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NUMERICAL ANALYSIS OF THE MIXED FINITE ELEMENT METHOD FOR THE NEUTRON DIFFUSION EIGENPROBLEM WITH HETEROGENEOUS COEFFICIENTS

P. Ciarlet Jr.[1], L. Giret[1,2], E. Jamelot[3,*] and F.D. Kpadonou[1,4]

Abstract. We study first the convergence of the finite element approximation of the mixed diffusion equations with a source term, in the case where the solution is of low regularity. Such a situation commonly arises in the presence of three or more intersecting material components with different characteristics. Then we focus on the approximation of the associated eigenvalue problem. We prove spectral correctness for this problem in the mixed setting. These studies are carried out without, and then with a domain decomposition method. The domain decomposition method can be non-matching in the sense that the traces of the finite element spaces may not fit at the interface between subdomains. Finally, numerical experiments illustrate the accuracy of the method.

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1. Introduction

The multigroup neutron diffusion equation, which is an approximation of the multigroup neutron transport equation, is important in nuclear industry since it allows to model many nuclear reactor cores [16]. In the steady state case, it corresponds to a generalized eigenvalue problem. We propose here the numerical analysis of this problem in the case of a discretization with mixed finite elements, possibly with a domain decomposition method. We focus on the one group of energy case which is the base block of the multigroup case. This paper is thus the extension of [13], where the authors proposed the numerical analysis of the one-group neutron diffusion equation with a source term, discretized with mixed finite elements, with matching and non-matching domain decomposition methods.

Nuclear reactor cores often have a Cartesian geometry. In Figure 1a, we draw a top view of a PWR-like core model. Each square, which represents a part of the reflector or an assembly, is made itself of cells, which are rectangular cuboids of $\mathbb{R}^3$. In Figure 1b, we make a zoom on a patch of six (3×2) assemblies: each

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[1] POEMS, ENSTA ParisTech, CNRS, INRIA, 828 Bd des Maréchaux, 91762 Palaiseau Cedex, France.
[2] DEN-Service d'Etudes des Réacteurs et de Mathématiques Appliquées-SERMA, LLPR, CEA, Université Paris-Saclay, 91191 Gif-sur-Yvette Cedex, France.
[3] DEN-Service de Thermo-hydraulique et de Mécanique des Fluides-STMF, LMSF, CEA, Université Paris-Saclay, 91191 Gif-sur-Yvette Cedex, France.
[4] Laboratoire de Mathématiques de Versailles, UVSQ, 45 Av des Etats-Unis, 78035 Versailles Cedex, France.

* Corresponding author: erell.jamelot@cea.fr

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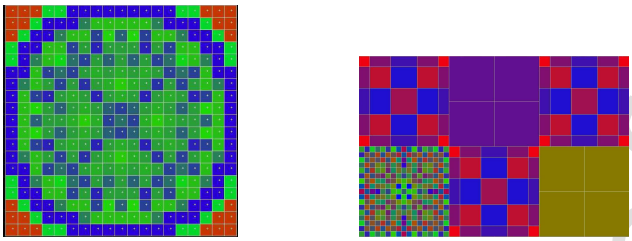


(a) Reactor core model (b) Patch of six assemblies

FIGURE 1. 2D depiction of a PWR core and a zoom on six assemblies.

colored square represents a cell containing fuel, absorbing or reflector material. In our model, the coefficients are polynomial (possibly constant) in each cell [16, 23, 24]. The global domain of the reactor core (see again Fig. 1a) is represented by a rectangular cuboid of $\mathbb{R}^3$. In practice the coefficients characterizing the materials may differ from one cell to another by a factor of order 10 or more.

The outline is as follows. In Section 2, we introduce the notations, and recall basic mathematical definitions. In the next section, we provide the main abstract tool that enables us to characterize the so-called low-regularity solutions, that is piecewise H^{1+r} solutions with an exponent $r > 0$ that can be (arbitrarily) small. Then in Section 4, we solve the diffusion equation written in mixed form, with either a source term, or as an eigenproblem. *We recall that the approximation of eigenvalue problems has been studied among others by Osborn* et al. *in [1, 27], and in particular by Boffi* et al. *[4, 5, 6] when the eigenproblem is in a mixed form. In our case however, their theory does not ensure the spectral correctness of the approximation so we design a new proof to obtain this result. On the other hand, we can adapt the work of Boffi* et al. *[8] to exhibit a convergence rate for the eigenvalues.* For the discretization, we choose the well-known Raviart-Thomas-Nédélec finite element. Then in Sections 5 and 6, we consider the same problems, solved now with the help of a Domain Decomposition method: the DD+L^2-jumps method. Finally, we analyze the numerical capabilities of the DD+L^2-jumps method, before giving some concluding remarks.

2. GEOMETRY, HILBERT SPACES AND NOTATIONS

Throughout the paper, C is used to denote a generic positive constant which is independent of the meshsize, the triangulation and the quantities/fields of interest. We also use the shorthand notation $A \lesssim B$ for the inequality $A \leq CB$, where A and B are two scalar quantities, and C is a generic constant. Respectively, $A \eqsim B$ for the inequalities $A \lesssim B$ and $B \lesssim A$.

Vector-valued (resp. tensor-valued) function spaces are written in boldface character (resp. blackboard characters); for the latter, the index *sym* indicates symmetric fields. Given an open set $O \in \mathbb{R}^d$, $d = 1, 2, 3$, we use the notation $(\cdot|\cdot)_{0,O}$ (respectively $\|\cdot\|_{0,O}$) for the $L^2(O)$ and $\mathbf{L}^2(O) := (L^2(O))^d$ scalar products (resp. norms). More generally, $(\cdot|\cdot)_{s,O}$ and $\|\cdot\|_{s,O}$ (respectively $|\cdot|_{s,O}$) denote the scalar product and norm (resp. semi-norm) of the Sobolev spaces $H^s(O)$ and $\mathbf{H}^s(O) := (H^s(O))^d$ for $s \in \mathbb{R}$ (resp. for $s > 0$).

If moreover the boundary ∂O is Lipschitz, $\mathbf{n}$ denotes the unit outward normal vector field to ∂O. Finally, it is assumed that the reader is familiar with vector-valued function spaces related to the diffusion equation, such as $\mathbf{H}(\mathrm{div}\,; O)$, $\mathbf{H}_0(\mathrm{div}\,; O)$, etc.

We let $\mathcal{R}$ be a bounded, connected and open subset of $\mathbb{R}^d$, having a Lipschitz boundary which is piecewise smooth. We split $\mathcal{R}$ into N open disjoint parts $\{\mathcal{R}_i\}_{1\leq i\leq N}$ with Lipschitz, piecewise smooth boundaries: $\overline{\mathcal{R}} = \cup_{1\leq i\leq N}\overline{\mathcal{R}_i}$ and the set $\{\mathcal{R}_i\}_{1\leq i\leq N}$ is called a partition of $\mathcal{R}$. For a field v defined over $\mathcal{R}$, we shall use the notations $v_i = v_{|\mathcal{R}_i}$, for $1 \leq i \leq N$.

Given a partition $\{\mathcal{R}_i\}_{1\le i\le N}$ of $\mathcal{R}$, we introduce function spaces with piecewise regular elements:

$$\mathcal{P}H^s(\mathcal{R}) = \left\{ \psi \in L^2(\mathcal{R}) \,|\, \psi_i \in H^s(\mathcal{R}_i),\, 1 \le i \le N \right\},\ s > 0;$$
$$\mathcal{P}W^{1,\infty}(\mathcal{R}) = \left\{ D \in L^\infty(\mathcal{R}) | D_i \in W^{1,\infty}(\mathcal{R}_i),\, 1 \le i \le N \right\}.$$

We recall that for a piecewise smooth $\psi \in \mathcal{P}H^s(\mathcal{R})$, $\|\psi\|^2_{\mathcal{P}H^s(\mathcal{R})} = \sum_{i=1}^N \|\psi\|^2_{s,\mathcal{R}_i}$. Similarly for elements of $\mathcal{P}W^{1,\infty}(\mathcal{R})$.

3. Setting of the model

Given a source term $S_f \in L^2(\mathcal{R})$, we consider the following neutron diffusion equation, with vanishing Dirichlet boundary condition. In its primal form, it is written:

Find $\phi \in H^1_0(\mathcal{R})$ such that:

$$-\operatorname{div} D\,\mathbf{grad}\,\phi + \Sigma_a\,\phi = S_f \text{ in } \mathcal{R} \tag{3.1}$$

where ϕ, D, and Σ_a denote respectively the neutron flux, the diffusion coefficient and the macroscopic absorption cross section. Finally, S_f denotes the fission source. When S_f depends on ϕ, the steady state neutron diffusion equation is a generalized eigenvalue problem. It reads (one group of energy):

Find $\phi \in H^1_0(\mathcal{R})\backslash\{0\}, \lambda \in \mathbb{R}$ such that:

$$-\operatorname{div} D\,\mathbf{grad}\,\phi + \Sigma_a\,\phi = \lambda\,\underline{\nu\Sigma}_f\phi \text{ in } \mathcal{R} \tag{3.2}$$

where $\underline{\nu\Sigma}_f$ is the fission yield times the macroscopic fission cross section. Under the assumption that the coefficients D, Σ_a and $\underline{\nu\Sigma}_f$ are positive, the physical solution corresponds to the smallest $\lambda \ge 0$ [12, 16]. When this problem is solved using the inverse power iteration, the source problem (3.1) corresponds to one iteration step, which further justifies its study.

When solving the neutron diffusion equation, D is scalar-valued. From now on and unless otherwise specified, we adopt the more general setting of a (symmetric) tensor-valued coefficient D. The coefficients defining Problems (3.1) and (3.2) satisfy the assumptions:

$$\begin{cases} (D, \Sigma_a, \underline{\nu\Sigma}_f) \in \mathbb{L}^\infty_{sym}(\mathcal{R}) \times L^\infty(\mathcal{R}) \times L^\infty(\mathcal{R}), \\ \exists D_*, D^* > 0,\ \forall \boldsymbol{z} \in \mathbb{R}^d,\ D_*\|\boldsymbol{z}\|^2 \le (D\boldsymbol{z}, \boldsymbol{z}) \le D^*\|\boldsymbol{z}\|^2 \text{ a.e. in } \mathcal{R}, \\ \exists (\Sigma_a)_*, (\Sigma_a)^* > 0,\ 0 < (\Sigma_a)_* \le \Sigma_a \le (\Sigma_a)^* \text{ a.e. in } \mathcal{R}, \\ 0 \le \underline{\nu\Sigma}_f \text{ a.e. in } \mathcal{R},\ \underline{\nu\Sigma}_f \ne 0. \end{cases} \tag{3.3}$$

In particular, it can happen that $\underline{\nu\Sigma}_f$ vanishes on some regions. Also, it is well known that Problem (3.1) is equivalent to the following variational formulation:

Find $\phi \in H^1_0(\mathcal{R})$ such that $\forall \psi \in H^1_0(\mathcal{R})$:

$$\int_{\mathcal{R}} D\,\mathbf{grad}\,\phi \cdot \mathbf{grad}\,\psi \;+\; \int_{\mathcal{R}} \Sigma_a \phi\,\psi \;=\; \int_{\mathcal{R}} S_f \psi. \tag{3.4}$$

Under the assumptions (3.3) on the coefficients, the primal problem (3.1) is well-posed, in the sense that for all $S_f \in L^2(\mathcal{R})$, there exists one and only one $\phi \in H^1_0(\mathcal{R})$ that solves (3.1), and in addition there holds $\|\phi\|_{1,\mathcal{R}} \lesssim \|S_f\|_{0,\mathcal{R}}$. We recall that under additional mild assumptions on the coefficients, the solution ϕ has some extra regularity (see [9, 14] and [13], Prop. 1).

Proposition 3.1. *Let $D \in \mathcal{P}\mathbb{W}^{1,\infty}_{sym}(\mathcal{R})$ and $\Sigma_a \in \mathcal{P}W^{1,\infty}(\mathcal{R})$ satisfy (3.3). There exists $r_{\max} \in]0,1]$, called the regularity exponent, such that for all source terms $S_f \in L^2(\mathcal{R})$, the solution $\phi \in H^1(\mathcal{R})$ to Problem (3.1) belongs to $\bigcap_{0\le r<r_{\max}} \mathcal{P}H^{1+r}(\mathcal{R})$ ($r_{\max} < 1$) or $\mathcal{P}H^2(\mathcal{R})$ ($r_{\max} = 1$) with continuous dependence:*
$\forall r \in [0, r_{\max}[$, $\|\phi\|_{\mathcal{P}H^{1+r}(\mathcal{R})} \lesssim \|S_f\|_{0,\mathcal{R}}$ ($r_{\max} < 1$) *or* $\|\phi\|_{\mathcal{P}H^2(\mathcal{R})} \lesssim \|S_f\|_{0,\mathcal{R}}$ ($r_{\max} = 1$).

In the following, we study the two different problems, the source problem (3.1) and the eigenvalue problem (3.2). Unless otherwise specified, we keep the assumptions of Proposition 3.1 *throughout the paper*. Since cross-points are allowed in our model, *cf.* Figure 1a, and in accordance with [9], the low-regularity case corresponds precisely to

$$r_{\max} < 1/2.$$

For the eigenvalue problem, the analysis is carried out for eigenfunctions which can be either low-regularity functions or "smooth" functions.

Remark 3.2. Instead of imposing a vanishing Dirichlet boundary condition in the model, one can consider a vanishing Neumann boundary condition $D\,\mathbf{grad}\,\phi\cdot\mathbf{n} = 0$ on $\partial\mathcal{R}$. Under some slight restrictions on the geometry, one can also consider a vanishing Fourier boundary condition $\mu_F\phi + D\,\mathbf{grad}\,\phi\cdot\mathbf{n} = 0$ on $\partial\mathcal{R}$, with $\mu_F > 0$. In the latter case, the restriction is that the coefficient D is smooth in a neighborhood of the boundary. The theory and numerical analysis written hereafter still apply.

4. The plain case

We start our study with the neutron diffusion problem without domain decomposition method: we call it the plain case. In this section, we use the function space:

$$\mathbf{X} = \left\{ \xi := (\mathbf{q},\psi) \in \mathbf{H}(\mathrm{div}\,,\mathcal{R}) \times L^2(\mathcal{R}) \right\}, \quad \|\xi\|_{\mathbf{X}} := \left(\|\mathbf{q}\|^2_{\mathbf{H}(\mathrm{div}\,,\mathcal{R})} + \|\psi\|^2_{0,\mathcal{R}} \right)^{1/2}.$$

From now on, we use the notations: $\zeta = (\mathbf{p},\phi)$ and $\xi = (\mathbf{q},\psi)$.

4.1. Setting of the mixed variational formulation

Starting from the solution ϕ to (3.1), if one lets $\mathbf{p} := -D\,\mathbf{grad}\,\phi \in \mathbf{L}^2(\mathcal{R})$, known as the neutron current, one may write the neutron diffusion problem as:
Find $(\mathbf{p},\phi) \in \mathbf{H}(\mathrm{div}\,,\mathcal{R}) \times H^1_0(\mathcal{R})$ such that:

$$\begin{cases} -D^{-1}\,\mathbf{p} - \mathbf{grad}\,\phi = 0 & \text{in } \mathcal{R}, \\ \mathrm{div}\,\mathbf{p} + \Sigma_a\phi = S_f & \text{in } \mathcal{R}. \end{cases} \tag{4.1}$$

Solving the mixed problem (4.1) is actually equivalent to solving (3.1), as the result below recalls.

Theorem 4.1. *Let D satisfy (3.3). The solution $(\mathbf{p},\phi) \in \mathbf{H}(\mathrm{div}\,,\mathcal{R}) \times H^1_0(\mathcal{R})$ to (4.1) is such that ϕ is a solution to (3.1) with the same data. Conversely, the solution $\phi \in H^1_0(\mathcal{R})$ to (3.1) is such that $(-D\,\mathbf{grad}\,\phi,\phi) \in \mathbf{H}(\mathrm{div}\,,\mathcal{R}) \times H^1_0(\mathcal{R})$ is a solution to (4.1) with the same data.*

In practice, writing the diffusion equation in its mixed form allows to compute precisely both the solution and its gradient: it avoids the propagation of the numerical error from the solution to its gradient. In order to obtain the variational formulation for the mixed problem (4.1), we consider any test functions $\mathbf{q} \in \mathbf{H}(\mathrm{div}\,,\mathcal{R})$ and $\psi \in L^2(\mathcal{R})$, we multiply the first equation of (4.1) by $\mathbf{q}$, the second equation of (4.1) by $\psi \in L^2(\mathcal{R})$, and

we integrate over $\mathcal{R}$. We sum the contributions to reach:

$$\int_{\mathcal{R}} \left(-D^{-1}\,\mathbf{p}\cdot\mathbf{q} - \mathbf{grad}\,\phi\cdot\mathbf{q} + \psi\,\mathrm{div}\,\mathbf{p} + \Sigma_a\phi\,\psi\right) = \int_{\mathcal{R}} S_f\psi. \tag{4.2}$$

We recall that $\phi \in H_0^1(\mathcal{R})$. One can integrate by parts to remove the first order derivatives of ϕ:

$$-\int_{\mathcal{R}} \mathbf{grad}\,\phi\cdot\mathbf{q} = \int_{\mathcal{R}} \phi\,\mathrm{div}\,\mathbf{q}.$$

Hence, the regularity requirement on the solution can be lowered to $\phi \in L^2(\mathcal{R})$, and we find that the solution to (4.1) also solves:

Find $(\mathbf{p},\phi) \in \mathbf{X}$, such that $\forall(\mathbf{q},\psi) \in \mathbf{X}$:

$$\int_{\mathcal{R}} \left(-D^{-1}\,\mathbf{p}\cdot\mathbf{q} + \phi\,\mathrm{div}\,\mathbf{q} + \psi\,\mathrm{div}\,\mathbf{p} + \Sigma_a\,\phi\,\psi\right) = \int_{\mathcal{R}} S_f\,\psi. \tag{4.3}$$

We define the bilinear forms:

$$a : \begin{cases} \mathbf{H}(\mathrm{div}\,,\mathcal{R}) \times \mathbf{H}(\mathrm{div}\,,\mathcal{R}) \to \mathbb{R} \\ (\mathbf{p},\mathbf{q}) \mapsto \displaystyle\int_{\mathcal{R}} -D^{-1}\,\mathbf{p}\cdot\mathbf{q} \end{cases} ; \tag{4.4}$$

$$b : \begin{cases} \mathbf{H}(\mathrm{div}\,,\mathcal{R}) \times L^2(\mathcal{R}) \to \mathbb{R} \\ (\mathbf{q},\psi) \mapsto \displaystyle\int_{\mathcal{R}} \psi\,\mathrm{div}\,\mathbf{q} \end{cases} ; \tag{4.5}$$

$$t : \begin{cases} L^2(\mathcal{R}) \times L^2(\mathcal{R}) \to \mathbb{R} \\ (\phi,\psi) \mapsto \displaystyle\int_{\mathcal{R}} \Sigma_a\,\phi\,\psi \end{cases} ; \tag{4.6}$$

and:

$$c : \begin{cases} \mathbf{X}\times\mathbf{X} \to \mathbb{R} \\ (\zeta,\xi) \mapsto a(\mathbf{p},\mathbf{q}) + b(\mathbf{q},\phi) + b(\mathbf{p},\psi) + t(\phi,\psi) \end{cases} . \tag{4.7}$$

Remark 4.2. The form $c(\cdot,\cdot)$ is symmetric as soon as the tensor field D is symmetric.

We consider the linear form:

$$f : \begin{cases} \mathbf{X} \to \mathbb{R} \\ \xi \mapsto \displaystyle\int_{\mathcal{R}} S_f\psi \end{cases} . \tag{4.8}$$

We may rewrite the variational formulation (4.3) as:

Find $\zeta \in \mathbf{X}$ such that $\forall\xi \in \mathbf{X}$:

$$c(\zeta,\xi) = f(\xi). \tag{4.9}$$

Theorem 4.3. *The solution $\zeta = (\mathbf{p}, \phi)$ to (4.9) satisfies (4.1). Hence, problems (4.9) and (4.1) are equivalent.*

4.2. Well-posedness of the mixed formulation

We now recall how to obtain the well-posedness of (4.9) by proving in particular an inf-sup condition.

Theorem 4.4. *Let D and Σ_a satisfy (3.3). Then, there exists a unique solution $\zeta \in \mathbf{X}$ to the mixed variational formulation (4.9).*

Proof. Since the form $c(\cdot,\cdot)$ is symmetric, the inf-sup condition yields the claim. This condition writes:

$$\exists \eta > 0, \quad \inf_{\zeta \in \mathbf{X}} \sup_{\xi \in \mathbf{X}} \frac{c(\zeta, \xi)}{\|\zeta\|_{\mathbf{X}} \, \|\xi\|_{\mathbf{X}}} \geq \eta. \tag{4.10}$$

To achieve (4.10), a possible choice is:

$$\begin{cases} \mathbf{q} = -\mathbf{p} \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}), \\ \psi = \dfrac{1}{2}\phi + \dfrac{1}{2}(\Sigma_a)^{-1} \mathrm{div}\, \mathbf{p} \in L^2(\mathcal{R}). \end{cases} \tag{4.11}$$

It holds $\|\zeta\|_{\mathbf{X}} \geq \upsilon \, \|\xi\|_{\mathbf{X}}$, with $\upsilon := \left(1 + \frac{1}{4}((\Sigma_a)_*)^{-2})\right)^{-1/2}$. The bound on c reads:

$$c(\zeta, \xi) \geq \gamma \, \upsilon \, \|\zeta\|_{\mathbf{X}} \|\xi\|_{\mathbf{X}},$$

with $\gamma := \min\left((D^*)^{-1}, \frac{1}{2}(\Sigma_a)_*, \frac{1}{2}((\Sigma_a)^*)^{-1}\right)$. □

4.3. Discretization

We study conforming discretizations of the variational formulation (4.9). To fix ideas, we use a family of triangulations, indexed by a parameter h, which is classically chosen as the largest diameter of elements of the triangulation. We introduce discrete, finite-dimensional, spaces indexed by h as follows:

$$\mathbf{Q}_h \subset \mathbf{H}(\mathrm{div}\,, \mathcal{R}), \text{ and } L_h \subset L^2(\mathcal{R}).$$

For approximation purposes, and following Definition 2.14 in [17], we assume that $(\mathbf{Q}_h)_h$, resp. $(L_h)_h$ have the *approximability property* in the sense that

$$\begin{aligned} &\forall \mathbf{q} \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}), \ \lim_{h \to 0} \left(\inf_{\mathbf{q}_h \in \mathbf{Q}_h} \|\mathbf{q} - \mathbf{q}_h\|_{\mathbf{H}(\mathrm{div}\,, \mathcal{R})} \right) = 0, \\ &\forall \psi \in L^2(\mathcal{R}), \ \lim_{h \to 0} \left(\inf_{\psi_h \in L_h} \|\psi - \psi_h\|_{0, \mathcal{R}} \right) = 0, \end{aligned} \tag{4.12}$$

and also that L_h includes the subspace L_h^0 of piecewise constant fields on the triangulation.

We impose: $\mathrm{div}\, \mathbf{Q}_h \subset L_h$.

We endow $\mathbf{Q}_h$ with the norm $\|\cdot\|_{\mathbf{H}(\mathrm{div}\,, \mathcal{R})}$, while L_h is endowed with $\|\cdot\|_{0, \mathcal{R}}$.

We finally define:

$$\mathbf{X}_h = \{ \xi_h := (\mathbf{q}_h, \psi_h) \in \mathbf{Q}_h \times L_h \}, \text{ endowed with } \|\cdot\|_{\mathbf{X}}.$$

The conforming discretization of the variational formulation (4.9) reads:

Find $(\mathbf{p}_h, \phi_h) \in \mathbf{X}_h$, such that $\forall (\mathbf{q}_h, \psi_h) \in \mathbf{X}_h$:

$$a(\mathbf{p}_h, \mathbf{q}_h) + b(\mathbf{q}_h, \phi_h) + b(\mathbf{p}_h, \psi_h) + t(\phi_h, \psi_h) = (S_f, \psi_h)_{0,\mathcal{R}}. \tag{4.13}$$

Or equivalently:

$$\text{Find } \zeta_h \in \mathbf{X}_h \text{ such that } \forall \xi_h \in \mathbf{X}_h,\ c(\zeta_h, \xi_h) = f(\xi_h). \tag{4.14}$$

For later use, we denote π^0 the $L^2(\mathcal{R})$ orthogonal projector on its subspace L^0_h. By construction, it holds $\text{range}(\pi^0) = L^0_h$ where π^0 is defined by:

$$\forall \psi \in L^2(\mathcal{R}),\ \forall \psi_h \in L^0_h, \quad (\pi^0 \psi - \psi, \psi_h)_{0,\mathcal{R}} = 0.$$

According to Proposition 1.135 of [17]:

$$\begin{aligned}
&\forall z \in L^2(\mathcal{R}), \quad \|z - \pi^0 z\|_{0,\mathcal{R}} \lesssim \|z\|_{0,\mathcal{R}},\\
&\forall z \in \mathcal{P}H^1(\mathcal{R}), \quad \|z - \pi^0 z\|_{0,\mathcal{R}} \lesssim h\, \|z\|_{\mathcal{P}H^1(\mathcal{R})},\\
&\forall z \in \mathcal{P}W^{1,\infty}(\mathcal{R}), \quad \|z - \pi^0 z\|_{\infty,\mathcal{R}} \lesssim h\, \|z\|_{\mathcal{P}W^{1,\infty}(\mathcal{R})}.
\end{aligned} \tag{4.15}$$

For the last two inequalities, the result holds provided that the triangulations are conforming with respect to the partition, namely for all triangulations, for all elements K of a triangulation, it holds that there exists $1 \le i \le N$ such that $K \subset \overline{\mathcal{R}_i}$. Similar results hold on subsets of $\mathcal{R}$.

4.4. Discrete inf-sup condition

The discrete inf-sup condition to be found writes:

$$\exists\, \eta_h > 0, \qquad \inf_{\zeta_h \in \mathbf{X}_h} \sup_{\xi_h \in \mathbf{X}_h} \frac{c(\zeta_h, \xi_h)}{\|\zeta_h\|_{\mathbf{X}}\, \|\xi_h\|_{\mathbf{X}}} \ge \eta_h. \tag{4.16}$$

Once (4.16) is achieved, one obtains existence and uniqueness of the discrete solution ζ_h, hence the corresponding linear system is well-posed. More generally, our aim is to obtain that $(\eta_h)_h$ is uniformly bounded away from 0. In this sense, one has at hand a *uniform discrete inf-sup condition* (udisc), from which the error analysis can classically be derived.

Theorem 4.5. *Let D, resp. $\Sigma_a \in \mathcal{P}W^{1,\infty}(\mathcal{R})$, satisfy (3.3). The discrete inf-sup condition (4.16) is fulfilled. Moreover, it is a uniform discrete inf-sup condition.*

Proof. In order to prove the discrete inf-sup condition, we use the same method as for the continuous inf-sup condition (*cf.* proof of Thm. 4.4). One can remark that if Σ_a is piecewise-constant, $\frac{1}{2}(\Sigma_a)^{-1} \operatorname{div} \mathbf{p}_h$ is automatically in L_h.

Otherwise, we project $(\Sigma_a)^{-1}$ on the piecewise-constant functions. One modifies (4.11) by choosing:

$$\begin{cases} \mathbf{q}_h = -\mathbf{p}_h \in \mathbf{Q}_h, \\ \psi_h = \dfrac{1}{2}\phi_h + \dfrac{1}{2}\pi^0((\Sigma_a)^{-1}) \operatorname{div} \mathbf{p}_h \in L_h. \end{cases}$$

Using (4.15) with $z = (\Sigma_a)^{-1}$ yields $\|(\Sigma_a)^{-1} - \pi^0((\Sigma_a)^{-1})\|_{\infty,\mathcal{R}} \lesssim h$, which allows us to derive again a udisc in this more general case. □

4.5. Numerical analysis of the source problem

We consider the neutron diffusion equation assuming that D, resp. $\Sigma_a \in \mathcal{P}W^{1,\infty}(\mathcal{R})$, satisfy (3.3). Under the assumptions of Section 4.3, it follows from the previous study that $\lim_{h\to 0} \|\zeta - \zeta_h\|_{\mathbf{X}} = 0$. We find below a sharper bound of the error $\|\zeta - \zeta_h\|_{\mathbf{X}}$ by using Proposition 3.1. In order to obtain optimal *a priori* error estimates, we must know the regularity of the solution to problem (3.1). Since we have assumed that the source term S_f belongs to $L^2(\mathcal{R})$, we already know that $\|\phi\|_{1,\mathcal{R}} \lesssim \|S_f\|_{0,\mathcal{R}}$. Moreover, under the assumptions of Proposition 3.1, the solution ϕ has some extra regularity, and the low-regularity case corresponds to $r_{\max} < 1/2$ there. This is the case that we are focusing on now. In this setting, the field $\mathbf{p} := -D\,\mathbf{grad}\,\phi$ automatically belongs to $\mathcal{P}\mathbf{H}^r(\mathcal{R})$, for $0 \le r < r_{\max}$. We suppose in addition that

$$\exists \mu \in]0, r_{\max}[, \quad S_f \in \mathcal{P}H^\mu(\mathcal{R}).$$

Then we have $\operatorname{div}\mathbf{p} \in \mathcal{P}H^\mu(\mathcal{R})$ (recall $\mathcal{P}H^\mu(\mathcal{R}) = H^\mu(\mathcal{R})$ for $\mu < 1/2$). We will use this hypothesis on S_f to carry on the calculations of the error estimates.

We recall below the definition of the Raviart-Thomas-Nédélec (or RTN) finite element [26, 28]. Let $(K_\ell)_{1\le\ell\le L}$ be a conforming mesh, or triangulation, of $\overline{\mathcal{R}}$ made of parallelepipeds (a mesh, or triangulation, is said to be conforming if in every K_ℓ, D and Σ_a are smooth). Let $P(K_\ell)$ be the set of polynomials defined over K_ℓ. For integer values $l, m, p \ge 0$, we consider the following subspace of $P(K_\ell)$:

$$Q_{l,m,p}(K_\ell) = \left\{ q(x,y,z) \in P(K_\ell) \,|\, q(x,y,z) = \sum_{e,j,k=0}^{l,m,p} a_{e,j,k}\, x^e\, y^j\, z^k,\ a_{e,j,k} \in \mathbb{R} \right\}.$$

For integer $k \ge 0$, let us set $k' = k+1$ and introduce the vector polynomial space:

$$\mathbf{D}_k(K_\ell) = [Q_{k',k,k}(K_\ell) \times \mathbf{0} \times \mathbf{0}] \oplus [\mathbf{0} \times Q_{k,k',k}(K_\ell) \times \mathbf{0}] \oplus [\mathbf{0} \times \mathbf{0} \times Q_{k,k,k'}(K_\ell)].$$

We can now define the $\mathrm{RTN}_{[k]}$ finite element subspace of $\mathbf{H}(\operatorname{div}, \mathcal{R}) \times L^2(\mathcal{R})$:

$$\begin{aligned} \mathbf{Q}_h^k &= \left\{ \mathbf{q} \in \mathbf{H}(\operatorname{div}, \mathcal{R}) \,|\, \forall \ell \in \{1, \dots, L\},\ \mathbf{q}_{|K_\ell} \in \mathbf{D}_k(K_\ell) \right\}, \\ L_h^k &= \left\{ \psi \in L^2(\mathcal{R}) \,|\, \forall \ell \in \{1, \dots, L\},\ \psi_{|K_\ell} \in Q_{k,k,k}(K_\ell) \right\}. \end{aligned} \tag{4.17}$$

As required, it holds $\operatorname{div}\mathbf{Q}_h^k \subset L_h^k$ and $L_h^0 \subset L_h^k$. We recall that for any $\mathbf{q}$ in $\mathbf{H}(\operatorname{div}, \mathcal{R})$, its $\mathrm{RTN}_{[k]}$-interpolant $\mathbf{q}_R^k \in \mathbf{Q}_h^k$ satisfies:

$$\forall \psi_h \in L_h^k,\ b(\mathbf{q} - \mathbf{q}_R^k, \psi_h) = 0. \tag{4.18}$$

In addition thanks to the commuting diagram property, *cf.* Section 2.5.2 of [7], it holds

$$\forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \mathcal{R}),\ \operatorname{div}\mathbf{q}_R^0 = \pi^0(\operatorname{div}\mathbf{q}). \tag{4.19}$$

Let $\mathbf{q} \in \mathbf{H}^r(\mathcal{R})$, such that $\operatorname{div}\mathbf{q} \in H^s(\mathcal{R})$, $0 < r, s < r_{\max}$. According to Lemma 3.3 of [3]:

$$\begin{aligned} \|\mathbf{q} - \mathbf{q}_R^0\|_{0,\mathcal{R}} &\lesssim (h^r |\mathbf{q}|_{r,\mathcal{R}} + h\, \|\operatorname{div}\mathbf{q}\|_{0,\mathcal{R}}), \\ \|\operatorname{div}(\mathbf{q} - \mathbf{q}_R^0)\|_{0,\mathcal{R}} &\lesssim h^s |\operatorname{div}\mathbf{q}|_{s,\mathcal{R}}. \end{aligned} \tag{4.20}$$

Similar results hold on subsets of $\mathcal{R}$, provided the discretizations are conforming.

Remark 4.6. If one chooses another discretization, all results presented hereafter hold provided the estimates (4.20) remain true. For instance, for the $\mathrm{RTN}_{[k]}$ finite element defined on tetrahedral triangulations of $\overline{\mathcal{R}}$, *cf.* Section 2.3.1 of [7]. To prove (4.20) in this case, one has simply to apply the results of Section 3.2 from [3]. On the other hand, provided that the field $\mathbf{q}$ and its divergence are "smooth" in the sense that they belong to $\mathcal{P}H^{m+1}(\mathcal{R})$ for some integer $m \geq 0$, using the $\mathrm{RTN}_{[m]}$ finite element one can recover interpolation estimates in $O(h^{m+1})$, *cf.* Section 2.5.5 in [7]. For meshes made of affine elements such as tetrahedra or parallelepipeds, the approximation estimate (4.20-top) does not require the term with the divergence (see, *e.g.* [7], Sect. 2.5.1).

4.5.1. A priori error estimates

Since we focus on the low-regularity case, we choose the $\mathrm{RTN}_{[0]}$ finite element, *i.e.* $\mathbf{X}_h = \mathbf{Q}_h^0 \times L_h^0$. If the solution is "smooth", one can increase the order of the RTN finite element. This will be used in particular in Section 4.6.2 for the study of the error on the eigenvalues. According to first Strang's Lemma [17] and because $(1 + \|c\|(\eta_h)^{-1}) \lesssim 1$, the error reads:

$$\|\zeta - \zeta_h\|_{\mathbf{X}} \lesssim \inf_{\xi_h \in \mathbf{X}_h} \|\zeta - \xi_h\|_{\mathbf{X}}. \tag{4.21}$$

Theorem 4.7. *Under the assumptions of Proposition 3.1, it holds, with* $r_{\max} < 1/2$*:*

$$\begin{aligned} &\forall \mu \in]0, r_{\max}[,\ \forall S_f \in H^{\mu}(\mathcal{R}), \\ &\|\mathbf{p} - \mathbf{p}_h\|_{\mathbf{H}(\mathrm{div},\mathcal{R})} + \|\phi - \phi_h\|_{0,\mathcal{R}} \lesssim h^{\mu} \|S_f\|_{\mu,\mathcal{R}}. \end{aligned} \tag{4.22}$$

Remark 4.8. In particular, for "smooth data" S_f, *i.e.* $S_f \in H^{r_{\max}}(\mathcal{R})$, one expects a convergence rate at least in $h^{r_{\max}-\eta}$ for $\eta > 0$ arbitrary small: by a slight *abuse of notation* there and in the sequel, we shall write $h^{r_{\max}}$. Also, the previous analysis can be extended to the case where $r_{\max}$ is in $[1/2, 1]$ and $\mu < r_{\max}$ (or $\mu \leq 1$ if $r_{\max} = 1$). Furthermore, for a "smooth" solution, one may recover a convergence rate like $O(h^{m+1})$ for an $\mathrm{RTN}_{[m]}$ discretization of order $m \geq 0$.

Proof. Choosing $\xi_h = (\mathbf{p}_R^0, \pi^0\phi) \in \mathbf{X}_h$, then thanks to the *a priori* estimates (4.15) and (4.20), it follows that:

$$\begin{aligned} \|\zeta - \xi_h\|_{\mathbf{X}}^2 &= \|\mathbf{p} - \mathbf{p}_R^0\|_{\mathbf{H}(\mathrm{div},\mathcal{R})}^2 + \|\phi - \pi^0\phi\|_{0,\mathcal{R}}^2 \\ &\lesssim h^{2\mu}(|\mathbf{p}|_{\mu,\mathcal{R}}^2 + \|\mathrm{div}\,\mathbf{p}\|_{\mu,\mathcal{R}}^2) + h^2\|\phi\|_{1,\mathcal{R}}^2 \\ &\lesssim h^{2\mu}\,\|S_f\|_{\mu,\mathcal{R}}^2. \end{aligned}$$

□

4.5.2. Aubin-Nitsche-type estimates

To derive improved estimates on the error $\|\phi - \phi_h\|_{0,\mathcal{R}}$ in $\mathbf{X}_h = \mathbf{Q}_h^0 \times L_h^0$, we shall rely on the illuminating work of Falk-Osborn [18]. Interestingly, one can obtain an improvement of the convergence rate, contrary to the case where the solution is "smooth". From the previous analysis, for all $\mu < r_{\max}$, we already have the estimate (4.22).

Lemma 4.9. *Let* $(\mathbf{p}, \phi)$ *(resp.* $(\mathbf{p}_h, \phi_h)$*) the solution of continuous (resp. discrete) variational problem* (4.3) *(resp.* (4.13)*). For all* $(\mathbf{q}_h, \psi_h)$ *in* $\mathbf{X}_h$*, it holds:*

$$a(\mathbf{p} - \mathbf{p}_h, \mathbf{q}_h) + b(\mathbf{q}_h, \phi - \phi_h) = 0, \tag{4.23}$$

$$b(\mathbf{p} - \mathbf{p}_h, \psi_h) + t(\phi - \phi_h, \psi_h) = 0. \tag{4.24}$$

Proof. Let $(\mathbf{q}_h, \psi_h)$ be in $\mathbf{X}_h$. The subtraction of (4.3) from (4.13), with $(\mathbf{q}, \psi) = (\mathbf{q}_h, \psi_h)$ in the former, gives

$$a(\mathbf{p} - \mathbf{p}_h, \mathbf{q}_h) + b(\mathbf{q}_h, \phi - \phi_h) + b(\mathbf{p} - \mathbf{p}_h, \psi_h) + t(\phi - \phi_h, \psi_h) = 0.$$

We obtain the first equality (4.23) (resp. the second equality (4.24)) with $\psi_h = 0$ (resp. $\mathbf{q}_h = \mathbf{0}$). □

Before improving the estimate, we need to introduce the adjoint problem:
For $d \in L^2(\mathcal{R})$, find $(\mathbf{y}_d, \eta_d) \in \mathbf{X}$ such that $\forall (\mathbf{q}, \psi) \in \mathbf{X}$:

$$a(\mathbf{y}_d, \mathbf{q}) + b(\mathbf{q}, \eta_d) + b(\mathbf{y}_d, \psi) + t(\eta_d, \psi) = (d, \psi)_{0,\mathcal{R}}. \tag{4.25}$$

Theorem 4.10. *Under the assumptions of Proposition 3.1, it holds, with* $r_{\max} < 1/2$*:*

$$\forall \mu \in]0, r_{\max}[,\ \forall S_f \in H^\mu(\mathcal{R}), \quad \|\phi - \phi_h\|_{0,\mathcal{R}} \lesssim h^{2\mu} \|S_f\|_{\mu,\mathcal{R}}. \tag{4.26}$$

Proof. Adapting the methodology of [18] and by using $(0, \phi - \phi_h)$ as a test function in the adjoint problem (4.25), we remark:

$$\|\phi - \phi_h\|_{0,\mathcal{R}} = \sup_{d \in L^2(\mathcal{R}) \setminus \{0\}} \frac{b(\mathbf{y}_d, \phi - \phi_h) + t(\eta_d, \phi - \phi_h)}{\|d\|_{0,\mathcal{R}}}. \tag{4.27}$$

We now look for an upper bound of the supremum in (4.27). We find that the numerator is successively equal to:

$$b(\mathbf{y}_d - (\mathbf{y}_d)^0_R, \phi - \phi_h) + b((\mathbf{y}_d)^0_R, \phi - \phi_h) + t(\eta_d, \phi - \phi_h);$$

using (4.18), for any ψ_h^*, ψ_h' in L_h:

$$b(\mathbf{y}_d - (\mathbf{y}_d)^0_R, \phi - \psi_h^*) + b((\mathbf{y}_d)^0_R, \phi - \phi_h) + t(\eta_d - \psi_h', \phi - \phi_h) + t(\psi_h', \phi - \phi_h);$$

using (4.23) with $\mathbf{q}_h = (\mathbf{y}_d)^0_R$:

$$b(\mathbf{y}_d - (\mathbf{y}_d)^0_R, \phi - \psi_h^*) - a(\mathbf{p} - \mathbf{p}_h, (\mathbf{y}_d)^0_R) + t(\eta_d - \psi_h', \phi - \phi_h) + t(\psi_h', \phi - \phi_h);$$

now we use (4.24) with $\psi_h = \psi_h'$:

$$b(\mathbf{y}_d - (\mathbf{y}_d)^0_R, \phi - \psi_h^*) - a(\mathbf{p} - \mathbf{p}_h, (\mathbf{y}_d)^0_R) + t(\eta_d - \psi_h', \phi - \phi_h) - b(\mathbf{p} - \mathbf{p}_h, \psi_h');$$

we add (4.25) with $(\mathbf{p} - \mathbf{p}_h, 0)$ as a test function:

$$b(\mathbf{y}_d - (\mathbf{y}_d)^0_R, \phi - \psi_h^*) + a(\mathbf{p} - \mathbf{p}_h, \mathbf{y}_d - (\mathbf{y}_d)^0_R) + t(\eta_d - \psi_h', \phi - \phi_h) + b(\mathbf{p} - \mathbf{p}_h, \eta_d - \psi_h'). \tag{4.28}$$

All terms[1] in the previous relation can be bounded with an h-dependent term:

$$\begin{aligned} \inf_{\psi_h^* \in L_h} |b(\mathbf{y}_d - (\mathbf{y}_d)^0_R, \phi - \psi_h^*)| &\lesssim \|\operatorname{div}(\mathbf{y}_d - (\mathbf{y}_d)^0_R)\|_{0,\mathcal{R}} \inf_{\psi_h^* \in L_h} \|\phi - \psi_h^*\|_{0,\mathcal{R}} \\ &\lesssim \|\operatorname{div} \mathbf{y}_d\|_{0,\mathcal{R}}\, h\, \|\phi\|_{1,\mathcal{R}} \\ &\lesssim h\, \|S_f\|_{\mu,\mathcal{R}} \|d\|_{0,\mathcal{R}}; \end{aligned}$$

[1] In particular, $\|\operatorname{div}(\mathbf{y}_d - (\mathbf{y}_d)^0_R)\|_{0,\mathcal{R}} \lesssim \|\operatorname{div} \mathbf{y}_d\|_{0,\mathcal{R}}$ according to (4.15) and (4.19).

$$
\begin{aligned}
|a(\mathbf{p}-\mathbf{p}_h, \mathbf{y}_d - (\mathbf{y}_d)^0_R)| &\lesssim \|\mathbf{p}-\mathbf{p}_h\|_{0,\mathcal{R}} \|\mathbf{y}_d - (\mathbf{y}_d)^0_R\|_{0,\mathcal{R}} \\
&\lesssim h^\mu \|S_f\|_{\mu,\mathcal{R}} \left(h^\mu |\mathbf{y}_d|_{\mu,\mathcal{R}} + h\, \|\operatorname{div} \mathbf{y}_d\|_{0,\mathcal{R}}\right) \\
&\lesssim h^{2\mu} \|S_f\|_{\mu,\mathcal{R}} \|d\|_{0,\mathcal{R}}.
\end{aligned}
$$

The last two terms in (4.28) are considered together.

$$
\begin{aligned}
&\inf_{\psi'_h \in L_h} |b(\mathbf{p}-\mathbf{p}_h, \eta_d - \psi'_h) + t(\phi-\phi_h, \eta_d - \psi'_h)| \\
&\lesssim \left(\|\operatorname{div}(\mathbf{p}-\mathbf{p}_h)\|_{0,\mathcal{R}} + \|\phi-\phi_h\|_{0,\mathcal{R}}\right) \inf_{\psi'_h \in L_h} \|\eta_d - \psi'_h\|_{0,\mathcal{R}} \\
&\lesssim h^\mu \|S_f\|_{\mu,\mathcal{R}} \inf_{\psi'_h \in L_h} \|\eta_d - \psi'_h\|_{0,\mathcal{R}} \\
&\lesssim h^\mu \|S_f\|_{\mu,\mathcal{R}}\, h\, \|\eta_d\|_{1,\mathcal{R}} \lesssim h^{\mu+1} \|S_f\|_{\mu,\mathcal{R}}\, \|d\|_{0,\mathcal{R}}.
\end{aligned}
$$

Thus, for low-regularity solutions ($\mu < 1/2$), we conclude that it holds:

$$
\|\phi-\phi_h\|_{0,\mathcal{R}} \lesssim \max(h, h^{2\mu}, h^{\mu+1})\, \|S_f\|_{\mu,\mathcal{R}} \eqsim h^{2\mu}\, \|S_f\|_{\mu,\mathcal{R}}.
$$

□

Corollary 4.11. *In the case of "smooth data"* S_f, i.e. $S_f \in H^{r_{\max}}(\mathcal{R})$, *the error estimate gives:*

$$
\|\phi-\phi_h\|_{0,\mathcal{R}} \lesssim h^{2r_{\max}} \|S_f\|_{r_{\max},\mathcal{R}}.
$$

4.6. Numerical analysis of the generalized eigenvalue problem

Let us focus on the approximation of the generalized eigenvalue problem (3.2) in our low-regularity setting, under the assumptions of Proposition 3.1, supplemented with $\underline{\nu\Sigma}_f \in \mathcal{P}W^{1,\infty}(\mathcal{R})$.

Let $0 \le \mu < r_{\max}$ be given, we introduce an operator B_μ associated to the source problem (4.3): given $f \in H^\mu(\mathcal{R})$, we call $B_\mu f = \phi \in H^1(\mathcal{R})$ the second component of the couple $(\mathbf{p}, \phi)$ that solves (4.3) with source $S_f = \underline{\nu\Sigma}_f f$. Since $\underline{\nu\Sigma}_f$ belongs to $\mathcal{P}W^{1,\infty}(\mathcal{R})$, it holds $\|S_f\|_{\mu,\mathcal{R}} \lesssim \|f\|_{\mu,\mathcal{R}}$ because $\mu < 1/2$. Hence, B_μ is a bounded operator from $H^\mu(\mathcal{R})$ to itself:

$$
\|B_\mu f\|_{\mu,\mathcal{R}} \lesssim \|B_\mu f\|_{1,\mathcal{R}} = \|\phi\|_{1,\mathcal{R}} \lesssim \|S_f\|_{0,\mathcal{R}} \lesssim \|S_f\|_{\mu,\mathcal{R}} \lesssim \|f\|_{\mu,\mathcal{R}};
$$

we write $B_\mu \in \mathcal{L}(H^\mu(\mathcal{R}))$ for short. In addition, since the second component of the solution actually belongs to $H^1(\mathcal{R})$ with continuous dependence ($\|\phi\|_{1,\mathcal{R}} \lesssim \|f\|_{\mu,\mathcal{R}}$), it follows that B_μ is a compact operator. Denote by $\sigma(B_\mu)$ its spectrum. By construction, $\lambda^{-1} \in \sigma(B_\mu)$ if, and only if, λ is an eigenvalue of (3.2).

Finally, we consider the discrete operator B^h_μ associated to the discrete source problem (4.13): given $f \in H^\mu(\mathcal{R})$, we call $B^h_\mu f$ the second component of the couple $(\mathbf{p}_h, \phi_h)$ that solves (4.13) with source $S_f = \underline{\nu\Sigma}_f f$.

Under the assumptions of Section 4.3 and as noted at the beginning of Section 4.5, it holds $\lim_{h\to 0} \|B_0 f - B^h_0 f\|_{0,\mathcal{R}} = 0$ for all $f \in L^2(\mathcal{R})$. This property is the so-called pointwise convergence. However, for a mixed formulation, the fact that the family $(B^h_0)_h$ converges pointwise towards the compact operator B_0 is not *sufficient* to guarantee that the family $(B^h_0)_h$ converges in operator norm towards B_0.

4.6.1. Convergence in operator norm

On the other hand, according to [27], proving that $\lim_{h\to 0} \|B_\mu - B^h_\mu\|_{\mathcal{L}(H^\mu(\mathcal{R}))} = 0$ for discrete approximants $(B^h_\mu)_h$ is a sufficient condition to obtain convergence of the eigenvalues. In order to ensure the convergence in operator norm of the family $(B^h_\mu)_h$ towards the compact operator B_μ, we need a technical assumption on the triangulations.

Definition 4.12. A family of triangulations $(\mathcal{T}_h)_h$ is *regular*$^+$ if it satisfies:

$$\exists \theta > 0,\ \forall h,\ h^{2-\theta} \lesssim \min_{K \in \mathcal{T}_h} diam(K). \tag{4.29}$$

In particular, a quasi-uniform family of triangulations is *regular*$^+$(take $\theta = 1$ in (4.29)). For a *regular*$^+$ family, one has the following inverse inequality, whose proof is given in Appendix A.

Lemma 4.13. *Let $\mu \in [0, 1/2[$. For a* regular$^+$ *family of triangulations, it holds:*

$$\forall h,\ \forall \psi_h \in L^k_h,\ \|\psi_h\|_{\mu,\mathcal{R}} \lesssim h^{-2\mu+\theta\mu} \|\psi_h\|_{0,\mathcal{R}}. \tag{4.30}$$

Theorem 4.14. *Under the assumptions of Proposition 3.1 with $r_{\max} < 1/2$ plus $\underline{\nu\Sigma}_f \in \mathcal{P}W^{1,\infty}(\mathcal{R})$, let $\mu \in [0, r_{\max}[$. Provided that the family of triangulations is* regular$^+$*, one has:*

$$\|B_\mu - B^h_\mu\|_{\mathcal{L}(H^\mu(\mathcal{R}))} \lesssim h^{\theta\mu}. \tag{4.31}$$

Proof. According to (4.26), we know that

$$\|(B_\mu - B^h_\mu) f\|_{0,\mathcal{R}} \lesssim h^{2\mu} \|f\|_{\mu,\mathcal{R}}. \tag{4.32}$$

It remains to estimate $\|(B_\mu - B^h_\mu) f\|_{\mu,\mathcal{R}}$: for that, we use the triangle inequality

$$\|(B_\mu - B^h_\mu) f\|_{\mu,\mathcal{R}} \le \|B_\mu f - \pi^0(B_\mu f)\|_{\mu,\mathcal{R}} + \|\pi^0(B_\mu f) - B^h_\mu f\|_{\mu,\mathcal{R}}.$$

To bound the first term, we have according to Theorem 2.3 in [2] that

$$\forall \psi \in \mathcal{P}H^1(\mathcal{R}),\ \|\psi - \pi^0\psi\|_{\mu,\mathcal{R}} \lesssim h^{1-\mu} \|\psi\|_{\mathcal{P}H^1(\mathcal{R})}.$$

Applying the result to $\psi = B_\mu f$, we find $\|B_\mu f - \pi^0(B_\mu f)\|_{\mu,\mathcal{R}} \lesssim h^{1-\mu}\|f\|_{\mu,\mathcal{R}}$.

To bound the second term, we use first the inverse inequality (4.30) on the discrete space L^k_h, valid for a *regular*$^+$ family of triangulations. Applying the result to $\psi_h = \pi^0(B_\mu f) - B^h_\mu f$ and using again the triangle inequality, we now find that

$$\begin{aligned}
\|\pi^0(B_\mu f) - B^h_\mu f\|_{\mu,\mathcal{R}} &\lesssim h^{-2\mu+\theta\mu} \|\pi^0(B_\mu f) - B^h_\mu f\|_{0,\mathcal{R}} \\
&\lesssim h^{-2\mu+\theta\mu} \left(\|\pi^0(B_\mu f) - B_\mu f\|_{0,\mathcal{R}} + \|B_\mu f - B^h_\mu f\|_{0,\mathcal{R}} \right) \\
&\lesssim \max(h^{1-2\mu+\theta\mu}, h^{\theta\mu}) \|f\|_{\mu,\mathcal{R}},
\end{aligned}$$

where we have used (4.15) and (4.32) to derive the final estimate. Since $\mu < 1/2$, we conclude by aggregating the results that (4.31) holds. □

Thanks to [27], convergence of the discrete eigenvalues to the exact ones is guaranteed, and so is the absence of spectral pollution:

- Given any closed, non-empty disk $D \subset \mathbb{C}$ such that $D \cap \sigma(B_\mu) = \emptyset$, there exists $h_0 > 0$ such that, for all $h < h_0$, $D \cap \sigma(B^h_\mu) = \emptyset$.
- Given any closed, non-empty disk $D \subset \mathbb{C}$ such that $D \cap \sigma(B_\mu) = \{\lambda\}$, with λ of multiplicity m_λ, there exists $h_0 > 0$ such that, for all $h < h_0$, $D \cap \sigma(B^h_\mu)$ contains exactly m_λ discrete eigenvalues.

4.6.2. Optimal convergence rate

Let the assumptions of Theorem 4.14 hold. We determine now the rate of convergence of the eigenvalues in the spirit of [8]. Let $\nu = \lambda^{-1}$ be an eigenvalue of B_μ. For simplicity, let us assume that ν is a simple eigenvalue, and denote by W the associated eigenspace. According to the absence of spectral pollution, for h small enough, the closest discrete eigenvalue, denoted by ν_h, is also simple; we denote by W_h the associated eigenspace.

Definition 4.15. Let $\omega_\nu > 0$ be the regularity exponent of the eigenfunction, *i.e.* either $W \subset \mathcal{P}H^{1+s}(\mathcal{R})$ for $s < \omega_\nu$ and $W \not\subset \mathcal{P}H^{1+\omega_\nu}(\mathcal{R})$, or $W \subset \mathcal{P}H^{1+\omega_\nu}(\mathcal{R})$ and $W \not\subset \mathcal{P}H^{1+s}(\mathcal{R})$ for $s > \omega_\nu$. Let $\omega = \min(\omega_\nu, m+1)$, where $m \geq 0$ is the order of the RTN finite element.

Clearly, ω_ν, and as a consequence ω, can be greater than $r_{\max}$. We shall prove that the approximation converges with a rate equal to twice the exponent ω defined above: this result is stated in Corollary 4.23 at the end of the subsection.

Let $\mu \in [0, r_{\max}[$ be given. As we defined B_μ (resp. B_μ^h), we define A_μ (resp. A_μ^h): for $f \in H^\mu(\mathcal{R})$, we call $A_\mu f = \mathbf{p} \in \mathbf{H}(\mathrm{div}\,, \mathcal{R})$ (resp. $A_\mu^h f = \mathbf{p}_h \in \mathbf{Q}_h$) the first component of the couple $(\mathbf{p}, \phi)$ (resp. $(\mathbf{p}_h, \phi_h)$) that solves (4.3) (resp. (4.13)) with source $S_f = \underline{\nu\Sigma}_f f$. The following lemma introduces some equalities that we will use later on.

Lemma 4.16. *Let φ and φ' be given in W. Then, it holds:*

$$\begin{aligned}(\underline{\nu\Sigma}_f\varphi, (B_\mu - B_\mu^h)\varphi')_{0,\mathcal{R}} &= a(A_\mu\varphi, (A_\mu - A_\mu^h)\varphi') \\ &+ b((A_\mu - A_\mu^h)\varphi', B_\mu\varphi) + b(A_\mu\varphi, (B_\mu - B_\mu^h)\varphi') + t(B_\mu\varphi, (B_\mu - B_\mu^h)\varphi');\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}0 &= a(A_\mu^h\varphi, (A_\mu - A_\mu^h)\varphi') + b((A_\mu - A_\mu^h)\varphi', B_\mu^h\varphi) \\ &+ b(A_\mu^h\varphi, (B_\mu - B_\mu^h)\varphi') + t(B_\mu^h\varphi, (B_\mu - B_\mu^h)\varphi').\end{aligned} \tag{4.34}$$

Proof. The definitions of A_μ, B_μ imply that for all $f \in H^\mu(\mathcal{R})$, for all $(\mathbf{q}, \psi) \in \mathbf{X}$:

$$(\underline{\nu\Sigma}_f f, \psi)_{0,\mathcal{R}} = a(A_\mu f, \mathbf{q}) + b(\mathbf{q}, B_\mu f) + b(A_\mu f, \psi) + t(B_\mu f, \psi), \tag{4.35}$$

whereas the definitions of A_μ^h, B_μ^h imply that for all $f \in H^\mu(\mathcal{R})$, for all $(\mathbf{q}, \psi) \in \mathbf{X}_h$:

$$(\underline{\nu\Sigma}_f f, \psi)_{0,\mathcal{R}} = a(A_\mu^h f, \mathbf{q}) + b(\mathbf{q}, B_\mu^h f) + b(A_\mu^h f, \psi) + t(B_\mu^h f, \psi). \tag{4.36}$$

The first equality (4.33) comes from (4.35) with:

$$f = \varphi;\ \mathbf{q} = (A_\mu - A_\mu^h)\varphi';\ \psi = (B_\mu - B_\mu^h)\varphi'.$$

The second one, (4.34), comes from the difference between (4.35) and (4.36) with:

$$f = \varphi';\ \mathbf{q} = A_\mu^h\varphi;\ \psi = B_\mu^h\varphi;$$

and with the symmetry of $a(\cdot,\cdot)$ and $t(\cdot,\cdot)$. □

We remark that $\varphi \mapsto \|\varphi\|_W = \|(\underline{\nu\Sigma}_f)^{\frac{1}{2}}\varphi\|_{0,\mathcal{R}}$ is a norm over W,[2] and this norm is induced by the inner product

$$(\varphi, \varphi')_W = (\underline{\nu\Sigma}_f\varphi, \varphi')_{0,\mathcal{R}}.$$

Proposition 4.17. *Let ω be as in Definition 4.15. For every φ in W, the following inequalities hold:*

$$\|(B_\mu - B_\mu^h)\varphi\|_{0,\mathcal{R}} \lesssim h^\omega \|\varphi\|_W$$
$$\|(A_\mu - A_\mu^h)\varphi\|_{\mathbf{H}(\mathrm{div},\mathcal{R})} \lesssim h^\omega \|\varphi\|_W.$$

Proof. These two inequalities come from the first Strang's Lemma. The method is the same as for Theorem 4.7 (see Rem. 4.8 for the "smooth" case). Here, we use the equivalence of all norms on W to state the result. □

Introducing $\delta(Z, Z') = \sup_{z\in Z,\, \|z\|_0=1} \inf_{z'\in Z'} \|z - z'\|_{0,\mathcal{R}}$ for Z, Z' closed subspaces of $L^2(\mathcal{R})$, the gap between W and W_h is defined by:

$$\hat{\delta}(W, W_h) = \max[\delta(W, W_h), \delta(W_h, W)].$$

It allows us to evaluate the approximation of the continuous eigenfunctions by their discrete counterparts. Classically, this gap can be bounded with the help of Proposition 4.17, following Theorem 1 from [27]:

$$\hat{\delta}(W, W_h) \lesssim h^\omega. \tag{4.37}$$

Let us now define E_h as the projector from $L^2(\mathcal{R})$ onto W_h such that

$$\forall\varphi \in L^2(\mathcal{R}),\ \forall\psi_h \in W_h,\ (\underline{\nu\Sigma}_f(\varphi - E_h\varphi), \psi_h)_{0,\mathcal{R}} = 0. \tag{4.38}$$

Lemma 4.18. *The operators E_h and B_μ^h commute.*

Proof. Let $\varphi \in L^2(\mathcal{R})$ be decomposed into $\varphi = E_h\varphi + \bar{\varphi}$. By construction $E_h\varphi \in W_h$, so that $B_\mu^h E_h\varphi \in W_h$, hence $E_h B_\mu^h E_h\varphi = B_\mu^h E_h\varphi$ because W_h is invariant through E_h. It follows $E_h B_\mu^h\varphi = E_h B_\mu^h E_h\varphi + E_h B_\mu^h\bar{\varphi} = B_\mu^h E_h\varphi + E_h B_\mu^h\bar{\varphi}$. This is equivalently expressed as

$$(E_h B_\mu^h - B_\mu^h E_h)\varphi = E_h B_\mu^h\bar{\varphi}.$$

By construction, $\psi_h = E_h B_\mu^h\bar{\varphi}$ belongs to W_h, with squared norm equal to

$$(\underline{\nu\Sigma}_f\psi_h, \psi_h)_{0,\mathcal{R}} = (\underline{\nu\Sigma}_f E_h B_\mu^h\bar{\varphi}, \psi_h)_{0,\mathcal{R}} = (\underline{\nu\Sigma}_f B_\mu^h\bar{\varphi}, \psi_h)_{0,\mathcal{R}} = (\underline{\nu\Sigma}_f\bar{\varphi}, B_\mu^h\psi_h)_{0,\mathcal{R}} = 0.$$

The penultimate equality stems from the fact that $c(\cdot,\cdot)$ is symmetric, and the last one comes from the definition of $\bar{\varphi}$ and E_h. □

Let F_h be the restriction of E_h to W. One has the following simple results as a consequence of the gap property.

Lemma 4.19. *For h small enough, F_h is a bijection from W to W_h. Moreover*

$$\forall\varphi \in W,\ \left\|(\underline{\nu\Sigma}_f)^{\frac{1}{2}}(\varphi - F_h\varphi)\right\|_{0,\mathcal{R}} \lesssim h^\omega\|\varphi\|_W. \tag{4.39}$$

[2] If $\|\varphi\|_W = 0$, then $\underline{\nu\Sigma}_f\varphi = 0$. By definition of W, φ is solution of (3.2) with zero right-hand side. Thus, by uniqueness of the solution it follows that $\varphi = 0$.

Let $\mathcal{S}_h = F_h^{-1}E_h - I \in \mathcal{L}(L^2(\mathcal{R}))$ for h small enough.

Lemma 4.20. *For h small enough, $W \subset \ker(\mathcal{S}_h)$; $(\mathcal{S}_h)_h$ is uniformly bounded.*

One can then prove an "orthogonality" result involving $\mathcal{S}_h$.

Proposition 4.21. *For all f in $L^2(\mathcal{R})$ and φ_h in W_h, one has for h small enough*

$$(\underline{\nu\Sigma}_f \mathcal{S}_h f, \varphi_h)_{0,\mathcal{R}} = 0.$$

Proof. Let f be in $L^2(\mathcal{R})$ and φ_h be in W_h. We find:

$$\begin{aligned}(\underline{\nu\Sigma}_f \mathcal{S}_h f, \varphi_h)_{0,\mathcal{R}} &= (\underline{\nu\Sigma}_f (F_h^{-1}E_h f - f), \varphi_h)_{0,\mathcal{R}} \\ &= (\underline{\nu\Sigma}_f (F_h^{-1}E_h f - E_h f), \varphi_h)_{0,\mathcal{R}} \\ &= (\underline{\nu\Sigma}_f (F_h^{-1}E_h f - F_h F_h^{-1}E_h f), \varphi_h)_{0,\mathcal{R}}.\end{aligned}$$

The second equality uses (4.38) with $\varphi = f$. One concludes by remarking that $\psi = F_h^{-1}E_h f \in W$ so $(\underline{\nu\Sigma}_f(\psi - F_h\psi), \varphi_h)_{0,\mathcal{R}} = 0$ using again (4.38), because $F_h\psi = E_h\psi$.

□

To obtain an optimal rate of convergence we restrict the operators B_μ and B_μ^h to the eigenspace W. We denote finally by $\hat{B}_\mu$ and $\hat{B}_\mu^h$ the operators, from W to itself, $\hat{B}_\mu = B_\mu|_W$ and $\hat{B}_\mu^h = F_h^{-1}B_\mu^h F_h$. Let us estimate

$$\|\hat{B}_\mu - \hat{B}_\mu^h\|_{\mathcal{L}(W)} = \sup_{\varphi,\varphi' \in W\setminus\{0\}} \frac{|(\varphi, (\hat{B}_\mu - \hat{B}_\mu^h)\varphi')_W|}{\|\varphi\|_W \, \|\varphi'\|_W}.$$

Theorem 4.22. *Let ω be as in Definition 4.15. Then for h small enough, the following estimate holds true*

$$\|\hat{B}_\mu - \hat{B}_\mu^h\|_{\mathcal{L}(W)} \lesssim h^{2\omega}. \tag{4.40}$$

Proof. Using the definition of F_h, Lemma 4.18 and finally Lemma 4.20, one checks that for all $\varphi' \in W$:

$$\begin{aligned}(\hat{B}_\mu - \hat{B}_\mu^h)\varphi' &= B_\mu\varphi' - F_h^{-1}B_\mu^h F_h\varphi' \\ &= B_\mu\varphi' - F_h^{-1}B_\mu^h E_h\varphi' \\ &= B_\mu\varphi' - F_h^{-1}E_h B_\mu^h\varphi' \\ &= (B_\mu - B_\mu^h)\varphi' + B_\mu^h\varphi' - F_h^{-1}E_h B_\mu^h\varphi' + \mathcal{S}_h B_\mu\varphi' \\ &= (B_\mu - B_\mu^h)\varphi' + \mathcal{S}_h(B_\mu - B_\mu^h)\varphi'.\end{aligned} \tag{4.41}$$

Hence, given $\varphi, \varphi' \in W$, we can bound $|(\varphi, (\hat{B}_\mu - \hat{B}_\mu^h)\varphi')_W| = |(\underline{\nu\Sigma}_f\varphi, (\hat{B}_\mu - \hat{B}_\mu^h)\varphi')_{0,\mathcal{R}}|$ by

$$|(\underline{\nu\Sigma}_f\varphi, (B_\mu - B_\mu^h)\varphi')_{0,\mathcal{R}}| + |(\underline{\nu\Sigma}_f\varphi, \mathcal{S}_h(B_\mu - B_\mu^h)\varphi')_{0,\mathcal{R}}|.$$

Let us bound each part separately below.

One obtains from the difference between (4.33) and (4.34)

$$\begin{aligned}(\underline{\nu\Sigma}_f\varphi, (B_\mu - B_\mu^h)\varphi')_{0,\mathcal{R}} = a((A_\mu - A_\mu^h)\varphi, (A_\mu - A_\mu^h)\varphi') + b((A_\mu - A_\mu^h)\varphi', (B_\mu - B_\mu^h)\varphi) \\ + b((A_\mu - A_\mu^h)\varphi, (B_\mu - B_\mu^h)\varphi') + t((B_\mu - B_\mu^h)\varphi, (B_\mu - B_\mu^h)\varphi').\end{aligned}$$

Then, one can bound the first part:

$$\begin{aligned}|(\underline{\nu\Sigma}_f\varphi,(B_\mu-B_\mu^h)\varphi')_{0,\mathcal{R}}| &\lesssim \|(A_\mu-A_\mu^h)\varphi\|_{0,\mathcal{R}}\|(A_\mu-A_\mu^h)\varphi'\|_{0,\mathcal{R}}\\ &\quad+\|\mathrm{div}\,(A_\mu-A_\mu^h)\varphi'\|_{0,\mathcal{R}}\|(B_\mu-B_\mu^h)\varphi\|_{0,\mathcal{R}}\\ &\quad+\|\mathrm{div}\,(A_\mu-A_\mu^h)\varphi\|_{0,\mathcal{R}}\|(B_\mu-B_\mu^h)\varphi'\|_{0,\mathcal{R}}\\ &\quad+\|(B_\mu-B_\mu^h)\varphi\|_{0,\mathcal{R}}\|(B_\mu-B_\mu^h)\varphi'\|_{0,\mathcal{R}}\\ &\lesssim h^{2\omega}\|\varphi\|_W\|\varphi'\|_W.\end{aligned}$$

The second part is bounded by:

$$\begin{aligned}|(\underline{\nu\Sigma}_f\varphi,\mathcal{S}_h(B_\mu-B_\mu^h)\varphi')_{0,\mathcal{R}}| &= |(\underline{\nu\Sigma}_f(\varphi-F_h\varphi),\mathcal{S}_h(B_\mu-B_\mu^h)\varphi')|\\ &\le \|\underline{\nu\Sigma}_f(\varphi-F_h\varphi)\|_{0,\mathcal{R}}\|\mathcal{S}_h(B_\mu-B_\mu^h)\varphi'\|_{0,\mathcal{R}}\\ &\lesssim \|\underline{\nu\Sigma}_f(\varphi-F_h\varphi)\|_{0,\mathcal{R}}\|(B_\mu-B_\mu^h)\varphi'\|_{0,\mathcal{R}}\\ &\lesssim h^{2\omega}\|\varphi\|_W\|\varphi'\|_W.\end{aligned}$$

In the first line we use Proposition 4.21 with $f=(B_\mu-B_\mu^h)\varphi'$ and $\varphi_h=F_h\varphi$. In the third line we use the uniform continuity of $\mathcal{S}_h$ in h, and in the last line we use the first inequality of Proposition 4.17 and the estimation (4.39). Therefore we have obtained (4.40). □

From this estimation and the work of Osborn in Theorem 2 of [27], one derives an optimal estimate on the error on the eigenvalues.

Corollary 4.23. *Let ω be as in Definition 4.15. Then for h small enough, the error on the eigenvalue is given by*

$$|\nu-\nu_h|\lesssim h^{2\omega}.$$

Remark 4.24. If ν has an algebraic multiplicity $m_\nu>1$, the previous analysis and the *a priori* estimate are still valid with $\nu_h=\frac{1}{m_\nu}\sum_{i=1}^{m_\nu}\nu_{h,i}$, where $(\nu_{h,i})_{i=1,m_\nu}$ are the m discrete eigenvalues closest to ν, see again Theorem 2 of [27].

5. The DD case

We continue by considering the neutron diffusion problem using a domain decomposition method: we call it the DD case. The diffusion problem with low-regularity solution in a mixed, multi-domain form has been analyzed in [13]. In this section, we first define some notations and spaces. Then we recall some results of [13], in which technical aspects on the choice and properties of the spaces and discretization are discussed. Finally, we define the variational formulation. The numerical analysis of the DD case is carried out in Section 6.

5.1. Setting of the DD spaces

Let us consider a partition $\{\widetilde{\mathcal{R}}_i\}_{1\le i\le\widetilde{N}}$ of $\mathcal{R}$ which can be independent from the physical partition of the materials in $\mathcal{R}$ (see *e.g.* [10, 11, 23]). In other words, it can happen that $\{\widetilde{\mathcal{R}}_i\}_{1\le i\le\widetilde{N}}\neq\{\mathcal{R}_i\}_{1\le i\le N}$. We denote by Γ_{ij} the interface between two subdomains $\widetilde{\mathcal{R}}_i$ and $\widetilde{\mathcal{R}}_j$, for $i\neq j$: if the Hausdorff dimension of $\overline{\widetilde{\mathcal{R}}_i}\cap\overline{\widetilde{\mathcal{R}}_j}$ is $d-1$, then $\Gamma_{ij}=int(\overline{\widetilde{\mathcal{R}}_i}\cap\overline{\widetilde{\mathcal{R}}_j})$; otherwise, $\Gamma_{ij}=\emptyset$. By construction, $\Gamma_{ij}=\Gamma_{ji}$. We define the interface Γ_S,

respectively the wirebasket $\partial\Gamma_W$ by

$$\Gamma_S = \bigcup_{i=1}^{\widetilde{N}} \bigcup_{j=i+1}^{\widetilde{N}} \overline{\Gamma_{ij}}, \quad \partial\Gamma_W = \bigcup_{i=1}^{\widetilde{N}} \bigcup_{j=i+1}^{\widetilde{N}} \partial\Gamma_{ij}.$$

It is stressed that the resulting interface Γ_S needs not necessarily coincide with the physical interface between cells.

When $d = 2$, the wirebasket consists of isolated crosspoints. When $d = 3$, the wirebasket consists of open edges and crosspoints. For a field v defined over $\mathcal{R}$, we shall use the notation $v_i = v_{|\widetilde{\mathcal{R}}_i}$, for $1 \leq i \leq \widetilde{N}$. Let us define the function space with zero Dirichlet boundary condition:

$$\widetilde{\mathcal{P}}H_0^1(\mathcal{R}) = \left\{ \psi \in L^2(\mathcal{R}) \,|\, \psi_i \in H^1(\widetilde{\mathcal{R}}_i),\, \psi_{|\partial\widetilde{\mathcal{R}}_i \setminus \overline{\Gamma_S}} = 0,\, 1 \leq i \leq \widetilde{N} \right\}.$$

When $\Gamma_{ij} \neq \emptyset$, let $H^{1/2}_{\Gamma_{ij}}$ be the set of $H^{1/2}(\Gamma_{ij})$ functions whose continuation by 0 to $\partial\mathcal{R}_i$ belongs to $H^{1/2}(\partial\mathcal{R}_i)$. On can prove that $H^{1/2}_{\Gamma_{ij}} = H^{1/2}_{\Gamma_{ji}}$. We also introduce the space of piecewise $\mathbf{H}(\mathrm{div}\,)$ vector-valued functions:

$$\widetilde{\mathcal{P}}\mathbf{H}(\mathrm{div}\,,\mathcal{R}) = \left\{ \mathbf{q} \in \mathbf{L}^2(\mathcal{R}) \,|\, \mathbf{q}_i \in \mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_i),\, 1 \leq i \leq \widetilde{N} \right\}, \quad ||\mathbf{q}||_{\widetilde{\mathcal{P}}\mathbf{H}(\mathrm{div}\,,\mathcal{R})} = \left(\sum_i \|\mathbf{q}_i\|^2_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_i)} \right)^{1/2}.$$

For $\mathbf{p} \in \widetilde{\mathcal{P}}\mathbf{H}(\mathrm{div}\,,\mathcal{R})$, let us set $[\mathbf{p}\cdot\mathbf{n}]_{ij} := \sum_{k=i,j} \mathbf{p}_k \cdot \mathbf{n}_{k|\Gamma_{ij}}$ the jump of the normal component of $\mathbf{p}$ on Γ_{ij} when $\Gamma_{ij} \neq \emptyset$. $[\mathbf{p}\cdot\mathbf{n}]_{ij}$ is well defined in $(H^{1/2}_{\Gamma_{ij}})'$ the dual space of $H^{1/2}_{\Gamma_{ij}}$ (see *e.g.* [19]). The *global jump* $[\mathbf{p}\cdot\mathbf{n}]$ of the normal component on the interface is defined by:

$$[\mathbf{p}\cdot\mathbf{n}]_{|\Gamma_{ij}} := [\mathbf{p}\cdot\mathbf{n}]_{ij}, \quad \text{for } 1 \leq i,j \leq \widetilde{N}.$$

By definition, it holds $[\mathbf{p}\cdot\mathbf{n}] \in \prod_{i<j} (H^{1/2}_{\Gamma_{ij}})'$. We recall that for $\mathbf{p} \in \mathbf{H}(\mathrm{div}\,,\mathcal{R})$, the global jump vanishes: $[\mathbf{p}\cdot\mathbf{n}] = 0$ (see *e.g.* [13], Lem. 1).

We introduce finally the following Hilbert spaces:

$$\begin{aligned}
M &= \left\{ \psi_S \in \prod_{i<j} L^2(\Gamma_{ij}) \right\}, \|\psi_S\|_M = \left(\sum_{i<j} \|\psi_S\|^2_{0,\Gamma_{ij}} \right)^{1/2}; \\
H^{1/2}_{-}(\Gamma_S) &= \left\{ \psi_S \in M \,|\, \psi_{S|\Gamma_{ij}} \in H^{1/2}(\Gamma_{ij}),\, \forall i<j \right\}, \text{ with graph norm;} \\
\widetilde{\mathbf{Q}} &= \left\{ \mathbf{q} \in \widetilde{\mathcal{P}}\mathbf{H}(\mathrm{div}\,,\mathcal{R}) \,|\, [\mathbf{q}\cdot\mathbf{n}] \in M \right\}, \\
\|\mathbf{q}\|_{\widetilde{\mathbf{Q}}} &= \left(||\mathbf{q}||^2_{\widetilde{\mathcal{P}}\mathbf{H}(\mathrm{div}\,,\mathcal{R})} + |||[\mathbf{q}\cdot\mathbf{n}]|||^2_M \right)^{1/2}; \\
\widetilde{\mathbf{X}} &= \left\{ \xi := (\mathbf{q},\psi) \in \widetilde{\mathbf{Q}} \times L^2(\mathcal{R}) \right\}, \|\xi\|_{\widetilde{\mathbf{X}}} := \left(\|\mathbf{q}\|^2_{\widetilde{\mathbf{Q}}} + \|\psi\|^2_{0,\mathcal{R}} \right)^{1/2}; \\
\mathtt{W} &= \left\{ \mathtt{w} := (\xi,\psi_S) \in \widetilde{\mathbf{X}} \times M \right\}, \|\mathtt{w}\|_{\mathtt{W}} := \left(\|\xi\|^2_{\widetilde{\mathbf{X}}} + \|\psi_S\|^2_M \right)^{1/2}.
\end{aligned}$$

By construction, one has $M \subset \prod_{i<j} (H^{1/2}_{\Gamma_{ij}})'$. We will next define a variational formulation which is conforming in $\widetilde{\mathbf{Q}} \times L^2(\mathcal{R})$.

5.2. Variational formulation and discretization in the DD case

The mixed form of the neutron diffusion problem (4.1) is now given by (see Sect. 3.2 from [13]):
Find $(\mathbf{p}, \phi, \phi_S) \in \widetilde{\mathbf{Q}} \times \widetilde{\mathcal{P}} H_0^1(\mathcal{R}) \times M$ such that:

$$\begin{cases} -D_i^{-1}\,\mathbf{p}_i \,-\, \mathbf{grad}\,\phi_i = 0 & \text{in } \widetilde{\mathcal{R}}_i, \quad \text{for } 1 \le i \le \widetilde{N}, \\ \operatorname{div}\mathbf{p}_i \,+\, \Sigma_{a,i}\phi_i = S_{f,i} & \text{in } \widetilde{\mathcal{R}}_i, \quad \text{for } 1 \le i \le \widetilde{N}, \\ \phi_i = \phi_S & \text{on } \partial\widetilde{\mathcal{R}}_i \cap \Gamma_S, \quad \text{for } 1 \le i \le \widetilde{N}, \\ [\mathbf{p}\cdot\mathbf{n}] = 0 & \text{on } \Gamma_S. \end{cases} \tag{5.1}$$

To solve this problem, we are looking for a solution $((\mathbf{p},\phi),\phi_S)$ in $\mathtt{W}$. Find $((\mathbf{p},\phi),\phi_S) \in \mathtt{W}$, such that $\forall((\mathbf{q},\psi),\psi_S) \in \mathtt{W}$:

$$\int_{\mathcal{R}} \left(-D^{-1}\,\mathbf{p}\cdot\mathbf{q} + \phi \operatorname{div}\mathbf{q} + \psi \operatorname{div}\mathbf{p} + \Sigma_a\,\phi\,\psi\right) + \int_{\Gamma_S} [\mathbf{p}\cdot\mathbf{n}]\psi_S - \int_{\Gamma_S} [\mathbf{q}\cdot\mathbf{n}]\,\phi_S = \int_{\mathcal{R}} S_f\,\psi. \tag{5.2}$$

In (5.1)–(5.2), ϕ_S, ψ_S play the role of Lagrange multipliers, with M the space of those Lagrange multipliers. To be mathematically precise, we should be integrating on $\cup_{i<j}\Gamma_{ij}$ instead of Γ_S. We make this slight abuse of notations from now on. This approach is called the DD+L^2-jumps method.

From now on, we use the notations:

- $\mathtt{u} = (\zeta, \phi_S)$, $\zeta = (\mathbf{p}, \phi)$, $\mathbf{p} = (\mathbf{p}_i)_{1\le i\le \widetilde{N}}$ and $\phi = (\phi_i)_{1\le i\le \widetilde{N}}$;
- $\mathtt{w} = (\xi, \psi_S)$, $\xi = (\mathbf{q}, \psi)$, $\mathbf{q} = (\mathbf{q}_i)_{1\le i\le \widetilde{N}}$ and $\psi = (\psi_i)_{1\le i\le \widetilde{N}}$;

and we define the bilinear forms:

$$\ell_S : \begin{cases} \mathtt{W}\times\mathtt{W} \to \mathbb{R} \\ (\mathtt{u},\mathtt{w}) \mapsto \displaystyle\int_{\Gamma_S} [\mathbf{p}\cdot\mathbf{n}]\,\psi_S \end{cases}, \tag{5.3}$$

and:

$$c_S : \begin{cases} \mathtt{W}\times\mathtt{W} \to \mathbb{R} \\ (\mathtt{u},\mathtt{w}) \mapsto c(\zeta,\xi) \,+\, \ell_S(\mathtt{u},\mathtt{w}) \,-\, \ell_S(\mathtt{w},\mathtt{u}) \end{cases}. \tag{5.4}$$

We consider the linear form:

$$f_S : \begin{cases} \mathtt{W} \to \mathbb{R} \\ \mathtt{w} \mapsto f(\xi) \end{cases}. \tag{5.5}$$

Above, we extended the definition (4.7) (resp. (4.8)) of the form c (resp. f), to elements of $\widetilde{\mathbf{X}}\times\widetilde{\mathbf{X}}$ (resp. $\widetilde{\mathbf{X}}$). We may rewrite the variational formulation (5.2) as:
Find $\mathtt{u} \in \mathtt{W}$ such that $\forall \mathtt{w} \in \mathtt{W}$:

$$c_S(\mathtt{u},\mathtt{w}) = f_S(\mathtt{w}). \tag{5.6}$$

We recall that c_S satisfies an inf-sup condition, so the variational problem is well-posed (see [13], Sect. 4), and that, under the assumptions of Proposition 3.1, the global jump of $\mathbf{p}$ vanishes: $[\mathbf{p}\cdot\mathbf{n}] = 0$ in M (see Lem. 1 of [13]).
We study abstract, conforming, discretization of the variational formulation (5.6) as it is done in Section 5 from

[13]. To that aim, we introduce discrete, finite-dimensional, spaces indexed by a (small) parameter h as follows: $\mathbf{Q}_{i,h} \subset \mathbf{H}(\mathrm{div}\,, \widetilde{\mathcal{R}}_i)$ and $L_{i,h} \subset L^2(\widetilde{\mathcal{R}}_i)$, for $1 \leq i \leq \widetilde{N}$. We impose the following requirements, for all $1 \leq i \leq \widetilde{N}$:

- $\mathbf{q}_{i,h} \cdot \mathbf{n}_{|\partial\widetilde{\mathcal{R}}_i} \in L^2(\partial\widetilde{\mathcal{R}}_i)$ for all $h > 0$, for all $\mathbf{q}_{i,h} \in \mathbf{Q}_{i,h}$;
- $\mathrm{div}\, \mathbf{Q}_{i,h} \subset L_{i,h}$ for all $h > 0$;
- $(\mathbf{Q}_{i,h})_h$ and $(L_{i,h})_h$ satisfy the *approximability property* (4.12) in $\widetilde{\mathcal{R}}_i$.

Then, let

$$\widetilde{\mathbf{Q}}_h = \prod_{1\leq i\leq \widetilde{N}} \mathbf{Q}_{i,h} \quad \text{and} \quad L_h = \prod_{1\leq i\leq \widetilde{N}} L_{i,h}.$$

In particular, the discretization $\widetilde{\mathbf{Q}}_h \times L_h$ is globally conforming in $\widetilde{\mathbf{Q}} \times L^2(\mathcal{R})$. We endow $\widetilde{\mathbf{Q}}_h$ with the norm $\|\cdot\|_{\widetilde{\mathbf{Q}}}$, while L_h is endowed with $\|\cdot\|_{0,\mathcal{R}}$.

We then define $T_{i,h}$ as the space of the normal traces of vectors of $\mathbf{Q}_{i,h}$ on $\partial\widetilde{\mathcal{R}}_i \cap \Gamma_S$:

$$T_{i,h} := \left\{ q_{i,h} \in L^2(\partial\widetilde{\mathcal{R}}_i \cap \Gamma_S) \,|\, \exists\, \mathbf{q}_{i,h} \in \mathbf{Q}_{i,h},\, q_{i,h} = \mathbf{q}_{i,h} \cdot \mathbf{n}_{i|\partial\widetilde{\mathcal{R}}_i \cap \Gamma_S} \right\}. \tag{5.7}$$

Classically, several situations can occur on a given interface Γ_{ij}, $1 \leq i, j \leq \widetilde{N}$:

(1) *non-nested meshes*: $T_{i,h|\Gamma_{ij}} \not\subset T_{j,h|\Gamma_{ij}}$ and $T_{j,h|\Gamma_{ij}} \not\subset T_{i,h|\Gamma_{ij}}$;
(2) *nested meshes*: $T_{i,h|\Gamma_{ij}} \subset T_{j,h|\Gamma_{ij}}$ or $T_{j,h|\Gamma_{ij}} \subset T_{i,h|\Gamma_{ij}}$;
(3) *matching meshes*: nested meshes with $T_{i,h|\Gamma_{ij}} = T_{j,h|\Gamma_{ij}}$.

Usually, the term *nested meshes* is used to describe a family of successively refined meshes. In this paper, we will use this expression to express that on all interfaces Γ_{ij}, case (5.2) described above holds. As an illustration, see the interfaces between the subdomains in Fig. 3a.

Let us denote by $M_h \subset M$ the discrete space of the Lagrange multipliers. We assume that M_h includes the subspace M_h^0 of piecewise constant fields. We introduce the discrete projection operators ([13], Sect. 5) from the spaces of normal traces $T_{i,h}$ to M_h, and vice versa, which are defined by:

$$\forall q_{i,h} \in T_{i,h},\, \forall \psi_{S,h} \in M_h \begin{cases} \displaystyle\int_{\partial\widetilde{\mathcal{R}}_i \cap \Gamma_S} (\Pi_i(q_{i,h}) - q_{i,h})\ \psi_{S,h} = 0 \\ \displaystyle\int_{\partial\widetilde{\mathcal{R}}_i \cap \Gamma_S} (\pi_i(\psi_{S,h}) - \psi_{S,h})\ q_{i,h} = 0 \end{cases}. \tag{5.8}$$

As the operators Π_i and π_i are orthogonal projections, they are continuous, with a continuity modulus equal to 1. We also introduce the orthogonal projection operator $\Pi_S^0 : M \to M_h^0$. According to Proposition 1.135 of [17], if we denote by h_S the meshsize on Γ_S:

$$\forall \psi_S \in H_-^{1/2}(\Gamma_S),\ \|\psi_S - \Pi_S^0(\psi_S)\|_M \lesssim h_S^{1/2} \|\psi_S\|_{H_-^{1/2}(\Gamma_S)}. \tag{5.9}$$

Next, let $\mathbf{p}_h \in \widetilde{\mathbf{Q}}_h$. We define the discrete jump of the normal component of $\mathbf{p}_h$ on the interface Γ_{ij} as $[\mathbf{p}_h \cdot \mathbf{n}]_{h,ij} := \sum_{l=i,j} \Pi_l(\mathbf{p}_{l,h} \cdot \mathbf{n}_{l|\Gamma_{ij}})$. The discrete global jump of the normal component, $[\mathbf{p}_h \cdot \mathbf{n}]_h \in M_h$, is defined by:

$$[\mathbf{p}_h \cdot \mathbf{n}]_{h|\Gamma_{ij}} := [\mathbf{p}_h \cdot \mathbf{n}]_{h,ij},\ \text{for } 1 \leq i, j \leq \widetilde{N}.$$

We finally define:

$$\widetilde{\mathbf{X}}_h = \left\{ \xi_h := (\mathbf{q}_h, \psi_h) \in \widetilde{\mathbf{Q}}_h \times L_h \right\}, \text{ endowed with } \|\cdot\|_{\widetilde{\mathbf{X}}},$$
$$\mathtt{W}_h = \left\{ \mathtt{w}_h := (\xi_h, \psi_{S,h}) \in \widetilde{\mathbf{X}}_h \times M_h \right\}, \text{ endowed with } \|\cdot\|_{\mathtt{W}}.$$

In the DD+L^2-jumps setting, the conforming discretization of the variational formulation (5.6) reads:

$$\text{Find } \mathtt{u}_h \in \mathtt{W}_h \text{ such that } \forall \mathtt{w}_h \in \mathtt{W}_h,\ c_S(\mathtt{u}_h, \mathtt{w}_h) = f_S(\mathtt{w}_h). \tag{5.10}$$

It is shown in Section 5 from [13] that c_S verifies a discrete inf-sup condition if the following conditions hold:

$$\exists \beta_h > 0,\ \forall \mathbf{q}_h \in \widetilde{\mathbf{Q}}_h,\ \int_{\Gamma_S} [\mathbf{q}_h \cdot \mathbf{n}]_h\, [\mathbf{q}_h \cdot \mathbf{n}] \geq \beta_h \int_{\Gamma_S} [\mathbf{q}_h \cdot \mathbf{n}]^2 \tag{5.11}$$

and

$$\exists \gamma_h > 0, \quad \forall\ \psi_{S,h} \in M_h,$$
$$\sum_{i=1}^{\widetilde{N}} \sum_{j=i+1}^{\widetilde{N}} \int_{\Gamma_{ij}} \left(\pi_i(\psi_{S,h})^2 + \pi_j(\psi_{S,h})^2 \right) \geq \gamma_h \|\psi_{S,h}\|_M^2, \tag{5.12}$$

Moreover, if β_h and γ_h can be chosen independently of h, the form c_S satisfies a udisc. For instance, conditions (5.11)–(5.12) are uniformly fulfilled when M_h is chosen as

$$M_h = \sum_{i=1}^{\widetilde{N}} T_{i,h}. \tag{5.13}$$

Last, under (5.11), one easily checks that $[\mathbf{p}_h \cdot \mathbf{n}] = 0$. In other words:

$$\mathbf{p}_h \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}) \cap \widetilde{\mathbf{Q}}_h. \tag{5.14}$$

In the DD case, we define $\mathbf{Q}_h = \mathbf{H}(\mathrm{div}\,, \mathcal{R}) \cap \widetilde{\mathbf{Q}}_h$.

6. Numerical analysis in the DD case

To carry out the numerical analysis in the low-regularity case, we first introduce a suitable discretization of the DD problem, and then we carry out the numerical analysis on this discretization. Again, if one chooses another discretization that fulfills those properties detailed in the previous section, one may recover similar convergence results.

6.1. Discretization

We consider (5.10) where the RTN finite element is used on each subdomain with a conforming mesh, or triangulation. For $1 \leq i \leq \widetilde{N}$, let h_i denote the local meshsize in $\widetilde{\mathcal{R}}_i$, and $h = \max_i h_i$ the global meshsize. Let us denote by $k_i \geq 0$ the order of the discretization in $\widetilde{\mathcal{R}}_i$, and $k = \min_i k_i$, the minimal order of the RTN finite element. The local RTN finite element subspace of $\mathbf{H}(\mathrm{div}\,, \widetilde{\mathcal{R}}_i) \times L^2(\widetilde{\mathcal{R}}_i)$ is defined as $\mathbf{Q}_{i,h_i}^{k_i} \times L_{i,h_i}^{k_i}$. With this choice, we have $\mathrm{div}\, \mathbf{Q}_{i,h_i}^{k_i} \subset L_{i,h_i}^{k_i}$ as required: local consistency is ensured. Now, if we set $\widetilde{\mathbf{Q}}_h^k = \prod_{1 \leq i \leq \widetilde{N}} \mathbf{Q}_{i,h_i}^{k_i}$ and $L_h^k = \prod_{1 \leq i \leq \widetilde{N}} L_{i,h_i}^{k_i}$, we have $\mathbf{q}_{i,h} \cdot \mathbf{n}_{|\partial \widetilde{\mathcal{R}}_i} \in L^2(\partial \widetilde{\mathcal{R}}_i)$ for all $\mathbf{q}_{i,h} \in \mathbf{Q}_{i,h_i}^{k_i}$, hence it follows that $\widetilde{\mathbf{Q}}_h^k \subset \widetilde{\mathbf{Q}}$:

the discretization $\widetilde{\mathbf{Q}}_h^k \times L_h^k$ is globally conforming in $\widetilde{\mathbf{Q}} \times L^2(\mathcal{R})$. For the reader's convenience, we omit the superscript k_i in the analysis below.

Finally, we choose M_h so that on the one hand (5.11)–(5.12) hold uniformly, and on the other hand it holds $h_S \lesssim h$: we refer to Section 5.2 from [13] for an extended discussion on suitable choices. According to the first Strang's Lemma [17] and because c_S verifies a udisc, the error reads:

$$\|\mathtt{u} - \mathtt{u}_h\|_{\mathtt{W}} \lesssim \inf_{\mathtt{w}_h \in \mathtt{W}_h} \|\mathtt{u} - \mathtt{w}_h\|_{\mathtt{W}}. \tag{6.1}$$

As a consequence $\lim_{h\to 0} \|\mathtt{u} - \mathtt{u}_h\|_{\mathtt{W}} = 0$. This result holds for nested and non-nested meshes. We study below how to improve the bound on the error, how to derive an Aubin-Nitsche estimate, and finally how to prove convergence for the generalized eigenvalue problem, for *nested* meshes.[3] As previously, those results hold under the assumptions of Proposition 3.1 (plus $\underline{\nu\Sigma}_f \in \mathcal{P}W^{1,\infty}(\mathcal{R})$ for the eigenproblem). We focus again on the low-regularity case.

6.2. *A priori* error estimates

Let $\mathbf{q} \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}) \cap \widetilde{\mathcal{P}}\mathbf{H}^\mu(\mathcal{R})$, with $0 < \mu$. A global RTN interpolant of $\mathbf{q}$ is defined on every subdomain $\widetilde{\mathcal{R}}_i$ *via* its restriction $\mathbf{q}_i$, and denoted by $\tilde{\mathbf{q}}_{i,R}$ for $1 \le i \le \widetilde{N}$. One may thus define the global interpolant of $\mathbf{q}$ in $\widetilde{\mathbf{Q}}_h$, denoted by $\tilde{\mathbf{q}}_R$ henceforth: $\tilde{\mathbf{q}}_{R|\widetilde{\mathcal{R}}_i} = \tilde{\mathbf{q}}_{i,R}$ for $1 \le i \le \widetilde{N}$. Below, we also use the orthogonal projection operators $\pi^0 : L^2(\mathcal{R}) \to L_h^0$ (see Sect. 4.5.1) and $\Pi_S^0 : M \to M_h^0$ (see Sect. 5.2). One has the following result, whose proof is given in Appendix A.

Lemma 6.1. *Assume that the meshes are nested, non-matching, on the interface Γ_{fc}, and that they are quasi-uniform on Γ_{fc}. To fix ideas, we assume $T_{c,h|\Gamma_{fc}} \subset T_{f,h|\Gamma_{fc}}$ with $T_{c,h|\Gamma_{fc}} \neq T_{f,h|\Gamma_{fc}}$ ([4]).*

Let $\mathbf{q} \in \mathbf{H}(\mathrm{div}\,, \mathcal{R}) \cap \mathbf{H}^\mu(\mathcal{R})$ with $0 < \mu < 1/2$, it holds:

$$\|[\tilde{\mathbf{q}}_R \cdot \mathbf{n}]\|_{0,\Gamma_{fc}} \lesssim h_f^{1/2} \| \mathbf{q}_f \|_{\mathbf{H}(\mathrm{div}\,, \widetilde{\mathcal{R}}_f)}.$$

Theorem 6.2. *Let the assumptions of Proposition 3.1 hold, with $r_{\max} < 1/2$. One has for matching meshes:*

$$\begin{aligned} &\forall \mu \in]0, r_{\max}[,\ \forall S_f \in H^\mu(\mathcal{R}), \\ &\|\mathbf{p} - \mathbf{p}_h\|_{\mathbf{H}(\mathrm{div}\,, \mathcal{R})} + \|\phi - \phi_h\|_{0,\mathcal{R}} + \|\phi_S - \phi_{S,h}\|_M \lesssim h^\mu \|S_f\|_{\mu,\mathcal{R}}. \end{aligned} \tag{6.2}$$

For nested, non-matching *meshes, the result holds under the assumption that on an interface Γ_{ij} where the meshes $T_{i,h|\Gamma_{ij}}$ and $T_{j,h|\Gamma_{ij}}$ are non-matching ($T_{i,h|\Gamma_{ij}} \neq T_{j,h|\Gamma_{ij}}$), the families of triangulations of $T_{i,h|\Gamma_{ij}}$ and $T_{j,h|\Gamma_{ij}}$ are quasi-uniform.*

Proof. We bound the different contributions in the right-hand side of (6.1) for some appropriately chosen discrete field $\mathtt{w}_h$. Recall that $\mathtt{u} = ((\mathbf{p}, \phi), \phi_S)$.

Matching meshes. We know that $[\mathbf{p} \cdot \mathbf{n}] = 0$. For matching meshes, one has also $[\tilde{\mathbf{p}}_R \cdot \mathbf{n}] = 0$, so $[(\mathbf{p} - \tilde{\mathbf{p}}_R) \cdot \mathbf{n}] = 0$. Starting from (6.1), the conclusion follows. Indeed, according to the *a priori* estimates (4.15), (4.20) and (5.9), $\mathtt{w}_h = (\tilde{\mathbf{p}}_R, \pi^0\phi, \Pi_S^0(\phi_S)) \in \mathtt{W}_h$ is such that

$$\begin{aligned} \|\mathtt{u} - \mathtt{w}_h\|_{\mathtt{W}}^2 &= \sum_{i=1}^{\widetilde{N}} \|\mathbf{p}_i - \mathbf{p}_{i,R}\|_{\mathbf{H}(\mathrm{div}\,, \widetilde{\mathcal{R}}_i)}^2 + \|\phi - \pi^0\phi\|_{0,\mathcal{R}}^2 + \|\phi_S - \Pi_S^0(\phi_S)\|_M^2 \\ &\lesssim h^{2\mu}(|\mathbf{p}|_{\mu,\mathcal{R}}^2 + \|\mathrm{div}\,\mathbf{p}\|_{\mu,\mathcal{R}}^2) + h^2\|\phi\|_{\mathcal{P}H^1(\mathcal{R})}^2 + h_S\|\phi_S\|_{H_-^{1/2}(\Gamma_S)}^2 \lesssim h^{2\mu} \|S_f\|_{\mu,\mathcal{R}}^2. \end{aligned}$$

[3] For non-nested meshes, numerical illustrations suggest that the convergence properties can be recovered in some situations (see [13], Tab. 2). See also Section 6.5.

[4] f refers to *fine discretization*, while c refers to *coarse discretization*.

Hence we conclude that for matching meshes it holds:

$$\|\mathtt{u}-\mathtt{u}_h\|_{\mathtt{W}} \lesssim h^{\mu}\,\|S_f\|_{\mu,\mathcal{R}}. \tag{6.3}$$

Nested meshes. In this case, $[\tilde{\mathbf{p}}_R\cdot\mathbf{n}]\neq 0$ in general. Nonetheless, one can use the result of Lemma 6.1, to find that

$$\|\,[(\mathbf{p}-\tilde{\mathbf{p}}_R)\cdot\mathbf{n}]\,\|_M \lesssim h^{1/2}\,\|\mathbf{p}\|_{\mathbf{H}(\mathrm{div}\,,\mathcal{R})},$$

provided that the meshes are *quasi-uniform* on the part of the interface where they are non-matching. One concludes that the estimate (6.3) still holds for nested meshes under this condition.

Conclusion. Noting that it always holds $[\mathbf{p}\cdot\mathbf{n}]=[\mathbf{p}_h\cdot\mathbf{n}]=0$ (*cf.* (5.14)), developing the norm $\|\mathtt{u}-\mathtt{u}_h\|_{\mathtt{W}}$, one concludes:

$$\|\mathbf{p}-\mathbf{p}_h\|_{\mathbf{H}(\mathrm{div}\,,\mathcal{R})}+\|\phi-\phi_h\|_{0,\mathcal{R}}+\|\phi_S-\phi_{S,h}\|_M \lesssim h^{\mu}\,\|S_f\|_{\mu,\mathcal{R}}.$$

In other words, we have the *a priori* error estimate (6.2). □

As in the plain case, for "smooth data" S_f, *i.e.* $S_f\in H^{r_{\max}}(\mathcal{R})$, one expects a convergence rate at least in $h^{r_{\max}}$.

Remark 6.3. Within our framework, we obtain error estimates that generalize those of [11, 32] for low-regularity solutions. In addition, the technical aspects we propose remain quite simple and natural.

6.3. Aubin-Nitsche-type estimates

To derive improved estimates on the error $\|\phi-\phi_h\|_{0,\mathcal{R}}$, we adapt the calculations of Section 4.5.2 to the DD case. Recall that $\mathbf{Q}_h=\widetilde{\mathbf{Q}}_h\cap\mathbf{H}(\mathrm{div}\,,\mathcal{R})$. We already know that when conditions (5.11)–(5.12) hold, the solution $((\mathbf{p}_h,\phi_h),\phi_{S,h})\in\widetilde{\mathbf{X}}_h\times M_h$ of (5.10) (discrete DD case) is such that $(\mathbf{p}_h,\phi_h)\in\mathbf{X}_h$, since $\mathbf{p}_h\in\mathbf{Q}_h$. Then restricting the test-fields in (5.10) to elements of $\mathbf{X}_h\times M_h$ we observe that $(\mathbf{p}_h,\phi_h)$ satisfies (4.14) too (discrete plain-case), because all interface terms *vanish*. Hence, to estimate $\|\phi-\phi_h\|_{0,\mathcal{R}}$ in the DD case, we explicitly consider that the discrete fields $(\mathbf{p}_h,\phi_h)$ are also the solution to the variational formulation of the plain-case (4.14). Let us begin by a technical result, whose proof is given in Appendix A.

Lemma 6.4. *Let the assumptions of Lemma 6.1 hold. Let* $\mathbf{q}\in\mathbf{H}(\mathrm{div}\,,\mathcal{R})\cap\mathbf{H}^{\mu}(\mathcal{R})$ *with* $0<\mu<1/2$, *and define* $\delta\mathbf{q}_{fc}\in\mathbf{Q}_{f,h}$ *by* $\delta\mathbf{q}_{fc}\cdot\mathbf{n}_{|\Gamma_{fc}}=(\tilde{\mathbf{q}}_{c,R}\cdot\mathbf{n}-\tilde{\mathbf{q}}_{f,R}\cdot\mathbf{n})_{|\Gamma_{fc}}$ *and zero extension in* $\widetilde{\mathcal{R}_f}\setminus\Gamma_{fc}$. *It holds*

$$\|\delta\mathbf{q}_{fc}\|_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_f)} \lesssim h^{\mu}\left(\|\mathbf{q}_f\|_{\mu,\widetilde{\mathcal{R}}_f}+\|\mathrm{div}\,\mathbf{q}_f\|_{0,\widetilde{\mathcal{R}}_f}\right).$$

Theorem 6.5. *Under the assumptions of Theorem 6.2 with* $r_{\max}<1/2$, *one has for nested meshes:*

$$\forall\mu\in]0,r_{\max}[,\ \forall S_f\in H^{\mu}(\mathcal{R}),\quad \|\phi-\phi_h\|_{0,\mathcal{R}}\lesssim h^{2\mu}\,\|S_f\|_{\mu,\mathcal{R}}. \tag{6.4}$$

Proof. Matching meshes. In this case, one can use the theory already developed in Section 4.5 for the plain case, to conclude that (6.4) holds.

Nested meshes. The difficulty for non-matching meshes is that one cannot define the global RTN-interpolant of $\mathbf{p}$ directly. Instead it is defined *via* its subdomain interpolants $(\widetilde{\mathbf{p}}_{i,R})_{1\le i\le\widetilde{N}}$. Introduce, for $1\le i\le\widetilde{N}$, $\mathcal{I}_i$ as the set of indices j such that $T_{j,h|\Gamma_{ij}}\subset T_{i,h|\Gamma_{ij}}$ (since we are dealing with nested meshes, it holds $T_{j,h|\Gamma_{ij}}\subset T_{i,h|\Gamma_{ij}}$ or $T_{i,h|\Gamma_{ij}}\subset T_{j,h|\Gamma_{ij}}$). We proceed as follows to obtain an $\mathbf{H}(\mathrm{div}\,,\mathcal{R})$-conforming approximant, *i.e.* an element of $\mathbf{Q}_h$. On all interfaces Γ_{ij}, introduce $\delta\mathbf{p}_{ij}\cdot\mathbf{n}=\tilde{\mathbf{p}}_{c,R}\cdot\mathbf{n}_{|\Gamma_{ij}}-\tilde{\mathbf{p}}_{f,R}\cdot\mathbf{n}_{|\Gamma_{ij}}$ where $\tilde{\mathbf{p}}_{f,R}$ is the interpolant from the finer discretization on Γ_{ij}, resp. $\tilde{\mathbf{p}}_{c,R}$ is the interpolant from the coarser discretization on Γ_{ij}. By construction,

$\delta\mathbf{p}_{ij}\cdot\mathbf{n}=0$ when $T_{i,h|\Gamma_{ij}}=T_{j,h|\Gamma_{ij}}$. Then $\delta\mathbf{p}_{ij}\cdot\mathbf{n}$ is extended by zero in $\widetilde{\mathcal{R}}_i$ to define an element of $\mathbf{Q}_{i,h}$; with a slight abuse of notation, we still denote the extension by $\delta\mathbf{p}_{ij}$. The $\mathbf{H}(\mathrm{div}\,,\mathcal{R})$-conforming approximant $\mathbf{p}_R\in\mathbf{Q}_h$ is then defined subdomain by subdomain as

$$\mathbf{p}_{i,R}=\widetilde{\mathbf{p}}_{i,R}+\sum_{j\in\mathcal{I}_i}\delta\mathbf{p}_{ij}\quad\text{for } 1\le i\le\widetilde{N}.$$

Indeed, $[\mathbf{p}_R\cdot\mathbf{n}]_{\Gamma_{ij}}=0$ for $1\le i,j\le\widetilde{N}$ by direct inspection. It remains to evaluate

$$\|\mathbf{p}-\mathbf{p}_R\|^2_{\mathbf{H}(\mathrm{div}\,,\mathcal{R})}=\sum_{1\le i\le\widetilde{N}}\|\mathbf{p}_i-\mathbf{p}_{i,R}\|^2_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_i)},\ \text{with}$$
$$\|\mathbf{p}_i-\mathbf{p}_{i,R}\|_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_i)}\le\|\mathbf{p}_i-\widetilde{\mathbf{p}}_{i,R}\|_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_i)}+\sum_{j\in\mathcal{I}_i}\|\delta\mathbf{p}_{ij}\|_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_i)}\quad\text{for } 1\le i\le\widetilde{N}.$$

Above, the fact that the index j belongs to $\mathcal{I}_i$ implies that if $\delta\mathbf{p}_{ij}\neq0$, then the finer discretization on Γ_{ij} automatically originates from $\widetilde{\mathcal{R}}_i$. To evaluate $\|\delta\mathbf{p}_{ij}\|_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_i)}$, one uses the results of Lemma 6.4 to find

$$\|\delta\mathbf{p}_{ij}\|_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_i)}\lesssim h^\mu\left(\|\mathbf{p}_i\|_{\mu,\widetilde{\mathcal{R}}_i}+\|\mathrm{div}\,\mathbf{p}_i\|_{0,\widetilde{\mathcal{R}}_i}\right).$$

Again, this bound holds under the condition that the meshes are *quasi-uniform* on the part of the interface where they are non-matching. Due to (4.20), one has $\|\mathbf{p}_i-\mathbf{p}_{i,R}\|_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_i)}\lesssim h^\mu\|S_f\|_{\mu,\mathcal{R}}$ for $1\le i\le\widetilde{N}$, and it follows that

$$\|\mathbf{p}-\mathbf{p}_R\|_{\mathbf{H}(\mathrm{div}\,,\mathcal{R})}\lesssim h^\mu\|S_f\|_{\mu,\mathcal{R}}.$$

As a consequence (follow Sect. 4.5.2) we conclude that the estimate (6.4) holds. □

6.4. Numerical analysis of the generalized eigenvalue problem

Let us focus on the approximation of the generalized eigenvalue problem (3.2) for low-regularity solutions with nested (matching or non-matching) meshes. We will follow the methodology of Section 4.6.

6.4.1. Convergence in operator norm

Let $0\le\mu<r_{\max}$ be given, we introduce an operator B_μ associated to the source problem (5.6): given $f\in H^\mu(\mathcal{R})$, we call $B_\mu f=\phi\in H^1(\mathcal{R})$ the second component of the triple $(\mathbf{p},\phi,\phi_S)$ that solves the source problem with $S_f=\underline{\nu\Sigma}_f f$. For the same reason as in the plain case Section 4.6.1, B_μ is a bounded and compact operator. Next, let us consider the discrete operator B^h_μ associated to the discrete source problem: given $f\in H^\mu(\mathcal{R})$, we call $B^h_\mu f$ the second component of the triple $(\mathbf{p}_h,\phi_h,\phi_{S,h})$ that solves (5.10) with source $S_f=\underline{\nu\Sigma}_f f$. Using estimate (6.4), we obtain, like in the plain case, the result below.

Theorem 6.6. *Under the assumptions of Theorem 6.2 with $r_{\max}<1/2$ plus $\underline{\nu\Sigma}_f\in\mathcal{P}W^{1,\infty}(\mathcal{R})$, let $\mu\in]0,r_{\max}[$. Provided that the families of triangulations are* regular$^+$ *on every subdomain, one has for nested meshes:*

$$\|B_\mu-B^h_\mu\|_{\mathcal{L}(H^\mu(\mathcal{R}))}\lesssim h^{\widetilde{\theta}\mu},\tag{6.5}$$

where $\widetilde{\theta}=\min_{i=1}^{\widetilde{N}}\theta_i>0$, and for $1\le i\le\widetilde{N}$, θ_i is defined by (4.29) on $\widetilde{\mathcal{R}}_i$.

We conclude to the absence of spectral pollution.

6.4.2. Optimal convergence rate

Let the assumptions of Theorems 6.2 and 6.6 hold, and in particular the conditions for nested, *non-matching* meshes. We use the same notations as in Section 4.6.2. In particular, let $\widetilde{\omega}_\nu > 0$ be the regularity exponent associated to ν with respect to $(\widetilde{\mathcal{P}}H^{1+s}(\mathcal{R}))_{s>0}$, and introduce $\widetilde{\omega} = \min(\widetilde{\omega}_\nu, k+1)$.

Let $\mu \in [0, r_{\max}[$ be given. As we defined B_μ (resp. B_μ^h), we define A_μ and C_μ (resp. A_μ^h and C_μ^h): for $f \in H^\mu(\mathcal{R})$, we call $A_\mu f = \mathbf{p} \in \widetilde{\mathbf{Q}}$ and $C_\mu f = \phi_S \in M$ (resp. $A_\mu^h f = \mathbf{p}_h \in \widetilde{\mathbf{Q}}_h$ and $C_\mu^h f = \phi_{S,h} \in M_h$) the first and the third components of the triple $(\mathbf{p}, \phi, \phi_S)$ (resp. $(\mathbf{p}_h, \phi_h, \phi_{S,h})$) that solves (5.6) (resp. (5.10)) with source $S_f = \underline{\nu\Sigma}_f f$.

For the DD+L^2-jumps method, the transposition of Lemma 4.16 reads:

Lemma 6.7. *Let φ and φ' be in W. Then, it holds:*

$$\begin{aligned}(\underline{\nu\Sigma}_f\varphi, (B_\mu - B_\mu^h)\varphi')_{0,\mathcal{R}} = a(A_\mu\varphi, (A_\mu - A_\mu^h)\varphi') + b((A_\mu - A_\mu^h)\varphi', B_\mu\varphi)\\ + b(A_\mu\varphi, (B_\mu - B_\mu^h)\varphi') + t(B_\mu\varphi, (B_\mu - B_\mu^h)\varphi');\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}0 = a(A_\mu^h\varphi, (A_\mu - A_\mu^h)\varphi') + b((A_\mu - A_\mu^h)\varphi', B_\mu^h\varphi)\\ + b(A_\mu^h\varphi, (B_\mu - B_\mu^h)\varphi') + t(B_\mu^h\varphi, (B_\mu - B_\mu^h)\varphi').\end{aligned} \tag{6.7}$$

The formulas (6.6) and (4.33), resp. (6.7) and (4.34), are identical. As Strang's Lemma holds for the DD+L^2-jumps method with nested meshes, we can also transpose Proposition 4.17. For that, we admit that the result of Lemma 6.1 can be improved for smooth functions $\mathbf{q}$. As a matter of fact, in this case one may directly compare the discrete normal traces $\Pi_{f,R}(\mathbf{q}\cdot\mathbf{n}_{|\Gamma_{fc}})$ and $\Pi_{c,R}(\mathbf{q}\cdot\mathbf{n}_{|\Gamma_{fc}})$ to the exact normal trace $\mathbf{q}\cdot\mathbf{n}_{|\Gamma_{fc}}$, and evaluate the difference in $L^2(\Gamma_{fc})$-norm, because for smooth functions the exact normal trace always belongs to $L^2(\Gamma_{fc})$.

Proposition 6.8. *For every φ in W, the following inequalities hold for the DD+L^2-jumps method with nested meshes:*

$$\begin{aligned}\|(B_\mu - B_\mu^h)\varphi\|_{0,\mathcal{R}} &\lesssim h^{\widetilde{\omega}}\|\varphi\|_W;\\ \|(A_\mu - A_\mu^h)\varphi\|_{\mathbf{H}(\mathrm{div},\mathcal{R})} &\lesssim h^{\widetilde{\omega}}\|\varphi\|_W.\end{aligned}$$

Estimate (4.37) on the gap between W and W_h is still valid: $\hat{\delta}(W, W_h) \lesssim h^{\widetilde{\omega}}$. Let E_h be the operator defined in (4.38). We recall that E_h and B_μ^h commute (Lem. 4.18 holds). The restriction of E_h to W, denoted by F_h is a bijection that satisfies estimate (4.39), for h small enough. We will also make use of $\mathcal{S}_h = F_h^{-1}E_h - I$ that satisfies Lemma 4.20 and Proposition 4.21. We recall that $\hat{B}_\mu = B_\mu|_W$ and $\hat{B}_\mu^h = F_h^{-1}B_\mu^h F_h$. The transposition of Theorem 4.22 is stated next. The proof is identical (replace ω by $\widetilde{\omega}$), so it is omitted.

Theorem 6.9. *For h small enough, one has for the DD+L^2-jumps method with nested meshes:*

$$\|\hat{B}_\mu - \hat{B}_\mu^h\|_{\mathcal{L}(W)} \lesssim h^{2\widetilde{\omega}}. \tag{6.8}$$

Corollary 6.10. *For h small enough, the error on the eigenvalue for the DD+L^2-jumps method with nested meshes is given by:*

$$|\nu - \nu_h| \lesssim h^{2\widetilde{\omega}}.$$

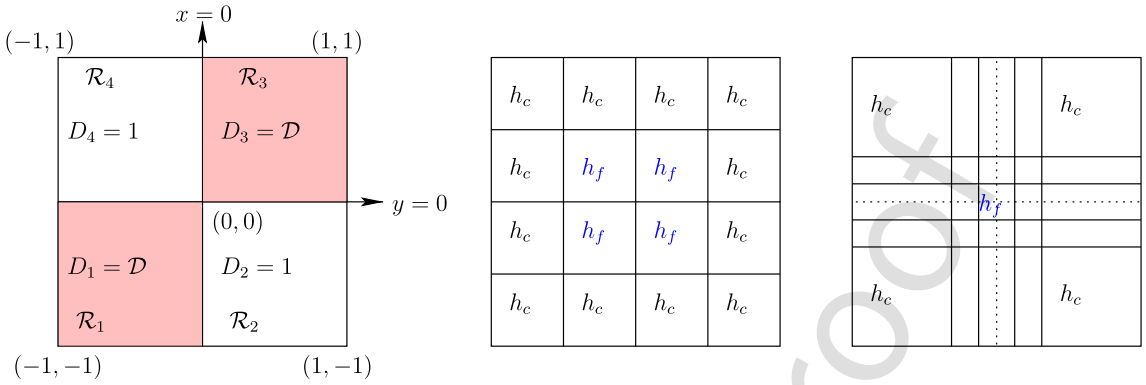


FIGURE 2. The domain of study, and the subdomain meshsizes.

6.5. About non-nested meshes

We recall that, for general non-nested meshes, one has convergence without explicit convergence rate, as soon as (5.11)–(5.12) hold uniformly. In the most general case however, it seems difficult to obtain a convergence error that depends explicitly on h.

On the other hand, let us consider the case where the meshes are non-nested, *with some structure*. By structure, it is understood that the non-nestedness can be described by a finite number of configurations (*e.g.* 3-face mesh *vs.* 5-face mesh, etc.) that are reproduced at smaller and smaller scales when the meshsize diminishes.

We note first that a result similar to Lemma 6.1 can be recovered. Going back to the reference configurations (by assumption there are a finite number of them) and taking the supremum in the upper bounds among all these configurations, we infer from (A.5) that $\|[\check{\mathbf{q}}_R \cdot \mathbf{n}]\|_{0,\Gamma_{fc}} \lesssim h_{c|\Gamma_{fc}} \| q_{f,h} \|_{0,\Gamma_{fc}}$, *i.e.* one can conclude the proof as before. As a consequence, an explicit convergence rate may be derived for the source problem as in Theorem 6.2.

Then, one may proceed in a similar fashion to prove Lemma 6.4, so as to derive an Aubin-Nitsche estimate as in Theorem 6.5. Finally, because interface terms are absent in the analysis of the convergence rate of the eigenvalues (see in particular (6.6)–(6.7)), such estimates can also be proved for non-nested meshes, with some structure.

7. NUMERICAL ILLUSTRATIONS

The tests are carried out in two dimensions: the cartesian coordinates are denoted by (x, y). We use $\mathrm{RTN}_{[0]}$ finite elements on rectangular meshes. We define the discrete space of Lagrange multipliers M_h as in (5.13).

7.1. Benchmark *square for transmission problems*

We study a singular toy problem described on Dauge's website [15] for a magnetic problem and adapted here for the neutron diffusion equation with Neuman boundary condition. Set $\mathcal{R} :=]-1, 1[^2$, and divide it into four subsquares (see Fig. 2 left). Let D, be a scalar, piecewise-constant, coefficient: $D := \mathcal{D} = 0.1$ in $\mathcal{R}_1 \cup \mathcal{R}_3$, and 1 elsewhere, $\Sigma_a = 1$ and $\underline{\nu\Sigma}_f = 1$. We consider the following problem:

$$\begin{cases} -\mathrm{div}\, D\, \mathbf{grad}\, \phi + \phi = \lambda\phi & \text{in } \mathcal{R} \\ \dfrac{\partial \phi}{\partial n} = 0 & \text{on } \partial\mathcal{R}. \end{cases} \tag{7.1}$$

The singularity exponent is $r_{\max} \approx 0.39$. Implementation is in MATLAB.

TABLE 1. Results with 16 subdomains.

$1/h$	N_ϕ	ε_{λ_1}	ε_{λ_2}	ε_{λ_3}	ε_{λ_4}
4	448	$2.88\,e-3$	$3.92\,e-2$	$5.49\,e-3$	$2.00\,e-2$
8	1 792	$7.22\,e-4$	$2.36\,e-2$	$1.38\,e-3$	$5.00\,e-3$
12	4 032	$3.22\,e-4$	$1.74\,e-2$	$6.12\,e-4$	$2.22\,e-3$
16	7 168	$1.81\,e-4$	$1.40\,e-2$	$3.44\,e-4$	$1.25\,e-3$
20	11 200	$1.16\,e-4$	$1.18\,e-2$	$2.20\,e-4$	$8.00\,e-4$
24	16 128	$8.05\,e-5$	$1.02\,e-2$	$1.53\,e-4$	$5.05\,e-4$
Rate		h^2	$h^{0.76}$	h^2	h^2

TABLE 2. Results with 25 subdomains using graded meshes.

$1/h$	N_ϕ	ε_{λ_1}	ε_{λ_2}	ε_{λ_3}	ε_{λ_4}
3	304	$7.47\,e-3$	$1.14\,e-2$	$1.92\,e-2$	$1.12\,e-1$
6	1 216	$1.92\,e-3$	$8.19\,e-3$	$4.90\,e-3$	$2.75,e-2$
12	4 864	$4.83\,e-4$	$5.28\,e-3$	$1.23\,e-3$	$6.85\,e-3$
15	7 600	$3.10\,e-4$	$4.42\,e-3$	$7.88\,e-4$	$4.38\,e-3$
18	10 944	$2.15\,e-4$	$3.86\,e-3$	$5.47\,e-4$	$3.04\,e-3$
21	14 896	$1.59\,e-4$	$6.68\,e-4$	$4.02\,e-4$	$2.24\,e-3$
Rate		h^2	$h^{0.71}$	h^2	h^2

We study the error on the four first eigenvalues (excluding $\lambda_0 = 1$), with two different partitions $\{\widetilde{\mathcal{R}}_i\}_{1\le i\le \widetilde{N}}$. The results are given in Tables 1 and 2, which data are:

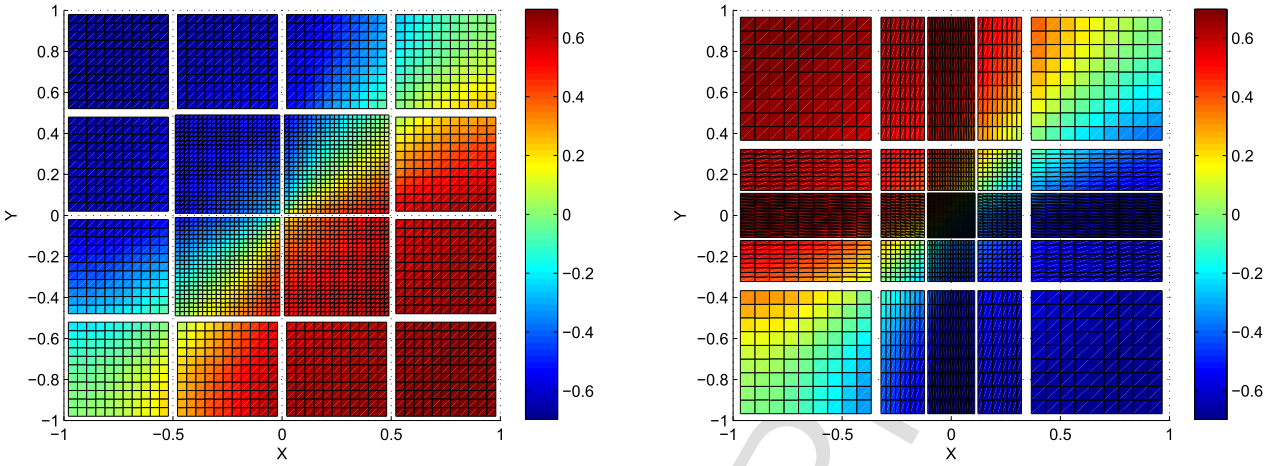
- h: the meshsize,
- N_ϕ: the number of degrees of freedom of ϕ,
- $\varepsilon_{\lambda_i} = |\lambda_{h,i} - \lambda_i|/|\lambda_i|$: the relative error for the eigenvalue λ_i, $i = 1, 4$.

In the last line, we report the average rate of convergence of the computations. In Figure 3a (resp. 3b), we represented the mesh for $1/h = 12$ (resp. $1/h = 18$) and the second non-constant eigenfunction ϕ_2, which is singular at the cross-point.

The first partition is based on $\widetilde{N} = 16$ square subdomains, represented in Figure 2 middle. As indicated in this figure, the four centered subdomains have a meshsize equal to h_f whereas the other subdomains have a meshsize equal to $h_c = 2h_f$, so that the parameter is $h = h_c$. The results are given in Table 1.

The second partition is based on $\widetilde{N} = 25$ subdomains, with graded meshes towards the cross-point, where the singular behaviour is expected. The subdomain in the center of $\mathcal{R}$ has a mesh size equal to h_f, whereas the four subdomains on the corners of $\mathcal{R}$ have a meshsize equal to $h_c = 6\,h_f$ (see Fig. 2 right). This is similar in spirit to the XFEM except there is only one mesh near the cross-point [20].

The results are given in Table 2. With this simple idea (the use of graded meshes), one derives an accurate approximation of the singular eigenfunction at low cost. Indeed, comparing Tables 1 and 2, one notices that the error ε_{λ_2} is comparable using the coarser mesh of the second partition (with $N_\phi = 304$) than using the finer mesh of the first partition (with $N_\phi = 16\,128$). However, the approximation of eigenvalues associated to smooth eigenfunctions is not improved by the use of graded meshes. On the contrary, as the order of the eigenvalues increases, their approximations seem to be more and more degraded, which is due to the difficulty to capture the faster and faster oscillations of the corresponding eigenfunctions.



(a) ϕ_2, 16 subdomains

(b) ϕ_2, 25 subdomains

FIGURE 3. The second non-constant eigenfunction.

7.2. PWR core

We give here some results of computations carried out with the MINOS solver of the APOLLO3®[5] neutronics code [30] developed at CEA. This industrial test models a pressurized water large reactor core with heavy-steel reflector similar to the one described in [29]. The neutron transport equation is discretized using the multigroup simplified P_N (SP_N) equations, with two groups of energy, and SP_1 and SP_3 angular orders. We recall that, for each group, the neutron SP_1 equation is similar to the neutron diffusion equation, whereas the neutron SP_3 equation corresponds to two coupled neutron diffusion equations. The different homogenization steps that allow to obtain the coefficients of this discretization on square cells lead to 229 different media. The coefficients are thus parametrized according to the medium, the energy group and the angular order, which depend respectively on the position, the energy and the direction of the neutrons. We refer to [23, 24, 25] for more details on the multigroup SP_N and diffusion neutron equations and the general algorithm to solve them.

The subdomains $\{\widetilde{\mathcal{R}}_i\}_{1\leq i\leq 361}$ of the partition correspond to the 19×19 cells of Figure 1a. In each subdomain, the coarser triangulation is also such that the coefficients are piecewise constant. The meshes of the subdomains are nested.

In neutronics, the quantity of interest is the inverse of the smallest eigenvalue, which is called the criticality, and is denoted by k_{eff}. Below, we make comparisons on the criticality, the reference value, denoted by k_{eff}^{ref}, being computed on a conforming mesh made of $1.5\,e+7$ (resp. $7.5\,e+6$) rectangles in SP_1 (resp. SP_3).

In Table 3, we present the results obtained with the MINOS solver for different levels of refinement, with $\mathrm{RTN}_{[0]}$ finite elements on rectangles. The data are:

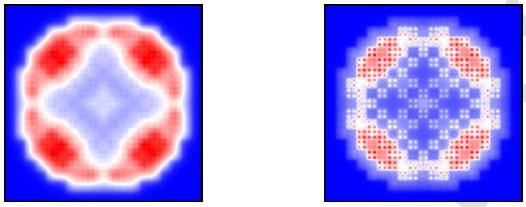
- h: the meshsize,
- N_ϕ: the spatial number of degrees of freedom of the neutron flux ϕ,
- ε_1 (resp. ε_3): the relative error made on the criticality $|k_{\mathit{eff}}-k_{\mathit{eff}}^{ref}|/k_{\mathit{eff}}^{ref}$, for a computation using the SP_1 (resp. SP_3) approximation.
- rate: the averaged rate of convergence.

Convergence rates are higher than 1, seemingly indicating the absence of strong singularities in the first eigenfunction. Instead, we hypothetize that we are still in the pre-asymptotic regime (for the first eigenfunction):

[5] APOLLO3 is a registered trademark in France.

TABLE 3. Results with 361 subdomains.

$1/h$	N_ϕ	ε_1	ε_3
285	$5.40\,e+5$	$1.35\,e-4$	$1.37\,e-4$
380	$9.60\,e+5$	$8.01\,e-5$	$8.79\,e-5$
570	$2.16\,e+6$	$4.10\,e-5$	$5.12\,e-5$
665	$2.94\,e+6$	$3.26\,e-5$	$4.30\,e-5$
950	$6.00\,e+6$	$2.09\,e-5$	$3.15\,e-5$
	Rate	$h^{1.55}$	$h^{1.22}$



(a) First group (b) Second group

FIGURE 4. Neutron flux.

on the one hand, the norm of the "more singular" part is small compared to the norm of the "more regular" part, and on the other hand there are only a few degrees of freedom per characteristic length (see Fig. 1b).

Note that the DD version is parallelized in the APOLLO3® code, contrary to the plain version. Hence, computational times are greatly reduced: we refer to [23] for the analyses of algorithms and their parallelization.

The neutron flux of the first (resp. second) group of energy are represented in Figure 4a (resp. Fig. 4b).

8. CONCLUSION

The solution of the steady-state one-group neutron diffusion equation being usually of low-regularity, the convergence of the eigenvalues and the error estimates are not straightforward to obtain. In particular, we provide new proofs:

- for the source and eigen-problems, with low-regularity solutions;
- for the eigenproblems, in mixed setting with non-vanishing zero-order term ($\Sigma_a \neq 0$).

Notice that our results are obtained under the *regular*$^+$ condition on the family of triangulations. For the DD case, we suggest the following strategies to take into account the apparently restrictive condition on quasi-uniform meshes on the interface, compared to the *regular*$^+$ condition on the family of triangulations:

- use $\{\widetilde{\mathcal{R}}_i\}_{1\leq i\leq \widetilde{N}}$ for DD as the orthogonal (*i.e.* Voronoï) tessellation of $\{\mathcal{R}_i\}_{1\leq i\leq N}$;
- use $\{\widetilde{\mathcal{R}}_i\}_{1\leq i\leq \widetilde{N}} = \{\mathcal{R}_i\}_{1\leq i\leq N}$ and compute the singular part of the solution (or eigenfunction) *via* some *ad hoc* technique (SCM, XFEM, etc.).

A possible continuation of this paper is the study of the steady-state multigroup neutron SP_N problem [21].

APPENDIX A. ADDITIONAL PROOFS

We provide here the proof of three technical lemmas.

Let $(\mathcal{T}_h)_h$ be a given regular family of triangulations. We call $\hat{K} := [0,1]^d$ the reference element. Let h be given. For every $K \in \mathcal{T}_h$, we denote by $\mathbf{x} = F_K(\hat{\mathbf{x}}) := \mathbb{A}_K\hat{\mathbf{x}} + \mathbf{b}_K$, $\mathbb{A}_K \in \mathbb{R}^{d\times d}$, $\mathbf{b}_K \in \mathbb{R}^d$, the map from $\hat{K}$ to

K. Introducing $h_K = diam(K)$ for all $K \in \mathcal{T}_h$, one may bound $\|\mathbb{A}_K\|$, $\|(\mathbb{A}_K)^{-1}\|$, $|det(\mathbb{A}_K)|$ with respect to h_K. The change of variable formulas from $\hat{K}$ to K, and vice versa, can be found *e.g.* in Section 1 of [17].

Proof of Lemma 4.13. We follow Section 2 of [2]. Given $\psi_h \in L^k_h$, one has $\psi_h \in H^\mu(\mathcal{R})$, for all $\mu < 1/2$. By the definition of the norm of $H^\mu(\mathcal{R})$, we have the following equalities:

$$\begin{aligned}\|\psi_h\|^2_{\mu,\mathcal{R}} &= \|\psi_h\|^2_{0,\mathcal{R}} + \int_{\mathcal{R}}\int_{\mathcal{R}} \frac{|\psi_h(\mathbf{x}) - \psi_h(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\mu}} \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \\ &= \sum_{K\in\mathcal{T}_h} \left(\|\psi_h\|^2_{0,K} + \int_K\int_{\mathcal{R}} \frac{|\psi_h(\mathbf{x}) - \psi_h(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\mu}} \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \right) \\ &= \sum_{K\in\mathcal{T}_h} \|\psi_h\|^2_{\mu,K} + \sum_{K\in\mathcal{T}_h} \int_K\int_{\mathcal{R}\setminus K} \frac{|\psi_h(\mathbf{x}) - \psi_h(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\mu}} \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}. \end{aligned} \tag{A.1}$$

Let us estimate first $\sum_{K\in\mathcal{T}_h} \|\psi_h\|^2_{\mu,K}$. According to Corollary 1.138 of [17], we know that

$$\sum_{K\in\mathcal{T}_h} \|\psi_h\|^2_{\mu,K} \lesssim \sum_{K\in\mathcal{T}_h} h_K^{-2\mu} \|\psi_h\|^2_{0,K} \lesssim h_{\min}^{-2\mu} \|\psi_h\|^2_{0,\mathcal{R}}, \tag{A.2}$$

where $h_{\min} = \min\limits_{K\in\mathcal{T}_h} h_K$. To estimate the remaining part, we recall that, for any $K \in \mathcal{T}_h$ and any $\mathbf{x} \in K$, it holds that, by going back the reference space, applying (*cf.* [22], 1.3.2.12) on $\hat{K}$ and then going to the physical space:

$$\int_{\mathcal{R}\setminus K} \frac{1}{|\mathbf{x}-\mathbf{y}|^{d+2\mu}} \mathrm{d}\mathbf{y} \lesssim \frac{1}{\rho_{\partial K}(\mathbf{x})^{2\mu}},$$

where $\rho_{\partial K}(\mathbf{x}) = \inf_{\mathbf{y}\in\partial K} |\mathbf{x}-\mathbf{y}|$. Thus we have:

$$\begin{aligned}\int_K\int_{\mathcal{R}\setminus K} \frac{|\psi_h(\mathbf{x}) - \psi_h(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\mu}} \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} &= \sum_{\substack{K'\in\mathcal{T}_h \\ K'\neq K}} \int_K\int_{K'} \frac{|\psi_h(\mathbf{x}) - \psi_h(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\mu}} \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \\ &\lesssim \sum_{\substack{K'\in\mathcal{T}_h \\ K'\neq K}} \int_K\int_{K'} \frac{\psi_h(\mathbf{x})^2 + \psi_h(\mathbf{y})^2}{|\mathbf{x}-\mathbf{y}|^{d+2\mu}} \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \\ &\lesssim \int_K \frac{\psi_h(\mathbf{x})^2}{\rho_{\partial K}(\mathbf{x})^{2\mu}} \mathrm{d}\mathbf{x}. \end{aligned} \tag{A.3}$$

Going back to the reference element $\hat{K}$ and introducing $\psi_{h|K}(\mathbf{x}) = \hat{\psi}(\hat{\mathbf{x}})$, it stands:

$$\int_K \frac{\psi_h(\mathbf{x})^2}{\rho_{\partial K}(\mathbf{x})^{2\mu}} \mathrm{d}\mathbf{x} \lesssim h_K^{d-2\mu} \int_{\hat{K}} \frac{\hat{\psi}(\hat{\mathbf{x}})^2}{\rho_{\partial\hat{K}}(\hat{\mathbf{x}})^{2\mu}} \mathrm{d}\hat{\mathbf{x}}.$$

Because $\mu < 1/2$ (*cf.* [22], Thm. 1.4.4.4), $\hat{\psi} \mapsto (\int_{\hat{K}} \hat{\psi}(\hat{\mathbf{x}})^2 \rho_{\partial\hat{K}}(\hat{\mathbf{x}})^{-2\mu}\, \mathrm{d}\hat{\mathbf{x}})^{1/2}$ is a norm on $\hat{L}^k = Q_{k,k,k}(\hat{K})$. Thanks to the equivalence of the norms on finite dimensional vector spaces, one gets

$$\int_K \frac{\psi_h(\mathbf{x})^2}{\rho_{\partial K}(\mathbf{x})^{2\mu}} \mathrm{d}\mathbf{x} \lesssim h_K^{d-2\mu} \|\hat{\psi}\|^2_{0,\hat{K}}.$$

Finally, going back to element K, we know that $\|\hat{\psi}\|^2_{0,\hat{K}} \lesssim h_K^{-d}\|\psi_h\|^2_{0,K}$. Hence using (A.3) and the results that follow, we have:

$$\int_K \int_{\mathcal{R}\setminus K} \frac{|\psi_h(\mathbf{x}) - \psi_h(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2\mu}} \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \lesssim h_K^{-2\mu}\|\psi_h\|^2_{0,K}. \tag{A.4}$$

Starting from (A.1) using (A.2) and (A.4), we obtain finally the global bound:

$$\|\psi_h\|_{\mu,\mathcal{R}} \lesssim h_{\min}^{-\mu}\|\psi_h\|_{0,\mathcal{R}}.$$

As the family of triangulations is *regular*$^+$, one has $h_{\min}^{-\mu} \lesssim h^{(\theta-2)\mu}$, which concludes the proof. □

Proof of Lemma 6.1. For $l = c, f$, we introduce the operators from the normal trace spaces $(\mathbf{H}(\mathrm{div}\,,\mathcal{R}) \cap \mathbf{H}^\mu(\mathcal{R})) \cdot \mathbf{n}_{|\Gamma_{fc}}$ to the discrete spaces of normal traces $T_{l,h}$ on Γ_{fc}:

$$\begin{cases} \Pi_{l,R} : (\mathbf{H}(\mathrm{div}\,,\mathcal{R}) \cap \mathbf{H}^\mu(\mathcal{R})) \cdot \mathbf{n}_{|\Gamma_{fc}} \to T_{l,h|\Gamma_{fc}} \\ \mathbf{q}' \cdot \mathbf{n}_{|\Gamma_{fc}} \mapsto \tilde{\mathbf{q}}'_{l,R} \cdot \mathbf{n}_{|\Gamma_{fc}}. \end{cases}$$

With a slight abuse of notations, we write $\Pi_{l,R}(\mathbf{q}'_l \cdot \mathbf{n}_{|\partial\widetilde{\mathcal{R}}_l}) = \tilde{\mathbf{q}}'_{l,R} \cdot \mathbf{n}_{|\partial\widetilde{\mathcal{R}}_l}$. We also introduce the operator $\Pi^0_{c,R}$ on the vector space of normal traces of elements of $\widetilde{\mathbf{Q}}_{c,h}$ with lowest-order RTN finite element, *i.e.* the vector space $T^0_{c,h|\Gamma_{fc}}$ of piecewise constant functions on the interface mesh defined as the trace on Γ_{fc} of the mesh used in $\widetilde{\mathcal{R}}_c$. Note that because the meshes are nested, the restriction of $\Pi_{f,R}$ (resp., $\Pi_{c,R}$ and $\Pi^0_{c,R}$) on $T_{f,h|\Gamma_{fc}}$ (resp., on the subspaces $T_{c,h|\Gamma_{fc}}$ and $T^0_{c,h|\Gamma_{fc}}$ where applicable) may also be considered as an orthogonal projection operator. Denoting $q_{f,h} = \Pi_{f,R}(\mathbf{q} \cdot \mathbf{n}_{|\Gamma_{fc}})$, we have:

$$\begin{aligned} \|[\tilde{\mathbf{q}}_R \cdot \mathbf{n}]\|_{0,\Gamma_{fc}} &= \|\Pi_{f,R}(\mathbf{q} \cdot \mathbf{n}_{|\Gamma_{fc}}) - \Pi_{c,R}(\mathbf{q} \cdot \mathbf{n}_{|\Gamma_{fc}})\|_{0,\Gamma_{fc}} \\ &= \|\Pi_{f,R}(\mathbf{q} \cdot \mathbf{n}_{|\Gamma_{fc}}) - \Pi_{c,R} \circ \Pi_{f,R}(\mathbf{q} \cdot \mathbf{n}_{|\Gamma_{fc}})\|_{0,\Gamma_{fc}} \\ &= \|(\mathbb{I} - \Pi_{c,R})q_{f,h}\|_{0,\Gamma_{fc}} \\ &\leq \|(\mathbb{I} - \Pi^0_{c,R})q_{f,h}\|_{0,\Gamma_{fc}}. \end{aligned} \tag{A.5}$$

As the meshes are quasi-uniform on the interface, one has $h_{c|\Gamma_{fc}} \eqsim h_{f|\Gamma_{fc}}$. Then, starting from (A.5), thanks to the quasi-uniform mesh assumption for the inverse inequalities on Γ_{fc}, *cf.* Lemma 10.10 of [31], we find

$$\begin{aligned} \|[\tilde{\mathbf{q}}_R \cdot \mathbf{n}]\|_{0,\Gamma_{fc}} &\lesssim h_{c|\Gamma_{fc}} \| q_{f,h} \|_{0,\Gamma_{fc}} \text{ ([3], Lem. 4.9)} \\ &\lesssim h_{c|\Gamma_{fc}} (h_{f|\Gamma_{fc}})^{-1/4} \| q_{f,h} \|_{-1/4,\Gamma_{fc}} \\ &\lesssim (h_{f|\Gamma_{fc}})^{3/4} \| \Pi_{f,R}(\mathbf{q} \cdot \mathbf{n}_{|\partial\widetilde{\mathcal{R}}_f}) \|_{-1/4,\partial\widetilde{\mathcal{R}}_f} \\ &\lesssim (h_{f|\Gamma_{fc}})^{3/4} (h_{f|\partial\widetilde{\mathcal{R}}_f})^{-1/4} \|\tilde{\mathbf{q}}_{f,R} \cdot \mathbf{n}_{|\partial\widetilde{\mathcal{R}}_f}\|_{-1/2,\partial\widetilde{\mathcal{R}}_f} \\ &\lesssim h_f^{1/2} \|\tilde{\mathbf{q}}_{f,R}\|_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_f)} \lesssim h_f^{1/2} \| \mathbf{q}_f \|_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_f)}. \end{aligned}$$

Above, we have used the continuity of the normal trace, resp. the stability of the RTN interpolant, to derive the last two inequalities. □

Proof of Lemma 6.4. First, let us bound the norm of $\|\delta\mathbf{q}_{fc}\|_{\mathbf{H}(\mathrm{div}\,,\widetilde{\mathcal{R}}_f)}$ by $\|\delta\mathbf{q}_{fc} \cdot \mathbf{n}\|_{0,\Gamma_{fc}}$. We use the notation $\mathbf{v} = \delta\mathbf{q}_{fc}$ below. Denoting by $(K_\ell)_\ell$ the parallelepipeds composing the mesh on $\widetilde{\mathcal{R}}_f$, and $\mathcal{N}_\Gamma$ the set of indices ℓ

such that $\Gamma_\ell := K_\ell \cap \Gamma_{fc}$ is of Hausdorff dimension $d-1$, because of the definition of $\mathbf{v}$ it now holds

$$\|\mathbf{v}\|^2_{\mathbf{H}(\mathrm{div},\widetilde{\mathcal{R}}_f)} = \sum_\ell \|\mathbf{v}_{|K_\ell}\|^2_{\mathbf{H}(\mathrm{div},K_\ell)} = \sum_{\ell\in\mathcal{N}_\Gamma} \|\mathbf{v}_{|K_\ell}\|^2_{\mathbf{H}(\mathrm{div},K_\ell)}.$$

Then, one can bound $\|\mathbf{v}_{|K_\ell}\|_{\mathbf{H}(\mathrm{div},K_\ell)}$ by $\|\mathbf{v}_{|K_\ell}\cdot\mathbf{n}\|_{0,\Gamma_\ell}$ for each index $\ell\in\mathcal{N}_\Gamma$. To that aim, one goes back to the reference element $\hat{K}$ *via* the Piola transform, which reads ([7], Sect. 2.1.3):

$$\mathbf{v}_{|K_\ell}(\mathbf{x}) = \frac{1}{|det(\mathbb{A}_{K_\ell})|}\mathbb{A}_{K_\ell}\hat{\mathbf{v}}(\hat{\mathbf{x}}), \quad \mathrm{div}\,\mathbf{v}_{|K_\ell}(\mathbf{x}) = \frac{1}{|det(\mathbb{A}_{K_\ell})|}\hat{\mathrm{div}}\,\hat{\mathbf{v}}(\hat{\mathbf{x}}).$$

With the help of a classical formula for the change of variables on Γ_ℓ ([7], Eq. (2.1.62)), one finds after a few elementary algebraic manipulations[6] that

$$h_{K_\ell}^{d-1}\int_{\Gamma_\ell}(\mathbf{v}_{|K_\ell}\cdot\mathbf{n})^2\,\mathrm{d}\Gamma \eqsim \int_{\hat{\Gamma}_\ell}(\hat{\mathbf{v}}\cdot\hat{\mathbf{n}})^2\,\mathrm{d}\hat{\Gamma},$$

where $\hat{\Gamma}_\ell$ is equal to $F_{K_\ell}^{-1}(\Gamma_\ell)$.

On the reference element, it holds

$$\|\hat{\mathbf{v}}\|^2_{\mathbf{H}(\hat{\mathrm{div}},\hat{K})} \lesssim \int_{\hat{\Gamma}_\ell}(\hat{\mathbf{v}}\cdot\hat{\mathbf{n}})^2\,\mathrm{d}\hat{\Gamma},$$

because the non-zero degrees of freedom are all located on $\hat{\Gamma}_\ell$. Finally, one has the classical bounds ([7], Lem. 2.1.7):

$$\|\mathbf{v}_{|K_\ell}\|^2_{0,K_\ell} \lesssim h_{K_\ell}^{2-d}\|\hat{\mathbf{v}}\|^2_{0,\hat{K}} \quad \|\mathrm{div}\,\mathbf{v}_{|K_\ell}\|^2_{0,K_\ell} \lesssim h_{K_\ell}^{-d}\|\hat{\mathrm{div}}\,\hat{\mathbf{v}}\|^2_{0,\hat{K}},$$

so that

$$\|\mathbf{v}_{|K_\ell}\|^2_{\mathbf{H}(\mathrm{div},K_\ell)} \lesssim h_{K_\ell}^{-d}\|\hat{\mathbf{v}}\|^2_{\mathbf{H}(\hat{\mathrm{div}},\hat{K})} \lesssim h_{K_\ell}^{-1}\int_{\Gamma_\ell}(\mathbf{v}_{|K_\ell}\cdot\mathbf{n})^2\,\mathrm{d}\Gamma.$$

Adding up the contributions for $\ell\in\mathcal{N}_\Gamma$, one finds:

$$\|\delta\mathbf{q}_{fc}\|_{\mathbf{H}(\mathrm{div},\widetilde{\mathcal{R}}_f)} \lesssim h_f^{-1/2}\|\delta\mathbf{q}_{fc}\cdot\mathbf{n}\|_{0,\Gamma_{fc}}. \tag{A.6}$$

By modifying the final computations in the proof of Lemma 6.1, one finds that for all $0<\epsilon<\mu$:

$$\begin{aligned}
\|\delta\mathbf{q}_{fc}\cdot\mathbf{n}\|_{0,\Gamma_{fc}} &\lesssim h_{c|\Gamma_{fc}}\,\|\,q_{f,h}\,\|_{0,\Gamma_{fc}} \text{ ([3], Lem. 4.9)}\\
&\lesssim h_{c|\Gamma_{fc}}\,(h_{f|\Gamma_{fc}})^{\epsilon-1/2}\|q_{f,h}\|_{\epsilon-1/2,\Gamma_{fc}} \text{ ([31], Lem. 10.10)}\\
&\lesssim h_f^{\epsilon+1/2}\,\|\,q_{f,h}\,\|_{\epsilon-1/2,\Gamma_{fc}}\\
&\lesssim h_f^{\epsilon+1/2}\,\|\,\Pi_{f,R}(\mathbf{q}_f\cdot\mathbf{n}_{|\partial\widetilde{\mathcal{R}}_f})\,\|_{\epsilon-1/2,\partial\widetilde{\mathcal{R}}_f}\\
&\lesssim h_f^{\epsilon+1/2}\,\|\,\mathbf{q}_f\cdot\mathbf{n}_{|\partial\widetilde{\mathcal{R}}_f}\,\|_{\epsilon-1/2,\partial\widetilde{\mathcal{R}}_f} \text{ ([2], Thm. 2.4 and Rem. 2.5)}\\
&\lesssim h_f^{\epsilon+1/2}\left(\|\mathbf{q}_f\|_{\epsilon,\widetilde{\mathcal{R}}_f}+\|\mathrm{div}\,\mathbf{q}_f\|_{0,\widetilde{\mathcal{R}}_f}\right).
\end{aligned}$$

[6] Since the meshes are quasi-uniform on Γ_{fc}, they are in particular regular.

Or, choosing $\epsilon = \mu - \eta$ for $\eta > 0$ arbitrary small, that

$$\|\delta \mathbf{q}_{fc} \cdot \mathbf{n}\|_{0,\Gamma_{fc}} \lesssim h_f^{\mu+1/2-\eta} \left(\|\mathbf{q}_f\|_{\mu,\widetilde{\mathcal{R}}_f} + \|\operatorname{div} \mathbf{q}_f\|_{0,\widetilde{\mathcal{R}}_f} \right).$$

Using (A.6), we conclude the proof. □

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