ through $\mathbf{0}$ and linear independent from the fully symmetric functions $b^{\text{sym}}_{p,k}$. We set

$$\begin{aligned} b^{\text{refl}}_{p,k} := \frac{1}{3}(2b_{p,2k}(x_1, x_2) &- b_{p,2k}(x_2, 1 - x_1 - x_2) \\ &- b_{p,2k}(1 - x_1 - x_2, x_1)) \quad 0 \leq k \leq d_{\text{refl}}(p) - 1, \end{aligned} \tag{21}$$

where

$$d_{\text{refl}}(p) := \left\lfloor \frac{p+2}{3} \right\rfloor. \tag{22}$$

Let K_1, K_2 denote two tetrahedrons which share a common facet, say T. The vertex of K_i which is opposite to T is denoted by $\mathbf{V}_i$. The procedure of lifting the nodal values to the facets of $\omega_T := K_1 \cup K_2$ is analogous as for the basis functions $B^{K,\text{nc}}_{n,k}$. However, it is necessary to choose the pullback $\chi_{i,\tilde{T}} : \widehat{T} \to \tilde{T}$ of a facet

[a]The superscript "refl" is a shorthand for "reflection" and explained in Sec. 5.3.1.

$\tilde{T} \subset \partial K_i \backslash \overset{\circ}{T}$ such that the origin is mapped to $\mathbf{V}_i$.

$$B_{p,k}^{T,\mathrm{nc}}(\mathbf{N}) := \begin{cases} b_{p,k}^{\mathrm{refl}} \circ \chi_{i,\tilde{T}}^{-1}(\mathbf{N}) & \forall \mathbf{N} \in \mathcal{N}^p \text{ s.t. } \mathbf{N} \in \tilde{T} \\ & \quad \text{for some facet } \tilde{T} \subset \partial K \backslash \overset{\circ}{T}_i, \\ 0 & \forall \mathbf{N} \in \mathcal{N}^p \cap \overset{\circ}{\omega}_T, \end{cases} \quad k = 0, 1, \ldots, d_{\mathrm{refl}}(p) - 1. \tag{23}$$

Again, the value 0 at the inner nodes of ω_T could be replaced by other values without changing the arising non-conforming space.

Definition 7. The non-conforming facet-oriented basis functions are given by

$$B_{p,k}^{T,\mathrm{nc}} = \sum_{\mathbf{N} \in \mathcal{N}^p \cap \partial \omega_T} B_{p,k}^{T,\mathrm{nc}}(\mathbf{N}) B_{p,\mathbf{N}}^{\mathcal{G}}|_{\omega_T} \quad \forall T \in \mathcal{F}_\Omega, \quad k = 0, 1, \ldots, d_{\mathrm{refl}}(p) - 1 \tag{24}$$

with values $B_{p,k}^{T,\mathrm{nc}}(\mathbf{N})$ as in (23) and span the space

$$S_{T,\mathrm{nc}}^p := \mathrm{span}\{B_{p,k}^{T,\mathrm{nc}} : k = 0, 1, \ldots, d_{\mathrm{refl}}(p) - 1\}. \tag{25}$$

The non-conforming finite element space of Crouzeix–Raviart type is given by

$$S_{\mathcal{G},\mathrm{nc}}^p := \left(\bigoplus_{E \in \mathcal{E}_\Omega} S_{E,\mathrm{c}}^p \right) \oplus \left(\bigoplus_{T \in \mathcal{F}_\Omega} S_{T,\mathrm{c}}^p \right) \oplus \left(\bigoplus_{K \in \mathcal{G}} S_{K,\mathrm{c}}^p \right) \oplus \left(\bigoplus_{K \in \mathcal{G}} S_{K,\mathrm{nc}}^p \right) \oplus \left(\bigoplus_{T \in \mathcal{F}_\Omega} \mathrm{span}\{B_{p,0}^{T,\mathrm{nc}}\} \right). \tag{26}$$

Remark 8. In Sec. 5.3.3, we will show that the polynomials $b_{p,k}^{\mathrm{refl}}$ are mirror symmetric with respect to the angular bisector in $\widehat{T}$ through $\mathbf{0}$. Thus, any of these functions can be lifted to the outer facets of two adjacent tetrahedrons via (oriented) affine pullbacks as employed in (23) and the resulting function on the surface is continuous. As a consequence, the value $B_{p,k}^{T,\mathrm{nc}}(\mathbf{N})$ in definition (23) is independent of the choice of T also for nodal points $\mathbf{N}$ which belong to different facets.

In Theorem 33, we will prove that (26), in fact, is a direct sum and a basis is given by the functions

$$B_{p,\mathbf{N}}^{\mathcal{G}} \quad \forall \mathbf{N} \in \mathcal{N}_\Omega \backslash \mathcal{V}, \quad B_{p,k}^{K,\mathrm{nc}} \quad \forall K \in \mathcal{G}, 0 \leq k \leq d_{\mathrm{triv}}(p) - 1, \quad B_{p,0}^{T,\mathrm{nc}} \quad \forall T \in \mathcal{F}_\Omega.$$

Also we will prove that $S_{\mathcal{G},\mathrm{c}}^p \subsetneq S_{\mathcal{G},\mathrm{nc}}^p \subseteq S_{\mathcal{G}}^p$. This condition implies that the convergence estimates as in Theorem 10 are valid for this space. We restricted the reflection-type non-conforming basis functions to the lowest-order $k = 0$ in order to keep the functions linearly independent.

Example 9. The lowest-order of p such that $d_{\mathrm{refl}}(p) \geq 1$ is $p = 1$. In this case, we get $d_{\mathrm{refl}}(p) = 1$. In Fig. 2, the function $b_{p,k}^{\mathrm{refl}}$ and corresponding basis functions $B_{p,k}^{T,\mathrm{nc}}$ are depicted for $(p, k) \in \{(1, 0), (2, 0), (4, 0), (4, 1)\}$.

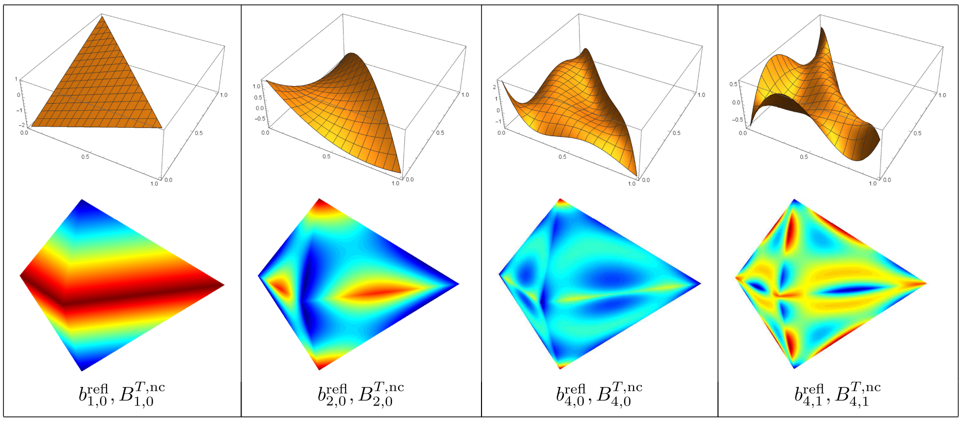


Fig. 2. (Color online) Orthogonal polynomials of reflection type and corresponding non-conforming basis functions which are supported on two adjacent tetrahedrons. The common facet is horizontal and the two tetrahedrons are on top of each other.

4.3. *Error analysis*

In this subsection, we present the error analysis for the Galerkin discretization (15) with the non-conforming finite element space $S_{\mathcal{G}}^p$ and subspaces thereof. The analysis is based on the second Strang lemma and has been presented for an intrinsic version of $S_{\mathcal{G}}^p$ in [9] following the framework developed in [12]. Here we briefly recall this analysis since the proof (step (30)) provides *the* important guideline for the construction of the non-conforming finite elements (and since, as a minor reason and to the best of our knowledge, the proof for the primal formulation for solutions with only piecewise higher-order regularity has not been treated in the literature).

For any inner facet $T \in \mathcal{F}$ and any $v \in S_{\mathcal{G}}^p$, condition (13) implies $\int_T [v]_T = 0$: hence, the jump $[v]_T$ is always zero-mean valued. Let h_T denote the diameter of T. The combination of a Poincaré inequality with a trace inequality then yields

$$\|[u]_T\|_{L^2(T)} \leq Ch_T|[u]_T|_{H^1(T)} \leq \tilde{C}h_T^{1/2}|u|_{H^1_{\mathrm{pw}}(\omega_T)}, \tag{27}$$

where

$$|u|_{H^p_{\mathrm{pw}}(\omega_T)} := \left(\sum_{K\subset\omega_T} |u|^2_{H^p(K)}\right)^{1/2}.$$

In a similar fashion we obtain for all boundary facets $T \in \mathcal{F}_{\partial\Omega}$ and all $u \in S_{\mathcal{G}}^p$ the estimate

$$\|u\|_{L^2(T)} \leq \tilde{C}h_T^{1/2}|u|_{H^1_{\mathrm{pw}}(\omega_T)}. \tag{28}$$

We say that the exact solution $u \in H_0^1(\Omega)$ is piecewise smooth over the partition $\mathcal{P} = (\Omega_j)_{j=1}^J$, if there exists some positive $s \in \mathbb{R}_{>0}$ such that

$$u_{|\Omega_j} \in H^{1+s}(\Omega_j) \quad \text{for } j = 1, 2, \ldots, J.$$

We write $u \in PH^{1+s}(\Omega)$ and refer for further properties, e.g., to [19, Sec. 4.1.9; 7].

For the approximation results, the finite element meshes $\mathcal{G}$ are assumed to be *compatible* with the partition $\mathcal{P}$ in the following sense: for all $K \in \mathcal{G}$, there exists a single index j such that $\overset{\circ}{K} \cap \Omega_j \neq \emptyset$.

The proof that $|\cdot|_{H^1_{\mathrm{pw}}(\Omega)}$ is a norm on $S^p_{\mathcal{G}}$ is similar as in [6, Sec. 10.3]: For $w \in H^1_0(\Omega)$ this follows from $|w|_{H^1_{\mathrm{pw}}(\Omega)} = \|\nabla w\|$ and a Friedrichs inequality; for $w \in S^p_{\mathcal{G}}$ the condition $\|\nabla_{\mathcal{G}} w\| = 0$ implies that $w|_K$ is constant on all simplices $K \in \mathcal{G}$. The combination with $\int_T w = 0$ for all $T \in \mathcal{F}_{\partial\Omega}$ leads to $w|_K = 0$ for the outmost simplex layer via a Poincaré inequality, i.e. $w|_K = 0$ for all $K \in \mathcal{G}$ having at least one facet on $\partial\Omega$. This argument can be iterated step by step over simplex layers toward the interior of Ω to finally obtain $w = 0$. Note that the norm $|\cdot|_{H^1_{\mathrm{pw}}(\Omega)}$ depends on the mesh $\mathcal{G}$ and, consequently, the scaling parameters of the mesh might enter the estimates. To exclude this, a piecewise Poincaré–Friedrichs inequality as developed in [4] can be employed which shows that the estimates only depend on the shape regularity of the mesh.

Theorem 10. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, polygonal $(d = 2)$ or polyhedral $(d = 3)$ Lipschitz domain and let $\mathcal{G}$ be a regular simplicial finite element mesh for Ω. Let the diffusion matrix $\mathbb{A} \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ satisfy assumption (2) and $\|\mathbb{A}\|_{PW^{1,\infty}(\Omega)} < \infty$.*

(a) *There exists $r_{\max} \in]0, 1]$ such that, for any $f \in L^2(\Omega)$, the solution $u \in H^1_0(\Omega)$ of (1) satisfies $u \in \bigcap_{0\leq r<r_{\max}} PH^{1+r}(\Omega)$ if $r_{\max} < 1$ and $u \in PH^2(\Omega)$ if $r_{\max} = 1$.*

(b) *Let $s \in \mathbb{R}_{>0}$ be such that $u \in PH^{1+s}(\Omega)$ holds and set $r := \min\{p, s\}$. We assume that $\|\mathbb{A}\|_{PW^{r,\infty}(\Omega)} < \infty$. Let the continuous problem (1) be discretized by the non-conforming Galerkin method (15) with a finite-dimensional space S which satisfies $S^p_{\mathcal{G},\mathrm{c}} \subset S \subset S^p_{\mathcal{G}}$ on a compatible mesh $\mathcal{G}$. Then, (15) has a unique solution which satisfies*

$$|u - u_S|_{H^1_{\mathrm{pw}}(\Omega)} \leq Ch^r \|u\|_{PH^{1+r}(\Omega)}.$$

The constant C only depends on $a_{\min}, a_{\max}, \|\mathbb{A}\|_{PW^{r,\infty}(\Omega)}, p, r,$ and the shape regularity of the mesh.

Proof. The regularity result (a) follows from [7, Proposition 1] (see also [3, Theorem 4.1; 10, Theorem 3.1]).

The second Strang lemma (cf. [8, Theorem 4.2.2]) applied to the non-conforming Galerkin discretization (15) implies the existence of a unique solution which satisfies the error estimate

$$|u - u_S|_{H^1_{\mathrm{pw}}(\Omega)} \leq \left(1 + \frac{a_{\max}}{a_{\min}}\right) \inf_{v\in S} |u - v|_{H^1_{\mathrm{pw}}(\Omega)} + \frac{1}{a_{\min}} \sup_{v\in S} \frac{|\mathcal{L}_u(v)|}{|v|_{H^1_{\mathrm{pw}}(\Omega)}},$$

where

$$\mathcal{L}_u(v) := a_{\mathcal{G}}(u, v) - (f, v).$$

The approximation properties of S are inherited from the approximation properties of $S^p_{\mathcal{G},\mathrm{c}}$ in the first infimum because of the inclusion $S^p_{\mathcal{G},\mathrm{c}} \subset S$. For the second term we obtain

$$\mathcal{L}_u(v) = (\mathbb{A}\nabla u, \nabla_{\mathcal{G}} v) - (f, v). \tag{29}$$

Note that $f \in L^2(\Omega)$ implies that $\operatorname{div}(\mathbb{A}\nabla u) \in L^2(\Omega)$ and, in turn, that the normal jump $[\mathbb{A}\nabla u \cdot \mathbf{n}_T]_T$ equals zero and the restriction $(\mathbb{A}\nabla u \cdot \mathbf{n}_T)|_T$ is well defined for all $T \in \mathcal{F}$. We may apply simplexwise integration by parts to (29) to obtain

$$\mathcal{L}_u(v) = -\sum_{T\in\mathcal{F}_\Omega} \int_T (\mathbb{A}\nabla u \cdot \mathbf{n}_T)[v]_T + \sum_{T\in\mathcal{F}_{\partial\Omega}} \int_T (\mathbb{A}\nabla u \cdot \mathbf{n}_T) v.$$

Let K_T be one simplex in ω_T. For $1 \leq i \leq d$, let $q_i \in \mathbb{P}_d^{p-1}(K_T)$ denote the best approximation of $w_i := (\sum_{j=1}^d A_{i,j}\partial_j u)|_{K_T}$ with respect to the $H^1(K_T)$ norm. Then, $q_i|_T n_{T,i} \in \mathbb{P}_{d-1}^{p-1}(T)$ for $1 \leq i \leq d$, and the inclusion $S \subset S_{\mathcal{G}}^p$ implies

$$\begin{aligned}
|\mathcal{L}_u(v)| &\leq \left| -\sum_{T\in\mathcal{F}_\Omega} \int_T \left(\sum_{i=1}^d (w_i - q_i) \cdot n_{T,i} \right) [v]_T \right| \\
&\quad + \left| \sum_{T\in\mathcal{F}_{\partial\Omega}} \int_T \left(\sum_{i=1}^d (w_i - q_i) \cdot n_{T,i} \right) v \right| \\
&\leq \sum_{T\in\mathcal{F}_\Omega} \|[v]_T\|_{L^2(T)} \sum_{i=1}^d \|w_i - q_i\|_{L^2(T)} \\
&\quad + \sum_{T\in\mathcal{F}_{\partial\Omega}} \|v\|_{L^2(T)} \sum_{i=1}^d \|w_i - q_i\|_{L^2(T)}.
\end{aligned} \tag{30}$$

Standard trace estimates and approximation properties lead to

$$\begin{aligned}
\|w_i - q_i\|_{L^2(T)} &\leq C(h_T^{-1/2}\|w_i - q_i\|_{L^2(K_T)} + h_T^{1/2}|w_i - q_i|_{H^1(K_T)}) \\
&\leq C h_T^{r-1/2}|w_i|_{H^r(K_T)} \leq C h_T^{r-1/2}\|u\|_{H^{1+r}(K_T)},
\end{aligned} \tag{31}$$

where C depends only on p, r, $\|\mathbb{A}\|_{W^r(K_T)}$, and the shape regularity of the mesh. The combination of (30), (31) and (27), (28) along with the shape regularity of the mesh leads to the consistency estimate

$$\begin{aligned}
|\mathcal{L}_u(v)| &\leq C \left(\sum_{T\in\mathcal{F}_\Omega} h_T^r \|u\|_{H^{1+r}(K_T)} |v|_{H^1_{\mathrm{pw}}(\omega_T)} + \sum_{T\in\mathcal{F}_{\partial\Omega}} h_T^r \|u\|_{H^{1+r}(K_T)} |v|_{H^1_{\mathrm{pw}}(\omega_T)} \right) \\
&\leq \tilde{C} h^r \|u\|_{PH^{1+r}(\Omega)} |v|_{H^1_{\mathrm{pw}}(\Omega)},
\end{aligned}$$

which completes the proof. □

Remark 11. If one chooses in (13) a degree $p' < p$ for the orthogonality relations in (13), then the order of convergence behaves like $h^{r'}\|e\|_{H^{1+r'}(\Omega)}$, with $r' := \min\{p', s\}$, because the best approximations q_i now belong to $P_{d-1}^{p'-1}(T)$.

5. Explicit Construction of Non-Conforming Crouzeix–Raviart Finite Elements

5.1. *Jacobi polynomials*

Let $\alpha, \beta > -1$. The *Jacobi polynomial* $P_n^{(\alpha,\beta)}$ is a polynomial of degree n such that

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\, q(x)(1-x)^{\alpha}(1+x)^{\beta}\, dx = 0$$

for all polynomials q of degree less than n, and (cf. [17, Table 18.6.1])

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}, \quad P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{(\beta+1)_n}{n!}. \tag{32}$$

Here the *shifted factorial* is defined by $(a)_n := a(a+1)\cdots(a+n-1)$ for $n > 0$ and $(a)_0 := 1$. The Jacobi polynomial has an explicit expression in terms of a *terminating Gauss hypergeometric series* (see (cf. [17, 18.5.7]))

$${}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix}; z\right) := \sum_{k=0}^{n} \frac{(-n)_k (b)_k}{(c)_k\, k!} z^k \tag{33}$$

as follows

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right). \tag{34}$$

5.2. *Orthogonal polynomials on triangles*

Recall that $\widehat{T}$ is the (closed) unit triangle in $\mathbb{R}^2$ with vertices $\widehat{\mathbf{A}}_0 = (0,0)^{\intercal}$, $\widehat{\mathbf{A}}_1 = (1,0)^{\intercal}$, and $\widehat{\mathbf{A}}_3 = (0,1)^{\intercal}$. An orthogonal basis for the space $\mathbb{P}_{n,n-1}^{\perp}(\widehat{T})$ was introduced in [18] and is given by the functions $b_{n,k}$, $0 \le k \le n$,

$$b_{n,k}(\mathbf{x}) := (x_1+x_2)^k\, P_{n-k}^{(0,2k+1)}(2(x_1+x_2)-1)\, P_k^{(0,0)}\left(\frac{x_1-x_2}{x_1+x_2}\right), \tag{35}$$

where $P_k^{(0,0)}$ are the Legendre polynomials (see [17, 18.7.9])[b] From (36), it follows that these polynomials satisfy the following symmetry relation:

$$b_{n,k}(x_1,x_2) = (-1)^k b_{n,k}(x_2,x_1) \quad \forall\, n \ge 0,\ \forall\,(x_1,x_2). \tag{37}$$

[b]The Legendre polynomials with normalization $P_k^{(0,0)}(1) = 1$ for all $k = 0,1,\ldots$ can be defined [17, Table 18.9.1] via the three-term recursion

$$\begin{aligned} &P_0^{(0,0)}(x) = 1; \quad P_1^{(0,0)}(x) = x; \quad \text{and} \\ &(k+1)P_{k+1}^{(0,0)}(x) = (2k+1)xP_k^{(0,0)}(x) - kP_{k-1}^{(0,0)}(x) \quad \text{for } k = 1,2,\ldots, \end{aligned} \tag{36}$$

from which the well-known relation $P_k^{(0,0)}(x) = (-1)^k P_k^{(0,0)}(x)$ for all $k \in \mathbb{N}_0$ follows.

By combining (33)–(35), an elementary calculation leads to[c] $b_{n,0}(0,0) = (-1)^n (n+1)$.

Let

$$E^{\mathrm{I}} := \overline{\widehat{\mathbf{A}}_0\widehat{\mathbf{A}}_1}, \quad E^{\mathrm{II}} := \overline{\widehat{\mathbf{A}}_0\widehat{\mathbf{A}}_2}, \quad \text{and} \quad E^{\mathrm{III}} := \overline{\widehat{\mathbf{A}}_1\widehat{\mathbf{A}}_2} \tag{38}$$

denote the edges of $\widehat{T}$. For $\mathrm{Z} \in \{\mathrm{I},\mathrm{II},\mathrm{III}\}$, we introduce the linear restriction operator for the edge E^{Z} by $\gamma^{\mathrm{Z}} : C^0(\widehat{T}) \to C^0([0,1])$ by

$$\gamma^{\mathrm{I}}u := u(\cdot,0), \quad \gamma^{\mathrm{II}}u := u(0,\cdot), \quad \gamma^{\mathrm{III}}u = u(1-\cdot,\cdot) \tag{39}$$

which allows to define

$$b^{\mathrm{I}}_{n,k} := \gamma^{\mathrm{I}}b_{n,k}, \quad b^{\mathrm{II}}_{n,k} := \gamma^{\mathrm{II}}b_{n,k}, \quad b^{\mathrm{III}}_{n,k} := \gamma^{\mathrm{III}}b_{n,k}, \quad \text{for } k = 0,1,\dots,n.$$

Lemma 12. *For any* $\mathrm{Z} \in \{\mathrm{I},\mathrm{II},\mathrm{III}\}$, *each of the systems* $(b^{\mathrm{Z}}_{n,k})_{k=0}^n$, *form a basis of* $\mathbb{P}_n([0,1])$.

Proof. First note that $\{x^j(x-1)^{n-j} : 0 \le j \le n\}$ is a basis for $\mathbb{P}_n([0,1])$; this follows from expanding the right-hand side of $x^m = x^m(x-(x-1))^{n-m}$. Specialize the formula [17, 18.5.8]

$$P_m^{(\alpha,\beta)}(s) = \frac{(\alpha+1)_m}{m!}\left(\frac{1+s}{2}\right)^m {}_2F_1\left(\begin{matrix}-m,-m-\beta\\ \alpha+1\end{matrix};\frac{s-1}{s+1}\right)$$

to $m = n-k$, $\alpha = 0, \beta = 2k+1, s = 2x-1$ to obtain

$$b^{\mathrm{I}}_{n,k}(x) \;=\; x^n\, {}_2F_1\left(\begin{matrix}k-n,-n-k-1\\ 1\end{matrix};\frac{x-1}{x}\right) \tag{40}$$

$$\overset{(33)}{=} \sum_{i=0}^{n-k}\frac{(k-n)_i(-n-k-1)_i}{i!i!}x^{n-i}(x-1)^i. \tag{41}$$

The highest index i of $x^{n-i}(x-1)^i$ in $b^{\mathrm{I}}_{n,k}(x)$ is $n-k$ with coefficient $\frac{(2k+2)_{n-k}}{(n-k)!} \neq 0$. Thus the matrix expressing $[b^{\mathrm{I}}_{n,0},\dots,b^{\mathrm{I}}_{n,n}]$ in terms of $[(x-1)^n, x(x-1)^{n-1},\dots,x^n]$ is triangular and nonsingular; hence $\{b^{\mathrm{I}}_{n,k} : 0 \le k \le n\}$ is a basis of $\mathbb{P}_n([0,1])$. The symmetry relation $b^{\mathrm{II}}_{n,k} = (-1)^k b^{\mathrm{I}}_{n,k}$ for $0 \le k \le n$ (cf. (37)) shows that $\{b^{\mathrm{II}}_{n,k} : 0 \le k \le n\}$ is also a basis of $\mathbb{P}_n([0,1])$. Finally substituting $x_1 = 1-x, x_2 = x$ in $b_{n,k}$ results in

$$b^{\mathrm{III}}_{n,k}(x) = P^{(0,2k+1)}_{n-k}(1)P^{(0,0)}_k(1-2x), \tag{42}$$

and $P^{(0,2k+1)}_{n-k}(1) = 1$ (from (32)). Clearly $\{P^{(0,0)}_k(1-2x) : 0 \le k \le n\}$ is a basis for $\mathbb{P}_n([0,1])$. $\square$

[c]Further special values are

$$b_{n,0}(0,0) = P_n^{(0,1)}(-1) = (-1)^n\frac{(2)_n}{n!} = (-1)^n(n+1), \quad b_{n,k}(0,0) = 0, \quad 1 \le k \le n,$$

$$b_{n,k}(1,0) = P^{(0,2k+1)}_{n-k}(1)P^{(0,0)}_k(1) = 1, \quad 0 \le k \le n, \quad b_{n,k}(0,1) = P^{(0,2k+1)}_{n-k}(1)P^{(0,0)}_k(-1) = (-1)^k, \quad 0 \le k \le n.$$

Lemma 13. *Let $v \in \mathbb{P}_n([0,1])$. Then, there exist unique orthogonal polynomials $u^{\mathrm{Z}} \in \mathbb{P}^{\perp}_{n,n-1}(\widehat{T}), \mathrm{Z} \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ with $v = \gamma^{\mathrm{Z}} u^{\mathrm{Z}}$. Thus, the linear extension operator $\mathcal{E}^{\mathrm{Z}} : \mathbb{P}_n([0,1]) \to \mathbb{P}^{\perp}_{n,n-1}(\widehat{T})$ is well defined by $\mathcal{E}^{\mathrm{Z}} v := u^{\mathrm{Z}}$.*

Proof. From Lemma 12, we conclude that γ^{Z} is surjective. Since the polynomial spaces are finite-dimensional the assertion follows from

$$\dim \mathbb{P}_n([0,1]) = n + 1 = \dim \mathbb{P}^{\perp}_{n,n-1}(\widehat{T}). \qquad \square$$

The orthogonal polynomials can be lifted to a general triangle T.

Definition 14. Let T denote a triangle and χ_T an affine pullback to the reference triangle $\widehat{T}$. Then, the space of orthogonal polynomials of degree n on T is

$$\mathbb{P}^{\perp}_{n,n-1}(T) := \left\{ v \circ \chi_T^{-1} : v \in \mathbb{P}^{\perp}_{n,n-1}(\widehat{T}) \right\}.$$

From the transformation rule for integrals one concludes that for any $u = v \circ \chi_T^{-1} \in \mathbb{P}^{\perp}_{n,n-1}(T)$ and all $q \in \mathbb{P}_{n-1}(T)$ it holds

$$\int_T uq = \int_T (v \circ \chi_T^{-1}) q = 2|T| \int_{\widehat{T}} v(q \circ \chi_T) = 0 \tag{43}$$

since $q \circ \chi_T \in \mathbb{P}_{n-1}(\widehat{T})$. Here $|T|$ denotes the area of the triangle T.

5.3. *Totally symmetric orthogonal polynomials*

In this section, we will decompose the space of orthogonal polynomials $\mathbb{P}^{\perp}_{n,n-1}(\widehat{T})$ into three irreducible modules (see Sec. 5.3.1) and thus, obtain a direct sum decomposition $\mathbb{P}^{\perp}_{n,n-1}(\widehat{T}) = \mathbb{P}^{\perp,\mathrm{sym}}_{n,n-1}(\widehat{T}) \oplus \mathbb{P}^{\perp,\mathrm{refl}}_{n,n-1}(\widehat{T}) \oplus \mathbb{P}^{\perp,\mathrm{sign}}_{n,n-1}(\widehat{T})$. We will derive an explicit representation for a basis of the space of totally symmetric polynomials $\mathbb{P}^{\perp,\mathrm{sym}}_{n,n-1}(\widehat{T})$ in Sec. 5.3.2 and of the space of reflection symmetric polynomials $\mathbb{P}^{\perp,\mathrm{refl}}_{n,n-1}(\widehat{T})$ in Sec. 5.3.3.

We start by introducing, for functions on triangles, the notation of total symmetry. For an arbitrary triangle T with vertices $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{A}_2$, we introduce the set of permutations $\Pi = \{(i,j,k) : i,j,k \in \{0,1,2\} \text{ pairwise disjoint}\}$. For $\pi = (i,j,k) \in \Pi$, define the affine mapping $\chi_\pi : T \to T$ by

$$\chi_\pi(\mathbf{x}) = \mathbf{A}_i + x_1(\mathbf{A}_j - \mathbf{A}_i) + x_2(\mathbf{A}_k - \mathbf{A}_i). \tag{44}$$

We say a function u, defined on T, has *total symmetry* if

$$u = u \circ \chi_\pi \quad \forall \pi \in \Pi.$$

The space of totally symmetric orthogonal polynomials is

$$\mathbb{P}^{\perp,\mathrm{sym}}_{n,n-1}(\widehat{T}) := \left\{ u \in \mathbb{P}^{\perp}_{n,n-1}(\widehat{T}) : u \text{ has total symmetry} \right\}. \tag{45}$$

The construction of a basis of $\mathbb{P}^{\perp,\mathrm{sym}}_{n,n-1}(\widehat{T})$ requires some algebraic tools which we develop in the following.

5.3.1. *The decomposition of* $\mathbb{P}^{\perp}_{n,n-1}(\widehat{T})$ *or* $\mathbb{P}_n([0,1])$ *into irreducible* $\mathcal{S}_3$ *modules*

We use the operator γ^{I} (cf. (39)) to set up an action of the symmetric group $\mathcal{S}_3$ on $\mathbb{P}_n([0,1])$ by transferring its action on $\mathbb{P}^{\perp}_{n,n-1}(\widehat{T})$ on the basis $\{b_{n,k}\}$. It suffices to work with two generating reflections. On the triangle $\chi_{\{0,2,1\}}(x_1,x_2)=(x_2,x_1)$ and thus $b_{n,k}\circ\chi_{\{0,2,1\}}=(-1)^k b_{n,k}$ (this follows from (37)). The action of $\chi_{\{0,2,1\}}$ is mapped to $\sum_{k=0}^{n}\alpha_k b^{\mathrm{I}}_{n,k}\mapsto\sum_{k=0}^{n}(-1)^k\alpha_k b^{\mathrm{I}}_{n,k}$, and denoted by R. For the other generator we use $\chi_{\{1,0,2\}}(x_1,x_2)=(1-x_1-x_2,x_2)$. Under γ^{I} this corresponds to the map $\sum_{k=0}^{n}\alpha_k b^{\mathrm{I}}_{n,k}(x)\mapsto\sum_{k=0}^{n}\alpha_k b^{\mathrm{I}}_{n,k}(1-x)$ which is denoted by M. We will return later to transformation formulae expressing

$$b_{n,k}\circ\chi_{\{1,0,2\}}(x_1,x_2)=(1-x_1)^k P^{(0,2k+1)}_{n-k}(1-2x_1)P^{(0,0)}_k\left(\frac{1-x_1-2x_2}{1-x_1}\right)$$

in the $\{b_{n,k}\}$-basis. Observe that $(MR)^3=I$ because $\chi_{\{1,0,2\}}\circ\chi_{\{0,2,1\}}(x_1,x_2)=(1-x_1-x_2,x_1)$ and this mapping is of period 3. It follows that each of $\{M,R\}$ and $\{\chi_{\{1,0,2\}},\chi_{\{0,2,1\}}\}$ generates (an isomorphic copy of) $\mathcal{S}_3$. It is a basic fact that the relations $M^2=I, R^2=I$ and $(MR)^3=I$ define $\mathcal{S}_3$. The representation theory of $\mathcal{S}_3$ informs us that there are three nonisomorphic irreducible representations:

$$\tau_{\mathrm{triv}}:\chi_{\{0,2,1\}}\to 1,\quad \chi_{\{1,0,2\}}\to 1;$$

$$\tau_{\mathrm{sign}}:\chi_{\{0,2,1\}}\to -1,\quad \chi_{\{1,0,2\}}\to -1;$$

$$\tau_{\mathrm{refl}}:\chi_{\{0,2,1\}}\to\sigma_1:=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\quad \chi_{\{1,0,2\}}\to\sigma_2:=\begin{bmatrix}\frac{1}{2} & 1\\ \frac{3}{4} & -\frac{1}{2}\end{bmatrix}$$

(The subscript "refl" designates the *reflection* representation). Then the eigenvectors of σ_1,σ_2 with -1 as eigenvalue are $(-1,0)^{\intercal}$ and $(2,-3)^{\intercal}$ respectively; these two vectors are a basis for $\mathbb{R}^2$. Similarly the eigenvectors of σ_1 and σ_2 with eigenvalue $+1$, namely $(0,1)^{\intercal}$, $(2,1)^{\intercal}$, form a basis. Form a direct sum

$$\mathbb{P}^{\perp}_{n,n-1}(\widehat{T}):=\left(\bigoplus_{j\ge 0}E^{(\mathrm{triv})}_j\right)\oplus\left(\bigoplus_{j\ge 0}E^{(\mathrm{sign})}_j\right)\oplus\left(\bigoplus_{j\ge 0}E^{(\mathrm{refl})}_j\right),$$

where the $E^{(\mathrm{triv})}_j, E^{(\mathrm{sign})}_j, E^{(\mathrm{refl})}_j$ are $\mathcal{S}_3$-irreducible and realizations of the representations $\tau_{\mathrm{triv}},\tau_{\mathrm{sign}},\tau_{\mathrm{refl}}$ respectively. Let $d_{\mathrm{triv}}(n), d_{\mathrm{sign}}(n), d_{\mathrm{refl}}(n)$ denote the respective multiplicities, so that $d_{\mathrm{triv}}(n)+d_{\mathrm{sign}}(n)+2d_{\mathrm{refl}}(n)=n+1$. The case n even or odd are handled separately. If $n=2m$ is even then the number of eigenvectors of R having -1 as eigenvalue equals m (the cardinality of $\{1,3,5,\ldots,2m-1\}$). The same property holds for M since the eigenvectors of M in the basis $\{x^{2m}(x-1)^{2m-j}\}$ are explicitly given by $\{x^{2m-2\ell}(x-1)^{2\ell}-x^{2\ell}(x-1)^{2m-2\ell}:0\le\ell\le m\}$. Each $E^{(\mathrm{refl})}_j$ contains one (-1)-eigenvector of $\chi_{\{1,0,2\}}$ and one of $\chi_{\{0,2,1\}}$ and each $E^{(\mathrm{sign})}_j$ consists of one (-1)-eigenvector of $\chi_{\{0,2,1\}}$. This gives the equation $d_{\mathrm{refl}}(n)+d_{\mathrm{sign}}(n)=m$.

Each $E_j^{(\mathrm{refl})}$ contains one $(+1)$-eigenvector of $\chi_{\{1,0,2\}}$ and one of $\chi_{\{0,2,1\}}$ and each $E_j^{(\mathrm{triv})}$ consists of one $(+1)$-eigenvector of $\chi_{\{0,2,1\}}$. There are $m+1$ eigenvectors with eigenvalue 1 of each of $\chi_{\{1,0,2\}}$ and $\chi_{\{0,2,1\}}$ thus $d_{\mathrm{refl}}(n) + d_{\mathrm{triv}}(n) = m+1$.

If $n = 2m+1$ is odd then the eigenvector multiplicities are $m+1$ for both eigenvalues $+1, -1$. By similar arguments we obtain the equations $d_{\mathrm{refl}}(n) + d_{\mathrm{sign}}(n) = m+1$, $d_{\mathrm{refl}}(n) + d_{\mathrm{triv}}(n) = m+1$. It remains to find one last relation for both, even and odd cases.

To finish the determination of the multiplicities $d_{\mathrm{triv}}(n), d_{\mathrm{sign}}(n), d_{\mathrm{refl}}(n)$ it suffices to find $d_{\mathrm{triv}}(n)$. This is the dimension of the space of polynomials in $\mathbb{P}_{n,n-1}^{\perp}(\widehat{T})$ which are invariant under both $\chi_{\{0,2,1\}}$ and $\chi_{\{1,0,2\}}$. Since these two group elements generate $\mathcal{S}_3$ this is equivalent to being invariant under each element of $\mathcal{S}_3$ .This property is called *totally symmetric*. Under the action of γ^{I} this corresponds to the space of polynomials in $\mathbb{P}_n([0,1])$ which are invariant under both R and M. We appeal to the classical theory of symmetric polynomials: suppose $\mathcal{S}_3$ acts on polynomials in (y_1, y_2, y_3) by permutation of coordinates then the space of symmetric (invariant under the group) polynomials is exactly the space of polynomials in $\{e_1, e_2, e_3\}$ the elementary symmetric polynomials, namely $e_1 = y_1 + y_2 + y_3$, $e_2 = y_1 y_2 + y_1 y_3 + y_2 y_3$, $e_3 = y_1 y_2 y_3$. To apply this we set up an affine map from $\widehat{T}$ to the triangle in $\mathbb{R}^3$ with vertices $(2,-1,-1), (-1,2,-1), (-1,-1,2)$. The formula for the map is

$$y(x) = (2 - 3x_1 - 3x_2, 3x_1 - 1, 3x_2 - 1).$$

The map takes $(0,0), (1,0), (0,1)$ to the three vertices, respectively. The result is

$$\begin{aligned} e_1(y(x)) &= 0, \\ e_2(y(x)) &= -9(x_1^2 + x_1 x_2 + x_2^2 - x_1 - x_2) - 3, \\ e_3(y(x)) &= (3x_1 - 1)(3x_2 - 1)(2 - 3x_1 - 3x_2). \end{aligned}$$

Thus any totally symmetric polynomial on $\widehat{T}$ is a linear combination of $e_2^a e_3^b$ with uniquely determined coefficients. The number of linearly independent totally symmetric polynomials in $(\oplus_{j=1}^n \mathbb{P}_{n,n-1}^{\perp}(\widehat{T})) \oplus \mathbb{P}_0(\widehat{T})$ equals the number of solutions of $0 \le 2a + 3b \le n$ with $a, b = 0, 1, 2, \ldots$. As a consequence $d_{\mathrm{triv}}(n) = \mathrm{card}\{(a,b) : 2a + 3b = n\}$. This number is the coefficient of t^n in the power series expansion of

$$\frac{1}{(1-t^2)(1-t^3)} = (1 + t^2 + t^3 + t^4 + t^5 + t^7)(1 + 2t^6 + 3t^{12} + \cdots).$$

From $d_{\mathrm{triv}}(n) = \mathrm{card}(\{0,2,4,\ldots\} \cap \{n, n-3, n-6, \ldots\})$ we deduce the formula (cf. (16))

$$d_{\mathrm{triv}}(n) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{3} \right\rfloor.$$

As a consequence: if $n = 2m$ then $d_{\mathrm{sign}}(n) = d_{\mathrm{triv}}(n) - 1$ and $d_{\mathrm{refl}}(n) = m + 1 - d_{\mathrm{triv}}(n)$; if $n = 2m+1$ then $d_{\mathrm{sign}}(n) = d_{\mathrm{triv}}(n)$ and $d_{\mathrm{refl}}(n) = m + 1 - d_{\mathrm{triv}}(n)$. From this the following can be derived: $d_{\mathrm{sign}}(n) = \lfloor \frac{n-1}{2} \rfloor - \lfloor \frac{n-1}{3} \rfloor$ and $d_{\mathrm{refl}}(n) = \lfloor \frac{n+2}{3} \rfloor$. Here is a table of values in terms of $n \bmod 6$:

n	$d_{\mathrm{triv}}(n)$	$d_{\mathrm{sign}}(n)$	$d_{\mathrm{refl}}(n)$
$6m$	$m+1$	m	$2m$
$6m+1$	m	m	$2m+1$
$6m+2$	$m+1$	m	$2m+1$
$6m+3$	$m+1$	$m+1$	$2m+1$
$6m+4$	$m+1$	m	$2m+2$
$6m+5$	$m+1$	$m+1$	$2m+2$

5.3.2. *Construction of totally symmetric polynomials*

Let M and R denote the linear maps $Mp(x_1, x_2) := p(1 - x_1 - x_2, x_2)$ and $Rp(x_1, x_2) := p(x_2, x_1)$ respectively. Both are automorphisms of $\mathbb{P}^{\perp}_{n,n-1}(\widehat{T})$. Note $Mp = p \circ \chi_{\{1,0,2\}}$ and $Rp = p \circ \chi_{\{0,2,1\}}$ (cf. Sec. 5.3.1).

Proposition 15. *Suppose* $0 \le k \le n$ *then*

$$Rb_{n,k} = (-1)^k b_{n,k}; \tag{46}$$

$$Mb_{n,k} = (-1)^n \sum_{j=0}^{n} {}_4F_3\left(\begin{matrix} -j, j+1, -k, k+1 \\ -n, n+2, 1 \end{matrix}; 1\right) \frac{2j+1}{n+1} b_{n,j}. \tag{47}$$

Proof. The ${}_4F_3$-sum is understood to terminate at k to avoid the $0/0$ ambiguities in the formal ${}_4F_3$-series. The first formula was shown in Sec. 5.3.1. The second formula is a specialization of transformations in [13, Theorem 1.7(iii)]: this paper used the shifted Jacobi polynomial $R^{(\alpha,\beta)}_m(s) = \frac{m!}{(\alpha+1)_m} P^{(\alpha,\beta)}_m(1-2s)$. Setting $\alpha = \beta = \gamma = 0$ in the formulas in [13, Theorem 1.7(iii)] results in $b_{n,k} = (-1)^k \frac{\theta_{n,k}}{k!(n-k)!}$ and $Mb_{n,k} = \frac{\phi_{n,k}}{k!(n-k)!}$, where $\theta_{n,k}$, $\phi_{n,k}$ are the polynomials introduced in [13, p. 690]. More precisely, the arguments v_1, v_2, v_3 in $\theta_{n,k}$ and $\phi_{n,k}$ are specialized to $v_1 = x_1, v_2 = x_2$ and $v_3 = 1 - x_1 - x_2$. $\square$

It is worthwhile to mention at this point that the transformation technique in [13] has been further developed in [16], where the connection coefficients for general d-variate Jacobi polynomials have been considered on the simplex for bases generated by elements of the symmetric group.

Proposition 16. *The range of* $I + RM + MR$ *is exactly the subspace* $\{p \in \mathbb{P}^{\perp}_{n,n-1}(\widehat{T}) : RMp = p\}$.

Proof. By direct computation $(MR)^3 = I$ (cf. Sec. 5.3.1). This implies $(RM)^2 = MR$. If p satisfies $RMp = p$ then $Mp = Rp$ and $p = MRp$. Now suppose $RMp = p$

then $(I + RM + MR)\frac{1}{3}p = p$; hence p is in the range of $I + RM + MR$. Conversely suppose $p = (I+RM+MR)p'$ for some polynomial p', then, $RM(I+RM+MR)p' = (RM + (RM)^2 + I)p' = p$.

Let $M_{i,j}^{(n)}, R_{i,j}^{(n)}$ denote the matrix entries of M, R with respect to the basis $\{b_{n,k} : 0 \le k \le n\}$, respectively (that is $Mb_{n,k} = \sum_{j=0}^{n} b_{n,j} M_{j,k}^{(n)}$) . Let $S_{i,j}^{(n)}$ denote the matrix entries of $MR + RM + I$. Then

$$R_{i,j}^{(n)} = (-1)^i \delta_{i,j};\ M_{i,j}^{(n)} = (-1)^n\ {}_4F_3\left(\begin{matrix}-i, i+1, -j, j+1\\ -n, n+2, 1\end{matrix}; 1\right)\frac{2i+1}{n+1};$$

$$S_{i,j}^{(n)} = ((-1)^j + (-1)^i) M_{i,j}^{(n)} + \delta_{i,j}.$$

Thus $S_{i,j}^{(n)} = 2M_{i,j}^{(n)} + \delta_{i,j}$ if both i, j are even, $S_{i,j}^{(n)} = -2M_{i,j}^{(n)} + \delta_{i,j}$ if both i, j are odd, and $S_{i,j}^{(n)} = 0$ if $i - j \equiv 1 \bmod 2$. □

Corollary 17. *For $0 \le k \le \frac{n}{2}$ each polynomial $r_{n,2k} := 2\sum_{0\le j\le n/2} M_{2j,2k}^{(n)} b_{n,2j} + b_{n,2k}$ is totally symmetric and for $0 \le k \le \frac{n-1}{2}$ each polynomial $r_{n,2k+1} = -2\sum_{0\le j\le (n-1)/2} M_{2j+1,2k+1}^{(n)} b_{n,2j+1} + b_{n,2k+1}$ satisfies $Mp = -p = Rp$ (the sign representation).*

Proof. The pattern of zeroes in $[M_{i,j}^{(n)}]$ shows that $r_{n,2k} = (MR + RM + I)b_{n,2k} \in \operatorname{span}\{b_{n,2j}\}$ and thus satisfies $Rr_{n,2k} = r_{n,2k}$; combined with $RMr_{n,2k} = r_{n,2k}$ this shows $r_{n,2k}$ is totally symmetric. A similar argument applies to $(MR + RM + I)b_{n,2k+1}$. □

Theorem 18. *The functions $b_{n,k}^{\mathrm{sym}}, 0 \le k \le d_{\mathrm{triv}}(n) - 1$, as in (17) form a basis for the totally symmetric polynomials in $\mathbb{P}_{n,n-1}^{\perp}(\widehat{T})$.*

Proof. We use the homogeneous form of the $b_{n,m}$ as in [13], that is, set

$$\begin{aligned} b'_{n,2m}(v) &= (v_1 + v_2 + v_3)^n b_{n,2m}\left(\frac{v_1}{v_1+v_2+v_3}, \frac{v_2}{v_1+v_2+v_3}\right)\\ &= (v_1+v_2+v_3)^{n-2m} P_{n-2m}^{(0,4m+1)}\left(\frac{v_1+v_2-v_3}{v_1+v_2+v_3}\right)\\ &\quad \times (v_1+v_2)^{2m} P_{2m}^{(0,0)}\left(\frac{v_1-v_2}{v_1+v_2}\right). \end{aligned}$$

Formally $b'_{n,j}(v) = (-1)^j (j!(n-j)!)^{-1}\theta_{n,j}(v)$ with $\theta_{n,j}$ as in [13, p. 690]. The expansion of such a polynomial is a sum of monomials $v_1^{n_1} v_2^{n_2} v_3^{n_3}$ with $\sum_{i=1}^{3} n_i = n$. Symmetrizing the monomial results in the sum of $v_1^{m_1} v_2^{m_2} v_3^{m_3}$ where (m_1, m_2, m_3) ranges over all permutations of (n_1, n_2, n_3). The argument is based on the occurrence of certain indices in $b_{n,m}$. For a more straightforward approach to the coefficients we use the following expansions (with $\ell = n-2k, \beta = 2k+1$):

$$\begin{aligned}
&(v_1+v_2+v_3)^{\ell} P_{\ell}^{(0,\beta)}\left(\frac{v_1+v_2-v_3}{v_1+v_2+v_3}\right) \\
&\quad = (-1)^{\ell}(v_1+v_2+v_3)^{\ell} P_{\ell}^{(\beta,0)}\left(\frac{-v_1-v_2+v_3}{v_1+v_2+v_3}\right) \\
&\quad = (-1)^{\ell}\frac{(\beta+1)_{\ell}}{\ell!}\sum_{i=0}^{\ell}\frac{(-\ell)_i(\ell+\beta+1)_i}{i!(\beta+1)_i} \\
&\qquad \times (v_1+v_2)^i(v_1+v_2+v_3)^{\ell-i};
\end{aligned} \tag{48}$$

and

$$(v_1+v_2)^{2k} P_{2k}^{(0,0)}\left(\frac{v_1-v_2}{v_1+v_2}\right) = \frac{1}{(2k)!}\sum_{j=0}^{2k}\frac{(-2k)_j(-2k)_j(-2k)_{2k-j}}{j!} v_2^j v_1^{2k-j}.$$

First let $n = 2m$. The highest power of v_3 that can occur in $b'_{2m,2m-2k}$ is $2k$, with corresponding coefficient $\frac{(4m-4k+1)_{2k}}{(2k)!}\sum_{j=0}^{2m-2k} c_j v_2^j v_1^{2m-j}$ for certain coefficients $\{c_j\}$. Recall that $d_{\mathrm{triv}}(n)$ is the number of solutions (i,j) of the equation $3j+2i = 2m$ (with $i,j = 0,1,2,\ldots$). The solutions can be listed as $(m,0),(m-3,2),(m-6,4)\cdots(m-3\ell,2\ell)$ where $\ell = d_{\mathrm{triv}}(n)-1$. By hypothesis $(m-3k,2k)$ occurs in the list and thus $m-3k \geq 0$ and $m-k \geq 2k$. There is only one possible permutation of $v_1^{m-k}v_2^{m-k}v_3^{2k}$ that occurs in $b'_{2m,2m-2k}$ and its coefficient is $\frac{(2k-2m)_{m-k}^3}{(2m-2k)!} \neq 0$. Hence there is a triangular pattern for the occurrence of $v_1^m v_2^m$, $v_1^{m-1}v_2^{m-1}v_3^2$, $v_1^{m-2}v_2^{m-2}v_3^4,\ldots$ in the symmetrizations of $b'_{2m,2m}$, $b'_{2m,2m-2},\ldots$ with nonzero numbers on the diagonal and this proves the basis property when $n = 2m$.

Now let $n = 2m+1$. The highest power of v_3 that can occur in $b'_{2m+1,2m-2k}$ is $2k+1$, with coefficient $\frac{(4m-4k+1)_{2k+1}}{(2k+1)!}\sum_{j=0}^{2m-2k} c_j v_2^j v_1^{2m-j}$ for certain coefficients $\{c_j\}$. The solutions of $3j+2i = 2m+1$ can be listed as $(m-1,1),(m-4,3),(m-7,5)\cdots(m-1-3\ell,2\ell+1)$ where $\ell = d_{\mathrm{triv}}(n)-1$. By hypothesis $(m-1-3k,2k+1)$ occurs in this list, thus $m-k \geq 2k+1$. There is only one possible permutation of $v_1^{m-k}v_2^{m-k}v_3^{2k+1}$ that occurs in $b'_{2m+1,2m-2k}$ and its coefficient is $\frac{(2k-2m)_{m-k}^3}{(2m-2k)!} \neq 0$. As above, there is a triangular pattern for the occurrence of $v_1^m v_2^m v_3$, $v_1^{m-1}v_2^{m-1}v_3^3$, $v_1^{m-2}v_2^{m-2}v_3^5,\ldots$ in the symmetrizations of $b'_{2m+1,2m}$, $b'_{2m+1,2m-2},\ldots$ with nonzero numbers on the diagonal and this proves the basis property when $n = 2m+1$. □

The totally symmetric orthogonal polynomials can be lifted to a general triangle T.

Definition 19. Let T denote a triangle. The space of totally symmetric, orthogonal polynomials of degree n is

$$\begin{aligned}
\mathbb{P}_{n,n-1}^{\perp,\mathrm{sym}}(T) &:= \left\{u \in \mathbb{P}_{n,n-1}^{\perp}(T) : u \text{ has total symmetry}\right\} \quad (49)\\
&= \mathrm{span}\{b_{n,m}^{T,\mathrm{sym}} : 0 \leq m \leq d_{\mathrm{triv}}(n)-1\}, \quad (50)
\end{aligned}$$

where the lifted symmetric basis functions are given by $b_{n,m}^{T,\mathrm{sym}} := b_{n,m}^{\mathrm{sym}} \circ \chi_T^{-1}$ for $b_{n,m}^{\mathrm{sym}}$ as in Theorem 18 and an affine pullback $\chi_T : \widehat{T} \to T$.

5.3.3. *A basis for the τ_{refl} component of $\mathbb{P}_{n,n-1}^{\perp}(T)$*

As explained in Sec. 5.3.1 the space $\mathbb{P}_{n,n-1}^{\perp}(\widehat{T})$ can be decomposed into the τ_{triv}-, the τ_{sign}- and the τ_{refl}-component. A basis for the τ_{triv} component are the fully symmetric basis functions (cf. Sec. 5.3.2).

Next, we will construct a basis for all of $\mathbb{P}_{n,n-1}^{\perp}(\widehat{T})$ by extending the totally symmetric one. It is straightforward to adjoin the $d_{\mathrm{sign}}(n)$ basis, using the same technique as for the fully symmetric ones: the monomials which appear in p with $Rp = -p = Mp$ must be permutations of $v_1^{n_1} v_2^{n_2} v_3^{n_3}$ with $n_1 > n_2 > n_3$. As in Theorem 18 for $n = 2m$ argue on monomials $v_1^{m-k} v_2^{m-1-k} v_3^{2k+1}$ and the polynomials $b'_{2m,2m-2k-1}$ with $0 \le k \le d_{\mathrm{sign}}(n) - 1 = d_{\mathrm{triv}}(n) - 2$, and for $n = 2m+1$ use the monomials $v_1^{m+1-k} v_2^{m-k} v_3^{2k}$ and $b_{2m+1,2m-2k}$ with $0 \le k \le d_{\mathrm{triv}}(n) - 1 = d_{\mathrm{sign}}(n) - 1$.

As we will see when constructing a basis for the non-conforming finite element space, the τ_{sign} component of $\mathbb{P}_{n,n-1}^{\perp}(\widehat{T})$ is not relevant, in contrast to the τ_{refl} component. In this section, we will construct a basis for the τ_{refl} polynomials in $\mathbb{P}_{n,n-1}^{\perp}(\widehat{T})$. Each such polynomial is an eigenvector of $RM + MR$ with eigenvalue -1. We will show that the polynomials

$$b_{n,k}^{\mathrm{refl}} = \frac{1}{3}(2I - RM - MR) b_{n,2k}, \quad 0 \le k \le \frac{n-1}{3}, \tag{51}$$

are linearly independent (and the same as introduced in (21)) and, subsequently, that the set

$$\left\{ RM b_{n,k}^{\mathrm{refl}}, MR b_{n,k}^{\mathrm{refl}} : 0 \le k \le \frac{n-1}{3} \right\} \tag{52}$$

is a basis for the τ_{refl} subspace of $\mathbb{P}_{n,n-1}^{\perp}(\widehat{T})$. (The upper limit of k is as in (52) $d_{\mathrm{refl}}(n) - 1$ (cf. (22)).) Note that

$$RM b_{n,k}^{\mathrm{refl}} = \frac{1}{3}(2RM - MR - I) b_{n,2k}, \quad MR b_{n,k}^{\mathrm{refl}} = \frac{1}{3}(2MR - I - RM) b_{n,2k}, \tag{53}$$

because $(RM)^2 = MR$. Thus the calculation of these polynomials follows directly from the formulae for $[M_{ij}]$ and $[R_{ij}]$. The method of proof relies on complex coordinates for the triangle.

Lemma 20. *For $k = 0, 1, 2, \ldots$*

$$P_{2k}^{(0,0)}(s) = (-1)^k \frac{\left(k + \frac{1}{2}\right)_k}{k!} \sum_{j=0}^{k} \frac{(-k)_j^2}{j! \left(\frac{1}{2} - 2k\right)_j} (1 - s^2)^{k-j},$$

$$(v_1+v_2)^{2k}P_{2k}^{(0,0)}\left(\frac{v_1-v_2}{v_1+v_2}\right)=(-1)^k\frac{\left(k+\frac{1}{2}\right)_k}{k!}$$
$$\times\sum_{j=0}^{k}\frac{(-k)_j^2}{j!\left(\frac{1}{2}-2k\right)_j}4^{k-j}(v_1v_2)^{k-j}(v_1+v_2)^{2j}.$$

Proof. Start with the formula (specialized from a formula for Gegenbauer polynomials [17, 18.5.10])

$$P_{2k}^{(0,0)}(s)=(2s)^{2k}\frac{\left(\frac{1}{2}\right)_{2k}}{(2k)!}\,{}_2F_1\left(\begin{matrix}-k,\frac{1}{2}-k\\ \frac{1}{2}-2k\end{matrix};\frac{1}{s^2}\right).$$

Apply the transformation (cf. [17, 15.8.1])

$${}_2F_1\left(\begin{matrix}-k,b\\ c\end{matrix};t\right)=(1-t)^k\,{}_2F_1\left(\begin{matrix}-k,c-b\\ c\end{matrix};\frac{t}{t-1}\right)$$

with $t=1/s^2$; then $\frac{t}{t-1}=\frac{1}{1-s^2}$ and $s^{2k}(1-\frac{1}{s^2})^k=(-1)^k(1-s^2)^k$. Also $2^{2k}\frac{(\frac{1}{2})_{2k}}{(2k)!}=\frac{(\frac{1}{2})_{2k}}{k!(\frac{1}{2})_k}=\frac{(k+\frac{1}{2})_k}{k!}$. This proves the first formula. Set $s=\frac{v_1-v_2}{v_1+v_2}$ then $1-s^2=\frac{4v_1v_2}{(v_1+v_2)^2}$ to obtain the second one. □

Introduce complex homogeneous coordinates:

$$z=\omega v_1+\omega^2v_2+v_3,$$
$$\overline{z}=\omega^2v_1+\omega v_2+v_3,$$
$$t=v_1+v_2+v_3.$$

Recall $\omega=e^{2\pi\mathrm{i}/3}=-\frac{1}{2}+\frac{\mathrm{i}}{2}\sqrt{3}$ and $\omega^2=\overline{\omega}$. The inverse relations are

$$v_1=\frac{1}{3}(-(\omega+1)z+\omega\overline{z}+t),$$
$$v_2=\frac{1}{3}(\omega z-(\omega+1)\overline{z}+t),$$
$$v_3=\frac{1}{3}(z+\overline{z}+t).$$

Suppose $f(z,\overline{z},t)$ is a polynomial in z and $\bar{z}$ then $Rf(z,\overline{z},t)=f(\overline{z},z,t)$ and $Mf(z,\overline{z},t)=f(\omega\overline{z},\omega^2z,t)$. Thus $RMf(z,\overline{z},t)=f(\omega^2z,\omega\overline{z},t)$ and $MRf(z,\overline{z},t)=f(\omega z,\omega^2\overline{z},t)$. The idea is to write $b_{n,2k}$ in terms of $z,\overline{z},t$ and apply the projection $\Pi:=\frac{1}{3}(2I-MR-RM)$. To determine linear independence it suffices to consider the terms of highest degree in $z,\overline{z}$ thus we set $t=v_1+v_2+v_3=0$ in the formula for $b_{n,2k}$ (previously denoted $b'_{n,2k}$ using the homogeneous coordinates, see proof of Theorem 18). From formula (48) and Lemma 20

$$\begin{aligned} b'_{n,2k}(v_1,v_2,0) &= (n-2k+2)_{n-2k}(v_1+v_2)^{n-2k}(-1)^k \frac{\left(k+\frac{1}{2}\right)_k}{k!} \\ &\quad \times \sum_{j=0}^{k} \frac{(-k)_j^2}{j!\left(\frac{1}{2}-2k\right)_j} 4^{k-j}(v_1v_2)^{k-j}(v_1+v_2)^{2j}. \end{aligned}$$

The coefficient of $(v_1v_2)^k(v_1+v_2)^{n-2k}$ in $b'_{n,2k}(v_1,v_2,0)$ is nonzero, and this is the term with highest power of v_1v_2. Thus $\{b'_{n,2k}(v_1,v_2,0) : 0 \le k \le \frac{n-2}{3}\}$ is a basis for $\operatorname{span}\{(v_1v_2)^k(v_1+v_2)^{n-2k} : 0 \le k \le \frac{n-2}{3}\}$. The next step is to show that the projection Π has trivial kernel. In the complex coordinates $v_1+v_2 = -\frac{1}{3}(z+z-t) = -\frac{1}{3}(z+z)$ and $v_1v_2 = \frac{1}{9}(z^2 - z\overline{z} + \overline{z}^2)$ (discarding terms of lower-order in $z, \overline{z}$, that is, set $t=0$).

Proposition 21. *If* $\Pi\sum_{k=0}^{\lfloor (n-1)/3\rfloor} c_k(z+\overline{z})^{n-2k}(z^2-z\overline{z}+\overline{z}^2)^k = 0$ *then* $c_k = 0$ *for all* k.

Proof. For any polynomial $f(z,\overline{z})$ we have $\Pi f(z,\overline{z}) = \frac{1}{3}(2f(z,\overline{z}) - f(\omega^2 z, \omega\overline{z}) - f(\omega z, \omega^2\overline{z}))$. In particular

$$\begin{aligned} \Pi(z+\overline{z})^{n-2k}(z^2-z\overline{z}+\overline{z}^2)^k &= \Pi(z+\overline{z})^{n-3k}(z^3+\overline{z}^3)^k \\ &= \frac{1}{3}\{2(z+\overline{z})^{n-3k} - (\omega^2 z+\omega\overline{z})^{n-3k} \\ &\quad - (\omega z+\omega^2\overline{z})^{n-3k}\}(z^3+\overline{z}^3)^k. \end{aligned}$$

By hypothesis $n-3k \ge 1$. Evaluate the expression at $z = e^{\pi\mathrm{i}/6}+\varepsilon$ where ε is real and near 0. Note $e^{\pi\mathrm{i}/6} = \frac{1}{2}(\sqrt{3}+\mathrm{i})$. Then

$$\begin{aligned} z+\overline{z} &= \sqrt{3}+2\varepsilon, \\ \omega^2 z+\omega\overline{z} &= -\varepsilon, \\ \omega z+\omega^2\overline{z} &= -\sqrt{3}-\varepsilon, \\ z^3+\overline{z}^3 &= 3\varepsilon+3\sqrt{3}\varepsilon^2+2\varepsilon^3, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{3}\{2(z+\overline{z})^{n-3k} - (\omega^2 z+\omega\overline{z})^{n-3k} - (\omega z+\omega^2\overline{z})^{n-3k}\}(z^3+\overline{z}^3)^k \\ &\quad = \frac{1}{3}\{(2-(-1)^{n-3k})\times 3^{(n-3k)/2} - (-\varepsilon)^{n-3k} + C\varepsilon + O(\varepsilon^2)\} \\ &\qquad \times \varepsilon^k(3+3\sqrt{3}\varepsilon+2\varepsilon^2)^k, \end{aligned}$$

where $C = 3^{(n-3k+1)/2}(n-3k)(4-2(-1)^{n-3k})$ (binomial theorem). The dominant term in the right-hand side is $(2-(-1)^{n-3k})3^{(n-k)/2-1}\varepsilon^k$. Now suppose

$\Pi \sum_{k=0}^{\lfloor (n-1)/3 \rfloor} c_k (z+\overline{z})^{n-2k}(z^2 - z\overline{z} + \overline{z}^2)^k = 0$. Evaluate the polynomial at $z = e^{\pi \mathrm{i}/6} + \varepsilon$. Let $\varepsilon \to 0$ implying $c_0 = 0$. Indeed write the expression as

$$\sum_{k=0}^{\lfloor (n-1)/3 \rfloor} c_k (2 - (-1)^{n-3k}) 3^{(n-k)/2-1} \varepsilon^k (1 + O(\varepsilon)) = 0.$$

Since $2 - (-1)^{n-3k} \geq 1$ this shows $c_k = 0$ for all k. $\square$

We have shown:

Proposition 22. *Suppose* $\Pi \sum_{k=0}^{\lfloor (n-1)/3 \rfloor} c_k b_{n,2k} = 0$ *then* $c_k = 0$ *for all* k; *the cardinality of the set* (52) *is* $d_{\mathrm{refl}}(n)$.

Theorem 23. (a) *The polynomials* $\{\Pi b_{n,2k} : 0 \leq k \leq \frac{n-1}{3}\}$ *are linearly independent.*

(b) *The set* $\{RM\Pi b_{n,2k}, MR\Pi b_{n,2k} : 0 \leq k \leq \frac{n-1}{3}\}$ *is linearly independent and defines a basis for the* τ_{refl} *component of* $\mathbb{P}^{\perp}_{n,n-1}(\widehat{T})$.

Proof. In general $\Pi z^a \overline{z}^b = z^a \overline{z}^b$ if $a - b \equiv 1, 2 \bmod 3$ and $\Pi z^a \overline{z}^b = 0$ if $a - b \equiv 0 \bmod 3$. Expand the polynomials $w_k(z, \overline{z}) := \Pi (z+\overline{z})^{n-3k}(z^3+\overline{z}^3)^k$ by the binomial theorem to obtain

$$\Pi (z+\overline{z})^{n-3k}(z^3+\overline{z}^3)^k = \sum_{\substack{j=0 \\ n-2j \equiv 1,2 \bmod 3}}^{n-3k} \binom{n-3k}{j} z^{n-3k-j} \overline{z}^j (z^3 + \overline{z}^3)^k.$$

Then

$$RMw_k(z, \overline{z}) = \sum_{j=0, n-2j \equiv 1,2 \bmod 3}^{n-3k} \binom{n-3k}{j} \omega^{2j-n} z^{n-3k-j} \overline{z}^j (z^3 + \overline{z}^3)^k,$$

$$MRw_k(z, \overline{z}) = \sum_{j=0, n-2j \equiv 1,2 \bmod 3}^{n-3k} \binom{n-3k}{j} \omega^{n-2j} z^{n-3k-j} \overline{z}^j (z^3 + \overline{z}^3)^k.$$

Firstly we show that $\{RMw_k, MRw_k\}$ is linearly independent for $0 \leq k \leq \frac{n-1}{3}$. For each value of $n \bmod 3$ we select the highest degree terms from RMw_k and MRw_k: (i) $n = 3m+1$, $\omega^2 z^{3m+1} + \omega \overline{z}^{3m+1}$ and $\omega z^{3m+1} + \omega^2 \overline{z}^{3m+1}$, (ii) $n = 3m+2$, $\omega z^{3m+2} + \omega^2 \overline{z}^{3m+2}$ and $\omega^2 z^{3m+2} + \omega \overline{z}^{3m+2}$, (iii) $n = 3m$, $(n-3k)(\omega^2 z^{3m}\overline{z} + \omega z \overline{z}^{3m})$ and $(n-3k)(\omega z^{3m}\overline{z} + \omega^2 z \overline{z}^{3m})$ (by hypothesis $n - 3k \geq 1$). In each case the two terms are linearly independent (the determinant of the coefficients is $\pm(\omega - \omega^2) = \mp \mathrm{i}\sqrt{3}$). Secondly the same argument as in the previous theorem shows that $\sum_{k=0}^{\lfloor (n-1)/3 \rfloor} \{c_k RMw_k + d_k MRw_k\} = 0$ implies $c_k RMw_k + d_k MRw_k = 0$ for all k. By the first part it follows that $c_k = 0 = d_k$. This completes the proof. $\square$

Remark 24. The basis $b_{n,k}$ for $\mathbb{P}^{\perp}_{n,n-1}(\widehat{T})$ in (35) is mirror symmetric with respect to the angular bisector in $\widehat{T}$ through the origin for even k and is mirror skew-symmetric for odd k. This fact makes the point $\mathbf{0}$ in $\widehat{T}$ special compared to the

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other vertices. As a consequence the functions defined in Theorem 23(a) reflects the special role of $\mathbf{0}$. Part (b) shows that it is possible to define a basis with functions which are either symmetric with respect to the angle bisector in $\widehat{T}$ through $(1,0)^\intercal$ or through $(0,1)^\intercal$ by "rotating" the functions $\Pi b_{n,2k}$ to these vertices:

$$RM(\Pi b_{n,2k})(x_1,x_2) = (\Pi b_{n,2k})(x_2, 1-x_1-x_2) \quad \text{and}$$

$$MR(\Pi b_{n,2k})(x_1,x_2) = (\Pi b_{n,2k})(1-x_1-x_2, x_1).$$

Since the dimension of $E^{(\text{refl})}$ is $2d_{\text{refl}}(n) = 2\lfloor\frac{n+2}{3}\rfloor$ is not (always) a multiple of 3, it is, in general, not possible to define a basis where all three vertices of the triangle are treated in a symmetric way.

Definition 25. Let

$$\mathbb{P}^{\perp,\text{refl}}_{n,n-1}(\widehat{T}) := \operatorname{span}\left\{RM\Pi b_{n,2k}, MR\Pi b_{n,2k} : 0 \leq k \leq \frac{n-1}{3}\right\}. \tag{54}$$

This space is lifted to a general triangle T by fixing a vertex $\mathbf{P}$ of T and setting

$$\mathbb{P}^{\perp,\text{refl}}_{n,n-1}(T) := \left\{u \circ \chi^{-1}_{\mathbf{P},T} : u \in \mathbb{P}^{\perp,\text{refl}}_{n,n-1}\left(\widehat{T}\right)\right\}, \tag{55}$$

where the lifting $\chi_{\mathbf{P},T}$ is an affine pullback $\chi_{\mathbf{P},T} : \widehat{T} \to T$ which maps $\mathbf{0}$ to $\mathbf{P}$.

The basis $b^{\text{refl}}_{n,k}$ to describe the restrictions of facet-oriented, non-conforming finite element functions to the facets is related to a reduced space and defined as in (51) with lifted versions

$$b^{\mathbf{P},T}_{n,k} := b^{\text{refl}}_{n,k} \circ \chi^{-1}_{\mathbf{P},T}, \quad 0 \leq k \leq \frac{n-1}{3}. \tag{56}$$

Remark 26. The construction of the spaces $\mathbb{P}^{\perp,\text{sym}}_{p,p-1}(T)$ and $\mathbb{P}^{\perp,\text{refl}}_{p,p-1}(T)$ (cf. Definitions 19 and 25) implies the direct sum decomposition

$$\operatorname{span}\left\{b_{p,2k} \circ \chi^{-1}_{\mathbf{P},T} : 0 \leq k \leq \lfloor p/2 \rfloor\right\} = \mathbb{P}^{\perp,\text{sym}}_{p,p-1}(T) \oplus \mathbb{P}^{\perp,\text{refl}}_{p,p-1}(T). \tag{57}$$

It is easy to verify that the basis functions $b^{\mathbf{P},T}_{p,k}$ are mirror symmetric with respect to the angle bisector in T through $\mathbf{P}$. However, the space $\mathbb{P}^{\perp,\text{refl}}_{n,n-1}(T)$ is independent of the choice of the vertex $\mathbf{P}$.

In Appendix A, we will define further sets of basis functions for the τ_{refl} component of $\mathbb{P}^{\perp}_{n,n-1}(\widehat{T})$ — different choices might be preferable for different kinds of applications.

5.4. *Simplex-supported and facet-oriented non-conforming basis functions*

In this section, we will define non-conforming Crouzeix–Raviart type functions which are supported either on one single tetrahedron or on two tetrahedrons which share a common facet. As a prerequisite, we study in Sec. 5.4.1 piecewise orthogonal polynomials on triangle stars, i.e. on a collection of triangles which share a common vertex and cover a neighborhood of this vertex (see Notation 27). We will derive conditions such that these functions are continuous across common edges and determine the dimension of the resulting space. This allows us to determine the non-conforming Courzeix–Raviart basis functions which are either supported on a single tetrahedron (see Sec. 5.4.2) or on two adjacent tetrahedrons (see Sec. 5.4.3) by "closing" triangle stars either by a single triangle or another triangle star.

5.4.1. *Orthogonal polynomials on triangle stars*

The construction of the functions $B_{p,k}^{K,\mathrm{nc}}$ and $B_{p,k}^{T,\mathrm{nc}}$ as in (20) and (24) requires some results of continuous, piecewise orthogonal polynomials on *triangle stars* which we provide in this section.

Notation 27. A subset $C \subset \Omega$ is a *triangle star* if C is the union of some, say $m_C \geq 3$, triangles $T \in \mathcal{F}_C \subset \mathcal{F}$, i.e. $C = \bigcup_{T\in\mathcal{F}_C} T$ and there exists some vertex $\mathbf{V}_C \in \mathcal{V}$ such that

$$\begin{aligned} &\mathbf{V}_C \text{ is a vertex of } T && \forall T \in \mathcal{F}_C, \\ &\exists \text{ a continuous, piecewise affine mapping } \chi : D_{m_C} \to C && \text{such that } \chi(0) = \mathbf{V}_C. \end{aligned} \tag{58}$$

Here, D_k denotes the regular closed k-gon (in $\mathbb{R}^2$).

For a triangle star C, we define

$$\mathbb{P}_{p,p-1}^{\perp}(C) := \left\{ u \in C^0(C) \,|\, \forall T \in \mathcal{F}_C : u|_T \in \mathbb{P}_{p,p-1}^{\perp}(T) \right\}.$$

In the next step, we will explicitly characterize the space $\mathbb{P}_{p,p-1}^{\perp}(C)$ by defining a set of basis functions. Set $\mathbf{A} := \mathbf{V}_C$ (cf. (58)) and pick an outer vertex in $\mathcal{F}_C$, denote it by $\mathbf{A}_1$, and number the remaining vertices $\mathbf{A}_2, \ldots, \mathbf{A}_{m_C}$ in $\mathcal{F}_C$ counterclockwise. We use the cyclic numbering convention $\mathbf{A}_{m_C+1} := \mathbf{A}_1$ and also for similar quantities.

For $1 \leq \ell \leq m_C$, let $\mathbf{e}_\ell := [\mathbf{A}, \mathbf{A}_\ell]$ be the straight line (convex hull) between and including $\mathbf{A}$, $\mathbf{A}_\ell$. Let $T_\ell \in \mathcal{F}_C$ be the triangle with vertices $\mathbf{A}$, $\mathbf{A}_\ell$, $\mathbf{A}_{\ell+1}$. Then we choose the affine pullbacks to the reference element $\widehat{T}$ by

$$\chi_\ell(x_1, x_2) := \begin{cases} \mathbf{A} + x_1(\mathbf{A}_\ell - \mathbf{A}) + x_2(\mathbf{A}_{\ell+1} - \mathbf{A}) & \text{if } \ell \text{ is odd}, \\ \mathbf{A} + x_1(\mathbf{A}_{\ell+1} - \mathbf{A}) + x_2(\mathbf{A}_\ell - \mathbf{A}) & \text{if } \ell \text{ is even}. \end{cases}$$

In this way, the common edges $\mathbf{e}_\ell$ are parametrized by $\chi_{\ell-1}(t, 0) = \chi_\ell(t, 0)$ if $3 \leq \ell \leq m_C$ is odd and by $\chi_{\ell-1}(0, t) = \chi_\ell(0, t)$ if $2 \leq \ell \leq m_C$ is even. The final edge $\mathbf{e}_1$ is parametrized by $\chi_1(t, 0) = \chi_{m_C}(t, 0)$ if m_C is even and by $\chi_1(t, 0) = \chi_{m_C}(0, t)$ (with interchanged arguments!) otherwise. We introduce the set

$$R_{p,C} := \begin{cases} \{0, \ldots, p\} & \text{if } m_C \text{ is even}, \\ \left\{ 2\ell : 0 \leq \ell \leq \left\lfloor \dfrac{p}{2} \right\rfloor \right\} & \text{if } m_C \text{ is odd} \end{cases}$$

and define the functions (cf. (49), (55), (57))

$$b_{p,k}^C|_{T_\ell} := b_{p,k} \circ \chi_\ell^{-1}, \quad \forall\, k \in R_{p,C}. \tag{59}$$

Lemma 28. *For a triangle star C, a basis for $\mathbb{P}_{p,p-1}^{\perp}(C)$ is given by $b_{p,k}^C, k \in R_{p,C}$. Further*

$$\dim \mathbb{P}_{p,p-1}^{\perp}(C) = \begin{cases} p+1 & \text{if } m_C \text{ is even,} \\ \left\lfloor \dfrac{p}{2} \right\rfloor + 1 & \text{if } m_C \text{ is odd.} \end{cases} \tag{60}$$

Proof. We show that $(b_{p,k}^C)_{k \in R_{p,C}}$ is a basis of $\mathbb{P}_{p,p-1}^{\perp}(C)$ and the dimension formula.

Continuity across $\mathbf{e}_\ell$ for odd $3 \leq \ell \leq m_C$.

The definition of the lifted orthogonal polynomials (see (49), (55), (57)) implies that the continuity across $\mathbf{e}_\ell$ for odd $3 \leq \ell \leq m_C$ is equivalent to

$$\sum_{k=0}^{p} \alpha_{p,k}^{(\ell-1)} b_{p,k}^{\mathrm{I}} = \sum_{k=0}^{p} \alpha_{p,k}^{(\ell)} b_{p,k}^{\mathrm{I}}.$$

From Lemma 12 we conclude that the continuity across such edges is equivalent to

$$\alpha_{p,k}^{(\ell-1)} = \alpha_{p,k}^{(\ell)} \quad \forall\, 0 \leq k \leq p. \tag{61}$$

Continuity across $\mathbf{e}_\ell$ for even $2 \leq \ell \leq m_C$.

Note that $\chi_2(0,t) = \chi_3(0,t)$. Taking into account (49), (55), (57) we see that the continuity across $\mathbf{e}_\ell$ is equivalent to

$$\sum_{k=0}^{p} \alpha_{p,k}^{(2)} b_{p,k}^{\mathrm{II}} = \sum_{k=0}^{p} \alpha_{p,k}^{(3)} b_{p,k}^{\mathrm{II}}.$$

From Lemma 12 we conclude that the continuity across $\mathbf{e}_\ell$ for even $2 \leq \ell \leq m_C$ is again equivalent to

$$\alpha_{p,k}^{(\ell-1)} = \alpha_{p,k}^{(\ell)} \quad \forall\, 0 \leq k \leq p. \tag{62}$$

Continuity across $\mathbf{e}_1$

For even m_C the previous argument also applies for the edge $\mathbf{e}_1$ and the functions $b_{p,k}^C$, $0 \leq k \leq p$, are continuous across $\mathbf{e}_1$. For odd m_C, note that $\chi_1(t,0) = \chi_{m_C}(0,t)$. Taking into account (49), (55), (57) we see that the continuity across $\mathbf{e}_1$ is equivalent to

$$\sum_{k=0}^{p} \alpha_{p,k}^{(1)} b_{p,k}^{\mathrm{I}} = \sum_{k=0}^{p} \alpha_{p,k}^{(m_C)} b_{p,k}^{\mathrm{II}}.$$

Using the symmetry relation (37) we conclude that this is equivalent to

$$\sum_{k=0}^{p} \alpha_{p,k}^{(1)} b_{p,k}^{\mathrm{I}} = \sum_{k=0}^{p} \alpha_{p,k}^{(m_C)} (-1)^k b_{p,k}^{\mathrm{I}}.$$

From Lemma 12, we conclude that this, in turn, is equivalent to

$$\begin{aligned} \alpha_{p,k}^{(1)} &= \alpha_{p,k}^{(m_C)} \quad && k \text{ is even}, \\ \alpha_{p,k}^{(1)} &= -\alpha_{p,k}^{(m_C)} \quad && k \text{ is odd}. \end{aligned} \tag{63}$$

From the above reasoning, the continuity of $b_{p,k}^{C}$ across $\mathbf{e}_1$ follows if $\alpha_{n,k}^{(\ell)} = 0$ for odd k and all $1 \leq \ell \leq m_C$.

The proof of the dimension formula (60) is trivial. □

5.4.2. *A basis for the symmetric non-conforming space* $S_{K,\mathrm{nc}}^{p}$

In this section, we will prove that $S_{K,\mathrm{nc}}^{p}$ (cf. (20)) satisfies

$$S_{K,\mathrm{nc}}^{p} \oplus S_{K,\mathrm{c}}^{p} = S_{K}^{p} := \{u \in S_{\mathcal{G}}^{p} : \operatorname{supp} u \subset K\},$$

where $S_{\mathcal{G}}^{p}$ is defined in (4) and, moreover, that the functions $B_{p,k}^{K,\mathrm{nc}}$, $k = 0, 1, \ldots, d_{\mathrm{triv}}(p) - 1$, as in (18), (20) form a basis of $S_{K,\mathrm{nc}}^{p}$.

Let T denote one facet of K and let $C := \partial K \backslash \overset{\circ}{T}$. Since C is a triangle star with $m_C = 3$, we can apply Lemma 28 to obtain that

$$S_{K}^{p}|_{C} := \{u|_{C} : u \in S_{K}^{p}\} \subset \operatorname{span}\left\{b_{p,2k}^{C} : 0 \leq k \leq \left\lfloor \frac{p}{2} \right\rfloor\right\}.$$

The continuity of $b_{p,2k}^{C}$ implies that the restriction $b_{p,2k}^{\partial T} := b_{p,2k}^{C}|_{\partial T}$ is continuous. From (42) we conclude that

$$b_{p,2k}^{\partial T}|_{E} = P_{2k}^{E} \quad \forall\, E \subset \partial T, \tag{64}$$

where P_{2k}^{E} is the Legendre polynomial of even degree $2k$ scaled to the edge E with endpoint values $+1$ and symmetry with respect to the midpoint of E. Hence, we are looking for orthogonal polynomials $\mathbb{P}_{p,p-1}^{\perp}(T)$ whose traces on ∂T are linear combination of $b_{p,2k}^{\partial T}$, $0 \leq k \leq \lfloor \frac{p}{2} \rfloor$. From (37) we deduce that they have total symmetry, i.e. belong to the space $\mathbb{P}_{p,p-1}^{\perp,\mathrm{sym}}(T)$ (cf. Definition 19). For $0 \leq m \leq d_{\mathrm{triv}}(p) - 1$, let $b_{p,m}^{\partial K,\mathrm{sym}} : \partial K \to \mathbb{R}$ be defined facet-wise for any $T \subset \partial K$ by

$$b_{p,m}^{\partial K,\mathrm{sym}}|_{T} := b_{p,m}^{T,\mathrm{sym}} \quad 0 \leq m \leq d_{\mathrm{triv}}(p) - 1. \tag{65}$$

Finally, we extend the function $b_{p,m}^{\partial K,\mathrm{sym}}$ to the total simplex K by polynomial extension (cf. (18), (19))

$$B_{p,m}^{K,\mathrm{nc}} = \sum_{\mathbf{N} \in \mathcal{N}^{p} \cap \partial K} b_{p,m}^{\partial K,\mathrm{sym}}(\mathbf{N}) B_{p,\mathbf{N}}^{\mathcal{G}}|_{K} \quad 0 \leq m \leq d_{\mathrm{triv}}(p) - 1. \tag{66}$$

These functions are the same as those introduced in Definition 5. The above reasoning leads to the following Proposition.

Proposition 29. *For a simplex K, the space of non-conforming, simplex-supported Crouzeix–Raviart finite elements can be chosen as in* (20) *and the functions $B_{p,k}^{K,\mathrm{nc}}, 0 \leq k \leq d_{\mathrm{triv}}(p) - 1$ are linearly independent.*

5.4.3. *A basis for* $S^p_{T,\mathrm{nc}}$

Let $T \in \mathcal{F}_\Omega$ be an inner facet and $K_1, K_2 \in \mathcal{G}$ such that $T = K_1 \cap K_2$ and $\omega_T = K_1 \cup K_2$ (cf. (9)) with the convention that the unit normal $\mathbf{n}_T$ points into K_2. In this section, we will prove that a space $\tilde{S}^p_{T,\mathrm{nc}}$ which satisfies

$$\tilde{S}^p_{T,\mathrm{nc}} \oplus \left(\bigoplus_{i=1}^{2} S^p_{K_i,\mathrm{nc}}\right) \oplus \left(\bigoplus_{i=1}^{2} S^p_{K_i,\mathrm{c}}\right) \oplus S^p_{T,\mathrm{c}} = S^p_T := \{u \in S^p_{\mathcal{G}} : \operatorname{supp} u \subset \omega_T\} \tag{67}$$

can be chosen as $\tilde{S}^p_{T,\mathrm{nc}} := S^p_{T,\mathrm{nc}}$ (cf. (25)) and, moreover, that the functions $B^{T,\mathrm{nc}}_{p,k}$, $k = 0, 1, \ldots, d_{\mathrm{refl}}(p) - 1$, as in (24) form a basis of $S^p_{T,\mathrm{nc}}$.

Let $C_i := (\partial K_i)\backslash \overset{\circ}{T}$, $i = 1, 2$, denote the triangle star (cf. Notation 27) formed by the three remaining triangles of ∂K_i. We conclude from Lemma 28 that a basis for $\mathbb{P}^{\perp}_{p,p-1}(C_i)$ is given by $b^{C_i}_{p,2k}$, $0 \leq k \leq \lfloor \frac{p}{2} \rfloor$ (cf. (59)). Any function u in S^p_T satisfies

$$\begin{aligned}
\gamma_{K_i} u &\in \mathbb{P}_p(K_i) \qquad i = 1, 2, \\
(\gamma_{K_i} u)|_{T'} &\in \mathbb{P}^{\perp}_{p,p-1}(T') \quad \forall T' \subset C_i, \quad i = 1, 2, \\
[u]_T &\in \mathbb{P}^{\perp}_{p,p-1}(T).
\end{aligned} \tag{68}$$

Since any function in S^p_T is continuous on C_i, we conclude from Lemma 28 (with $m_{C_i} = 3$) that

$$u|_{C_i} \in \mathbb{P}^{\perp}_{p,p-1}(C_i) \quad \text{and} \quad \gamma_{K_i} u|_{\partial T} \in \operatorname{span}\left\{ b^{\partial T}_{p,2k} : 0 \leq k \leq \left\lfloor \frac{p}{2} \right\rfloor \right\} \quad i = 1, 2 \tag{69}$$

with $b^{\partial T}_{p,2k}$ as in (64).

To identify a space $\tilde{S}^p_{T,\mathrm{nc}}$ which satisfies (67) we consider the jump condition in (68) restricted to the boundary ∂T. The symmetry of the functions $b^{\partial T}_{p,2k}$ implies that $[u]_T \in \mathbb{P}^{\perp,\mathrm{sym}}_{p,p-1}(T)$, i.e. there is a function $q_1 \in S^p_{K_1,\mathrm{nc}}$(see (20)) such that $[u]_T = q_1|_T$ and $\tilde{u}$, defined by $\tilde{u}|_{K_1} = u_1 + q_1$ and $\tilde{u}|_{K_2} = u_2$, is continuous across T. On the other hand, all functions $u \in S^p_T$ whose restrictions $u|_{\omega_T}$ are discontinuous can be found in $S^p_{K_1,\mathrm{nc}} \oplus S^p_{K_2,\mathrm{nc}}$. In view of the direct sum in (67) we may thus assume that the functions in $\tilde{S}^p_{T,\mathrm{nc}}$ are continuous in ω_T.

To finally arrive at a direct decomposition of the space in the right-hand side of (67) we have to split the spaces $\mathbb{P}^{\perp}_{p,p-1}(C_i)$ into a direct sum of the spaces of totally symmetric orthogonal polynomials and the spaces introduced in Definition 25 and glue them together in a continuous way. We introduce the functions $b^{C_i,\mathrm{sym}}_{p,k} := b^{\partial K_i,\mathrm{sym}}_{p,k}|_{C_i}$, $0 \leq k \leq d_{\mathrm{triv}}(p) - 1$, with $b^{\partial K_i,\mathrm{sym}}_{p,k}$ as in (65) and define $b^{C_i,\mathrm{refl}}_{p,k}$, $0 \leq k \leq d_{\mathrm{refl}}(p) - 1$, piecewise by $b^{C_i,\mathrm{refl}}_{p,k}|_{T'} := b^{\mathbf{A}_i,T'}_{p,k}$ for $T' \subset C_i$ with $b^{\mathbf{A}_i,T'}_{p,k}$ as in (56). The mirror symmetry of $b^{\mathbf{A}_i,T'}_{p,k}$ with respect to the angular bisector in T' through $\mathbf{A}_i$ implies the continuity of $b^{C_i,\mathrm{refl}}_{p,k}$. Hence,

$$\begin{aligned}
\mathbb{P}^{\perp}_{p,p-1}(C_i) = \operatorname{span}&\left\{ b^{C_i,\mathrm{sym}}_{p,k}\Big|_{C_i} : 0 \leq k \leq d_{\mathrm{triv}}(p) - 1 \right\} \\
&\oplus \operatorname{span}\left\{ b^{C_i,\mathrm{refl}}_{p,k} : 0 \leq k \leq d_{\mathrm{refl}}(p) - 1 \right\}.
\end{aligned} \tag{70}$$

Since the traces of $b_{p,k}^{C_i,\mathrm{sym}}$ and $b_{p,k}^{C_i,\mathrm{refl}}$ at ∂T are continuous and are, from both sides, the same linear combinations of edge-wise Legendre polynomials of even degree, the gluing $b_{p,k}^{\partial\omega_T,\mathrm{sym}}|_{C_i} := b_{p,k}^{C_i,\mathrm{sym}}$ and $b_{p,k}^{\partial\omega_T,\mathrm{refl}}|_{\dot{C}_i} := b_{p,k}^{C_i,\mathrm{refl}}$, $i = 1, 2$, defines continuous functions on $\partial\omega_T$. Since the space $S_{T,\mathrm{nc}}^p$ must satisfy a direct sum decomposition (cf. (67)), it suffices to consider the functions $b_{p,k}^{\partial\omega_T,\mathrm{refl}}$ for the definition of $S_{T,\mathrm{nc}}^p$. The resulting non-conforming facet-oriented space $S_{T,\mathrm{nc}}^p$ was introduced in Definition 7 and $\tilde{S}_{T,\mathrm{nc}}^p$ can be chosen to be $S_{T,\mathrm{nc}}^p$.

Proposition 30. *For any* $u \in S_{T,\mathrm{nc}}^p$, *the following implication holds*

$$u|_T \in S_{T,\mathrm{nc}}^p\Big|_T \cap \mathbb{P}_{p,p-1}^{\perp}(T) \Rightarrow u = 0.$$

Proof. Assume there exists $u \in S_{T,\mathrm{nc}}^p$ with $u|_T \in S_{T,\mathrm{nc}}^p|_T \cap \mathbb{P}_{p,p-1}^{\perp}(T)$. Let K be a simplex adjacent to T. Then $u_K = u|_K$ satisfies $u_K|_{T'} \in \mathbb{P}_{p,p-1}^{\perp}(T')$ for all $T' \subset \partial K$ and, thus, $u_K \in S_{K,\mathrm{nc}}^p$. Since $S_{K,\mathrm{nc}}^p|_{T'} \cap S_{T,\mathrm{nc}}^p|_{T'} = \{0\}$ for $T' \in \partial K \backslash \overset{\circ}{T}$ we conclude that $u_K = 0$. $\square$

Note that Definition 7 and Proposition 30 neither imply *a priori* that the functions

$$B_{p,k}^{T,\mathrm{nc}}\Big|_K, \quad \forall T \subset \partial K, \quad k = 0, \ldots, d_{\mathrm{refl}}(p) - 1$$

are linearly independent nor that

$$\forall T \subset \partial K \quad \text{it holds} \sum_{T' \subset C} B_{p,m}^{T',\mathrm{nc}}\Big|_T = \mathbb{P}_{p,p-1}^{\perp,\mathrm{refl}}(T) \quad \text{for the triangle star } C = \partial K \backslash \overset{\circ}{T} \tag{71}$$

holds. These properties will be proved next. Recall the projection $\Pi = \frac{1}{3}(2I - MR - RM)$ from Proposition 21. We showed (Theorem 23(a)) that $\{b_{p,k}^{\mathrm{refl}} : 0 \leq k \leq \frac{p-1}{3}\}$ is linearly independent, where $b_{p,k}^{\mathrm{refl}} := \Pi b_{p,2k}$. Additionally $Rb_{p,k}^{\mathrm{refl}} = b_{p,k}^{\mathrm{refl}}$ which implies $b_{p,k}^{\mathrm{refl}}(0, x_1) = b_{p,k}^{\mathrm{refl}}(x_1, 0)$, and the restriction $x_1 \mapsto b_{p,k}^{\mathrm{refl}}(x_1, 1-x_1)$ is invariant under $x_1 \mapsto 1 - x_1$. For four non-coplanar points A_0, A_1, A_2, A_3 let K denote the tetrahedron with these vertices. For any k such that $0 \leq k \leq \frac{p-1}{3}$ define a piecewise polynomial on the faces of K as follows: choose a local (x_1, x_2)-coordinate system for $A_0A_1A_2$ so that the respective coordinates are $(0, 0), (1, 0), (0, 1)$, and define $Q_k^{(0)}$ on the facet equal to $b_{p,k}^{\mathrm{refl}}$. Similarly define $Q_k^{(0)}$ on $A_0A_2A_3$ and $A_0A_3A_1$ (with analogously chosen local (x_1, x_2)-coordinate systems), by the property $b_{p,k}^{\mathrm{refl}}(0, x_1) = b_{p,k}^{\mathrm{refl}}(x_1, 0)$. $Q_k^{(0)}$ is continuous at the edges A_0A_1, A_0A_2, and A_0A_3. The values at the boundary of the triangle star equal $b_{p,k}^{\mathrm{refl}}(x_1, 1-x_1)$; note the symmetry and thus the orientation of the coordinates on the edges A_1A_2, A_2A_3, A_3A_1 is immaterial. The value of $Q_k^{(0)}$ on the triangle $A_1A_2A_3$ is taken to be a degree p polynomial, totally symmetric, with values agreeing with $b_{p,k}^{\mathrm{refl}}(x_1, 1 - x_1)$ on each edge.

Similarly $Q_k^{(1)}, Q_k^{(2)}, Q_k^{(3)}$ are defined by taking A_1, A_2, A_3 as the center of the construction, respectively.

Theorem 31. (a) *The functions* $Q_k^{(i)}, 0 \leq k \leq d_{\mathrm{refl}}(p) - 1, i = 0, 1, 2, 3$ *are linearly independent.*

(b) *Property* (71) *holds.*

The proof involves a series of steps. The argument will depend on the values of the functions on the three rays A_0A_1, A_0A_2, A_0A_3, each one of them is given coordinates t so that $t = 0$ at A_0 and $t = 1$ at the other end-point. For a fixed k let $q(t) = b_{p,k}^{\mathrm{refl}}(t, 0)$, $\widehat{q}(t) = b_{p,k}^{\mathrm{refl}}(1 - t, 0)$ and $\widetilde{q}(t) = b_{p,k}^{\mathrm{refl}}(t, 1 - t)$.

Lemma 32. *Suppose* $0 \leq k \leq \frac{p-1}{3}$ *and* $0 \leq t \leq 1$ *then* $q(t) + \widehat{q}(t) + \widetilde{q}(t) = 0$.

Proof. The actions of RM and MR on polynomials $f(x_1, x_2)$ are given by $MRf(x_1, x_2) = f(1 - x_1 - x_2, x_1)$ and $RMf(x_1, x_2) = f(x_2, 1 - x_1 - x_2)$. Polynomials of τ_{refl}-type satisfy $f + RMf + MRf = 0$. Apply this relation to $b_{p,k}^{\mathrm{refl}}$ with $x_1 = t$ and $x_2 = 0$ with the result

$$b_{p,k}^{\mathrm{refl}}(t, 0) + b_{p,k}^{\mathrm{refl}}(1 - t, t) + b_{p,k}^{\mathrm{refl}}(0, 1 - t) = 0.$$

The fact that $b_{p,k}^{\mathrm{refl}}(x_1, x_2) = b_{p,k}^{\mathrm{refl}}(x_2, x_1)$ finishes the proof. □

Proof of Theorem 31. Consider the contribution of $Q_k^{(1)}$ to the values on the ray A_0A_1: because $Q_k^{(1)}$ is constructed taking the origin at A_1 and because of the reverse orientation of the ray we see that the value of $Q_k^{(1)}$ is given by $\widehat{q}$. The value of $Q_k^{(1)}$ on the ray A_0A_2 is $\widetilde{q}$ (by the symmetry of $\widetilde{q}$ the orientation of the ray does not matter). The other functions are handled similarly, and the contributions to the three rays are given in this table:

$$\left| \begin{array}{lcccc} & Q_k^{(0)} & Q_k^{(1)} & Q_k^{(2)} & Q_k^{(3)} \\ A_0A_1 & q & \widehat{q} & \widetilde{q} & \widetilde{q} \\ A_0A_2 & q & \widetilde{q} & \widehat{q} & \widetilde{q} \\ A_0A_3 & q & \widetilde{q} & \widetilde{q} & \widehat{q} \end{array} \right|.$$

We use $q_k, \widetilde{q}_k, \widehat{q}_k$ to denote the polynomials corresponding to $b_{p,k}^{\mathrm{refl}}$. Suppose that the linear combination $\sum_{k=0}^{\lfloor (p-1)/3 \rfloor} \sum_{i=0}^{3} c_{k,i} Q_k^{(i)} = 0$. Evaluate the sum on the three rays to obtain the equations:

$$\begin{aligned} 0 &= \sum_{k=0}^{\lfloor (p-1)/3 \rfloor} \{c_{k,0} q_k + c_{k,1} \widehat{q}_k + (c_{k,2} + c_{k,3}) \widetilde{q}_k\} \\ &= \sum_{k=0}^{\lfloor (p-1)/3 \rfloor} \{(c_{k,1} - c_{k,0}) \widehat{q}_k + (c_{k,2} + c_{k,3} - c_{k,0}) \widetilde{q}_k\}, \end{aligned}$$

$$\begin{aligned}0 &= \sum_{k=0}^{\lfloor (p-1)/3\rfloor} \{c_{k,0}q_k + c_{k,2}\widehat{q}_k + (c_{k,1}+c_{k,3})\widetilde{q}_k\}\\ &= \sum_{k=0}^{\lfloor (p-1)/3\rfloor} \{(c_{k,2}-c_{k,0})\widehat{q}_k + (c_{k,1}+c_{k,3}-c_{k,0})\widetilde{q}_k\},\\ 0 &= \sum_{k=0}^{\lfloor (p-1)/3\rfloor} \{c_{k,0}q_k + c_{k,3}\widehat{q}_k + (c_{k,1}+c_{k,2})\widetilde{q}_k\}\\ &= \sum_{k=0}^{\lfloor (p-1)/3\rfloor} \{(c_{k,3}-c_{k,0})\widehat{q}_k + (c_{k,1}+c_{k,2}-c_{k,0})\widetilde{q}_k\}.\end{aligned}$$

We used Lemma 32 to eliminate q_k from the equations. In Theorem 23(b) we showed the linear independence of $\{RMb^{\mathrm{refl}}_{p,k}, MRb^{\mathrm{refl}}_{p,k} : 0 \le k \le \frac{p-1}{3}\}$, and in Lemma 12 that the restriction map $f \mapsto f(x_1, 0)$ is an isomorphism from the orthogonal polynomials $\mathbb{P}^{\perp}_{p,p-1}$ to $\mathbb{P}_p([0,1])$. Thus the projection of the set is also linearly independent, that is, $\{\widetilde{q}_k, \widehat{q}_k : 0 \le k \le \frac{p-1}{3}\}$ is a linearly independent set of polynomials on $0 \le t \le 1$. This implies all the coefficients in the above equations vanish: the $\widehat{q}_k$ terms show $c_{k,0} = c_{k,1} = c_{k,2} = c_{k,3}$ and then the $\widetilde{q}_k$-terms show $2c_{k,0} - c_{k,0} = c_{k,0} = 0$.

To prove (71) it suffices to transfer the statement to the reference element $\widehat{T}$. The pullbacks of the restrictions $B^{T',\mathrm{nc}}_{p,m}|_T$, $T' \subset C$, are given by

$$b^{\mathrm{refl}}_{n,k} = \Pi b_{n,2k}, \quad \widetilde{b}^{\mathrm{refl}}_{n,k} := RM\Pi b_{n,2k}, \quad \widehat{b}^{\mathrm{refl}}_{n,k} := MR\Pi b_{n,2k}, \quad k = 0, \dots d_{\mathrm{refl}}(n) - 1. \tag{72}$$

Since $b^{\mathrm{refl}}_{n,k} \in \mathbb{P}^{\perp,\mathrm{refl}}_{n,n-1}(\widehat{T})$ (cf. (A.1)) it follows

$$\begin{aligned}\mathbb{P}^{\perp,\mathrm{refl}}_{n,n-1}(\widehat{T}) &\overset{(54)}{=} \operatorname{span}\left\{\widetilde{b}^{\mathrm{refl}}_{n,k}, \widehat{b}^{\mathrm{refl}}_{n,k} : 0 \le k \le d_{\mathrm{refl}}(n) - 1\right\}\\ &= \operatorname{span}\left\{b^{\mathrm{refl}}_{n,k}, \widetilde{b}^{\mathrm{refl}}_{n,k}, \widehat{b}^{\mathrm{refl}}_{n,k} : 0 \le k \le d_{\mathrm{refl}}(n) - 1\right\}.\end{aligned}$$ □

6. Properties of Non-Conforming Crouzeix–Raviart Finite Elements

The non-conforming Crouzeix–Raviart finite element space $S^p_{\mathcal{G},\mathrm{nc}}$ satisfies $S^p_{\mathcal{G},\mathrm{c}} \subsetneq S^p_{\mathcal{G},\mathrm{nc}} \subset S^p_{\mathcal{G}}$ (cf. Sec. 4.2). In this section, we will present a basis for $S^p_{\mathcal{G},\mathrm{nc}}$ and discuss whether the inclusion $S^p_{\mathcal{G},\mathrm{nc}} \subset S^p_{\mathcal{G}}$, in fact, is an equality.

6.1. *A basis for non-conforming Crouzeix–Raviart finite elements*

We have defined conforming and non-conforming sets of functions which are spanned by functions with local support. In this section, we will investigate the linear independence of these functions. We introduce the following spaces

$$S^p_{\mathrm{sym,nc}} := \bigoplus_{K\in\mathcal{G}} S^p_{K,\mathrm{nc}}, \quad S^p_{\mathrm{refl,nc}} := \bigoplus_{T\in\mathcal{F}_\Omega} S^p_{T,\mathrm{nc}},$$

where $S^p_{K,\mathrm{nc}}$ and $S^p_{T,\mathrm{nc}}$ are as in Definitions 5 and 7. For some $0 \le k \le d_{\mathrm{refl}}(p) - 1$, we introduce the subspace $S^{p,k}_{\mathrm{refl,nc}} \subset S^p_{\mathrm{refl,nc}}$ by

$$S^{p,k}_{\mathrm{refl,nc}} := \bigoplus_{T\in\mathcal{F}_\Omega} \left\{ B^{T,\mathrm{nc}}_{p,m} : 0 \le m \le k \right\}.$$

Further we will need the conforming finite element space (cf. (4), Definition 1), where the vertex-oriented functions are removed, i.e.

$$\tilde{S}^p_{\mathcal{G},\mathrm{c}} := \left(\bigoplus_{E\in\mathcal{E}_\Omega} S^p_{E,\mathrm{c}} \right) \oplus \left(\bigoplus_{T\in\mathcal{F}_\Omega} S^p_{T,\mathrm{c}} \right) \oplus \left(\bigoplus_{K\in\mathcal{G}} S^p_{K,\mathrm{c}} \right).$$

Theorem 33. *The sums*

$$S^p_{\mathcal{G},\mathrm{c}} \oplus S^p_{\mathrm{sym,nc}}, \quad S^p_{\mathrm{sym,nc}} \oplus S^p_{\mathrm{refl,nc}} \tag{73}$$

are direct. The sum

$$S^p_{\mathcal{G},\mathrm{c}} + S^p_{\mathrm{refl,nc}} \tag{74}$$

*is **not** direct. The sum*

$$\tilde{S}^p_{\mathcal{G},\mathrm{c}} \oplus S^p_{\mathrm{sym,nc}} \oplus S^{p,0}_{\mathrm{refl,nc}} \tag{75}$$

is direct.

Proof. Part 1. We prove that the sum $S^p_{\mathrm{sym,nc}} \oplus S^p_{\mathrm{refl,nc}}$ is direct.

From Proposition 30 we know that the sum $S^p_{T,\mathrm{nc}}|_T \oplus \mathbb{P}^\perp_{p,p-1}(T)$ is direct. Let $\Pi_T : L^2(T) \to \mathbb{P}_{p-1}(T)$ denote the $L^2(T)$ orthogonal projection. Since $\mathbb{P}_{p-1}(T)$ is the orthogonal complement of $\mathbb{P}^\perp_{p,p-1}(T)$ in $\mathbb{P}_p(T)$ and since $\mathbb{P}^\perp_{p,p-1}(T) \cap S^p_{T,\mathrm{nc}}|_T = \{0\}$, the restricted mapping $\Pi_T : S^p_{T,\mathrm{nc}}|_T \to \mathbb{P}_{p-1}(T)$ is injective and the functions $q^T_{p,k} := \Pi_T(B^{T,\mathrm{nc}}_{p,k}|_T)$, $0 \le k \le d_{\mathrm{refl}}(p) - 1$, are linearly independent and belong to $\mathbb{P}_{p-1}(T)$. We define the functionals

$$J^T_{p,k}(w) := \int_T w q^T_{p,k} \quad 0 \le k \le d_{\mathrm{refl}}(p) - 1.$$

Next we consider a general linear combination and show that the condition

$$\sum_{K\subset\mathcal{G}} \sum_{i=0}^{d_{\mathrm{triv}}(p)-1} \alpha^K_i B^{K,\mathrm{nc}}_{p,i} + \sum_{K\subset\mathcal{G}} \sum_{T'\subset\partial K} \sum_{j=0}^{d_{\mathrm{refl}}(p)-1} \beta^{T'}_j B^{T',\mathrm{nc}}_{p,j} \overset{!}{=} 0 \tag{76}$$

implies that all coefficients are zero.

We apply the functionals $J_{p,k}^T$ to (76) and use the orthogonality between $\mathbb{P}_{p,p-1}^{\perp}(T)$ and $q_{p,k}^T$ to obtain

$$\sum_{K\subset\mathcal{G}}\sum_{T'\subset\partial K}\sum_{j=0}^{d_{\mathrm{refl}}(p)-1}\beta_j^{T'}J_{p,k}^T(B_{p,j}^{T',\mathrm{nc}})\stackrel{!}{=}0. \tag{77}$$

For $T'\neq T$ it holds $J_{p,k}^T(B_{p,i}^{T',\mathrm{nc}})=0$ since $B_{p,i}^{T',\mathrm{nc}}|_K|_T$ is an orthogonal polynomial. Thus, Eq. (77) is equivalent to

$$\sum_{j=0}^{d_{\mathrm{refl}}(p)-1}\beta_j^T J_{p,k}^T\left(B_{p,j}^{T,\mathrm{nc}}\right)\stackrel{!}{=}0. \tag{78}$$

The matrix $(J_{p,k}^T(B_{p,j}^{T,\mathrm{nc}}))_{k,j=0}^{d_{\mathrm{refl}}(p)-1}$ is regular because

$$J_{p,k}^T\left(B_{p,j}^{T,\mathrm{nc}}\right)=\int_T B_{p,j}^{T,\mathrm{nc}}q_{p,k}^T=\int_T B_{p,j}^{T,\mathrm{nc}}\Pi_T\left(B_{p,k}^{T,\mathrm{nc}}\Big|_T\right)=\int_T B_{p,j}^{T,\mathrm{nc}}B_{p,k}^{T,\mathrm{nc}}$$

and $(B_{p,k}^{T,\mathrm{nc}}|_T)_k$ are linearly independent. Hence we conclude from (78) that all coefficients β_j^T are zero and the condition (76) reduces to

$$\sum_{K\subset\mathcal{G}}\sum_{i=0}^{d_{\mathrm{triv}}(p)-1}\alpha_i^K B_{p,i}^{K,\mathrm{nc}}\stackrel{!}{=}0.$$

The left-hand side is a piecewise continuous function so that the condition is equivalent to $\sum_{i=0}^{d_{\mathrm{triv}}(p)-1}\alpha_i^K B_{p,i}^{K,\mathrm{nc}}\stackrel{!}{=}0$ for all $K\in\mathcal{G}$. Since $B_{p,i}^{K,\mathrm{nc}}$ is a basis for $S_{K,\mathrm{nc}}^p|_K$ we conclude that all α_i^K are zero.

Part 2. Next we prove that $(S_{\mathrm{sym,nc}}^p\oplus S_{\mathrm{refl,nc}}^{p,0})\cap\tilde{S}_{\mathcal{G},\mathrm{c}}^p=\{0\}$ and we show this by contradiction. Let $u\in(S_{\mathrm{sym,nc}}^p\oplus S_{\mathrm{refl,nc}}^{p,0})\cap\tilde{S}_{\mathcal{G},\mathrm{c}}^p$ which satisfies $u\neq0$. We decompose $u=u_{\mathrm{sym}}+u_{\mathrm{refl}}$ with $u_{\mathrm{sym}}\in S_{\mathrm{sym,nc}}^p$ and $u_{\mathrm{refl}}\in S_{\mathrm{refl,nc}}^p$. We prove by contradiction that $u_{\mathrm{sym}}\in C^0(\Omega)$. Assume that $u_{\mathrm{sym}}\notin C^0(\Omega)$. Then, there exists a facet $T\subset\mathcal{F}_\Omega$ such that $[u_{\mathrm{sym}}]_T\neq0$. Then, $[u_{\mathrm{refl}}]_T=-[u_{\mathrm{sym}}]_T$ is a necessary condition for the continuity of u. However, $[u_{\mathrm{sym}}]_T\in\mathbb{P}_{p,p-1}^{\perp,\mathrm{sym}}(T)$ while $[u_{\mathrm{refl}}]_T\in\mathbb{P}_{p,p-1}^{\perp,\mathrm{refl}}(T)$ and there is a contradiction because $\mathbb{P}_{p,p-1}^{\perp,\mathrm{sym}}(T)\cap\mathbb{P}_{p,p-1}^{\perp,\mathrm{refl}}(T)=\{0\}$. Hence, $u_{\mathrm{sym}}\in C^0(\Omega)$ and, in turn, $u_{\mathrm{refl}}\in C^0(\Omega)$.

Since $u\neq0$, at least, one of the functions u_{sym} and u_{refl} must be different from the zero function.

Case (a) We show $u_{\mathrm{sym}}=0$ by contradiction: Assume $u_{\mathrm{sym}}\neq0$. Then, $u_{\mathrm{sym}}|_T\neq0$ for all facets $T\in\mathcal{F}$. (Proof by contradiction: If $u_{\mathrm{sym}}|_T=0$ for some $T\in\mathcal{F}$, we pick some $K\in\mathcal{F}$ which has T as a facet. Since $u_{\mathrm{sym}}|_K\in S_{K,\mathrm{nc}}^p|_K$ we have $u_{\mathrm{sym}}|_{T'}=0$ for all facets T' of K and $u_{\mathrm{sym}}|_K=0$. Since u_{sym} is continuous in Ω, the restriction $u_{\mathrm{sym}}|_{K'}$ is zero for any $K'\in\mathcal{G}$ which shares a facet with K. This argument can be applied inductively to show that $u_{\mathrm{sym}}=0$ in Ω. This is a contradiction.) We pick a boundary facet $T\in\mathcal{F}_{\partial\Omega}$. The condition $u\in\tilde{S}_{\mathcal{G},\mathrm{c}}^p$ implies $u=0$ on $\partial\Omega$ and, in particular, $u|_T=u_{\mathrm{sym}}|_T+u_{\mathrm{refl}}|_T=0$. We use again the argument $\mathbb{P}_{p,p-1}^{\perp,\mathrm{sym}}(T) \cap \mathbb{P}_{p,p-1}^{\perp,\mathrm{refl}}(T) = \{0\}$ which implies $u_{\mathrm{sym}} = 0$ and this is a contradiction to the assumption $u_{\mathrm{sym}} \neq 0$.

Case (b) From Case (a) we know that $u_{\mathrm{sym}} = 0$, i.e. $u_{\mathrm{refl}} = u$, and it remains to show $u_{\mathrm{refl}} = 0$. The condition $u_{\mathrm{refl}} \in \tilde{S}_{\mathcal{G},\mathrm{c}}^{p}$ implies $u_{\mathrm{refl}}|_{\partial\Omega} = 0$ and $u_{\mathrm{refl}}(\mathbf{V}) = 0$ for all vertices $\mathbf{V} \in \mathcal{V}$.

The proof of Case (b) is similar than the proof of Case (a) and we start by showing for a tetrahedron, say K, with a facet on the boundary that $u_{\mathrm{refl}}|_K = 0$ and employ an induction over adjacent tetrahedrons to prove that $u_{\mathrm{refl}} = 0$ on every tetrahedron in $\mathcal{G}$.

We consider a boundary facet $T_0 \in \mathcal{F}_{\partial\Omega}$ with adjacent tetrahedron $K \subset \mathcal{G}$. We denote the three other facets of K by T_i, $1 \leq i \leq 3$, and for $0 \leq i \leq 3$, the vertex of K which is opposite to T_i by $\mathbf{A}_i$.

Case (b1) First we consider the case that there is one and only one *other* facet, say, T_1 which lies in $\partial\Omega$. Then $u_{\mathrm{refl}}|_T = u_2|_T + u_3|_T$ for some $u_i \in S_{T_i,\mathrm{nc}}^{p,0} := \mathrm{span}\{B_{p,0}^{T_i,\mathrm{nc}}\}$, $i = 2, 3$. From Theorem 23(b) we conclude that the sum $S_{T_2,\mathrm{nc}}^{p,0}|_T \oplus S_{T_3,\mathrm{nc}}^{p,0}|_T$ is direct. The condition $u_{\mathrm{refl}}|_T = 0$ then implies $u_2 = u_3 = 0$. Thus, we have proved $u_{\mathrm{refl}}|_K = 0$.

Case (b2) The case that there are exactly two *other* facets which are lying in $\partial\Omega$ can be treated in a similar way.

Case (b3) Next, we consider the case that $T_i \in \mathcal{F}_\Omega$ for $i = 1, 2, 3$. Note that $u_{\mathrm{refl}}|_T = \sum_{i=1}^{3} u_i|_T$ for some $u_i \in S_{T_i,\mathrm{nc}}^{p,0}$. On T we choose a local (x_1, x_2)-coordinate system such that $\mathbf{A}_1 = \mathbf{0}$, $\mathbf{A}_2 = (1, 0)^\intercal$, $\mathbf{A}_3 = (0, 1)^\intercal$. From (51) and (53) we conclude that

$$b_{n,k}^{\mathrm{refl}} + RMb_{n,k}^{\mathrm{refl}} + MRb_{n,k}^{\mathrm{refl}} = 0.$$

This implies $u_2|_T = RM(u_1|_T) = u_1|_T \circ \chi_{\{3,2,1\}}$ and $u_3|_T = MR(u_1|_T) = u_1|_T \circ \chi_{\{2,3,1\}}$ (cf. (44)) and, in turn, that the restrictions u_i^E of u_i to the edge $E_i = T_i \cap T_0$, $1 \leq i \leq 3$, are the "same", more precisely, the affine pullbacks of u_i^E to the interval $[0, 1]$ are the same. From Lemma 13, we obtain that

$$u_1|_{T_1} \circ \chi_1 = u_2|_{T_2} \circ \chi_2 = u_3|_{T_3} \circ \chi_3, \tag{79}$$

where $\chi_i : \widehat{T} \to T_i$ are affine pullbacks to the reference triangle such that $\chi_i(\mathbf{0}) = \mathbf{A}_0$.

This implies that the functions u_i at $\mathbf{A}_0$ have the same value (say w_0) and, from the condition $u_{\mathrm{refl}}(\mathbf{A}_0) = 3w_0 = 0$, we conclude that $u_i(\mathbf{A}_0) = 0$. The values of u_i at the vertex $\mathbf{A}_i$ of K (which is opposite to T_i) also coincide and we denote this value by v_0. Since $u_{\mathrm{refl}}|_T = 0$ it holds $u_{\mathrm{refl}}(\mathbf{A}_i) = 2w_0 + v_0 = 0$. From $w_0 = 0$ we conclude that also $v_0 = 0$. Let $\chi_{i,T_0} : \widehat{T} \to T_0$ denote an affine pullback with the property $\chi_{i,T_0}(\mathbf{0}) = \mathbf{A}_i$. Hence,

$$\widehat{u}_i := u_i|_{T_0} \circ \chi_{i,T_0}^{-1} \in \mathrm{span}\{b_{p,0}^{\mathrm{refl}}\} \tag{80}$$

with values zero at the vertices of $\hat{T}$. Note that

$$b_{p,0}(0,0) = (-1)^p(p+1) \quad \text{and} \quad b_{p,0}(1,0) = b_{p,0}(0,1) = 1. \tag{81}$$

The vertex properties (81) along the definition of $b_{p,k}^{\text{refl}}$ (cf. (51)) imply that

$$\begin{aligned} b_{p,0}^{\text{refl}}(1,0) &= b_{p,0}^{\text{refl}}(0,1) = \frac{1}{3}(1-(-1)^p(p+1)) = c_p, \\ b_{p,0}^{\text{refl}}(0,0) &= -2b_{p,0}^{\text{refl}}(1,0). \end{aligned} \tag{82}$$

Since $c_p \neq 0$ for $p \geq 1$ we conclude that $\widehat{u}_i = 0$ holds. Relation (80) implies $u_i|_{T_0} = 0$ and thus $u_i = 0$. From $u_{\text{refl}}|_T = \sum_{i=1}^3 u_i|_T$ we deduce that $u_{\text{refl}}|_K = 0$.

The Cases (b1)–(b3) allow to proceed with the same induction argument as for Case (a) and $u_{\text{refl}} = 0$ follows by induction.

Part 3. An inspection of Part 2 shows that, for the proof of Case (a), it was never used that the vertex-oriented basis functions have been removed from $S_{\mathcal{G},\text{c}}^p$ and Case (a) holds verbatim for $S_{\mathcal{G},\text{c}}^p$. This implies that the first sum in (73) is direct.

Part 4. The fact that the sum $S_{\mathcal{G},\text{c}}^p + S_{\text{refl,nc}}^p$ is *not* direct is postponed to Proposition 34. ◻

Proposition 34. *For any vertex* $\mathbf{V} \in \mathcal{V}_\Omega$ *it holds* $B_{p,\mathbf{V}}^{\mathcal{G}} \in S_{\text{sym,nc}}^p \oplus S_{\text{refl,nc}}^{p,0} \oplus \tilde{S}_{\mathcal{G},\text{c}}^p$.

Proof. We will show the stronger statement $B_{p,\mathbf{V}}^{\mathcal{G}} \in S_{\text{refl,nc}}^{p,0} \oplus \tilde{S}_{\mathcal{G},\text{c}}^p$. It suffices to construct a continuous function $u_{\mathbf{V}} \in S_{\text{refl,nc}}^p$ which coincides with $B_{p,\mathbf{V}}^{\mathcal{G}}$ at all vertices $\mathbf{V}' \in \mathcal{V}$ and vanishes at $\partial\Omega$; then, $B_{p,\mathbf{V}}^{\mathcal{G}} - u_{\mathbf{V}} \in \tilde{S}_{\mathcal{G},\text{c}}^p$ and the assertion follows. Recall the known values of $b_{p,0}^{\text{refl}}$ at the vertices of the reference triangle and the definition of c_p as in (82). Let $K \in \mathcal{G}$ be a tetrahedron with $\mathbf{V}$ as a vertex. The facets of K are denoted by T_i, $0 \leq i \leq 3$, and the vertex which is opposite to T_i is denoted by $\mathbf{A}_i$. As a convention we assume that $\mathbf{A}_0 = \mathbf{V}$. For every T_i, $1 \leq i \leq 3$, we define the function $u_{T_i} \in S_{T_i,\text{nc}}^p$ by setting (cf. (56))

$$u_{T_i}|_{T_0} = b_{p,0}^{\text{refl}} \circ \chi_{\mathbf{A}_i,T_0}^{-1},$$

where $\chi_{\mathbf{A}_i,T_0} : \widehat{T} \to T_0$ is an affine pullback which satisfies $\chi_{\mathbf{A}_i,T_0}(\mathbf{0}) = \mathbf{A}_i$. (It is easy to see that the definition of u_{T_i} is independent of the side of T_i, where the tetrahedron K is located.) From (51) and (53) we conclude that $\sum_{i=1}^3 u_{T_i}|_{T_0} = 0$ holds. We proceed in the same way for all tetrahedrons $K \in \mathcal{G}_{\mathbf{V}}$ (cf. (9)). This implies that

$$\tilde{u}_{\mathbf{V}} := \sum_{\substack{T \in \mathcal{F}_\Omega \\ \mathbf{V} \in T}} u_T \tag{83}$$

vanishes at $\Omega \backslash \overset{\circ}{\omega}_{\mathbf{V}}$ (cf. (9)). By construction the function $\tilde{u}_{\mathbf{V}}$ is continuous. At $\mathbf{V}$, the function u_{T_i} has the value (cf. (82))

$$u_{T_i}(\mathbf{V}) = c_p$$

so that $\tilde{u}_{\mathbf{V}}(\mathbf{V}) = Cc_p$, where C is the number of terms in the sum (83). Since $c_p > 0$ for all $p \geq 1$, the function $u_{\mathbf{V}} := \frac{1}{Cc_p}\tilde{u}_V$ is well defined and has the desired properties. □

Remark 35. We have seen that the extension of the basis functions of $S^p_{\mathcal{G},\mathrm{c}}$ by the basis functions of $S^p_{\mathrm{refl,nc}}$ leads to linearly depending functions. On the other hand, if the basis functions of the subspace $S^{p,0}_{\mathrm{refl,nc}}$ are added and the vertex-oriented basis functions in $S^p_{\mathcal{G},\mathrm{c}}$ are simply removed, one arrives at a set a linear independent functions which span a larger space than $S^p_{\mathcal{G},\mathrm{c}}$. Note that $S^{p,0}_{\mathrm{refl,nc}} = S^p_{\mathrm{refl,nc}}$ for $p = 1, 2, 3$.

One could add more basis functions from $S^p_{\mathrm{refl,nc}}$ but then has to remove further basis functions from $\tilde{S}^p_{\mathcal{G},\mathrm{c}}$ or formulate side constraints in order to obtain a set of linearly independent functions.

We finish this section by an example which shows that there exist meshes with fairly special topology, where the inclusion

$$S^p_{\mathcal{G},\mathrm{c}} + S^p_{\mathrm{sym,nc}} + S^p_{\mathrm{refl,nc}} \subset S^p_{\mathcal{G}} \tag{84}$$

is strict. We emphasize that the left-hand side in (84), for $p \geq 4$, defines a larger space than the space in (75) since it contains *all* non-conforming functions of reflection type.

Example 36. Let us consider the octahedron Ω with vertices $\mathbf{A}^{\pm} := (0, 0, \pm 1)^{\intercal}$ and $\mathbf{A}_1 := (1, 0, 0)^{\intercal}$, $\mathbf{A}_2 := (0, 1, 0)^{\intercal}$, $\mathbf{A}_3 := (-1, 0, 0)^{\intercal}$, $\mathbf{A}_4 := (0, -1, 0)^{\intercal}$. Ω is subdivided into a mesh $\mathcal{G} := \{K_i : 1 \leq i \leq 8\}$ consisting of eight congruent tetrahedrons sharing the origin $\mathbf{0}$ as a common vertex. The six vertices at $\partial\Omega$ have the special topological property that each one belongs to exactly four surface facets.

Note that the space defined by the left-hand side of (84) does not contain functions whose restriction to a surface facet, say T, belongs to the τ_{sign} component of $\mathbb{P}^{\perp}_{n,n-1}(T)$. Hence, the inclusion in (84) is strict if we identify a function in $S^p_{\mathcal{G}}$ whose restriction to some surface facet is an orthogonal polynomial of "sign type". Let $\widehat{q} \neq 0$ be a polynomial which belongs to the τ_{sign} component of $\mathbb{P}^{\perp}_{n,n-1}(T)$ on the reference element. Denote the (eight) facets on $\partial\Omega$ with the vertices $\mathbf{A}^{\pm}$, $\mathbf{A}_i$, $\mathbf{A}_{i+1}$ by $T_i^{\pm}$ for $1 \leq i \leq 4$ (with cyclic numbering convention) and choose affine pullbacks $\chi_{\pm,i} : \widehat{T} \to T_i^{\pm}$ as $\chi_{\pm,i}(\mathbf{x}) := \mathbf{A}^{\pm} + x_1(\mathbf{A}_i - \mathbf{A}^{\pm}) + x_2(\mathbf{A}_{i+1} - \mathbf{A}^{\pm})$. Then, it is easy to verify (use Lemma 28 with even m_C) that the function $q : \partial\Omega \to \mathbb{R}$, defined by $q|_{T_i^{\pm}} := \widehat{q} \circ \chi^{-1}_{\pm,i}$ is continuous on $\partial\Omega$. Hence the "finite element extension" to the interior of Ω via

$$Q := \sum_{\mathbf{N} \in \mathcal{N}^p \cap \partial\Omega} q(\mathbf{N}) B^{\mathcal{G}}_{p,\mathbf{N}}$$

defines a function in $S^p_{\mathcal{G}}$ which is not in the space defined by the left-hand side of (84).

We state in passing that the space $S_{\mathcal{G}}^{p}$ does not contain any function whose restriction to a boundary facet, say T, belongs to the τ_{sign} component of $\mathbb{P}_{p,p-1}^{\perp}(T)$ if there exists at least one surface vertex which belongs to an *odd* number of surface facets. In this sense, the topological situation considered in this example is fairly special.

7. Conclusion

In this paper, we developed explicit representation of a local basis for non-conforming finite elements of the Crouzeix–Raviart type. As a model problem we have considered Poisson-type equations in three-dimensional domains; however, this approach is by no means limited to this model problem. Using theoretical conditions in the spirit of the second Strang lemma, we have derived conforming and non-conforming finite element spaces of arbitrary order. For these spaces, we also derived sets of local basis functions. To the best of our knowledge, such explicit representation for general polynomial order p is not available in the existing literature. The derivation requires some deeper tools from orthogonal polynomials of triangles, in particular, the splitting of these polynomials into three irreducible $\mathcal{S}_3$ modules.

Based on these orthogonal polynomials, simplex- and facet-oriented non-conforming basis functions are defined. There are two types of non-conforming basis functions: those whose supports consist of one tetrahedron and those whose supports consist of two adjacent tetrahedrons. The first type can be simply added to the conforming hp basis functions. It is important to note that the span of the functions of the second type contains also conforming functions and one has to remove some conforming functions in order to obtain a linearly independent set of functions. We have proposed a non-conforming space which consists of (a) all basis functions of the first type and (b) a reduced set of basis functions of the second type and (c) of the conforming basis functions without the vertex-oriented ones. This leads to a set of linearly independent functions and is in analogy to the well known lowest-order Crouzeix–Raviart element.

It is interesting to compare these results with high-order Crouzeix–Raviart finite elements for the two-dimensional case which have been presented in [9]. Facets T of tetrahedrons in 3D correspond to edges E of triangles in 2D. As a consequence the dimension of the space of orthogonal polynomials $\mathbb{P}_{p,p-1}^{\perp}(E)$ equals one. For even degree p, one has only non-conforming basis functions of "symmetric' type (which are supported on a single triangle) and for odd degree p, one has only non-conforming basis functions of "reflection" type (which are supported on two adjacent triangles). It turns out that adding the non conforming symmetric basis function to the conforming hp finite element space leads to a set of linearly independent functions which is the analogue of the first sum in (73). If the non-conforming basis functions of reflection type are added, the set of vertex-oriented conforming basis functions have to be removed from the conforming space. This is in analogy to the properties (74) and (75).

Finally, we compare some non-conforming elements from the literature with our family of finite elements (14): In the original paper [12], the definition of non-conforming Crouzeix–Raviart spaces is not identical to our definition (14): the condition $\gamma_K u \in \mathbb{P}_p(K)$ in (14) is replaced by the condition $\gamma_K u \in P_K$ for some space which satisfies $\mathbb{P}_p(K) \subset P_K$. For two-dimensional problems, the following space for $p = 3$ has been introduced in the original paper [12]:

$$P_K := \mathbb{P}_3(K) \oplus \operatorname{span}\{\lambda_1^2\lambda_2\lambda_3, \lambda_1\lambda_2^2\lambda_3, \lambda_1\lambda_2\lambda_3^2\},$$

where λ_i are the barycentric coordinates; P_K contains some polynomials of degree 4 and this space is different from (75) for $p = 3$. In [14] a three-dimensional version of the non-conforming quadratic Fortin–Soulie element [15] is presented which is exactly our space in (75) for $p = 2$. The construction of this element in [14] employs a different theory which might be the reason that it was restricted to $p = 2$. In 2D, basis functions for Crouzeix–Raviart elements are known for general polynomial order p: for $p = 1$, this is the standard $\mathbb{P}_1$ Crouzeix–Raviart element, for $p = 2$ it is the quadratic Fortin–Soulie element, for $p = 3$, this is the Crouzeix–Falk element [11], while the family of Gauss–Legendre elements is a 2D element for general polynomial degree p (see [1]). To the best of our knowledge there is no analogue for these elements in three spatial dimension.

Future research is devoted on numerical experiments and the application of these functions to system of equations such as, e.g., Stokes equation and the Lamé system.

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Appendix A. Alternative Sets of "Reflection-Type" Basis Functions

In this Appendix, we define further sets of basis functions for the τ_{refl} component of $\mathbb{P}^{\perp}_{n,n-1}(\widehat{T})$ — different choices might be preferable for different kinds of applications. All these sets have in common that two vertices of $\widehat{T}$ are special — any basis function is symmetric/skew symmetric with respect to the angular bisector of one of these two vertices.

Remark A.1. The functions $b_{n,2k}$ can be characterized as the range of $I + R$. We project these functions onto τ_{refl}, that is, the space $E^{(\text{refl})} := \{p : RMp + MRp = -p\}$. Let

$$T_1 := I - MR \quad \text{and} \quad T_2 := I - RM.$$

The range of both is $E^{(\text{refl})}$. We will show that $\{T_1 b_{n,2k}, T_2 b_{n,2k}, 0 \le k \le (n-2)/3\}$ is a basis for $E^{(\text{refl})}$. Previously we showed $\{RMq_k, MRq_k\}$ is a basis, where

$q_k = (2I - MR - RM)b_{n,2k} = (T_1 + T_2)b_{n,2k}$ (cf. (51)). Observe that

$$RM(2I - MR - RM) = 2RM - I - MR = T_1 - 2T_2,$$
$$MR(2I - MR - RM) = 2MR - RM - I = -2T_1 + T_2$$

hold, so the basis is made up out of linear combinations of $\{T_1 b_{n,2k}, T_2 b_{n,2k}, 0 \leq k \leq (n-1)/3\}$. These can be written as elements of the range of $T_1(I+R)$ and $T_2(I+R)$. Different linear combinations will behave differently under the reflections R, M, RMR (that is $(x,y) \to (y,x), (1-x-y, y), (x, 1-x-y)$ respectively). After some computations we find

$$\begin{aligned} R(T_1 + T_2)(I+R) &= (T_1 + T_2)(I+R),\\ R(T_1 - T_2)(I+R) &= -(T_1 - T_2)(I+R),\\ M(T_1 - 2T_2)(I+R) &= (T_1 - 2T_2)(I+R),\\ MT_1(I+R) &= -T_1(I+R),\\ RMR(2T_1 - T_2)(I+R) &= (2T_1 - T_2)(I+R),\\ RMRT_2(I+R) &= -T_2(I+R). \end{aligned} \tag{A.1}$$

Any two of these types can be used in producing bases from the $b_{n,2k}$. Also each pair (first two, second two, third two) are orthogonal to each other. Note R fixes $(0,0)$ and reflects in the line $x = y$, M fixes $(0,1)$, reflects in $2x + y = 1$, and RMR fixes $(1,0)$, reflects in $x + 2y = 1$.

If we allow for a complex valued basis, the three vertices of $\widehat{T}$ can be treated more equally as can be seen from the following remark.

Remark A.2. The basis functions can be complexified: set $\omega = \mathrm{e}^{2\pi\,\mathrm{i}/3}$; any polynomial in $E^{(\mathrm{refl})}$ can be expressed as $p = p_1 + p_2$ such that $MRp = \omega p_1 + \omega^2 p_2$ (consequently $RMp = \omega^2 p_1 + \omega p_2$), then

$$-\frac{1}{3}(\omega T_1 + \omega^2 T_2)p = p_1,$$

$$-\frac{1}{3}(\omega^2 T_1 + \omega T_2)p = p_2.$$

These lead to another basis built up from the $b_{n,2k}$. Let

$$S_1 = -\frac{1}{3}(\omega T_1 + \omega^2 T_2)(I+R),$$

$$S_2 = -\frac{1}{3}(\omega^2 T_1 + \omega T_2)(I+R).$$

Applying these operators to $b_{n,2k}$ produces a basis $\{S_1 b_{n,2k}, S_2 b_{n,2k} : 0 \leq k \leq (n-1)/3\}$ satisfying

$$RS_1 b_{n,2k} = S_2 b_{n,2k},\ RS_2 b_{n,2k} = S_1 b_{n,2k},$$

$$MS_1 b_{n,2k} = \omega S_2 b_{n,2k},\ MS_2 b_{n,2k} = \omega^2 S_1 b_{n,2k},$$

$$RMRS_1 b_{n,2k} = \omega^2 S_2 b_{n,2k},\ RMRS_2 b_{n,2k} = \omega S_1 b_{n,2k}.$$

This is a basis which behaves similarly at each vertex.

References

[1] Á. Baran and G. Stoyan, Gauss-Legendre elements: A stable, higher-order nonconforming finite element family, *Computing* **79**(1) (2007) 1–21.

[2] D. Boffi, F. Brezzi and M. Fortin, *Mixed Finite Element Methods and Applications*, Springer Series in Computational Mathematics, Vol. 44 (Springer, Heidelberg, 2013).

[3] A. Bonito, J.-L. Guermond and F. Luddens, Regularity of the Maxwell equations in heterogeneous media and Lipschitz domains, *J. Math. Anal. Appl.* **408**(2) (2013) 498–512.

[4] S. C. Brenner, Poincaré–Friedrichs inequalities for piecewise H^1 functions, *SIAM J. Numer. Anal.* **41**(1) (2003) 306–324.

[5] S. C. Brenner, Forty years of the Crouzeix–Raviart element, *Numer. Methods Partial Differential Equations* **31**(2) (2015) 367–396.

[6] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd edn., Vol. 15 (Springer, New York, 2008).

[7] P. Ciarlet Jr., E. Jamelot and F. Kpadonou, Domain decomposition methods for the diffusion equation with low-regularity solution, *Comput. Math. Appl.* **74**(10) (2017) 2369–2384.

[8] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems* (North-Holland, 1978).

[9] P. G. Ciarlet, P. Ciarlet Jr., S. A. Sauter and C. Simian, Intrinsic finite element methods for the computation of fluxes for Poisson's equation, *Numer. Math.* **132**(3) (2016) 433–462.

[10] M. Costabel, M. Dauge and S. Nicaise, Singularities of Maxwell interface problems, *Math. Model. Numer. Anal.* **33**(3) (1999) 627–649.

[11] M. Crouzeix and R. Falk, Nonconforming finite elements for Stokes problems, *Math. Comp.* **186** (1989) 437–456.

[12] M. Crouzeix and P. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, *Revue Française d'Automatique, Informatique et Recherche Opérationnelle* **3** (1973) 33–75.

[13] C. F. Dunkl, Orthogonal polynomials with symmetry of order three, *Canad. J. Math.* **36**(4) (1984) 685–717.

[14] M. Fortin, A three-dimensional quadratic nonconforming element, *Numer. Math.* **46**(2) (1985) 269–279.

[15] M. Fortin and M. Soulie, A nonconforming quadratic finite element on triangles, *Internat. J. Numer. Methods Eng.* **19** (1983) 505–520.

[16] P. Iliev and Y. Xu, Connection coefficients for classical orthogonal polynomials of several variables, *Adv. Math.* **310** (2017) 290–326.

[17] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders (eds.), *NIST Digital Library of Mathematical Functions*, Release 1.0.17 of 2017-12-22; http://dlmf.nist.gov/.

[18] J. Proriol, Sur une famille de polynomes à deux variables orthogonaux dans un triangle, *C. R. Acad. Sci. Paris* **245** (1957) 2459–2461.

[19] S. Sauter and C. Schwab, *Boundary Element Methods* (Springer, Heidelberg, 2010).

[20] C. Schwab, *p- and hp-Finite Element Methods*: *Theory and Applications in Solid and Fluid Mechanics* (The Clarendon Press, New York, 1998).

[21] G. Stoyan and Á. Baran, Crouzeix–Velte decompositions for higher-order finite elements, *Comput. Math. Appl.* **51**(6–7) (2006) 967–986.