ANALYSIS OF TIME-HARMONIC MAXWELL IMPEDANCE PROBLEMS IN ANISOTROPIC MEDIA

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Abstract. We consider the time-harmonic Maxwell's equations in anisotropic media. The problem to be solved is an approximation of the diffraction problem, or scattering from bounded objects, that is usually set in some exterior domain in $\mathbb{R}^3$. We consider perfectly conducting objects, so the equations are supplemented with a Dirichlet boundary condition on those objects, and we truncate the exterior domain by imposing an impedance condition on an artificial boundary, to model an approximate radiation condition. The resulting problem is then posed in a bounded domain, with Dirichlet and impedance boundary conditions. In this work, we focus on the mathematical meaning of the impedance condition, precisely in which function space it holds. This relies on a careful analysis of the regularity of the traces of electromagnetic fields, which can be derived thanks to the study of the regularity of the solution to second-order surface PDEs. Then, we prove well-posedness of the model, and we determine the a priori regularity of the fields in the domain and on the boundaries, depending on the geometry, the coefficients and the data. Finally, the discretization of the formulations is presented, with an approximation based on edge finite elements. Error estimates are derived, and a benchmark is provided to discuss those estimates.

Key words. Maxwell's equations; anisotropic media; impedance boundary condition; surface PDEs; edge finite elements

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1. Introduction.

In electromagnetic theory, a popular field is the study of time-harmonic electromagnetic wave propagation models. The analysis of these models is well-known for isotropic materials (e.g. [31]) and certain classes of anisotropic materials, mostly hermitian positive definite ones [24, 5, 14]. We recall that the electromagnetic materials are usually characterized by two tensor-valued coefficients, the electric permittivity $\underline{\boldsymbol{\varepsilon}}$ and the magnetic permeability $\underline{\boldsymbol{\mu}}$ (the coefficients will be respectively denoted ε and μ if they are scalar-valued). In [13], with A. Modave, we have proposed the mathematical and numerical analyses of the model for elliptic, anisotropic and possibly non-hermitian coefficients $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$, when the time-harmonic Maxwell's equations are supplemented with either a Dirichlet boundary condition, or a Neumann boundary condition.

In this manuscript, we consider a model including an impedance boundary condition on (some part of) the boundary. In principle, this condition can include a third coefficient that can be related to the impedance, and denoted $\underline{\boldsymbol{\alpha}}$ afterwards if it is tensor-valued, and α if it is scalar-valued. For this model, a challenging issue is to find in which sense this boundary condition holds. As a matter of fact, even for models with scalar, constant coefficients ε, μ and α, although results are available when the boundary is smooth, see eg. [2], much less is known when the boundary is piecewise smooth. We refer to [1, Section 5.1] for some pointers on this issue. This issue is crucial, because it drives the choice of the norm to measure the electromagnetic fields, and how to build a variational formulation that is equivalent to the model. Precisely, one has to characterize the regularity of the traces of those fields on the boundary. This can be achieved through the a priori regularity analysis of the solution to second-order surface PDEs. For a scalar, constant coefficient α, this analysis can be traced back to [10, 8].

Regarding the numerical solution, the difficulty is to verify that the boundary condition is variationally expressed in a way that is compatible with the use of edge finite element methods. Explicit conditions (on the geometry, and on the coefficient $\underline{\boldsymbol{\alpha}}$) have to be derived. And, in the case where the variational formulation involves a coercive form, an *a priori* error estimate can be derived, which seems to be an improvement over classical theory [31, Section 5.8].

The paper is organized as follows. In the next Section we introduce the notation and the model to be solved and recall the mathematical framework, especially concerning traces function spaces. In Section 3 we focus on function spaces considerations for traces on the (part of the) boundary where the impedance condition is imposed, and prove a generic embedding result. In Section 4 we derive the *a priori* regularity of the traces on this boundary. In Section 5 we study the well-posedness of problem (2.6) and the regularity of the solution. Section 6 is devoted to numerical analysis. Finally, we provide a numerical illustration in Section 7.

2. Model, notation, and framework.

2.1. Notation.

In the manuscript, vector fields are written in boldface character, and tensor fields are written in underlined bold characters. Whenever applicable, in a function space H, the notation H_{zmv} is used to denote the

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subspace of H with zero-mean-value elements, respectively $H_{\rm per}$ is used to denote the subspace of H with periodic elements. Additionally, $\langle \cdot, \cdot \rangle_H$ is used to denote the duality product between H' and H. It is assumed that the reader is familiar with Sobolev spaces $H^{\mathtt{s}}(\mathcal{O})$, for $\mathtt{s} \in \mathbb{R}$, and with the function spaces related to Maxwell's equations, such as $\boldsymbol{H}(\mathbf{curl};\mathcal{O})$, $\boldsymbol{H}_0(\mathbf{curl};\mathcal{O})$, $\boldsymbol{H}(\mathrm{div};\mathcal{O})$ and $\boldsymbol{H}_0(\mathrm{div};\mathcal{O})$, where $\mathcal{O}$ is a non-empty open set of $\mathbb{R}^3$. For $\mathtt{s} > 0$ we denote by $\mathrm{P}H^{\mathtt{s}}(\mathcal{O})$ the subset of $L^2(\mathcal{O})$ composed of piecewise-$H^{\mathtt{s}}$ fields. A priori, $\boldsymbol{H}(\mathbf{curl};\mathcal{O})$ is endowed with the "natural" norm $\boldsymbol{v} \mapsto \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\mathcal{O})} := (\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\mathcal{O})} + \|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\mathcal{O})})^{1/2}$, etc. We refer to the monographs of Monk [31], Kirsch and Hettlich [26], and Assous *et al.* [1] for further details.

Throughout the manuscript, Ω is assumed to be an open, connected, bounded domain of $\mathbb{R}^3$, with a Lipschitz-continuous boundary $\partial\Omega$ that is either of class $\mathcal{C}^2$, or polyhedral. The unit outward normal vector field to $\partial\Omega$ is denoted $\boldsymbol{n}$. Lebesgue function spaces of tangential fields defined on a subset Γ of the boundary are denoted by $\mathbf{L}^p_t(\Gamma)$, for ad hoc p. We recall the classical integration by parts formula (see Eq. (35) in Ref. [11]):

$$(\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} - (\mathbf{curl}\,\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = {}_\gamma\langle \gamma^T \boldsymbol{u}, \pi^T \boldsymbol{v}\rangle_\pi, \;\; \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega), \tag{2.1}$$

where $\gamma^T : \boldsymbol{u} \mapsto (\boldsymbol{u} \times \boldsymbol{n})_{|\partial\Omega}$ denotes the tangential trace operator, $\pi^T : \boldsymbol{v} \mapsto \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n})_{|\partial\Omega}$ denotes the tangential components trace operator, and ${}_\gamma\langle \gamma^T \boldsymbol{u}, \pi^T \boldsymbol{v}\rangle_\pi$ expresses duality between the *ad hoc* trace spaces (see Section 4 for more details).
We let $\partial\Omega$ be partitioned into two closed, disjoint, subsets Γ_A and $\tilde{\Gamma}$. We use $\gamma^T_A : \boldsymbol{u} \mapsto (\boldsymbol{u} \times \boldsymbol{n})_{|\Gamma_A}$ and $\pi^T_A : \boldsymbol{v} \mapsto \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n})_{|\Gamma_A}$ to denote respectively the tangential trace operator and the tangential components trace operator on Γ_A. Similarly, we use the notation $\tilde{\gamma}^T$ and $\tilde{\pi}^T$ to denote those traces on $\tilde{\Gamma}$. Then, we define

$$\boldsymbol{H}_{0,\tilde{\Gamma}}(\mathbf{curl}, \Omega) := \left\{ \mathbf{v} \in \boldsymbol{H}(\mathbf{curl};\Omega), \;\tilde{\gamma}^T \boldsymbol{v} = 0 \right\}.$$

By default, we use the tensor notation to denote the coefficients, ie. $\underline{\boldsymbol{\varepsilon}}$, $\underline{\boldsymbol{\mu}}$ and $\underline{\boldsymbol{\alpha}}$.
Finally, the symbol C is used to denote a generic positive constant which is independent of the mesh and the fields of interest ; C may depend on the geometry or on the coefficients defining the model. We use the notation $A \lesssim B$ for the inequality $A \leq CB$, where A and B are two scalar fields, and C is a generic positive constant.

2.2. Motivation and model. We study the time-harmonic Maxwell system

$$-i\omega\underline{\boldsymbol{\mu}}\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = 0 \tag{2.2}$$

$$i\omega\underline{\boldsymbol{\varepsilon}}\boldsymbol{E} + \mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J} \tag{2.3}$$

posed in a domain Ω of $\mathbb{R}^3$, where our unknowns are the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$, which we assume *a priori* to belong to $\boldsymbol{H}(\mathbf{curl};\Omega)$; $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ are respectively the electric permittivity tensor and the magnetic permeability tensor; $\omega > 0$ is the angular frequency, and $\boldsymbol{J}$ is the current density.

The problem is supplemented by appropriate boundary conditions on $\partial\Omega$. As mentioned previously, $\partial\Omega$ is made of two closed *non-intersecting* parts, $\tilde{\Gamma}$ and Γ_A. $\tilde{\Gamma}$ is the part of boundary that surrounds perfectly conducting objects (if any), so a Dirichlet condition

$$\tilde{\gamma}^T \boldsymbol{E} = 0 \text{ on } \tilde{\Gamma} \tag{2.4}$$

is prescribed there. On the other hand, Γ_A is an artificial boundary. An impedance condition (also called Robin condition)

$$\pi^T_A((i\omega)^{-1}\boldsymbol{H}) + \underline{\boldsymbol{\alpha}}\gamma^T_A \boldsymbol{E} = \mathbf{g} \text{ on } \Gamma_A \tag{2.5}$$

is prescribed there. This condition is inherited from the classical Silver-Müller boundary condition, which is an absorbing condition. Indeed, assume that, in a neighbourhood of Γ_A, $\underline{\boldsymbol{\varepsilon}} = \varepsilon\underline{\mathbf{I}}$ and $\underline{\boldsymbol{\mu}} = \mu\underline{\mathbf{I}}$ with $\varepsilon, \mu > 0$. Then the Silver-Müller condition writes (see eg. [1, p. 57])

$$\pi^T_A \boldsymbol{H} + \sqrt{\frac{\varepsilon}{\mu}}\gamma^T_A \boldsymbol{E} = \mathbf{g}_{SM} \text{ on } \Gamma_A.$$

While possibly one can have $\tilde{\Gamma} = \emptyset$ (if there are no perfectly conducting objects), we impose $\Gamma_A \neq \emptyset$. We also assume Γ_A to be connected.

Remark 2.1. In principle, one could consider more general classes of boundaries. First, that the artificial boundary Γ_A is piecewise smooth. But then computations below, see eg. Section 3, are not explicit anymore (we refer to [18] for some pointers on this issue). Then, that the other part of the boundary $\tilde{\Gamma}$ is only Lipschitz continuous, but then the results of Subsection 5.3 are not known (we refer to [16]).

Classically, in the above, one can eliminate the magnetic field $\boldsymbol{H}$. It follows that the problem to be solved can be expressed in terms of a second-order PDE plus boundary conditions, in the electric field $\boldsymbol{E}$ only:

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}) - \omega^2\underline{\boldsymbol{\varepsilon}}\boldsymbol{E} = \boldsymbol{f} & \text{in } \Omega, \\ \pi_A^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}) + \underline{\boldsymbol{\alpha}}\gamma_A^T\boldsymbol{E} = \mathbf{g} & \text{on } \Gamma_A, \\ \tilde{\gamma}^T\boldsymbol{E} = 0 & \text{on } \tilde{\Gamma}, \end{cases} \tag{2.6}$$

where $\boldsymbol{f} = i\omega\boldsymbol{J}$. We refer to Subection 5.2, for the problem expressed in terms of the magnetic field $\boldsymbol{H}$ only.

We now express the *ellipticity condition* on the coefficients $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\mu}}, \underline{\boldsymbol{\alpha}}$ (cf. Definition 3.1.1 in [12]).

Definition 2.2. *Let $D \in \{\Omega, \Gamma_A\}$.*
We say that the scalar-valued field $\xi \in L^\infty(D)$ is elliptic *iff*

$$\exists\theta_\xi \in \mathbb{R},\ \exists\xi_- > 0,\quad a.e.\ in\ D,\quad \xi_- \leq \Re[e^{i\theta_\xi}\xi]. \tag{2.7}$$

We say that the 3-by-3 tensor-valued field $\underline{\boldsymbol{\xi}} \in \underline{\mathbf{L}}^\infty(\Omega)$, resp. 2-by-2 tensor-valued field $\underline{\boldsymbol{\xi}} \in \underline{\mathbf{L}}_t^\infty(\Gamma_A)$, is elliptic *iff*

$$\exists\theta_\xi \in \mathbb{R},\ \exists\xi_- > 0,\quad a.e.\ in\ D,\ \forall\mathbf{z} \in \mathbb{C}^d,\quad \xi_-|\mathbf{z}|^2 \leq \Re[e^{i\theta_\xi}\cdot\mathbf{z}^*\underline{\boldsymbol{\xi}}\mathbf{z}], \tag{2.8}$$

where $d = 3$ if $D = \Omega$, resp. $d = 2$ if $D = \Gamma_A$.
In (2.7) or (2.8), we call θ_ξ an ellipticity direction *and note that, in general, it is not unique. Therefore, we introduce $\Theta_\xi := \{\theta_\xi \in]-\pi,\pi],\ \theta_\xi$ fulfills (2.7) or (2.8)$\}$, and call it the set of* admissible (ellipticity) directions*.*

Note that the tensor-valued case encompasses the scalar-valued one. Indeed, given a scalar-valued coefficient ξ, it is equivalent to consider the tensor-valued coefficient $\underline{\boldsymbol{\xi}} := \xi\underline{\mathbf{I}}$, where $\underline{\mathbf{I}}$ is the $d \times d$ identity tensor. And, given $\underline{\boldsymbol{\xi}} \in \underline{\mathbf{L}}^\infty(D)$ for $D \in \{\Omega, \Gamma_A\}$, we will use the notation $\xi_+ := \|\underline{\boldsymbol{\xi}}\|_{\underline{\mathbf{L}}^\infty(D)}$.

Proposition 2.3. *Let $\underline{\boldsymbol{\xi}} \in \underline{\mathbf{L}}^\infty(D)$: one has $\underline{\boldsymbol{\xi}}^\star \in \underline{\mathbf{L}}^\infty(D)$ with $(\xi^\star)_+ = \xi^\star$.*
If $\underline{\boldsymbol{\xi}}$ is elliptic, then $\underline{\boldsymbol{\xi}}^\star$ satisfies the ellipticity condition with $\theta_{\xi^\star} = -\theta_\xi$, and $(\xi^\star)_- = \xi_-$. In this case, one has $\underline{\boldsymbol{\xi}}^{-1} \in \underline{\mathbf{L}}^\infty(D)$ with $(\xi^{-1})_+ = (\xi_-)^{-1}$. Moreover, $\underline{\boldsymbol{\xi}}^{-1}$ satisfies the ellipticity condition with $\theta_{\xi^{-1}} = -\theta_\xi$, and $(\xi^{-1})_- := \xi_-(\xi_+)^{-2}$.

Proposition 2.4. *Let $\underline{\boldsymbol{\xi}} \in \underline{\mathbf{L}}^\infty(\Omega)$ be elliptic. For any $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, one has the following inequalities:*

$$\xi_-\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \leq \Re\left[e^{i\theta_\xi}\left(\underline{\boldsymbol{\xi}}\boldsymbol{v},\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)}\right] \leq \left|\left(\underline{\boldsymbol{\xi}}\boldsymbol{v},\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)}\right| \leq \xi_+\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{2.9}$$

Similarly, let $\underline{\boldsymbol{\xi}} \in \underline{\mathbf{L}}_t^\infty(\Gamma_A)$ be elliptic, one has, for any $\mathbf{v} \in \mathbf{L}_t^2(\Gamma_A)$,

$$\xi_-\|\mathbf{v}\|^2_{\mathbf{L}_t^2(\Gamma_A)} \leq \Re\left[e^{i\theta_\xi}\left(\underline{\boldsymbol{\xi}}\mathbf{v},\mathbf{v}\right)_{\mathbf{L}_t^2(\Gamma_A)}\right] \leq \left|\left(\underline{\boldsymbol{\xi}}\mathbf{v},\mathbf{v}\right)_{\mathbf{L}_t^2(\Gamma_A)}\right| \leq \xi_+\|\mathbf{v}\|^2_{\mathbf{L}_t^2(\Gamma_A)}. \tag{2.10}$$

In all of the following, we consider that the coefficients fulfill Assumption 1 below.

Assumption 1. *The coefficients $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\mu}}$ belong to $\underline{\mathbf{L}}^\infty(\Omega)$ and are elliptic. The coefficient $\underline{\boldsymbol{\alpha}}$ belongs to $\underline{\mathbf{L}}_t^\infty(\Gamma_A)$ and is elliptic.*

2.3. Recap on traces function spaces. Before going to the core of our work, let us recall a few well-known results regarding surface or tangential operators and function spaces defined on the boundary. For more details, we refer to eg. [1, Section 3.1] and References therein for the case of a polyhedral boundary, respectively to [33, Sections 2.5.2 and 2.5.6] for the case of a boundary of class $\mathcal{C}^2$. See also [10] for a nice study. Below, we define operators and function spaces on $\partial\Omega$, however it is understood that one may give identical definitions for operators and function spaces on $\tilde{\Gamma}$, or on Γ_A. First of all, let us introduce the subspace of tangential fields of $\boldsymbol{L}^2(\partial\Omega)$,

$$\mathbf{L}_t^2(\partial\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\partial\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n} = 0\right\}.$$

Then, we introduce fractional order Sobolev spaces, for $s \in]0, \frac{1}{2}]$,

$$\mathbf{H}_t^s(\partial\Omega) := \boldsymbol{H}^s(\partial\Omega) \cap \mathbf{L}_t^2(\partial\Omega), \quad \mathbf{H}_\perp^s(\partial\Omega) := \gamma^T\left(\boldsymbol{H}^{1/2+s}(\Omega)\right), \quad \mathbf{H}_\parallel^s(\partial\Omega) := \pi^T\left(\boldsymbol{H}^{1/2+s}(\Omega)\right).$$

Their dual spaces are denoted, respectively, $\mathbf{H}_t^{-s}(\partial\Omega)$, $\mathbf{H}_\perp^{-s}(\partial\Omega)$ and $\mathbf{H}_\parallel^{-s}(\partial\Omega)$, with $\mathbf{L}_t^2(\partial\Omega)$ as the pivot space.

Remark 2.5. For $s \in]0, \frac{1}{2}[$, one may check that $\mathbf{H}_\perp^s(\partial\Omega) = \mathbf{H}_\parallel^s(\partial\Omega) = \mathbf{H}_t^s(\partial\Omega)$. Then, for $s = \frac{1}{2}$, two different situations may occur. On one hand, if $\partial\Omega$ is smooth, there holds $\mathbf{H}_\perp^{1/2}(\partial\Omega) = \mathbf{H}_\parallel^{1/2}(\partial\Omega) = \mathbf{H}_t^{1/2}(\partial\Omega)$. On the other hand, if $\partial\Omega$ is polyhedral, the function spaces do not coincide anymore ; instead $\mathbf{H}_t^{1/2}(\partial\Omega)$ is a strict proper subset of $\mathbf{H}_\perp^{1/2}(\partial\Omega)$, and of $\mathbf{H}_\parallel^{1/2}(\partial\Omega)$, with $\mathbf{H}_\perp^{1/2}(\partial\Omega) \neq \mathbf{H}_\parallel^{1/2}(\partial\Omega)$.

Using a localization argument and local maps, one may also introduce the Sobolev scale $(H^s(\partial\Omega))_{s\in[-1,1]}$.
The classical tangential operators are then:

$$\nabla_\Gamma\ :\ H^1(\partial\Omega) \to \mathbf{L}_t^2(\partial\Omega), \mathbf{curl}_\Gamma\ :\ H^1(\partial\Omega) \to \mathbf{L}_t^2(\partial\Omega).$$

These operators can also be defined from larger, or smaller, functions spaces defined on $\partial\Omega$.
The adjoint operators are

$$\mathrm{div}_\Gamma\ :\ \mathbf{L}_t^2(\partial\Omega) \to H_{\mathrm{zmv}}^{-1}(\partial\Omega), \mathrm{curl}_\Gamma\ :\ \mathbf{L}_t^2(\partial\Omega) \to H_{\mathrm{zmv}}^{-1}(\partial\Omega),$$

where we denote $H_{\mathrm{zmv}}^{-1}(\partial\Omega) = \{q \in H^{-1}(\partial\Omega),\ \langle q, 1\rangle_{H^1(\partial\Omega)} = 0\}$ by a slight abuse of notation. They can also be defined from larger, or smaller, functions spaces defined on $\partial\Omega$.
Then, let us introduce

$$\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega) := \{\mathbf{v} \in \mathbf{H}_\parallel^{-1/2}(\partial\Omega),\ \mathrm{div}_\Gamma\,\mathbf{v} \in H^{-1/2}(\partial\Omega)\};$$
$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega) := \{\mathbf{v} \in \mathbf{H}_\perp^{-1/2}(\partial\Omega),\ \mathrm{curl}_\Gamma\,\mathbf{v} \in H^{-1/2}(\partial\Omega)\}.$$

They are Hilbert spaces endowed with their natural norm. For short, the natural norm of the space $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$ is denoted $\|\cdot\|_\gamma$: $\|\mathbf{v}\|_\gamma^2 := \|\mathbf{v}\|^2_{\mathbf{H}_\parallel^{-1/2}(\partial\Omega)} + \|\,\mathrm{div}_\Gamma\,\mathbf{v}\|^2_{H^{-1/2}(\partial\Omega)}$. The norm of the space $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega)$ is denoted $\|\cdot\|_\pi$.
Then, one has the fundamental results.

THEOREM 2.6. *The mapping γ^T is continuous and surjective from $\mathbf{H}(\mathbf{curl}, \Omega)$ to $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$. The mapping π^T is continuous and surjective from $\mathbf{H}(\mathbf{curl}, \Omega)$ to $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega)$.*
Moreover, $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega)$ and $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega)$ are dual spaces. If one denotes by ${}_\gamma\langle\cdot,\cdot\rangle_\pi$ their duality product, one has the integration by parts formula (2.1).

Let us introduce

$$\mathcal{H}(\partial\Omega) = \{v \in H_{\mathrm{zmv}}^1(\partial\Omega),\ \ \Delta_\Gamma\, v \in H^{-1/2}(\partial\Omega)\}. \tag{2.11}$$

Moreover, one has the next surfacic Helmholtz (Hodge) decompositions:

$$\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \partial\Omega) = \mathbf{curl}_\Gamma(H^{1/2}(\partial\Omega)) \oplus \nabla_\Gamma(\mathcal{H}(\partial\Omega)); \tag{2.12}$$

$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega) = \nabla_\Gamma(H^{1/2}(\partial\Omega)) \oplus \mathbf{curl}_\Gamma(\mathcal{H}(\partial\Omega)); \tag{2.13}$$

that hold when $\partial\Omega$ is topologically trivial. For the more general case, we refer to [7] (see also (3.24)-(3.25) below).

Let us focus on the artificial boundary Γ_A. One can define the function spaces on Γ_A as above. As a consequence of remark 2.5, we observe that $\gamma_A^T : \boldsymbol{H}^{s+1/2}(\Omega) \to \mathbf{H}_t^s(\Gamma_A)$ is surjective for all $s \in]0, \frac{1}{2}[$. We still denote by ${}_\gamma\langle\cdot,\cdot\rangle_\pi$ the duality product between $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A)$ and $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A)$. Then, when Γ_A is polyhedral, we shall use some equivalent definitions to carry out explicit computations. Let us denote its faces by $(\Gamma_i)_i$. We define, for $s \in]0, \frac{1}{2}[$,

$$
\begin{aligned}
\mathbf{H}_t^s(\Gamma_i) &:= \mathbf{L}_t^2(\Gamma_i) \cap \mathbf{H}^s(\Gamma_i),\ \forall i\,;\\
\mathbf{H}_t^s(\Gamma_A) &:= \left\{\mathbf{v} \in \mathbf{L}_t^2(\Gamma_A),\ \mathbf{v}_{|\Gamma_i} \in \mathbf{H}_t^s(\Gamma_i),\ \forall i\right\}\,;\\
H^{1+s}(\Gamma_A) &:= \left\{v \in H^1(\Gamma_A),\ \nabla_\Gamma v \in \mathbf{H}_t^s(\Gamma_A)\right\}.
\end{aligned}
$$

3. Further considerations of traces function spaces. For some general classes of coefficients, our aim is now to find a mathematical framework which guarantees that the Robin condition

$$\pi_A^T \boldsymbol{C} + \underline{\boldsymbol{\alpha}}\gamma_A^T \boldsymbol{E} = \mathbf{g} \quad \text{on } \Gamma_A, \tag{3.1}$$

with $\boldsymbol{C} := \underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}$, holds in $\mathbf{L}_t^2(\Gamma_A)$. To start with, mathematically speaking, one has $\boldsymbol{E},\ \boldsymbol{C} \in \boldsymbol{H}(\mathbf{curl},\Omega)$, but the traces $\gamma_A^T\boldsymbol{E}$ and $\pi_A^T\boldsymbol{C}$ belong to different trace spaces: $\gamma_A^T\boldsymbol{E} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A)$, whereas $\pi_A^T\boldsymbol{C} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A)$.

Therefore, the function space in which the Robin condition of (2.6) may hold is unclear. In the literature, it is generally assumed or stated without proper justification (see, e.g., [20, 34, 25, 21]), that the Robin condition of (2.6) holds in $\mathbf{L}_t^2(\Gamma_A)$; and, in particular, that one should look for the solution of the associated time-harmonic Maxwell problem in the space

$$\boldsymbol{H}^+(\mathbf{curl},\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\Omega),\ \tilde{\gamma}\boldsymbol{v} = 0 \text{ and } \gamma_A^T\boldsymbol{v} \in \mathbf{L}_t^2(\Gamma_A)\right\}, \tag{3.2}$$

endowed with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^+(\mathbf{curl},\Omega)} := \left(\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\gamma_A^T\boldsymbol{v}\|^2_{\mathbf{L}_t^2(\Gamma_A)}\right)^{1/2}.$$

When the impedance coefficient α is scalar and constant, if one assumes that $\mathbf{g}$ belongs to $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A)$, then both traces $\gamma_A^T\boldsymbol{E}$ and $\pi_A^T\boldsymbol{C}$ also belong to this function space. Thus, to determine whether the Robin condition of (2.6) holds in $\mathbf{L}_t^2(\Gamma_A)$ reduces to determining whether the continuous embedding

$$\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A) \subset \mathbf{L}_t^2(\Gamma_A), \tag{3.3}$$

is true, which is a general assertion of functional analysis, independent of the boundary condition. However, this assumption is not always justified. It has been proven in the case of smooth domains, eg. with a boundary of class $\mathcal{C}^2$ by Barucq and Hanouzet [2]. For domains with a piecewise smooth boundary, however, the result does not always hold. We refer to Theorem 8 of [10], Proposition 4.11 of [8], and sections 5.1.2.1 and 5.1.2.2 in [1].

To the best of our knowledge, only the case of scalar constant coefficients has been addressed in the literature. In more general settings, we shall investigate whether the Robin condition of (2.6) holds in $\mathbf{L}_t^2(\Gamma_A)$, under appropriate assumptions on Γ_A and $\underline{\boldsymbol{\alpha}}$. In the first subsection, we find necessary and sufficient conditions for the assertion (3.3) to hold. In fact, we prove a functional analysis result (Theorem 3.5) which is actually more precise than statement (3.3). To that aim, we follow the path proposed in sections 5.1.2.1 and 5.1.2.2 of [1]. Then, we discuss whether (3.1) holds in $\mathbf{L}_t^2(\Gamma_A)$, for different classes of impedance coefficient $\underline{\boldsymbol{\alpha}}$ in the following subsections.

Preliminary discussion. We assume for simplicity that Γ_A is topologically trivial (this assumption, however, is not restrictive; see the end of the subsection. Let $\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A)$, we write the surface Helmholtz decompositions of $\mathbf{u}$, first considered as an element of $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A)$, and then as an element of $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A)$ (see [9, 11]):

$$\mathbf{u} = \mathbf{curl}_\Gamma\,\phi^- + \nabla_\Gamma\,\psi^+,\ \text{ where } \phi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma_A),\ \psi^+ \in \mathcal{H}(\Gamma_A)\,; \tag{3.4}$$

(3.5) $$\mathbf{u} = \nabla_\Gamma \psi^- + \mathbf{curl}_\Gamma \phi^+, \text{ where } \psi^- \in H^{1/2}_{\text{zmv}}(\Gamma_A),\ \phi^+ \in \mathcal{H}(\Gamma_A).$$

If either $\phi^- \in H^1(\Gamma_A)$ or $\psi^- \in H^1(\Gamma_A)$, then it follows that $\mathbf{u} \in \mathbf{L}^2_t(\Gamma_A)$, and so the claim (3.3) is proven.

Therefore, we investigate whether ϕ^- or ψ^- are actually more regular than $H^{1/2}(\Gamma_A)$. Subtracting the two decompositions of $\mathbf{u}$, one has

(3.6) $$\mathbf{curl}_\Gamma \left(\phi^- - \phi^+\right) + \nabla_\Gamma \left(-\psi^- + \psi^+\right) = \mathbf{0} \quad \text{on } \Gamma_A.$$

We introduce $\phi^{\text{sing}} := \phi^- - \phi^+ \in H^{1/2}_{\text{zmv}}(\Gamma_A)$ and $\psi^{\text{sing}} := -\psi^- + \psi^+ \in H^{1/2}_{\text{zmv}}(\Gamma_A)$. Taking respectively the curl_Γ and the div_Γ of equation (3.6), and recalling that $\Delta_\Gamma = -\,\text{curl}_\Gamma\, \mathbf{curl}_\Gamma = \text{div}_\Gamma \nabla_\Gamma$, one gets that $\phi^{\text{sing}}, \psi^{\text{sing}}$ are governed by

(3.7) $$\text{Find } \phi^{\text{sing}}, \psi^{\text{sing}} \in H^{1/2}_{\text{zmv}}(\Gamma_A) \text{ s.t. } \begin{cases} \Delta_\Gamma \phi^{\text{sing}} = 0 \\ \Delta_\Gamma \psi^{\text{sing}} = 0 \end{cases} \text{ on } \Gamma_A.$$

Thus the question is actually to determine whether the homogeneous Laplace-Beltrami problem admits *singular solutions*, i.e. solutions in $H^{1/2}(\Gamma_A) \setminus H^1(\Gamma_A)$. If not, one would have $\phi^{\text{sing}} \in H^1_{\text{zmv}}(\Gamma_A)$, hence, because the Laplace-Beltrami problem is well-posed in $H^1_{\text{zmv}}(\Gamma_A)$, $\phi^{\text{sing}} = 0$.

Study of Laplace-Beltrami problems. When Γ_A is of class $\mathcal{C}^2$, there are no singular solutions (we refer to the beginning of the proof of lemma 3.3). On the other hand, when Γ_A is polyhedral, the existence of singular solutions for the Laplace-Beltrami problems (3.7) has to be addressed carefully. To that aim, we introduce the notion of *(semi-)pathological vertices* of Γ_A, first considered in [10].

DEFINITION 3.1. *Let v be a vertex of Γ_A, that stands at the intersection of K faces denoted Γ_k. Locally, each face Γ_k can be described by polar coordinates (r, θ), with $\theta \in]\theta_k, \theta_{k+1}[$, $\theta_0 = 0$ and $\theta_K = \theta_v$, where θ_v is the sum of all face angles at the vertex v. If $\theta_v > 4\pi$, the vertex v is said to be* pathological *(see Figure 1 for an illustration). In this case, we define $I_v := \max\{q \in \mathbb{N} \mid \theta_v > 4\pi q\} \geq 1$. In the limit case $\theta_v = 4\pi$, the vertex v is said to be* semi-pathological.

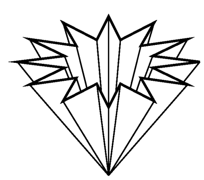

Fig. 1: Example of pathological vertex (cf. [10, §5])

Remark 3.2. Let v be a vertex of Γ_A. We observe that if the domain Ω itself, or if $\mathbb{R}^3 \setminus \overline{\Omega}$, is convex in a neighborhood of v, then $\theta_v < 2\pi$.

LEMMA 3.3. *The solutions to the problem*

(3.8) $$\textit{Find } \phi^{\text{sing}} \in H^{1/2}_{\text{zmv}}(\Gamma_A) \textit{ s.t. } \Delta_\Gamma \phi^{\text{sing}} = 0 \quad \textit{on } \Gamma_A$$

are characterized as follows:
- *If Γ_A is of class $\mathcal{C}^2$, then $\phi^{\text{sing}} = 0$;*
- *If Γ_A is polyhedral without pathological vertex, then $\phi^{\text{sing}} = 0$;*
- *If Γ_A is polyhedral with pathological vertices $(v_p)_{p=1,P}$, then ϕ^{sing} spans a vector space $\mathcal{S}$ of dimension $\sum_{p=1,P} I_{v_p}$.*

If it exists, the vector space $\mathcal{S}$ is called the *space of singularities.*

Proof. When Γ_A is of class $\mathcal{C}^2$, no solution is in $H^{1/2}(\Gamma_A) \setminus H^1(\Gamma_A)$. In this case, the result is a consequence of elliptic regularity. It is obtained by localization (via a finite covering of Γ_A), and then going back to the parametric plane locally via a smooth mapping, and finally using the standard theory of singularities, see for instance [22, Chapter 2].

In the case of a polyhedral boundary, one also studies the solution locally. One can define a finite covering $(\Gamma_o)_o$ of Γ_A where all open subsets Γ_o are of one of the following three types: Γ_o is included in one face ; Γ_o contains one edge ; or Γ_o contains one vertex and the adjacent edges. The proof makes intensive use of results from Grisvard [22].

1. When Γ_o is included in one face. We conclude that ${\phi^{\rm sing}}_{|\Gamma_o} \in H^1(\Gamma_o)$ as above.
2. When Γ_o contains an edge e_{ij}, at the intersection of two faces Γ_i and Γ_j. We denote $\boldsymbol{\tau}_{ij}$ the unit vector tangent to e_{ij}, and, for $k = i, j$, $\boldsymbol{\tau}_k$ the unit vector s.t. $(\boldsymbol{\tau}_{ij}, \boldsymbol{\tau}_k)$ defines an orthonormal basis on Γ_k. One has $\phi_k := {\phi^{\rm sing}}_{|\Gamma_k} \in L^2(\Gamma_k)$ with $\Delta_\Gamma \phi_k = 0$ on Γ_k. Then, according to Theorem 1.5.2 in [22], both traces of ϕ_i, ϕ_j belong to $(\widetilde{H}^{1/2}(e_{ij}))'$, and both normal derivatives belong to $(\widetilde{H}^{3/2}(e_{ij}))'$, where $\widetilde{H}^s(e_{ij})$ denotes fields whose continuation by 0 on $\partial\Gamma_k$ belong to $H^s(\partial\Gamma_k)$. Moreover, as ${\phi^{\rm sing}}_{|\Gamma_o} \in H^{1/2}(\Gamma_o)$ and $\Delta_\Gamma({\phi^{\rm sing}}_{|\Gamma_o}) \in L^2(\Gamma_o)$, Dirichlet and Neumann traces on e_{ij} match, so there holds $\phi_{i|e_{ij}} = \phi_{j|e_{ij}}$, as well as $\partial_{\boldsymbol{\tau}_i}\phi_i = \partial_{\boldsymbol{\tau}_j}\phi_j$ on e_{ij}.
   Then, one maps Γ_o *locally* around e_{ij} into a subset $\Gamma_o^\star$ of the parametric plane. This is done by a piecewise affine, bijective mapping $F : \Gamma_o \to \Gamma_o^\star$. Introducing $\phi_k^\star := \phi_k \circ F^{-1}$ on each $\Gamma_k^\star := F(\Gamma_k)$, and finally $\phi^\star \in L^2(\Gamma_o^\star)$ defined by $\phi^\star_{|\Gamma_k^\star} = \phi_k^\star$ for $k = i, j$, one gets that $\phi^\star \in H^{1/2}(\Gamma_o^\star)$ with $\Delta_\Gamma \phi^\star = 0$ on $\Gamma_o^\star$ thanks to the trace matchings. Therefore, using the result of item 1., we have that $\phi^\star \in H^1(\Gamma_o^\star)$. Coming back to ${\phi^{\rm sing}}_{|\Gamma_o} = F \circ \phi^\star$, it also belongs to $H^1(\Gamma_o)$.
3. When Γ_o contains a vertex v, at the intersection of K faces denoted Γ_k. We denote $e_{k,k+1}$ the edge between Γ_k and Γ_{k+1}, and use the same notation as above. As before, the ϕ_k satisfy $\phi_k \in L^2(\Gamma_k)$ with $\Delta_\Gamma \phi_k = 0$ on Γ_k, and trace matchings: $\phi_k = \phi_{k+1}$ as well as $\partial_{\boldsymbol{\tau}_k}\phi_k = \partial_{\boldsymbol{\tau}_{k+1}}\phi_{k+1}$ on $e_{k,k+1}$ in a weak sense. Thus, outside any neighbourhood of the vertex v, we conclude thanks to items 1. and 2. that $\phi^{\rm sing}$ is of H^1-regularity.
   In a neighbourhood Γ_v of the vertex, we follow Kondrat'ev's theory [27, 17]. We use polar coordinates (r, θ) in each Γ_k as in definition 3.1. Expressing the problem governing ${\phi^{\rm sing}}_{|\Gamma_v}$ in this coordinates system, we find by direct computations (see [22, §2.3]) that the solution to this problem locally belongs to $\mathrm{span}_{\lambda\in\Lambda}(r^\lambda \varphi_\lambda(\theta))$, where $(\varphi_\lambda)_\lambda$ are eigenfunctions of the operator $\varphi \mapsto -\varphi''$ on $[0, \theta_v]$ with periodic boundary conditions, and $\lambda \in \mathbb{C}$ is such that $\phi_\lambda \ : \ (r,\theta) \mapsto r^\lambda \varphi_\lambda(\theta)$ solves locally our problem. It follows that $\varphi_\lambda(\theta) = \exp(\pm i\lambda\theta)$ and, since the admissible value of λ are constrained by the trace matching conditions on $e_{K,1}$, one has
   $$\phi_\lambda^\pm(r,\theta) = r^\lambda e^{\pm i\lambda\theta}, \quad \lambda \in \Lambda := \frac{2\pi}{\theta_v}\mathbb{Z}. \tag{3.9}$$
   The regularity of each $\phi_\lambda^\pm$ is given by Theorem 1.2.18 in [22]:
   $$\text{for all } s \in]0,1[,\ \phi_\lambda^\pm \in H^s(\Gamma_v) \iff \lambda > s - 1; \tag{3.10}$$
   in particular, $\phi_\lambda^\pm \in H^{1/2}(\Gamma_v) \iff \lambda > -\frac{1}{2}$. On the other hand, $\phi_\lambda^\pm \in H^1(\Gamma_v) \iff \lambda \geq 0$. So, (non-constant) local singular solutions exist if, and only if, there exists $\lambda \in \Lambda$ s.t. $-\frac{1}{2} < \lambda < 0$. That is when $-\frac{1}{2} < -\frac{2\pi}{\theta_v}$, i.e. $\theta_v > 4\pi$; in other words, when v is *pathological*.
   In this case, to reconstruct a global solution of (3.8) starting from $\phi_\lambda^\pm$, we introduce a smooth cut-off function χ whose value is 1 in Γ_v and 0 on $\partial\Gamma_o$: one has $\Delta_\Gamma(\chi\phi_\lambda^\pm) \in H^{-1}(\Gamma_A)$. Then, one solves the (well-posed) problem
   $$\text{Find } w_\lambda^\pm \in H^1_{\rm zmv}(\Gamma_A) \text{ s.t. } \ \Delta_\Gamma w = \Delta_\Gamma(\chi\phi_\lambda^\pm) \quad \text{in } \Gamma_A. \tag{3.11}$$
   Introducing $s_{v,\lambda}^\pm := w_\lambda^\pm - \chi\phi_\lambda^\pm \in H^{1/2}(\Gamma_A)$, we have found $2I_v$ independent singular solutions that do not vanish at the neighbourhood of the vertex v. We proceed similarly with the other vertices. Outside the neighbourhood of the vertices, there are no singular solutions because of items 1. and 2..

One concludes that there exist singular solutions of (3.8) as soon as Γ_A has at least one pathological vertex. More precisely, there exist a basis of $2\sum_{p=1}^{P} I_{v_p}$ (independent) singular solutions $s_{v,\lambda}^\pm \in H^{1/2}(\Gamma_A) \setminus H^1(\Gamma_A)$ arising from the different pathological vertices v_p. On the other hand, in the absence of pathological vertices, we conclude that there are no singular solutions. Thus, we have obtained and characterized all the singular solutions to the homogenous Laplace-Beltrami problem, by spanning all singular solutions for each pathological vertex of Γ_A. ∎

So far, semi-pathological vertices do not play any role. This is not the case below, as one has now to differentiate between pathological and semi-pathological vertices.

LEMMA 3.4. *If Γ_A is of class $\mathcal{C}^2$, or if Γ_A is polyhedral without pathological or semi-pathological vertex, then*

$$\mathcal{H}(\Gamma_A) \subset H^{3/2}_{\mathrm{zmv}}(\Gamma_A), \tag{3.12}$$

with continuous embedding.
Otherwise, there exists $s_{\max} \in]0, \frac{1}{2}]$ s.t., $\forall s \in [0, s_{\max}[$,

$$\mathcal{H}(\Gamma_A) \subset H^{1+s}_{\mathrm{zmv}}(\Gamma_A). \tag{3.13}$$

The embeddings are continuous, and the value of $s_{\max}$ depends only on geometry:

- *If Γ_A is polyhedral without pathological vertex, then $s_{\max} = \frac{1}{2}$;*
- *If Γ_A is polyhedral with pathological vertices $(v_p)_{p=1,P}$, then $s_{\max} = \min_{p=1,P}\left(\frac{2\pi}{\theta_{v_p}}\right) < \frac{1}{2}$.*

Proof. Let $\varphi \in \mathcal{H}(\Gamma_A)$: there holds $\varphi \in H^1_{\mathrm{zmv}}(\Gamma_A)$, and $g := \Delta_\Gamma \varphi \in H^{-1/2}(\Gamma_A)$. Therefore, φ can be interpreted as the (regular) solution of a Laplace-Beltrami problem with data in $H^{-1/2}(\Gamma_A)$. The extra H^{1+s}-regularity of φ is then limited to $s \in [0, \frac{1}{2}]$, the value $\frac{1}{2}$ coming from the Shift Theorem for the Laplace-Beltrami operator with data in $H^{-1/2}(\Gamma_A)$. In particular, if Γ_A is of class $\mathcal{C}^2$, then $\varphi \in H^{3/2}_{\mathrm{zmv}}(\Gamma_A)$ with continuous dependence (see eg. [33, §5.4.1]). However, if Γ_A is polyhedral, this extra-regularity is also driven by the regularity at the vertices v of Γ_A. As before, the solutions are locally given at a vertex v by

$$\phi_\lambda(r, \theta) = r^\lambda e^{\pm i\lambda\theta}, \quad \lambda \in \Lambda := \frac{2\pi}{\theta_v}\mathbb{Z}, \tag{3.14}$$

now with $\lambda > 0$ because $\varphi \in H^1_{zmv}(\Gamma_A)$. So, the minimal exponent of local regularity at a vertex v is $\lambda = \frac{2\pi}{\theta_v}$. If the vertex is pathological, this value is smaller than $\frac{1}{2}$. Hence, if there are pathological vertices, (3.13) holds with $s_{\max} = \min_{p=1,P}\left(\frac{2\pi}{\theta_{v_p}}\right) < \frac{1}{2}$. While, if there is no pathological vertex, the regularity is now limited to $\frac{1}{2}$ because of the Shift Theorem: we conclude that, if moreover there is no semi-pathological vertex, the embedding (3.12) holds, whereas in the presence of semi-pathological vertices, (3.13) holds with $s_{\max} = \frac{1}{2}$. Finally, in all of the above, one has continuous dependence of the embeddings. ∎

With this, one can derive the embedding results below, that generalize the result of [1, Remark 5.1.5] for a polyhedral boundary.

THEOREM 3.5. *If Γ_A is of class $\mathcal{C}^2$, or if Γ_A is polyhedral without pathological or semi-pathological vertex, then*

$$\mathbf{H}^{-1/2}_{\parallel}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}_\Gamma, \Gamma_A) \subset \mathbf{H}^{1/2}_t(\Gamma_A). \tag{3.15}$$

Otherwise:

- *If Γ_A is polyhedral without pathological vertex, then $\forall s \in [0, \frac{1}{2}[$,*

$$\mathbf{H}^{-1/2}_{\parallel}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}_\Gamma, \Gamma_A) \subset \mathbf{H}^{s}_t(\Gamma_A). \tag{3.16}$$

- *If Γ_A is polyhedral with pathological vertices $(v_p)_{p=1,P}$, then $\forall s \in [0, \min_{p=1,P}\left(\frac{2\pi}{\theta_{v_p}}\right)[$,*

$$\mathbf{H}^{-1/2}_{\parallel}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}_\Gamma, \Gamma_A) \subset \mathbf{H}^{s}_t(\Gamma_A) \oplus \nabla_\Gamma \mathcal{S}, \tag{3.17}$$

$$\mathbf{H}^{-1/2}_{\parallel}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}_\Gamma, \Gamma_A) \subset \mathbf{H}^{s}_t(\Gamma_A) \oplus \mathbf{curl}_\Gamma \mathcal{S}, \tag{3.18}$$

where $\mathcal{S}$ is the space of singularities.

Moreover, all those embeddings are continuous.

Proof. Let us consider $\mathbf{u} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A)$, with the surface Helmholtz decompositions (3.4)-(3.5) of $\mathbf{u}$. Taking the curl_Γ of (3.4) and the div_Γ of (3.5), one gets that ϕ^-, ψ^- are governed by

$$\text{Find } \phi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma_A) \text{ such that } \Delta_\Gamma \phi^- = -\,\mathrm{curl}_\Gamma \mathbf{u} \text{ on } \Gamma_A, \tag{3.19}$$

$$\text{Find } \psi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma_A) \text{ such that } \Delta_\Gamma \psi^- = \mathrm{div}_\Gamma \mathbf{u} \text{ on } \Gamma_A. \tag{3.20}$$

Likewise, ϕ^+, ψ^+ are governed by

$$\text{Find } \phi^+ \in H^{1}_{\mathrm{zmv}}(\Gamma_A) \text{ such that } \Delta_\Gamma \phi^+ = -\,\mathrm{curl}_\Gamma \mathbf{u} \text{ on } \Gamma_A, \tag{3.21}$$

$$\text{Find } \psi^+ \in H^{1}_{\mathrm{zmv}}(\Gamma_A) \text{ such that } \Delta_\Gamma \psi^+ = \ \mathrm{div}_\Gamma \mathbf{u} \text{ on } \Gamma_A, . \tag{3.22}$$

□

Note that ϕ^+, ψ^+ are actually the unique solutions to the problems (3.21) and (3.22), because the Laplace-Beltrami problem is well-posed in $H^1_{\mathrm{zmv}}(\Gamma_A)$. The regularity of ϕ^+, ψ^+ enters the scope of Lemma 3.4. In particular, for any admissible exponent s defined there, one has the stability estimates $\|\phi^+\|_{H^{1+s}(\Gamma_A)} \lesssim \|\,\mathrm{curl}_\Gamma \mathbf{u}\|_{H^{-1/2}(\Gamma_A)}$ and $\|\psi^+\|_{H^{1+s}(\Gamma_A)} \lesssim \|\,\mathrm{div}_\Gamma \mathbf{u}\|_{H^{-1/2}(\Gamma_A)}$. And, by definition of the norms $\|\cdot\|_\pi$ and $\|\cdot\|_\gamma$, we obviously have the stability estimates $\|\phi^+\|_{H^{1+s}(\Gamma_A)} \lesssim \|\mathbf{u}\|_\pi$ and $\|\psi^+\|_{H^{1+s}(\Gamma_A)} \lesssim \|\mathbf{u}\|_\gamma$.
Then, to analyse the regularity of ϕ^- and ψ^-, we recall that $\phi^{\mathrm{sing}} = \phi^- - \phi^+$ and $\psi^{\mathrm{sing}} = \psi^- - \psi^+$ are solutions of the problem (3.7). These solutions are characterized by Lemma 3.3. Applying the results obtained there, we conclude that:

- If Γ_A is of class $\mathcal{C}^2$, or if Γ_A is polyhedral without pathological or semi-pathological vertex, then $\phi^{\mathrm{sing}} = 0$, $\psi^{\mathrm{sing}} = 0$, because of Lemma 3.3. Therefore, $\phi^- = \phi^+$, $\psi^- = \psi^+$. Moreover, according to Lemma 3.4, both fields $\phi^+, \psi^+ \in H^{3/2}(\Gamma_A)$. Using either decomposition (3.4) or (3.5) together with the stability estimates, we find that the continuous embedding (3.15) holds.
- Otherwise: if Γ_A is polyhedral without pathological vertex, then $\phi^{\mathrm{sing}} = 0$, $\psi^{\mathrm{sing}} = 0$ again, because of Lemma 3.3: $\phi^- = \phi^+$, $\psi^- = \psi^+$. Moreover, according to Lemma 3.4, both fields $\phi^+, \psi^+ \in H^{1+s}(\Gamma_A)$ for all $s < \frac{1}{2}$. We conclude as in the previous case that the (continuous) embeddings (3.16) hold.
- Otherwise: if Γ_A is polyhedral with pathological vertices, then $\phi^- = \phi^{\mathrm{sing}} - \phi^+$, where ϕ^{sing} is a singular solution which belongs to $\mathcal{S}$ (Lemma 3.3). Besides, $\phi^+, \psi^+ \in H^{1+s}(\Gamma_A)$ for all $0 \le s < s_{\max} = \min_{p=1,P}\left(\frac{2\pi}{\theta_{v_p}}\right)$, because of Lemma 3.4. So, thanks to decomposition (3.4), we conclude that, for all $s \in [0, s_{\max}[$, $\mathbf{u}$ belongs to $\mathbf{H}^s_t(\Gamma_A) + \mathbf{curl}_\Gamma\, \mathcal{S}$. This is the embedding (3.18).
  Similarly, there also holds $\psi^- = \psi^{\mathrm{sing}} + \psi^+$ with $\psi^{\mathrm{sing}} \in \mathcal{S}$ and $\phi^+, \psi^+ \in H^{1+s}(\Gamma_A)$ for all $0 \le s < s_{\max}$. Using now decomposition (3.5), we conclude that for all $s \in [0, s_{\max}[$, $\mathbf{u}$ belongs to $\mathbf{H}^s_t(\Gamma_A) + \nabla_\Gamma\, \mathcal{S}$. This is the embedding (3.17).
  All the results come with stability estimates on ϕ^+ and ψ^+. If $\mathbf{u}$ has a singular part expressed as a gradient, ie. $\mathbf{u} = \nabla_\Gamma\, \psi^{\mathrm{sing}} + \nabla_\Gamma\, \psi^+ + \mathbf{curl}_\Gamma\, \phi^+$, there holds by triangle inequality that $\|\,\nabla_\Gamma\, \psi^{\mathrm{sing}}\|_{H^{-1/2}(\Gamma_A)} \lesssim \|\mathbf{u}\|_\gamma + \|\mathbf{u}\|_\pi$. Because $\mathcal{S}$ is finite dimensional, all norms are equivalent, and we have obtained a stability estimate for the singular part. Hence embedding (3.17) is continuous. One can proceed similarly if $\mathbf{u}$ has a singular part expressed as a curl to conclude that embedding (3.18) is continuous.

Remark 3.6. In fact, there holds

$$\mathbf{H}^s_t(\Gamma_A) \oplus \nabla_\Gamma\, \mathcal{S} = \mathbf{H}^s_t(\Gamma_A) \oplus \mathbf{curl}_\Gamma\, \mathcal{S}. \tag{3.23}$$

Indeed, looking at the singularities at a pathological vertex v, they are expressed locally as $\phi^+_\lambda(r,\theta) = r^\lambda e^{i\lambda\theta}$ and $\phi^-_\lambda(r,\theta) = r^\lambda e^{-i\lambda\theta}$, and one can observe that the $\mathbf{curl}_\Gamma$ of one is equal to the ∇_Γ of the other, up to a $+$ or $-$ sign.

Comments. When Γ_A is polyhedral without pathological vertex, the result obtained in Theorem 3.5 is stronger than the conjectured embedding (3.3). As a matter of fact, a by-product of the continuous embedding (3.15) or (3.16), is that the embedding in $\mathbf{L}^2_t(\Gamma_A)$ is *compact*. On the other hand, *the embedding* (3.3) *does not hold if* Γ_A *is polyhedral with pathological vertices*. However, the singularities that appear in (3.17)-(3.18) belong to a finite-dimensional vector space and, once they are taken into account, the remaining "regular" function space $\mathbf{H}^s_t(\Gamma_A)$ (for some $s > 0$) still embeds compactly in $\mathbf{L}^2_t(\Gamma_A)$.

All previous proofs can be adapted to a topologically non-trivial boundary Γ_A. In this case [7], one has extra-terms in the decompositions (3.4) and (3.5):

$$\mathbf{u} = \mathbf{curl}_\Gamma \, \phi^- + \nabla_\Gamma \, \psi^+ + \mathbf{h}_1, \tag{3.24}$$

$$\mathbf{u} = \nabla_\Gamma \, \psi^- + \mathbf{curl}_\Gamma \, \phi^+ + \mathbf{h}_2, \tag{3.25}$$

where $\mathbf{h}_1$, $\mathbf{h}_2$ belong to the finite dimensional vector space

$$\mathbb{H} := \left\{ \mathbf{u} \in \mathbf{L}^2_t(\Gamma_A), \ \mathrm{curl}_\Gamma \, \mathbf{u} = 0 \text{ and } \mathrm{div}_\Gamma \, \mathbf{u} = 0 \text{ on } \Gamma_A \right\}. \tag{3.26}$$

However, we observe that these terms vanish when taking the curl_Γ or the div_Γ of the decompositions. So, this has no impact on the above lines about the regularity of ϕ^-, ψ^-, or of ϕ^+, ψ^+. Since by definition one has $\mathbb{H} \subset \mathbf{L}^2_t(\Gamma_A)$, it follows that the embedding

$$\mathbf{H}^{-1/2}_{\|}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}_\Gamma, \Gamma_A) \subset \mathbf{L}^2_t(\Gamma_A)$$

occurs under the same conditions than in Theorem 3.5. Finally, one can study the regularity of elements of $\mathbb{H}$ with the help of their characterization, which is given by Theorem 4 in [7]. One easily checks that they have the same a priori regularity than the fields $\nabla_\Gamma \, \psi^+$ that appear in (3.24), or the fields $\mathbf{curl}_\Gamma \, \phi^+$ that appear in (3.25). Hence the same conclusions as in theorem 3.5 can be drawn, when the boundary Γ_A is topologically non-trivial.

4. A study of traces regularity on the artificial boundary.

With these results at hand, we are now in position to study the regularity of the traces $\gamma_A^T \boldsymbol{E}$ and $\pi_A^T \boldsymbol{C}$ involved in the Robin condition (3.1). We do so for smooth or piecewise constant, scalar, impedance coefficient α. We also sketch a result for tensor-valued coefficient $\underline{\boldsymbol{\alpha}}$. In all of this section, we assume that the coefficient is elliptic (cf. Assumption 1).

4.1. The case of a smooth scalar impedance coefficient.

The abstract result of Theorem 3.5 allows one to conclude directly on the regularity of traces, when the coefficient α admits a smooth lifting in Ω. This is summarized in the next theorem.

THEOREM 4.1. *Assume α is an invertible scalar coefficient, and either constant on Γ_A, or admits a smooth lifting $\tilde{\alpha} \in W^{1,\infty}(\Omega)$. Let $\gamma_A^T \boldsymbol{E}$ and $\pi_A^T \boldsymbol{C}$ be governed by (3.1), with $\mathbf{g} \in \mathbf{H}^{-1/2}_{\|}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}_\Gamma, \Gamma_A)$.*
If Γ_A is of class $\mathcal{C}^2$, or if Γ_A is polyhedral without pathological or semi-pathological vertex, then $\gamma_A^T \boldsymbol{E}, \pi_A^T \boldsymbol{C} \in \mathbf{H}^{1/2}_t(\Gamma_A)$, with the stability estimates

$$\|\gamma_A^T \boldsymbol{E}\|_{\mathbf{H}^{1/2}_t(\Gamma_A)} \lesssim \|\mathbf{g}\|_\pi + \|\pi_A^T \boldsymbol{C}\|_\pi + \|\gamma_A^T \boldsymbol{E}\|_\gamma \quad \text{and} \quad \|\pi_A^T \boldsymbol{C}\|_{\mathbf{H}^{1/2}_t(\Gamma_A)} \lesssim \|\mathbf{g}\|_\gamma + \|\pi_A^T \boldsymbol{C}\|_\pi + \|\gamma_A^T \boldsymbol{E}\|_\gamma.$$

Otherwise:
- *If Γ_A is polyhedral without pathological vertex, then $\forall s \in [0, \frac{1}{2}[$, $\gamma_A^T \boldsymbol{E}, \pi_A^T \boldsymbol{C} \in \mathbf{H}^s_t(\Gamma_A)$, with the stability estimates*
$$\|\gamma_A^T \boldsymbol{E}\|_{\mathbf{H}^s_t(\Gamma_A)} \lesssim \|\mathbf{g}\|_\pi + \|\pi_A^T \boldsymbol{C}\|_\pi + \|\gamma_A^T \boldsymbol{E}\|_\gamma \quad \text{and} \quad \|\pi_A^T \boldsymbol{C}\|_{\mathbf{H}^s_t(\Gamma_A)} \lesssim \|\mathbf{g}\|_\gamma + \|\pi_A^T \boldsymbol{C}\|_\pi + \|\gamma_A^T \boldsymbol{E}\|_\gamma.$$
- *If Γ_A is polyhedral with pathological vertices $(v_p)_{p=1,P}$, then $\forall s \in [0, \min_{p=1,P}\left(\frac{2\pi}{\theta_{v_p}}\right)[$,*
$$\gamma_A^T \boldsymbol{E} \in \mathbf{H}^s_t(\Gamma_A) \oplus \mathbf{curl}_\Gamma \, \mathcal{S} \quad \text{and} \quad \pi_A^T \boldsymbol{C} \in \mathbf{H}^s_t(\Gamma_A) \oplus \nabla_\Gamma \, \mathcal{S},$$
where $\mathcal{S}$ is the space of singularities introduced in Lemma 3.3, with stability estimates like above.

Proof. The proof is a direct consequence of Theorem 3.5 when α is constant. When α is smooth, one introduces $\tilde{\boldsymbol{E}} := \tilde{\alpha} \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$, and there holds $\alpha \gamma_A^T \boldsymbol{E} = \gamma_A^T(\tilde{\alpha} \boldsymbol{E})$. So, the Robin condition (3.1) rewrites

$$\pi_A^T \boldsymbol{C} + \gamma_A^T \tilde{\boldsymbol{E}} = \mathbf{g} \quad \text{on } \Gamma_A, \tag{4.1}$$

and one concludes as above. ∎

Remark 4.2. Note that, even when there is a singular part (ie. when the artificial boundary is polyhedral with pathological vertices), we obtain that the regular part depends continuously on $\|\mathbf{g}\|_\gamma + \|\mathbf{g}\|_\pi$, $\|\pi_A^T \boldsymbol{C}\|_\pi$, and $\|\gamma_A^T \boldsymbol{E}\|_\gamma$. An alternate proof, that does not make use of a lifting of α, is proposed in [12] (in this case, one assumes that $\alpha \in W^{2,\infty}(\Gamma_A)$).

4.2. The case of a piecewise constant scalar impedance coefficient. In this subsection, we assume that α is an elliptic piecewise constant scalar field on Γ_A, and that Γ_A is polyhedral (the latter assumption to simplify a little bit the presentation). Then, one has to deal with the singularities arising from the presence of discontinuity lines and vertices, where discontinuity is understood with respect to the value of the coefficient. This is done in the spirit of Lemma 3.3. The plane regions where α is constant are denoted $(\Gamma_k)_k$ and called *coefficient faces*. They will play the same role as the faces of Γ_A in Lemma 3.3. They are naturally plane, because included in faces of Γ_A. Similarly, one will have to deal with *coefficient edges* (lines where two *coefficient faces* meet), and with *coefficient vertices* (points where three or more *coefficient faces* meet). Again for simplicity, we assume that the coefficient edges are straight lines.

As before, the condition is (3.1), with $\mathbf{g} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A)$. We write the surface Helmholtz decompositions of $\gamma_A^T \boldsymbol{E}$ and $\pi_A^T \boldsymbol{C}$:

$$\gamma_A^T \boldsymbol{E} = \mathbf{curl}_\Gamma\, \phi^- + \nabla_\Gamma\, \psi^+, \qquad \phi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma_A), \quad \psi^+ \in \mathcal{H}(\Gamma_A); \tag{4.2}$$

$$\pi_A^T \boldsymbol{C} = \nabla_\Gamma\, \psi^- + \mathbf{curl}_\Gamma\, \phi^+, \qquad \psi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma_A), \quad \phi^+ \in \mathcal{H}(\Gamma_A). \tag{4.3}$$

Thus, we get

$$\left(\nabla_\Gamma\, \psi^- + \mathbf{curl}_\Gamma\, \phi^+\right) + \alpha\left(\mathbf{curl}_\Gamma\, \phi^- + \nabla_\Gamma\, \psi^+\right) = \mathbf{g} \text{ on } \Gamma_A. \tag{4.4}$$

Then, taking the curl_Γ of (4.4), $\phi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma_A)$ is governed by

$$\mathrm{curl}_\Gamma(\alpha\, \mathbf{curl}_\Gamma\, \phi^-) = \mathrm{curl}_\Gamma\, \mathbf{g} + \Delta_\Gamma\, \phi^+ - \mathrm{curl}_\Gamma(\alpha\, \nabla_\Gamma\, \psi^+) \text{ on } \Gamma_A. \tag{4.5}$$

We observe that $\mathrm{curl}_\Gamma\, \mathbf{g}$, $\Delta_\Gamma\, \phi^+ \in H^{-1/2}(\Gamma_A)$, and $\mathrm{curl}_\Gamma(\alpha\, \nabla_\Gamma\, \psi^+) \in H^{-1}(\Gamma_A)$. In this case, one has to study the (singular) solutions of the problem

$$\text{Find } \phi^{\mathrm{sing}} \in H^{1/2}_{\mathrm{zmv}}(\Gamma_A) \text{ s.t. } \mathrm{curl}_\Gamma(\alpha\, \mathbf{curl}_\Gamma\, \phi^{\mathrm{sing}}) = 0 \text{ on } \Gamma_A. \tag{4.6}$$

This is done, as in Lemma 3.3 with the help of localization techniques. For a coefficient vertex v, we now define θ_v as the sum of all coefficient face angles at the vertex v, and we use the face-by-face polar coordinates like in definition 3.1.

LEMMA 4.3. *Assume α is a piecewise constant, elliptic, scalar coefficient. Let v a coefficient vertex, and define the eigenproblem*

$$\left| \begin{aligned} &\textit{Find } (\lambda, \varphi) \in \mathbb{C} \times H^1_{\mathrm{per}}([0,\theta_v]) \textit{ s.t. } ,\forall \psi \in H^1_{\mathrm{per}}([0,\theta_v]), \\ &\qquad \int_0^{\theta_v} \alpha \varphi' \bar{\psi}' \mathrm{d}\theta - \lambda^2 \int_0^{\theta_v} \alpha \varphi \bar{\psi} \mathrm{d}\theta = 0. \end{aligned} \right. \tag{4.7}$$

The solution to the problem (4.6) vanishes if, for all coefficient vertices, there is no λ solution to (4.7) with $\Re(\lambda) \in]-\frac{1}{2}, 0[$. Otherwise, ϕ^{sing} spans a finite dimensional vector space $\mathcal{S}_1$.

Proof. As in Lemma 3.3, we study the regularity locally. We define a finite covering $(\Gamma_o)_o$ of Γ_A where all open subsets Γ_o are of one of the following two types: Γ_o is included in one coefficient face ; Γ_o contains either one coefficient edge, or one coefficient vertex and the adjacent coefficient edges. Indeed, the edge type can be seen as a special case of the vertex type by picking an arbitrary point v on the edge: it corresponds to a vertex type with two faces, each of them with a face angle equal to π (so $\theta_v = 2\pi$).

1. It is included in a *coefficient face* Γ_i. In this case, α is constant in Γ_o, and there holds $\phi^{\mathrm{sing}}{}_{|\Gamma_o} \in H^1(\Gamma_o)$ as in the proof of Lemma 3.3.
2. It contains a *coefficient vertex* v at the intersection of K coefficient faces Γ_k, possibly with $K = 2$ and both face angles equal to π to cover also the *coefficient edge* as a particular case. In a neighbourhood Γ_v of the vertex, we use polar coordinates in each Γ_k as in definition 3.1.
   We follow again Kondrat'ev's theory (we refer to [30] in which similar 2D problems are studied), and look for non-constant solutions that write *locally* $\phi_\lambda(r,\theta) = r^\lambda \varphi_\lambda(\theta)$, where $\varphi_{\lambda|\Gamma_k}(\theta) = c_k e^{\pm i\lambda\theta}$ for some $\lambda \in \mathbb{C}$ and a coefficient $c_k \in \mathbb{C}$. The explicit form of $\varphi_{\lambda|\Gamma_k}$ is a consequence of the fact that α depends only on the angular

coordinate θ, and that α is constant on each Γ_k.
One can notice that ϕ_λ satisfies $2K$ linear compatibility conditions corresponding to the (weak) continuity of the Dirichlet and Neumann traces at each coefficient edge. The existence of a non-trivial solution to this set of $2K$ equations (this is with at least one non-zero coefficient) leads to a dispersion equation governing λ.
For the particular case of a *coefficient edge*, one finds by direct computations on the 4 equations that $\lambda \in \mathbb{Z}$, and in addition $\lambda > 0$ because $\varphi_\lambda \in H^{1/2}(\Gamma_A)$. Hence ${\phi^{\rm sing}}_{|\Gamma_o} \in H^1(\Gamma_o)$.
On the other hand, in the general case, going back to the condition $\mathrm{curl}_\Gamma(\alpha\,\mathbf{curl}_\Gamma\,\phi^{\rm sing}) = 0$ leads to the eigenproblem governing $\varphi_\lambda \in H^1_{\rm per}([0,\theta_v])$ for some $\lambda \in \mathbb{C}$: this is exactly problem (4.7) with $\varphi = \varphi_\lambda$. Now, we recall that the regularity of ϕ_λ is determined only by λ (see again Theorem 1.2.18 in [22]):

$$\text{for all } s \in]0,1[,\ \phi_\lambda \in H^s(\Gamma_v) \Longleftrightarrow \Re(\lambda) > s-1.$$

Moreover, one can note that $\lambda = 0$ leads to the constant solution; and that $\lambda \in i\mathbb{R}\setminus\{0\}$ is not an eigenvalue (in this case, we note that the sesquilinear form in problem (4.7) is coercive, because α is elliptic). Therefore, the problem (4.6) admits singular solutions if, and only if, there are solutions to (4.7) s.t. $\Re(\lambda) \in]-\frac{1}{2},0[$. Furthermore, as in Lemma 3.3, we note that a local singularity ϕ_λ can be continued on Γ_A by the means of a cut-off function. This concludes the first part of the proof.

There remains to prove that the singular solutions belong to a finite dimensional vector space.
First, even if there are eigenvalues leading to singular solutions, one can always state that they are isolated. Indeed, introducing the operator $\mathcal{L}_\lambda$ defined on $H^1_{\rm per}([0,\theta_v])$ by:

$$\forall \varphi,\psi \in H^1_{\rm per}([0,\theta_v]),\ (\mathcal{L}_\lambda\varphi,\psi)_{H^1_{\rm per}([0,\theta_v])} := \int_0^{\theta_v} \alpha\varphi'\bar{\psi}'{\rm d}\theta - \lambda^2\int_0^{\theta_v}\alpha\varphi\bar{\psi}{\rm d}\theta \tag{4.8}$$

it is clear that $\mathcal{L}_\lambda - \mathcal{L}_\mu$ is a compact operator $\forall\lambda,\mu$, thanks to Rellich theorem and we already noted that $\mathcal{L}_\mu$ is an isomorphism for all $\mu \in i\mathbb{R}\setminus\{0\}$. Hence, $\mathcal{L}_\lambda = \mathcal{L}_i + (\mathcal{L}_\lambda - \mathcal{L}_i)$ is a Fredholm operator. As moreover the family of operators $(\mathcal{L}_\lambda)_\lambda$ is analytic w.r.t. λ, one can apply the analytic Fredholm Theorem (see, e.g., [28, Theorem 1.1.1]): the spectrum of $\mathcal{L}_\lambda$ consists only of isolated eigenvalues, which are of finite multiplicities, and do not have accumulation points.
Let us show next that there is at most a finite number of λ in the strip $\{\lambda \in \mathbb{C} \text{ s.t. } \Re(\lambda) \in]-\frac{1}{2},0[\}$. Let $\lambda = a+ib$, one has

$$\mathcal{L}_\lambda(\varphi,\varphi) = \int_0^{\theta_v}\left[\alpha\,|\varphi'|^2 + \big(\alpha(b^2-a^2) - 2\alpha iab\big)\,|\varphi|^2\right]{\rm d}\theta.$$

On one hand, there holds

$$\Re\left[e^{i\theta_\alpha}\int_0^{\theta_v}\alpha(b^2-a^2)|\varphi|^2\right] \geq \alpha_-(b^2-a^2)\|\varphi\|^2_{L^2(]0,\theta_v[)}$$

as soon as $|b| > |a|$. On the other hand,

$$\left|-2iab\int_0^{\theta_v}\alpha\,|\varphi|^2{\rm d}\theta\right| \leq 2|a||b|\alpha_+\|\varphi\|^2_{L^2(]0,\theta_v[)}.$$

We conclude that $\mathcal{L}_\lambda$ is coercive as soon as

$$|\Im(\lambda)| > |\Re(\lambda)| \quad \text{and} \quad \alpha_-|\Im(\lambda)|^2 \geq 2|\Re(\lambda)||\Im(\lambda)|\alpha_+ + \alpha_-|\Re(\lambda)|^2. \tag{4.9}$$

In (4.9), the second condition can be seen as a polynomial of second order on $|\Im(\lambda)|$ (parameterized by $|\Re(\lambda)|$). So, clearly, for imaginary parts large enough, $\mathcal{L}_\lambda$ is an isomorphism. Therefore, the conditions $\Re(\lambda) \in]-\frac{1}{2},0[$ and the negation of (4.9) define a bounded region of $\mathbb{C}$. Thanks to the analytic Fredholm Theorem, there is a finite number of eigenvalues inside this region. Out of this region, there are no eigenvalues in the strip $\{\lambda \in \mathbb{C} \text{ s.t. } \Re(\lambda) \in]-\frac{1}{2},0[\}$. The second claim follows. ∎

We note that, in all configurations, the solution to

$$\text{Find } \phi \in H^1_{\text{zmv}}(\Gamma_A) \text{ s.t. } \operatorname{curl}_\Gamma(\alpha\, \mathbf{curl}_\Gamma\, \phi^{\text{sing}}) = g \text{ on } \Gamma_A \tag{4.10}$$

actually exhibits extra-regularity as soon as the data g does too. As a matter of fact, we have proved in passing the *increased regularity* result below, which can be viewed as a Shift theorem.

Proposition 4.4. *Assume α is a piecewise constant, elliptic, scalar coefficient. There exists a* limiting regularity exponent $\mathtt{r}_{\max} \in]0,1]$ *such that, given $g \in H^{-1+\epsilon}(\Gamma_A)$ for some $\epsilon \in]0,1]$, the solution ϕ to (4.10) belongs to $H^{1+s}(\Gamma_A)$, for all $s \in [0, \min(\epsilon, \mathtt{r}_{\max})[$.*

Proof. Going through the proof of Lemma 4.3, one checks easily that the value of $\mathtt{r}_{\max}$ is equal to the smallest strictly positive real part of λ, where λ solves (4.7). ∎

Remark 4.5. It is possible to estimate the regularity exponent in a given configuration by solving numerically the eigenproblems (4.7) at each coefficient vertex, see e.g. [30, 29, 6].

With these results, we propose a definition of the pathological coefficient vertices.

Definition 4.6. *Given an elliptic piecewise constant scalar impedance coefficient α, a coefficient vertex $v \in \Gamma_A$ is* pathological *if there is at least one solution to (4.7) with a value of λ such that $\Re(\lambda) \in]-\frac{1}{2}, 0[$.*

Having studied the singularities of operator $\operatorname{curl}_\Gamma(\alpha\, \mathbf{curl}_\Gamma\, \cdot)$ on Γ_A, we are now in position to state our result.

Theorem 4.7. *Assume that Γ_A has no pathological vertex. Let α be a piecewise constant, elliptic, scalar coefficient. Let $\gamma_A^T \boldsymbol{E}$ and $\pi_A^T \boldsymbol{C}$ be governed by (3.1), with $\mathbf{g} \in \mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma, \Gamma_A) \cap \mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma, \Gamma_A)$.*
If, for all its coefficient vertices, the eigenproblem (4.7) has no solution λ s.t. $\Re(\lambda) \in]-\frac{1}{2}, 0[$, then one has an increased regularity result: there exists $s_{\max} \in]0, \frac{1}{2}[$ s.t., $\forall s \in [0, s_{\max}[$,

$$\gamma_A^T \boldsymbol{E} \in \mathbf{H}_t^s(\Gamma_A) \quad \textit{and} \quad \pi_A^T \boldsymbol{C} \in \mathbf{H}_t^s(\Gamma_A), \tag{4.11}$$

with moreover

$$\|\gamma_A^T \boldsymbol{E}\|_{\mathbf{H}_t^s(\Gamma_A)} \lesssim \|\mathbf{g}\|_\pi + \|\pi_A^T \boldsymbol{C}\|_\pi + \|\gamma_A^T \boldsymbol{E}\|_\gamma; \tag{4.12}$$

$$\|\pi_A^T \boldsymbol{C}\|_{\mathbf{H}_t^s(\Gamma_A)} \lesssim \|\mathbf{g}\|_\gamma + \|\pi_A^T \boldsymbol{C}\|_\pi + \|\gamma_A^T \boldsymbol{E}\|_\gamma. \tag{4.13}$$

Else, there exists $s_{\max} \in]0, \frac{1}{2}[$ s.t., $\forall s \in [0, s_{\max}[$,

$$\gamma_A^T \boldsymbol{E} \in \mathbf{H}_t^s(\Gamma_A) \oplus \mathbf{curl}_\Gamma\, \mathcal{S}_1, \tag{4.14}$$

$$\pi_A^T \boldsymbol{C} \in \mathbf{H}_t^s(\Gamma_A) \oplus \nabla_\Gamma\, \mathcal{S}_2, \tag{4.15}$$

where $\mathcal{S}_1, \mathcal{S}_2$ are finite dimensional spaces of sigularities.

Proof. Let us assume first that the hypotheses of the theorem are satisfied. If, for all coefficient vertices, there is no λ of real part in $]-\frac{1}{2}, 0[$, then there is no non-zero solution to (4.6), and the solution ϕ^- to (4.5) belongs to $H^1(\Gamma_A)$. Moreover, one has actually extra-regularity for ϕ^-, since ϕ^- solves an elliptic PDE set in $H^1_{\text{zmv}}(\Gamma_A)$, cf. Proposition 2.4). This extra-regularity is driven, on one hand, by the regularity of the right-hand side in (4.5). For all $s < \frac{1}{2}$, as $\psi^+ \in H^{3/2}(\Gamma_A)$ (because of Lemma 3.4), and α is piecewise constant, there holds $\alpha\, \nabla_\Gamma\, \psi^+ \in \mathbf{H}_t^s(\Gamma_A)$, with $\|\alpha\, \nabla_\Gamma\, \psi^+\|_{\mathbf{H}_t^s(\Gamma_A)} \lesssim \|\psi^+\|_{H^{1+s}(\Gamma_A)}$; thus the right-hand side of (4.5) belongs to $H^{s-1}(\Gamma_A)$. On the other hand, it is also driven by the lowest (strictly) positive value of $\Re(\lambda)$ where λ solves (4.7) (since we know that $\lambda \notin i\mathbb{R}$, see the proof of lemma 4.3). Therefore, there holds $\phi^- \in H^{1+s}(\Gamma_A)$ for $s < s_{\max}$, with $s_{\max} = \min(\frac{1}{2}, \min_{\lambda \text{ st } \Re(\lambda)>0} \Re(\lambda))$. Thus, $\gamma_A^T \boldsymbol{E} \in \mathbf{H}_t^s(\Gamma_A)$. The estimates are obtained as in Theorem 4.1 (case without pathological vertex).

To get the result for $\pi_A^T \boldsymbol{C}$, we take the $\operatorname{div}_\Gamma$ of (3.1). Then, $\psi^- \in H^{1/2}_{\text{zmv}}(\Gamma_A)$ is governed by

$$\Delta_\Gamma\, \psi^- = \operatorname{div}_\Gamma \mathbf{g} - \operatorname{div}_\Gamma \left(\alpha\, \mathbf{curl}_\Gamma\, \phi^-\right) - \operatorname{div}_\Gamma \left(\alpha\, \nabla_\Gamma\, \psi^+\right) \text{ on } \Gamma_A. \tag{4.16}$$

This problem involves the standard Laplace-Beltrami operator on Γ_A. So, since Γ_A has no pathological vertex, there are no singular solutions (Lemma 3.3). Besides, the right-hand side of (4.16) is meaningful in $H^{s-1}(\Gamma_A)$, because

$\psi^+ \in H^{3/2}(\Gamma_A)$ and we know that $\phi^- \in H^{1+s}(\Gamma_A)$. We conclude that $\psi^- \in H^{1+s}(\Gamma_A)$, and $\pi_A^T \boldsymbol{C} \in \mathbf{H}_t^s(\Gamma_A)$, with estimates obtained as in Theorem 4.1 (case without pathological vertex).

On the other hand, if there are singular solutions to (4.6), then $\phi^- \in H^1(\Gamma_A) \oplus \mathcal{S}_1$, where $\mathcal{S}_1$ is the space of solutions of (4.6) in $H^{1/2}(\Gamma_A) \setminus H^1(\Gamma_A) + \{0\}$. Moreover, this corresponds also to the space of solutions in $H^{1/2}(\Gamma_A) \setminus H^{1+s}(\Gamma_A) + \{0\}$, for all $s > 0$ smaller than the lowest (strictly) positive value of $\Re(\lambda)$ where λ solves (4.7). Therefore, $\phi^- \in H^{1+s}(\Gamma_A) \oplus \mathcal{S}_1$, where $\mathcal{S}_1$ is the finite-dimensional subspace of singularities. To deal with ψ^-, we multiply (3.1) by α^{-1} (α being elliptic, so is α^{-1}) and take its $\operatorname{div}_\Gamma$, which gives

$$\operatorname{div}_\Gamma(\alpha^{-1}\nabla\psi^-) = \operatorname{div}_\Gamma(\alpha^{-1}\mathbf{g}) - \operatorname{div}_\Gamma\left(\alpha^{-1}\,\mathbf{curl}_\Gamma\,\phi^+\right) - \Delta_\Gamma\,\psi^+ \text{ on } \Gamma_A. \tag{4.17}$$

One has then to study the singularities of operator $\operatorname{div}_\Gamma(\alpha^{-1}\,\nabla_\Gamma\,\cdot)$ on Γ_A, namely the solutions to

$$\text{Find } \psi^{\text{sing}} \in H^{1/2}_{\text{zmv}}(\Gamma_A) \text{ s.t. } \operatorname{div}_\Gamma(\alpha^{-1}\nabla\psi^{\text{sing}}) = 0 \text{ on } \Gamma_A. \tag{4.18}$$

This is done as previously, in a neighborhood of the coefficient vertices. One considers now the eigenproblem

$$\left|\begin{array}{l}\text{Find } (\lambda,\varphi) \in \mathbb{C} \times H^1_{\text{per}}([0,\theta_v]) \text{ s.t. } ,\forall\psi \in H^1_{\text{per}}([0,\theta_v]),\\ \qquad\displaystyle\int_0^{\theta_v} \alpha^{-1}\varphi'\bar{\psi}'\mathrm{d}\theta - \lambda^2 \int_0^{\theta_v} \alpha^{-1}\varphi\bar{\psi}\mathrm{d}\theta = 0.\end{array}\right. \tag{4.19}$$

One concludes that $\psi^- \in H^{1+s}(\Gamma_A) \oplus \mathcal{S}_2$, where $\mathcal{S}_2$ is the finite-dimensional vector space of those singularities that solve (4.18). ∎

Remark 4.8. The proof can be adapted to more complex geometries. In particular, in the case where Γ_A has pathological vertices, one has to take in consideration both types of singularities: those arising from the pathological geometrical vertices (Lemma 3.3); and those arising from the pathological coefficient vertices (Lemma 4.3). In principle, it can also be adapted to a boundary Γ_A of class $\mathcal{C}^2$, again with the idea of using [18].

4.3. The case of a piecewise smooth tensor-valued impedance coefficient. We consider briefly the more general boundary condition

$$\pi_A^T \boldsymbol{C} + \underline{\boldsymbol{\alpha}}\gamma_A^T \boldsymbol{E} = \mathbf{g} \quad \text{ on } \Gamma_A, \tag{4.20}$$

where the impedance coefficient is tensor-valued. Precisely, we assume that $\underline{\boldsymbol{\alpha}} \in \underline{\mathbf{L}}_t^\infty(\Gamma_A)$ is piecewise smooth, and is elliptic in the sense of definition 2.2. Again, a legitimate goal is to determine whether this boundary condition may be meaningful in $\mathbf{L}_t^2(\Gamma_A)$. Proceeding as in the previous subsection, we have the

PROPOSITION 4.9. *Let Γ_A of class $\mathcal{C}^2$, or polyhedral without pathological vertex. Let $\gamma_A^T \boldsymbol{E}$ and $\pi_A^T \boldsymbol{C}$ be governed by (4.20), with $\mathbf{g} \in \mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma, \Gamma_A) \cap \mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma, \Gamma_A)$. Given a piecewise smooth and elliptic $\underline{\boldsymbol{\alpha}} \in \underline{\mathbf{L}}_t^\infty(\Gamma_A)$, if $\underline{\boldsymbol{\alpha}}$ is such that the problem*

$$\text{Find } \phi^{\text{sing}} \in H^{1/2}_{\text{zmv}}(\Gamma_A) \text{ s.t. } \operatorname{curl}_\Gamma(\underline{\boldsymbol{\alpha}}\,\mathbf{curl}_\Gamma\,\phi) = 0 \text{ on } \Gamma_A$$

has no solution in $H^{1/2}(\Gamma_A) \setminus H^1(\Gamma_A)$, then

$$\gamma_A^T \boldsymbol{E} \in \mathbf{L}_t^2(\Gamma_A) \quad \text{and} \quad \pi_A^T \boldsymbol{C} \in \mathbf{L}_t^2(\Gamma_A).$$

Remark 4.10. Once again, one would have to study the singularities of the operator $\operatorname{curl}_\Gamma(\underline{\boldsymbol{\alpha}}\,\mathbf{curl}_\Gamma\,\cdot)$ on Γ_A in a given configuration to get a more precise result, and in particular an increased regularity result. Because one is dealing with a PDE involving a 2-by-2 tensor valued coefficient, singularities may occur now not only at a coefficient vertex, but also at a coefficient edge.

We extend the notion of absence of pathological coefficient vertices on Γ_A via a slight twist of definition 4.6.

DEFINITION 4.11. *Given a piecewise smooth and elliptic $\underline{\boldsymbol{\alpha}} \in \underline{\mathbf{L}}_t^\infty(\Gamma_A)$, if the results of propositions 4.9 (absence of singular solutions) and 4.4 (increased regularity) hold, we say that the boundary Γ_A is* without pathological coefficient vertices.

To conclude the study of the boundary condition, and of the traces, on Γ_A, we define a suitable framework to be able to carry on the analysis of the time-harmonic Maxwell equations.

Assumption 2. *The boundary Γ_A:*
- *is of class $\mathcal{C}^2$, or is polyhedral without pathological geometrical vertex;*
- *is without pathological coefficient vertices in the sense of Definition 4.6 (scalar-valued impedance coefficient) or of Definition 4.11 (tensor-valued impedance coefficient).*

Under Assumption 2, one has the same conclusions as in Theorem 4.7.

Remark 4.12. In practice, because Γ_A is an artificial boundary, it is legitimate to choose a boundary of class $\mathcal{C}^2$, or polyhedral without (semi-)pathological vertices. We already observed that, if Γ_A is the polyhedral boundary of a convex domain, it has no pathological vertices. On the other hand, as observed previously, the absence of pathological coefficient vertices is a more delicate issue.

5. Well-posedness and regularity.

Now, let us come back to the Maxwell problem expressed in terms of the electric field only:

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}) - \omega^2\underline{\boldsymbol{\varepsilon}}\boldsymbol{E} = \boldsymbol{f} & \text{in } \Omega, \\ \pi_A^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}) + \underline{\boldsymbol{\alpha}}\gamma_A^T\boldsymbol{E} = \mathbf{g} & \text{on } \Gamma_A, \\ \tilde{\gamma}^T\boldsymbol{E} = 0 & \text{on } \tilde{\Gamma}, \end{cases} \tag{5.1}$$

with data $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A)$. To study problem (5.1), we complement Assumption 1.

Assumption 1bis. *In addition to Assumption 1, the coefficients $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\alpha}}$ are* simultaneously elliptic*, ie. $\Theta_{\mu^{-1}} \cap \Theta_{-\alpha} \neq \emptyset$. In other words,*

$$\exists\theta \in \mathbb{R}, \left\{ \begin{array}{ll} \exists\mu_-^{\mathrm{inv}} > 0, \text{ a.e. in } \Omega,\ \forall\mathbf{z} \in \mathbb{C}^3, & \mu_-^{\mathrm{inv}}|\mathbf{z}|^2 \leq \Re[e^{i\theta}\cdot\mathbf{z}^*\underline{\boldsymbol{\mu}}^{-1}\mathbf{z}] \\ \exists\alpha_- > 0, \text{ a.e. in } \Gamma_A,\ \forall\mathbf{y} \in \mathbb{C}^2, & \alpha_-|\mathbf{y}|^2 \leq \Re[e^{i\theta}\cdot\mathbf{y}^*(-\underline{\boldsymbol{\alpha}})\mathbf{y}] \end{array} \right. . \tag{5.2}$$

5.1. Well-posedness.

Thanks to Assumption 2, we know that the boundary condition on Γ_A implies that both traces $\pi_A^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E})$ and $\gamma_A^T\boldsymbol{E}$ belong to $\mathbf{L}_t^2(\Gamma_A)$. This observation allows us to look for $\boldsymbol{E}$ in the function space $\boldsymbol{H}^+(\mathbf{curl}, \Omega)$, introduced in (3.2). In fact, $\boldsymbol{H}^+(\mathbf{curl}, \Omega)$ is the appropriate space to deal with the problem (5.1), just as $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$ is for the purely Dirichlet problem or as $\boldsymbol{H}(\mathbf{curl}, \Omega)$ is for the purely Neumann problem.

Then, let us derive the variational formulation of (5.1). To that aim, we define the sesquilinear form on $\boldsymbol{H}^+(\mathbf{curl}, \Omega) \times \boldsymbol{H}^+(\mathbf{curl}, \Omega)$:

$$a(\boldsymbol{u}, \boldsymbol{v}) := \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} - \omega^2\left(\underline{\boldsymbol{\varepsilon}}\boldsymbol{u}, \boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} - \left(\underline{\boldsymbol{\alpha}}\gamma_A^T\boldsymbol{u}, \gamma_A^T\boldsymbol{v}\right)_{\mathbf{L}_t^2(\Gamma_A)}. \tag{5.3}$$

We also introduce ℓ_{R} the antilinear continuous form on $\boldsymbol{H}^+(\mathbf{curl}, \Omega)$ defined by the right-hand side,

$$\ell_{\mathrm{R}} : \boldsymbol{F} \mapsto (\boldsymbol{f}, \boldsymbol{F})_{\boldsymbol{L}^2(\Omega)} - \left(\mathbf{g}, \gamma_A^T\boldsymbol{F}\right)_{\mathbf{L}_t^2(\Gamma_A)}, \tag{5.4}$$

with

$$\|\ell_{\mathrm{R}}\|_{(\boldsymbol{H}^+(\mathbf{curl},\Omega))'} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{g}\|_{\mathbf{L}_t^2(\Gamma_A)}. \tag{5.5}$$

THEOREM 5.1. *Under Assumption 2, the problem* (5.1) *is equivalent to the variational formulation*

$$\textit{Find } \boldsymbol{E} \in \boldsymbol{H}^+(\mathbf{curl}, \Omega) \textit{ s.t., } \forall\boldsymbol{F} \in \boldsymbol{H}^+(\mathbf{curl}, \Omega), \quad a(\boldsymbol{E}, \boldsymbol{F}) = \ell_R(\boldsymbol{F}). \tag{5.6}$$

Proof. Direct. Let us multiply the volume equation of (5.1) by a test function $\boldsymbol{F} \in \boldsymbol{H}^+(\mathbf{curl}, \Omega)$ and integrate by parts:

$$\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{F}\right)_{\boldsymbol{L}^2(\Omega)} - \omega^2\left(\underline{\boldsymbol{\varepsilon}}\boldsymbol{E}, \boldsymbol{F}\right)_{\boldsymbol{L}^2(\Omega)} + {}_{\pi}\langle\pi_A^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}), \gamma_A^T\boldsymbol{F}\rangle_{\gamma} = (\boldsymbol{f}, \boldsymbol{F})_{\boldsymbol{L}^2(\Omega)} \tag{5.7}$$

Using the boundary condition on Γ_A, and the fact that it holds in $\mathbf{L}^2_t(\Gamma_A)$, one gets that

$$
\begin{aligned}
{}_\pi\langle \pi_A^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}), \gamma_A^T\boldsymbol{F}\rangle_\gamma &= {}_\pi\langle \mathbf{g} - \underline{\boldsymbol{\alpha}}\gamma_A^T\boldsymbol{E}, \gamma_A^T\boldsymbol{F}\rangle_\gamma \\
&= \left(\mathbf{g} - \underline{\boldsymbol{\alpha}}\gamma_A^T\boldsymbol{E}, \gamma_A^T\boldsymbol{F}\right)_{\mathbf{L}^2_t(\Gamma_A)}
\end{aligned}
$$

and then (5.6) holds.

Reverse. Taking $\boldsymbol{F} \in \boldsymbol{\mathcal{D}}(\Omega)$ in (5.6) and differentiating in $\boldsymbol{\mathcal{D}}'(\Omega)$, we find $\mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}) - \omega^2\underline{\boldsymbol{\varepsilon}}\boldsymbol{E} = \boldsymbol{f}$ in $\boldsymbol{\mathcal{D}}'(\Omega)$, hence in $\mathbf{L}^2(\Omega)$. Once again, we multiply by a test function $\boldsymbol{F} \in \boldsymbol{H}^+(\mathbf{curl},\Omega)$ and integrate by parts to recover (5.7). Subtracting it to (5.6), we get

$$
{}_\pi\langle \pi_A^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}), \gamma_A^T\boldsymbol{F}\rangle_\gamma - \left(\mathbf{g}, \gamma_A^T\boldsymbol{F}\right)_{\mathbf{L}^2_t(\Gamma_A)} = -\left(\underline{\boldsymbol{\alpha}}\gamma_A^T\boldsymbol{E}, \gamma_A^T\boldsymbol{F}\right)_{\mathbf{L}^2_t(\Gamma_A)},
$$

and, because $\mathbf{g} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A)$,

$$
{}_\pi\langle \pi_A^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}), \gamma_A^T\boldsymbol{F}\rangle_\gamma - {}_\pi\langle \mathbf{g}, \gamma_A^T\boldsymbol{F}\rangle_\gamma = -\left(\underline{\boldsymbol{\alpha}}\gamma_A^T\boldsymbol{E}, \gamma_A^T\boldsymbol{F}\right)_{\mathbf{L}^2_t(\Gamma_A)}.
$$

Moreover, this holds for all $\boldsymbol{F} \in \boldsymbol{\mathcal{C}}^\infty(\overline{\Omega}) \subset \boldsymbol{H}^+(\mathbf{curl},\Omega)$. As $\boldsymbol{\mathcal{C}}^\infty(\overline{\Omega})$ is dense in $\boldsymbol{H}(\mathbf{curl},\Omega)$, and γ_A^T is surjective from $\boldsymbol{H}(\mathbf{curl},\Omega)$ to $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A)$, we have that $\gamma_A^T(\boldsymbol{\mathcal{C}}^\infty(\overline{\Omega}))$ is dense in $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A)$. Therefore, we conclude by density that

$$
\pi_A^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}) - \mathbf{g} = -\underline{\boldsymbol{\alpha}}\gamma_A^T\boldsymbol{E}
$$

holds in $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A) = (\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A))'$. ∎

The well-posedness analysis of the variational formulation (5.6) relies on two main ingredients. The first one is a Helmholtz decomposition of the space $\boldsymbol{H}^+(\mathbf{curl},\Omega)$, which can be obtained thanks to the ellipticity condition on $\underline{\boldsymbol{\varepsilon}}$.

Lemma 5.2. *Under Assumption 1 (on $\underline{\boldsymbol{\varepsilon}}$), one has the direct decomposition:*

$$
\boldsymbol{H}^+(\mathbf{curl},\Omega) = \nabla H_0^1(\Omega) \oplus \mathbf{W}_N(\underline{\boldsymbol{\varepsilon}},\Omega), \tag{5.8}
$$

where

$$
\mathbf{W}_N(\underline{\boldsymbol{\varepsilon}},\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{H}^+(\mathbf{curl},\Omega),\ \mathrm{div}\,\underline{\boldsymbol{\varepsilon}}\boldsymbol{v} = 0\right\}. \tag{5.9}
$$

In addition, the embedding of $\mathbf{W}_N(\underline{\boldsymbol{\varepsilon}},\Omega)$ into $\boldsymbol{L}^2(\Omega)$ is compact.

The detailed proof of those results can be found in [12] (Theorems 3.2.4 and 3.3.5). We are now in position to analyze the problem (5.1) and its associated variational formulation (5.6).

Theorem 5.3. *Under Assumption 2, the formulation* (5.6) *can be equivalently recast as follows: set $\boldsymbol{E} = \nabla p + \tilde{\boldsymbol{E}}$, with $p \in H_0^1(\Omega)$ and $\tilde{\boldsymbol{E}} \in \mathbf{W}_N(\underline{\boldsymbol{\varepsilon}};\Omega)$, respectively governed by*

$$
\text{Find } p \in H_0^1(\Omega) \text{ s.t., } \forall q \in H_0^1(\Omega), \quad -\omega^2\left(\underline{\boldsymbol{\varepsilon}}\nabla p, \nabla q\right) = (\boldsymbol{f}, \nabla q) \tag{5.10}
$$

and

$$
\text{Find } \tilde{\boldsymbol{E}} \in \mathbf{W}_N(\underline{\boldsymbol{\varepsilon}};\Omega) \text{ s.t., } \forall \tilde{\boldsymbol{F}} \in \mathbf{W}_N(\underline{\boldsymbol{\varepsilon}};\Omega), \quad a(\tilde{\boldsymbol{E}},\tilde{\boldsymbol{F}}) = \left(\boldsymbol{f},\tilde{\boldsymbol{F}}\right)_{\boldsymbol{L}^2(\Omega)} - \left(\mathbf{g},\gamma_A^T\tilde{\boldsymbol{F}}\right)_{\mathbf{L}^2_t(\Gamma_A)} + \omega^2\left(\underline{\boldsymbol{\varepsilon}}\nabla p,\tilde{\boldsymbol{F}}\right)_{\boldsymbol{L}^2(\Omega)}. \tag{5.11}
$$

Furthermore, under Assumptions 1bis and 2, the problem (5.1) and its associated variational formulation (5.6) *satisfy a Fredholm alternative, and*

- *either the problem admits a unique solution $\boldsymbol{E}$ in $\boldsymbol{H}(\mathbf{curl},\Omega)$, which depends continuously on the data $\boldsymbol{f}$ and $\mathbf{g}$:*

$$
\|\boldsymbol{E}\|_{\boldsymbol{H}^+(\mathbf{curl},\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{g}\|_{\mathbf{L}^2_t(\Gamma_A)}; \tag{5.12}
$$

- *or, the problem has solutions if, and only if, $\boldsymbol{f}$ and $\mathbf{g}$ satisfy a finite number of compatibility conditions; in this case, the space of solutions is an affine space of finite dimension. Additionally, the component of the solution which is orthogonal (in the sense of the $\boldsymbol{H}^+(\mathbf{curl},\Omega)$ inner product) to the corresponding linear vector space, depends continuously on the data $\boldsymbol{f}$ and $\mathbf{g}$.*

Proof. We begin by proving that (5.6) is equivalent to (5.10)-(5.11) under Assumption 2.
Direct. Taking $\boldsymbol{F} = \nabla q$ for any $q \in H_0^1(\Omega)$ in (5.6) yields $-\omega^2\left(\underline{\boldsymbol{\varepsilon}}(\nabla p + \tilde{\boldsymbol{E}}), \nabla q\right) = (\boldsymbol{f}, \nabla q)$. Because $\operatorname{div}\underline{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{E}} = 0$, one gets (5.10). Then, for $\tilde{\boldsymbol{E}} = \boldsymbol{E} - \nabla p \in \mathbf{W}_N(\underline{\varepsilon}, \Omega)$, there holds

$$\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\tilde{\boldsymbol{E}}, \mathbf{curl}\,\boldsymbol{F}\right)_{\boldsymbol{L}^2(\Omega)} - \omega^2\left(\underline{\boldsymbol{\varepsilon}}(\nabla p + \tilde{\boldsymbol{E}}), \boldsymbol{F}\right)_{\boldsymbol{L}^2(\Omega)} - \left(\underline{\boldsymbol{\alpha}}\gamma_A^T\tilde{\boldsymbol{E}}, \gamma_A^T\boldsymbol{F}\right)_{\mathbf{L}_t^2(\Gamma_A)} = (\boldsymbol{f}, \boldsymbol{F})_{\boldsymbol{L}^2(\Omega)} - \left(\mathbf{g}, \gamma_A^T\boldsymbol{F}\right)_{\mathbf{L}_t^2(\Gamma_A)}$$

for any $\boldsymbol{F} \in \boldsymbol{H}^+(\mathbf{curl},\Omega)$, hence in particular for any $\tilde{\boldsymbol{F}} \in \mathbf{W}_N(\underline{\varepsilon};\Omega)$.

Reverse. We sum (5.10) and (5.11), and pose $\boldsymbol{E} = \nabla p + \tilde{\boldsymbol{E}} \in \boldsymbol{H}^+(\mathbf{curl},\Omega)$. Adding null terms, one gets easily

$$a(\boldsymbol{E}, \nabla q + \tilde{\boldsymbol{F}}) = \ell_R(\nabla q + \tilde{\boldsymbol{F}}).$$

As q and $\tilde{\boldsymbol{F}}$ span respectively $H_0^1(\Omega)$ and $\mathbf{W}_N(\underline{\varepsilon};\Omega)$, the sum $\nabla q + \tilde{\boldsymbol{F}}$ spans the whole space $\boldsymbol{H}^+(\mathbf{curl},\Omega)$, according to the direct decomposition (5.8). Hence (5.6) holds. This ends the first part of the proof.

We now study successively the formulations (5.10) and (5.11), under Assumptions 1bis and 2.

1. The formulation (5.10) is clearly well-posed by Lax-Milgram lemma, because $\underline{\boldsymbol{\varepsilon}}$ is elliptic.
2. For the formulation (5.11), we recall that $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\alpha}}$ are simultaneously elliptic. Using the notation $\theta, \mu_-^{\mathrm{inv}}, \alpha_-$ as in (5.2), we introduce the sesquilinear form defined on $\mathbf{W}_N(\underline{\varepsilon};\Omega)$

$$a_{coer}(\boldsymbol{u}, \boldsymbol{v}) := \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} + e^{-i\theta}\left(\boldsymbol{u}, \boldsymbol{v}\right)_{\boldsymbol{L}^2(\Omega)} - \left(\underline{\boldsymbol{\alpha}}\gamma_A^T\boldsymbol{u}, \gamma_A^T\boldsymbol{v}\right)_{\mathbf{L}_t^2(\Gamma_A)},$$

which is continuous and coercive on $\mathbf{W}_N(\underline{\varepsilon};\Omega)$. Clearly, $|a_{coer}(\boldsymbol{u},\boldsymbol{v})| \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}^+(\mathbf{curl},\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^+(\mathbf{curl},\Omega)}$, and one has (see proposition 2.4)

$$\begin{aligned} |a_{coer}(\boldsymbol{v},\boldsymbol{v})| &\geq \Re[e^{i\theta}a(\boldsymbol{v},\boldsymbol{v})] \\ &\geq \mu_-^{\mathrm{inv}}\|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \alpha_-\|\gamma_A^T\boldsymbol{v}\|^2_{\mathbf{L}_t^2(\Gamma_A)}, \\ &\geq \min(\mu_-^{\mathrm{inv}}, 1, \alpha_-)\,\|\boldsymbol{v}\|^2_{\boldsymbol{H}^+(\mathbf{curl},\Omega)}, \end{aligned}$$

with $\mathbf{W}_N(\underline{\varepsilon};\Omega)$ and $\boldsymbol{H}^+(\mathbf{curl},\Omega)$ sharing the same norm. Then, we observe that the complementary form

$$a_{comp}(\boldsymbol{u},\boldsymbol{v}) := -\omega^2(\underline{\boldsymbol{\varepsilon}}\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + e^{-i\theta}(\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}, \tag{5.13}$$

it is continuous on $\boldsymbol{L}^2(\Omega) \times \mathbf{W}_N(\underline{\varepsilon};\Omega)$. Moreover, the embedding of $\mathbf{W}_N(\underline{\varepsilon};\Omega)$ into $\boldsymbol{L}^2(\Omega)$ is compact according to Lemma 5.2. Hence the formulation (5.11) enters the coercive + compact framework, and the conclusions of Fredholm alternative apply to problem (5.11).

Grouping both results, one has the result for problem (5.6). ∎

Remark 5.4. Note that although one requires a compatibilty condition between $\underline{\boldsymbol{\mu}}$ and $-\underline{\boldsymbol{\alpha}}$, no condition on $\underline{\boldsymbol{\varepsilon}}$ is required (this is Assumption 1bis). Above, we proposed a proof that is valid for scalar-valued, or tensor-valued, impedance coefficient $\underline{\boldsymbol{\alpha}}$, provided it allows the Robin boundary condition to hold in $\mathbf{L}_t^2(\Gamma_A)$ (this is Assumption 2).

In what preceeds, we have assumed interplay only between $\underline{\boldsymbol{\mu}}$ and $\underline{\boldsymbol{\alpha}}$, in order to get the Fredholm character of the problem. One can go further if one assumes interplay between all three coefficients $\underline{\boldsymbol{\varepsilon}}$, $\underline{\boldsymbol{\mu}}$ and $\underline{\boldsymbol{\alpha}}$. If $\underline{\boldsymbol{\mu}}^{-1}$, $-\underline{\boldsymbol{\varepsilon}}$ and $-\underline{\boldsymbol{\alpha}}$ are simultaneously elliptic, i.e. if $\Theta_{\mu^{-1}} \cap \Theta_{-\varepsilon} \cap \Theta_{-\alpha} \neq \emptyset$, then

$$\exists\theta \in \mathbb{R}, \left\{\begin{array}{lll} \exists\mu_-^{\mathrm{inv}} > 0, & \text{a.e. in } \Omega,\ \forall\mathbf{z} \in \mathbb{C}^3, & \mu_-^{\mathrm{inv}}|\mathbf{z}|^2 \leq \Re[e^{i\theta}\cdot\mathbf{z}^*\underline{\boldsymbol{\mu}}^{-1}\mathbf{z}] \\ \exists\varepsilon_- > 0, & \text{a.e. in } \Omega,\ \forall\mathbf{z} \in \mathbb{C}^3, & \varepsilon_-|\mathbf{z}|^2 \leq \Re[e^{i\theta}\cdot\mathbf{z}^*(-\underline{\boldsymbol{\varepsilon}})\mathbf{z}] \\ \exists\alpha_- > 0, & \text{a.e. in } \Gamma_A,\ \forall\mathbf{y} \in \mathbb{C}^2, & \alpha_-|\mathbf{y}|^2 \leq \Re[e^{i\theta}\cdot\mathbf{y}^*(-\underline{\boldsymbol{\alpha}})\mathbf{y}] \end{array}\right. . \tag{5.14}$$

As a consequence, the sesquilinear form (5.3) is coercive. Indeed,

$$\begin{aligned}\Re\left[e^{i\theta}\left(a(\boldsymbol{v},\boldsymbol{v})\right)\right] &\geq \mu_-^{\mathrm{inv}}\|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \omega^2\varepsilon_-\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \alpha_-\|\gamma_A^T\boldsymbol{v}\|^2_{\mathbf{L}^2_t(\Gamma_A)} \\ &\geq \min(\mu_-^{\mathrm{inv}},\,\omega^2\varepsilon_-,\,\alpha_-)\|\boldsymbol{v}\|^2_{\boldsymbol{H}^+(\mathbf{curl},\Omega)}.\end{aligned}$$

THEOREM 5.5. *Under Assumption 2, if* $\underline{\boldsymbol{\mu}}^{-1}$*,* $-\underline{\boldsymbol{\varepsilon}}$ *and* $-\underline{\boldsymbol{\alpha}}$ *are simultaneously elliptic, i.e.* $\Theta_{\mu^{-1}}\cap\Theta_{-\varepsilon}\cap\Theta_{-\alpha}\neq\emptyset$*, then there exists a unique solution* $\boldsymbol{E}\in\boldsymbol{H}^+(\mathbf{curl},\Omega)$ *to the problem* (5.1)*, with moreover*

$$\|\boldsymbol{E}\|_{\boldsymbol{H}^+(\mathbf{curl},\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{g}\|_{\mathbf{L}^2_t(\Gamma_A)}. \tag{5.15}$$

5.2. Solving the magnetic problem. Alternatively, one could choose to solve the second-order time-harmonic magnetic problem, that is, by eliminating $\boldsymbol{E}$ instead of $\boldsymbol{H}$ in (2.2)-(2.5). Here, we specifically assume that the impedance coefficient is scalar-valued. When it is normal-tensor-valued, these computations can be extended, whereas in the most general case they are a bit tedious. First, the second-order PDE reads

$$\mathbf{curl}\,\underline{\boldsymbol{\varepsilon}}^{-1}\,\mathbf{curl}\,\boldsymbol{H} - \omega^2\underline{\boldsymbol{\mu}}\boldsymbol{H} = \tilde{\boldsymbol{f}}, \tag{5.16}$$

with $\tilde{\boldsymbol{f}} = \mathbf{curl}\,\underline{\boldsymbol{\varepsilon}}^{-1}\boldsymbol{J}$. In other words, the roles of $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ are permuted compared to the electric problem. Then, on $\tilde{\Gamma}$, the Dirichlet boundary condition for the electric field becomes a Neumann condition for the magnetic field. Considering finally the Robin condition, expressing it in terms of the magnetic field, one gets

$$\pi_A^T(\underline{\boldsymbol{\varepsilon}}^{-1}\,\mathbf{curl}\,\boldsymbol{H}) - \alpha^{-1}\omega^2\gamma_A^T\boldsymbol{H} = \tilde{\mathbf{g}} \text{ on } \Gamma_A, \tag{5.17}$$

where $\tilde{\mathbf{g}} = \pi^T(\underline{\boldsymbol{\varepsilon}}^{-1}\boldsymbol{J}) + i\omega\alpha^{-1}(\mathbf{g}\times\mathbf{n})$. To sum up, the problem to be solved, expressed in term of a second-order problem in the magnetic field $\boldsymbol{H}$, is the following:

$$\begin{cases} \mathbf{curl}\,\underline{\boldsymbol{\varepsilon}}^{-1}\,\mathbf{curl}\,\boldsymbol{H} - \omega^2\underline{\boldsymbol{\mu}}\boldsymbol{H} = \tilde{\boldsymbol{f}} & \text{in } \Omega, \\ \pi_A^T(\underline{\boldsymbol{\varepsilon}}^{-1}\,\mathbf{curl}\,\boldsymbol{H}) - \alpha^{-1}\omega^2\gamma_A^T\boldsymbol{H} = \tilde{\mathbf{g}} & \text{on } \Gamma_A, \\ \tilde{\pi}^T(\underline{\boldsymbol{\varepsilon}}^{-1}\,\mathbf{curl}\,\boldsymbol{H}) = \tilde{\pi}^T(\underline{\boldsymbol{\varepsilon}}^{-1}\boldsymbol{J}) & \text{on } \tilde{\Gamma}. \end{cases} \tag{5.18}$$

Under Assumption 2, the equivalent variational formulation of the problem (5.18) reads:

(5.19)

$$\left|\begin{array}{l}\text{Find } \boldsymbol{H}\in\boldsymbol{H}^+(\mathbf{curl},\Omega) \text{ s.t., } \forall\boldsymbol{F}\in\boldsymbol{H}^+(\mathbf{curl},\Omega), \\ \left(\underline{\boldsymbol{\varepsilon}}^{-1}\,\mathbf{curl}\,\boldsymbol{H},\mathbf{curl}\,\boldsymbol{F}\right)_{\boldsymbol{L}^2(\Omega)} - \omega^2\left(\underline{\boldsymbol{\mu}}\boldsymbol{H},\boldsymbol{F}\right)_{\boldsymbol{L}^2(\Omega)} + \left(\omega^2\alpha^{-1}\gamma_A^T\boldsymbol{H},\gamma_A^T\boldsymbol{F}\right)_{\mathbf{L}^2_t(\Gamma_A)} = \left(\tilde{\boldsymbol{f}},\boldsymbol{F}\right)_{\boldsymbol{L}^2(\Omega)} - \left(\tilde{\mathbf{g}},\gamma_A^T\boldsymbol{F}\right)_{\mathbf{L}^2_t(\Gamma_A)}.\end{array}\right.$$

Comparing the electric problem to the magnetic problem, we make the following observations.

- On the one hand, to ensure the Fredholm character of problem (5.19), one needs apparently a different condition than for the electric problem (5.6), although both problems are supposed to be equivalent. In particular, the coefficients involved are not the same: here, one requires simultaneous ellipticity between $\underline{\boldsymbol{\varepsilon}}^{-1}$ and α^{-1}, independently of $\underline{\boldsymbol{\mu}}$, while for the electric problem we required simultaneous ellipticity between $\underline{\boldsymbol{\mu}}^{-1}$ and $-\alpha$, independently of $\underline{\boldsymbol{\varepsilon}}$ (this is Assumption 1bis).
- On the other hand, in order to get the coercivity of the magnetic problem (5.19), one has to assume interplay between all three coefficients: namely, that $\underline{\boldsymbol{\varepsilon}}^{-1}$, $-\underline{\boldsymbol{\mu}}$ and α^{-1} are simultaneously elliptic (ie. $\Theta_{\varepsilon^{-1}}\cap\Theta_{-\mu}\cap\Theta_{\alpha^{-1}}\neq\emptyset$). Whereas, for the electric problem, we asked that $\underline{\boldsymbol{\mu}}^{-1}$, $-\underline{\boldsymbol{\varepsilon}}$ and $-\alpha$ are simultaneously elliptic (ie. $\Theta_{\mu^{-1}}\cap\Theta_{-\varepsilon}\cap\Theta_{-\alpha}\neq\emptyset$). Now, we observe that, to go from the electric to the magnetic problem boils down to change the triplet of coefficients $(\underline{\boldsymbol{\mu}}^{-1},-\underline{\boldsymbol{\varepsilon}},-\alpha)$ into $(\underline{\boldsymbol{\varepsilon}}^{-1},-\underline{\boldsymbol{\mu}},\alpha^{-1})$, that is, all coefficients are changed by the transformation $\xi\mapsto-\xi^{-1}$. Hence, we find that both conditions $\Theta_{\mu^{-1}}\cap\Theta_{-\varepsilon}\cap\Theta_{-\alpha}\neq\emptyset$ and $\Theta_{\varepsilon^{-1}}\cap\Theta_{-\mu}\cap\Theta_{\alpha^{-1}}\neq\emptyset$ are equivalent. Therefore, we conclude both electric and magnetic problems are coercive under *the same conditions on the coefficients*.

5.3. Regularity. We come back to the electric problem (5.1). Let us state a generic Shift Theorem for scalar problems set in the domain Ω, with homogeneous Dirichlet or Neumann boundary condition:

$$\text{Find } p \in H^1_0(\Omega) \text{ such that } \forall q \in H^1_0(\Omega), \quad \left(\underline{\boldsymbol{\xi}}\nabla p, \nabla q\right)_{\boldsymbol{L}^2(\Omega)} = \ell(q)\,; \tag{5.20}$$

$$\text{Find } p' \in H^1_{\text{zmv}}(\Omega) \text{ such that } \forall q' \in H^1_{\text{zmv}}(\Omega), \quad \left(\underline{\boldsymbol{\xi}}\nabla p', \nabla q'\right)_{\boldsymbol{L}^2(\Omega)} = (\ell', q')_{L^2(\Omega)}. \tag{5.21}$$

THEOREM 5.6 (generic Shift Theorem). *Let Ω be a domain, and $\underline{\boldsymbol{\xi}} \in \underline{\mathbf{L}}^\infty(\Omega)$ be an elliptic tensor-valued field.*
Dirichlet boundary condition. *Let ℓ in $\left(H^1_0(\Omega)\right)'$, and let p be governed by (5.20): $\forall\sigma \in [1,2]\setminus\{\frac{3}{2}\}$, $\ell \in \left(H^{2-\sigma}_0(\Omega)\right)'$ implies $p \in H^\sigma(\Omega)$; additionally,*

$$\exists C_\sigma > 0 \text{ such that } \forall \ell \in \left(H^{2-\sigma}_0(\Omega)\right)', \ \|p\|_{H^\sigma(\Omega)} \le C_\sigma \|\ell\|_{\left(H^{2-\sigma}_0(\Omega)\right)'}.$$

Neumann boundary condition. *Let ℓ' in $L^2_{\text{zmv}}(\Omega)$, and let p' be governed by (5.21): $p' \in H^2(\Omega)$; additionally,*

$$\exists C' > 0 \text{ such that } \forall \ell' \in L^2_{\text{zmv}}(\Omega), \ \|p'\|_{H^2(\Omega)} \le C' \|\ell'\|_{L^2(\Omega)}.$$

Remark 5.7. Such a Shift Theorem is valid, in particular, when $\partial\Omega$ is of class $\mathcal{C}^2$, and if $\underline{\boldsymbol{\xi}} \in \underline{\boldsymbol{\mathcal{C}}}^1(\overline{\Omega})$ (we refer to [16, Theorem 3.4.5]).

Below, we recall that $\mathtt{r}_{\max} \in]0,1]$ is the limiting regularity exponent of the scalar problems (4.10) defined on Γ_A.

THEOREM 5.8. *Under Assumptions 1bis and 2, assume moreover that $\underline{\boldsymbol{\varepsilon}} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$ and Theorem 5.6 holds for $\underline{\boldsymbol{\xi}} = \underline{\boldsymbol{\varepsilon}}$ for the Dirichlet boundary condition, and $\underline{\boldsymbol{\mu}} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$ and Theorem 5.6 holds for $\underline{\boldsymbol{\xi}} = \underline{\boldsymbol{\mu}}$ for both Dirichlet and Neumann boundary conditions. Let $\mathtt{s} \in [0,1] \setminus \{\frac{1}{2}\}$ be given.*
Then for all $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ with $\operatorname{div} \boldsymbol{f} \in H^{\mathtt{s}-1}(\Omega)$, and for all $\mathbf{g} \in \mathbf{H}^{-1/2}_{\parallel}(\operatorname{div}_\Gamma, \Gamma_A) \cap \mathbf{H}^{-1/2}_{\perp}(\operatorname{curl}_\Gamma, \Gamma_A)$, one has

$$\forall \mathtt{r} \in [0, \min(\frac{1}{2}, \mathtt{r}_{\max})[, \qquad \boldsymbol{E} \in \boldsymbol{H}^{\min(\mathtt{r}+1/2,\mathtt{s})}(\Omega) \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{H}^{\mathtt{r}+1/2}(\Omega) \tag{5.22}$$

with the bounds

$$\|\boldsymbol{E}\|_{\boldsymbol{H}^{\min(\mathtt{r}+1/2,\mathtt{s})}(\Omega)} \lesssim \|\operatorname{div} \boldsymbol{f}\|_{H^{\mathtt{s}-1}(\Omega)} + \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|\gamma^T_A \boldsymbol{E}\|_{\mathbf{H}^{\mathtt{r}}_t(\Gamma_A)} \tag{5.23}$$

$$\|\mathbf{curl}\,\boldsymbol{E}\|_{\boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|\pi^T_A(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E})\|_{\mathbf{H}^{\mathtt{r}}_t(\Gamma_A)}. \tag{5.24}$$

Proof. Let $\mathtt{r} \in [0, \min(\frac{1}{2}, \mathtt{r}_{\max})[$. Under Assumption 2, there holds $\gamma^T_A \boldsymbol{E}, \pi^T_A(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}) \in \mathbf{H}^{\mathtt{r}}_t(\Gamma_A)$ thanks to theorem 4.7. Moreover, since $\tilde{\gamma}\boldsymbol{E} = 0$, $\gamma^T\boldsymbol{E}$ admits a lifting $\boldsymbol{E}_d \in \boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)$, s.t. $\gamma^T\boldsymbol{E} = \gamma^T\boldsymbol{E}_d$ on $\partial\Omega$, and $\|\boldsymbol{E}_d\|_{\boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)} \lesssim \|\gamma^T_A\boldsymbol{E}\|_{\mathbf{H}^{\mathtt{r}}_t(\Gamma_A)}$. Following the steps of Theorem 3.9 in [13], we introduce $\boldsymbol{E}_0 := \boldsymbol{E} - \boldsymbol{E}_d \in \boldsymbol{H}_0(\mathbf{curl},\Omega)$. Then, one uses a continuous splitting of $\boldsymbol{E}_0$ into regular and singular part, cf. Lemma 2.4 in [24] or Theorem 3.6.7 in [1]:

$$\boldsymbol{E}_0 = \boldsymbol{E}^{\text{reg}} + \nabla\phi_E \tag{5.25}$$

where $\boldsymbol{E}^{\text{reg}} \in \boldsymbol{H}^1(\Omega)$, $\phi_E \in H^1_0(\Omega)$, and

$$\|\boldsymbol{E}^{\text{reg}}\|_{\boldsymbol{H}^1(\Omega)} + \|\phi_E\|_{H^1_0(\Omega)} \lesssim \|\boldsymbol{E}_0\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}. \tag{5.26}$$

Since $\omega^2 \operatorname{div} \underline{\boldsymbol{\varepsilon}}\boldsymbol{E} = -\operatorname{div} \boldsymbol{f}$, one finds that the singular part ϕ_E is governed by a scalar Dirichlet problem:

$$\left|\begin{array}{l}\text{Find } \phi_E \in H^1_0(\Omega) \text{ s.t., } \forall q \in H^1_0(\Omega),\\ \omega^2\left(\underline{\boldsymbol{\varepsilon}}\nabla\phi_E, \nabla q\right)_{\boldsymbol{L}^2(\Omega)} = \langle \operatorname{div} \boldsymbol{f} + \omega^2 \operatorname{div} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{E}_d + \boldsymbol{E}^{\text{reg}}), q\rangle_{H^1_0(\Omega)}.\end{array}\right. \tag{5.27}$$

Because $\underline{\varepsilon} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$, on has $\underline{\varepsilon}(\boldsymbol{E}_d + \boldsymbol{E}^{\text{reg}}) \in \boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)$, so that $\operatorname{div}\underline{\varepsilon}(\boldsymbol{E}_d + \boldsymbol{E}^{\text{reg}}) \in H^{\mathtt{r}-1/2}(\Omega)$. Then, the extra-regularity of ϕ_E is given by Theorem 5.6 (Dirichlet boundary condition) with $\underline{\boldsymbol{\xi}} = \underline{\varepsilon}$. In other words, $\phi_E \in H^{\min(\mathtt{s}+1,\mathtt{r}+3/2)}(\Omega)$, with

$$\|\phi_E\|_{H^{\min(\mathtt{s}+1,\mathtt{r}+3/2)}(\Omega)} \lesssim \|\operatorname{div}\boldsymbol{f}\|_{H^{\mathtt{s}-1}(\Omega)} + \|\boldsymbol{E}_d\|_{\boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)} + \|\boldsymbol{E}_0\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}.$$

We conclude that $\boldsymbol{E} \in \boldsymbol{H}^{\min(\mathtt{s},\mathtt{r}+1/2)}(\Omega)$, with the bounds

$$\begin{aligned}
\|\boldsymbol{E}\|_{\boldsymbol{H}^{\min(\mathtt{s},\mathtt{r}+1/2)}(\Omega)} &\lesssim \|\boldsymbol{E}^{\text{reg}}\|_{\boldsymbol{H}^1(\Omega)} + \|\phi_E\|_{H^{\min(\mathtt{s}+1,\mathtt{r}+3/2)}(\Omega)} + \|\boldsymbol{E}_d\|_{\boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)} \\
&\lesssim \|\boldsymbol{E}_0\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|\operatorname{div}\boldsymbol{f}\|_{H^{\mathtt{s}-1}(\Omega)} + \|\boldsymbol{E}_d\|_{\boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)} \\
&\lesssim \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|\operatorname{div}\boldsymbol{f}\|_{H^{\mathtt{s}-1}(\Omega)} + \|\boldsymbol{E}_d\|_{\boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)} + \|\boldsymbol{E}_d\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \\
&\lesssim \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|\operatorname{div}\boldsymbol{f}\|_{H^{\mathtt{s}-1}(\Omega)} + \|\gamma_A^T\boldsymbol{E}\|_{\mathbf{H}_t^{\mathtt{r}}(\Gamma_A)} + \|\gamma_A^T\boldsymbol{E}\|_{\gamma_A} \\
&\lesssim \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|\operatorname{div}\boldsymbol{f}\|_{H^{\mathtt{s}-1}(\Omega)} + \|\gamma_A^T\boldsymbol{E}\|_{\mathbf{H}_t^{\mathtt{r}}(\Gamma_A)}.
\end{aligned}$$

For the curl of the solution, we study $\boldsymbol{C} = \underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E}$. By construction, $\boldsymbol{C}$ belongs to $\boldsymbol{H}(\mathbf{curl},\Omega)$, with $\|\boldsymbol{C}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}$, and $\operatorname{div}\underline{\boldsymbol{\mu}}\boldsymbol{C} = 0$ in Ω. In addition, one has

$$\begin{aligned}
&\pi_A^T\boldsymbol{C} \in \mathbf{H}_t^{\mathtt{r}}(\Gamma_A), \text{ or equivalently } \gamma_A^T\boldsymbol{C} \in \mathbf{H}_t^{\mathtt{r}}(\Gamma_A) \text{ since } \mathtt{r} < \frac{1}{2}\,; \\
&\underline{\boldsymbol{\mu}}\boldsymbol{C}\cdot\boldsymbol{n} = \mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{n} = \operatorname{div}_\Gamma(\tilde{\gamma}\boldsymbol{E}) = 0 \text{ on } \tilde{\Gamma}.
\end{aligned}$$

We now proceed by localization. Let $\chi \in \mathcal{C}^\infty(\overline{\Omega})$ such that $\chi = 1$ in a neighborhood of $\tilde{\Gamma}$, and $\chi = 0$ in a neighborhood of Γ_A. We introduce $\boldsymbol{C}_A = (1-\chi)\boldsymbol{C}$ and $\tilde{\boldsymbol{C}} = \chi\boldsymbol{C}$. Obviously, $\boldsymbol{C} = \boldsymbol{C}_A + \tilde{\boldsymbol{C}}$ in Ω. Let us study the two fields separately.

First, $\boldsymbol{C}_A \in \boldsymbol{H}(\mathbf{curl},\Omega)$, with the bound $\|\boldsymbol{C}_A\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \lesssim \|\boldsymbol{C}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}$, and

$$\gamma_A^T\boldsymbol{C}_A \in \mathbf{H}_t^{\mathtt{r}}(\Gamma_A) \text{ and } \tilde{\gamma}\boldsymbol{C}_A = 0,$$

where the boundary condition on $\tilde{\Gamma}$ is a consequence of the property $(1-\chi) = 0$ in a neighborhood of $\tilde{\Gamma}$. Also $\operatorname{div}\underline{\boldsymbol{\mu}}\boldsymbol{C}_A = \operatorname{div}((1-\chi)\,\mathbf{curl}\,\boldsymbol{E}) = -\nabla\chi\cdot\mathbf{curl}\,\boldsymbol{E} \in L^2(\Omega)$ with the bound $\|\operatorname{div}\underline{\boldsymbol{\mu}}\boldsymbol{C}_A\|_{L^2(\Omega)} \lesssim \|\mathbf{curl}\,\boldsymbol{E}\|_{\boldsymbol{L}^2(\Omega)}$. Hence, $\boldsymbol{C}_A$ exhibits properties that are identical to those of $\boldsymbol{E}$, with regularity exponent $\mathtt{s}_A$ on $\operatorname{div}\underline{\boldsymbol{\mu}}\boldsymbol{C}_A$ (defined by $\operatorname{div}\underline{\boldsymbol{\mu}}\boldsymbol{C}_A \in H^{\mathtt{s}_A-1}(\Omega)$) equal to 1. So, one can follow all steps of the proof on the regularity of $\boldsymbol{E}$. We introduce : first, a lifting $\boldsymbol{C}_{Ad} \in \boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)$, s.t. $\gamma^T\boldsymbol{C}_A = \gamma^T\boldsymbol{C}_{Ad}$ on $\partial\Omega$; then, $\boldsymbol{C}_{A0} := \boldsymbol{C}_A - \boldsymbol{C}_{Ad} \in \boldsymbol{H}_0(\mathbf{curl},\Omega)$; finally, a continuous splitting of $\boldsymbol{C}_{A0}$ into regular and singular parts. In the present case, one obtains the extra-regularity of the singular part through Theorem 5.6 (Dirichlet boundary condition), now with $\underline{\boldsymbol{\xi}} = \underline{\boldsymbol{\mu}}$. One concludes that $\boldsymbol{C}_A \in \boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)$, with the bound

$$\|\boldsymbol{C}_A\|_{\boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|\pi_A^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E})\|_{\mathbf{H}_t^{\mathtt{r}}(\Gamma_A)}.$$

Second, $\tilde{\boldsymbol{C}} \in \boldsymbol{H}(\mathbf{curl},\Omega)$, $\operatorname{div}\underline{\boldsymbol{\mu}}\tilde{\boldsymbol{C}} = \operatorname{div}(\chi\,\mathbf{curl}\,\boldsymbol{E}) = +\nabla\chi\cdot\mathbf{curl}\,\boldsymbol{E} \in L^2(\Omega)$ with the bounds $\|\tilde{\boldsymbol{C}}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \lesssim \|\boldsymbol{C}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}$ and $\|\operatorname{div}\underline{\boldsymbol{\mu}}\tilde{\boldsymbol{C}}\|_{L^2(\Omega)} \lesssim \|\mathbf{curl}\,\boldsymbol{E}\|_{\boldsymbol{L}^2(\Omega)}$. On the other hand,

$$\underline{\boldsymbol{\mu}}\tilde{\boldsymbol{C}}\cdot\boldsymbol{n} = \underline{\boldsymbol{\mu}}\boldsymbol{C}\cdot\boldsymbol{n} = 0 \text{ on } \tilde{\Gamma} \quad\text{and}\quad \underline{\boldsymbol{\mu}}\tilde{\boldsymbol{C}}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_A,$$

where the boundary condition on Γ_A is a consequence of the property $\chi = 0$ in a neighborhood of Γ_A. Hence, $\tilde{\boldsymbol{C}} \in \boldsymbol{H}(\mathbf{curl},\Omega) \cap \boldsymbol{H}_0(\operatorname{div},\Omega)$. One uses a continuous splitting of $\tilde{\boldsymbol{C}}$ into regular and singular part, cf. Theorem 3.6.7 in [1]:

$$\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{C}}^{\text{reg}} + \nabla\psi, \text{ where } \tilde{\boldsymbol{C}}^{\text{reg}} \in \boldsymbol{H}^1(\Omega) \text{ with } \tilde{\boldsymbol{C}}^{\text{reg}}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega,\ \psi \in H^1_{\text{zmv}}(\Omega). \tag{5.28}$$

Because $\underline{\boldsymbol{\mu}} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$, one has $\operatorname{div}\underline{\boldsymbol{\mu}}\tilde{\boldsymbol{C}}^{\text{reg}} \in L^2(\Omega)$ with the bound $\|\operatorname{div}\underline{\boldsymbol{\mu}}\tilde{\boldsymbol{C}}^{\text{reg}}\|_{L^2(\Omega)} \lesssim \|\tilde{\boldsymbol{C}}^{\text{reg}}\|_{\boldsymbol{H}^1(\Omega)}$. Hence, for all $q' \in H^1_{\text{zmv}}(\Omega)$, one may use integration by parts to derive

$$\big(\underline{\boldsymbol{\mu}}\nabla\psi, \nabla q'\big)_{\boldsymbol{L}^2(\Omega)} = (\underline{\boldsymbol{\mu}}(\tilde{\boldsymbol{C}} - \tilde{\boldsymbol{C}}^{\text{reg}}), \nabla q')_{\boldsymbol{L}^2(\Omega)} = (\operatorname{div}(\underline{\boldsymbol{\mu}}(\tilde{\boldsymbol{C}}^{\text{reg}} - \tilde{\boldsymbol{C}})), q')_{L^2(\Omega)}.$$

In other words, ψ' solves (5.21) with $\ell' = \operatorname{div}(\underline{\boldsymbol{\mu}}(\tilde{\boldsymbol{C}}^{\mathrm{reg}} - \tilde{\boldsymbol{C}})) \in L^2(\Omega)$. According to Theorem 5.6 (Neumann boundary condition) with $\underline{\boldsymbol{\xi}} = \underline{\boldsymbol{\mu}}$, $\psi' \in H^2(\Omega)$ with the bound $\|\psi'\|_{H^2(\Omega)} \lesssim \|\tilde{\boldsymbol{C}}^{\mathrm{reg}}\|_{\boldsymbol{H}^1(\Omega)} + \|\operatorname{div} \underline{\boldsymbol{\mu}}\tilde{\boldsymbol{C}}\|_{L^2(\Omega)}$. One concludes that $\tilde{\boldsymbol{C}} \in \boldsymbol{H}^1(\Omega)$, with the bound

$$\|\tilde{\boldsymbol{C}}\|_{\boldsymbol{H}^1(\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}.$$

Aggregating the results on $\boldsymbol{C}_A$ and $\tilde{\boldsymbol{C}}$, one concludes $\boldsymbol{C} \in \boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)$, with the bound

$$\|\boldsymbol{C}\|_{\boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|\pi_A^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\boldsymbol{E})\|_{\mathbf{H}_t^{\mathtt{r}}(\Gamma_A)}.$$

Finally, because $\mathtt{r} + \frac{1}{2} < 1$, the multiplication by $\underline{\boldsymbol{\mu}} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$ is stable in $\boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)$, so that the results on $\mathbf{curl}\,\boldsymbol{E} = \underline{\boldsymbol{\mu}}\boldsymbol{C}$ follow. □

One can go further to get estimates with continuous dependence w.r.t. the data only, making use of the continuous dependence of the solution (if the problem is well-posed) and of the estimates coming with the trace regularity considerations of section 4.

COROLLARY 5.9. *Assume that all assumptions of Theorem 5.8 hold, and that the problem (5.1) has a unique solution. Then there holds*

$$\|\boldsymbol{E}\|_{\boldsymbol{H}^{\min(\mathtt{r}+1/2,\mathtt{s})}(\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div}\boldsymbol{f}\|_{H^{\mathtt{s}-1}(\Omega)} + \|\mathbf{g}\|_\pi + \|\mathbf{g}\|_\gamma; \tag{5.29}$$

$$\|\mathbf{curl}\,\boldsymbol{E}\|_{\boldsymbol{H}^{\mathtt{r}+1/2}(\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{g}\|_\gamma + \|\mathbf{g}\|_\pi. \tag{5.30}$$

Proof. It results from combining the estimates of theorems 5.8, 5.3 and 4.1 or 4.7 (recall that $\|\mathbf{g}\|_{\mathbf{L}_t^2(\Gamma_A)} \lesssim \|\mathbf{g}\|_\pi + \|\mathbf{g}\|_\gamma$). □

6. Discretisation and numerical analysis.

In this section, we focus on a $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming discretization of variational formulation (5.6). In addition to Assumptions 1bis and 2, we assume now that the three coefficients $\underline{\boldsymbol{\mu}}^{-1}$, $-\underline{\boldsymbol{\varepsilon}}$ and $-\underline{\boldsymbol{\alpha}}$ are simultaneously elliptic (with also $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\mu}} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$): as a consequence, the sesquilinear form a defined by (5.3) is coercive on $\boldsymbol{H}^+(\mathbf{curl},\Omega)$. And, for simplicity, we choose a right-hand side $\boldsymbol{f} \in \boldsymbol{H}(\operatorname{div},\Omega)$, so that $\mathtt{s} = 1$ in Theorem 5.8. In the first subsection, we define some ad hoc discretization of the exact problem (5.6). In subsection 6.2, we apply Céa's lemma to prove convergence and we derive error estimates.

Remark 6.1. One can also carry out the numerical analysis of the magnetic problem (5.18). As a matter of fact, we already observed that the sesquilinear form in the equivalent variational formulation (5.19) is coercive under the same simultaneous ellipticity conditions on the coefficients than the ones above.

6.1. Discretisation.

For the ease of exposition[1], we assume that Ω is a Lipschitz polyhedron: $\overline{\Omega}$ is triangulated by a shape regular family of meshes $(\mathcal{T}_h)_h$, made up of (closed) simplices, generically denoted by K. Each mesh is indexed by $h := \max_K h_K$ (the meshsize), where h_K is the diameter of K. We choose the Nédélec's first family of edge finite elements [32, 31] to define finite dimensional subspaces $(\boldsymbol{V}_h)_h$ of $\boldsymbol{H}^+(\mathbf{curl},\Omega)$. More precisely, we consider first-order finite elements. Let $\mathcal{R}_1(K)$ be the vector space of polynomials on K defined by

$$\mathcal{R}_1(K) := \{\boldsymbol{v} \in \boldsymbol{P}_1(K) \;:\; \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x},\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}.$$

For given h, the finite element spaces taking into account a Dirichlet boundary condition on $\tilde{\Gamma}$ are classically defined by

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}_{0,\tilde{\Gamma}}(\mathbf{curl},\Omega) \;:\; \boldsymbol{v}_{h|K} \in \mathcal{R}_1(K),\ \forall K \in \mathcal{T}_h\}.$$

We observe that, since it holds that $\gamma_A^T \boldsymbol{v}_h, \pi_A^T \boldsymbol{v}_h \in \mathbf{L}_t^2(\Gamma_A)$ for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$, one has $\boldsymbol{V}_h \subset \boldsymbol{H}^+(\mathbf{curl},\Omega)$ by construction. The discrete variational formulation of the Maxwell problem with a generalised impedance boundary condition (5.6) is then

$$\text{Find } \boldsymbol{E}_h \in \boldsymbol{V}_h \text{ s.t., } \forall \boldsymbol{F}_h \in \boldsymbol{V}_h, \quad a(\boldsymbol{E}_h, \boldsymbol{F}_h) = (\boldsymbol{f}, \boldsymbol{F}_h)_{\boldsymbol{L}^2(\Omega)} - (\mathbf{g}, \gamma_A^T \boldsymbol{F}_h)_{\mathbf{L}_t^2(\Gamma_A)}. \tag{6.1}$$

[1]The results obtained below still hold if the boundary is of class $\mathcal{C}^2$. See [19] for the discretization by first-order edge finite elements. It is proven there that optimal interpolation properties hold, ie. one may recover up to $O(h)$ accuracy, provided the field to be interpolated is sufficiently smooth.

We introduce $\Pi_h^{\mathbf{curl}}$, the classical global Raviart-Thomas-Nédélec interpolant in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ with values in $\boldsymbol{V}_h$ [32]. In addition, there is an important commuting property that holds for surface fields. Let $\Pi_h^{\mathrm{curl}_\Gamma}$ be the surface Raviart-Thomas-Nédélec interpolant on Γ_A (see eg. [8], §4), where the surface meshes are equal to $\left(\mathcal{T}_{h|\Gamma_A}\right)_h$. That is, where the surface meshes are made of those triangles that are boundary faces on Γ_A of the meshes $(\mathcal{T}_h)_h$.

Proposition 6.2. *Let h be given, and let $\boldsymbol{v} \in \boldsymbol{H}^+(\mathbf{curl}, \Omega)$. Then if $\Pi_h^{\mathbf{curl}}\boldsymbol{v}$ is well-defined, one has*

$$\Pi_h^{\mathrm{curl}_\Gamma}(\pi_A^T \boldsymbol{v}) = \pi_A^T(\Pi_h^{\mathbf{curl}}\boldsymbol{v}). \tag{6.2}$$

As a side-product, for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$, one has

$$\Pi_h^{\mathrm{curl}_\Gamma}(\pi_A^T \boldsymbol{v}_h) = \pi_A^T \boldsymbol{v}_h. \tag{6.3}$$

Proof. Assume that $\Pi_h^{\mathbf{curl}}\boldsymbol{v}$ is well-defined, and let us check that property (6.2) stems from the definition of the degrees of freedom for the two interpolants.
On the one hand, for a tetrahedron K in $\overline{\Omega}$, any element $\boldsymbol{v}$ in $\mathcal{R}_1(K)$ is uniquely determined by the degrees of freedom in the moment set $M_K(\boldsymbol{v})$:

$$M_K(\boldsymbol{v}) := \left(\int_e \boldsymbol{v}\cdot\boldsymbol{t}\,dl\right)_{e\in A_K},$$

where A_K is the set of edges of K, and $\boldsymbol{t}$ is a unit vector along the edge e. As $\Pi_h^{\mathbf{curl}}\boldsymbol{v}$ is well-defined, all degrees of freedom $(M_K(\boldsymbol{v}))_{K\in\mathcal{T}_h}$ are meaningful. In particular, $\pi_A^T(\Pi_h^{\mathbf{curl}}\boldsymbol{v})$ is characterized by those degrees of freedom on the edges that belongs to Γ_A, ie. edges of $\mathcal{T}_{h|\Gamma_A}$.
On the other hand, on Γ_A, let a triangle be denoted by T with (local) coordinates system (x_1^T, x_2^T). Then, $\mathcal{R}_1(T)$ is the vector space of polynomials on T defined by

$$\mathcal{R}_1(T) := \{\mathbf{v} \in \mathbf{P}_1(T)\ :\ \mathbf{v}(x_1^T, x_2^T) = (a_1 - bx_2^T, a_2 + bx_1^T),\ a_1, a_2, b \in \mathbb{R}\}.$$

And any element $\mathbf{v}$ in $\mathcal{R}_1(T)$ is uniquely determined by the degrees of freedom in the moment set $M_T^\Gamma(\mathbf{v})$:

$$M_T^\Gamma(\mathbf{v}) := \left(\int_e \mathbf{v}\cdot\boldsymbol{t}\,dl\right)_{e\in A_T}.$$

Above, A_T is the set of edges of T, and $\boldsymbol{t}$ is again a unit vector along the edge e.
From the above, we conclude that all degrees of freedom $(M_T(\pi_A^T\boldsymbol{v}))_{T\in(\mathcal{T}_{h|\Gamma_A})}$ are meaningful, because by definition they coincide with those of $\boldsymbol{v}$ on all edges of $\mathcal{T}_{h|\Gamma_A}$. Hence $\Pi_h^{\mathrm{curl}_\Gamma}(\pi_A^T\boldsymbol{v})$ is well-defined too, and moreover $\Pi_h^{\mathrm{curl}_\Gamma}(\pi_A^T\boldsymbol{v}) = \pi_A^T(\Pi_h^{\mathbf{curl}}\boldsymbol{v})$: (6.2) holds.
Finally, if one recalls that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$, one has the property $\Pi_h^{\mathbf{curl}}\boldsymbol{v}_h = \boldsymbol{v}_h$, one gets (6.3). □

6.2. Convergence. Because the form a is coercive, Céa's lemma ensures that

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}^+(\mathbf{curl};\Omega)} \lesssim \inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h} \|\boldsymbol{E} - \boldsymbol{v}_h\|_{\boldsymbol{H}^+(\mathbf{curl};\Omega)}. \tag{6.4}$$

According to the basic approximability property of $(\boldsymbol{V}_h)_h$ in $\boldsymbol{H}^+(\mathbf{curl};\Omega)$, convergence follows:

$$\lim_{h\to 0} \|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}^+(\mathbf{curl};\Omega)} = 0.$$

To obtain explicit error estimates on $\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}^+(\mathbf{curl};\Omega)}$, a natural idea is to use the interpolation of its solution $\boldsymbol{E}$. Indeed, if one interpolates $\boldsymbol{E}$, one finds

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}^+(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{E} - \Pi_h^{\mathbf{curl}}\boldsymbol{E}\|_{\boldsymbol{H}^+(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{E} - \Pi_h^{\mathbf{curl}}\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\pi_A^T\boldsymbol{E} - \Pi_h^{\mathrm{curl}_\Gamma}(\pi_A^T\boldsymbol{E})\|_{\mathbf{L}_t^2(\Gamma_A)}, \tag{6.5}$$

where we use Proposition 6.2 to reformulate the surface term.
First, in the domain Ω, one can define the Raviart-Thomas-Nédélec interpolant of a field $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega)$ as soon as

$\boldsymbol{v} \in \mathbf{P}\boldsymbol{H}^{\mathtt{t}}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{v} \in \mathbf{P}\boldsymbol{H}^{\mathtt{t}'}(\Omega)$ for some $\mathtt{t} > \frac{1}{2}$, $\mathtt{t}' > 0$. Then one can define $\Pi_h^{\mathbf{curl}}\boldsymbol{v}$ and, in addition, one has the approximation result [4]:

$$\|\boldsymbol{v} - \Pi_h^{\mathbf{curl}}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \lesssim h^{\min(\mathtt{t},\mathtt{t}',1)}\{\|\boldsymbol{v}\|_{\mathbf{P}\boldsymbol{H}^{\mathtt{t}}(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{v}\|_{\mathbf{P}\boldsymbol{H}^{\mathtt{t}'}(\Omega)}\}. \tag{6.6}$$

In our case, according to Theorem 5.8 and its Corollary 5.9, for $\boldsymbol{v} = \boldsymbol{E}$ we can choose $\mathtt{t} = \mathtt{t}' = \mathtt{r} + \frac{1}{2}$ for any $\mathtt{r} \in [0, \min(\frac{1}{2}, \mathtt{r}_{\max})[$, and it holds that $\boldsymbol{E} \in \boldsymbol{H}^{\mathtt{t}}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{H}^{\mathtt{t}'}(\Omega)$. So interpolating $\boldsymbol{E}$ in Ω is possible, with the estimate:

$$\|\boldsymbol{E} - \Pi_h^{\mathbf{curl}}\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \lesssim h^{\mathtt{r}+1/2}\{\|\boldsymbol{f}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} + \|\mathbf{g}\|_{\pi} + \|\mathbf{g}\|_{\gamma}\}. \tag{6.7}$$

Next, because Γ_A is a (piecewise smooth) two-dimensional manifold, we recall that one can define the Raviart-Thomas-Nédélec interpolant of a field in $\mathbf{H}(\mathrm{curl}_\Gamma, \Gamma_A) := \{\mathbf{v} \in \mathbf{L}_t^2(\Gamma_A) \;:\; \mathrm{curl}_\Gamma\,\mathbf{v} \in L^2(\Gamma_A)\}$ as soon as it belongs to $\mathbf{PH}_t^{\mathtt{t}''}(\Gamma_A)$ for some $\mathtt{t}'' \in]0,1]$ (there is no requirement on the regularity of $\mathrm{curl}_\Gamma\,\mathbf{v}$). This result is proven in [3] for fields in $\mathbf{H}(\mathrm{div}, \Omega_2)$ for a domain Ω_2 of $\mathbb{R}^2$, and it obviously carries over to fields in $\mathbf{H}(\mathrm{curl}_\Gamma, \Gamma_A)$ by appropriate coordinates transform. Further, one has the approximation result:

$$\|\mathbf{v} - \Pi_h^{\mathrm{curl}_\Gamma}\mathbf{v}\|_{\mathbf{H}(\mathrm{curl}_\Gamma,\Gamma_A)} \lesssim h^{\min(\mathtt{t}'',1)}\big(\|\mathbf{v}\|_{\mathbf{PH}_t^{\mathtt{t}''}(\Gamma_A)} + \|\,\mathrm{curl}_\Gamma\,\mathbf{v}\|_{L^2(\Gamma_A)}\big). \tag{6.8}$$

Regarding the regularity needed on Γ_A for $\pi_A^T\boldsymbol{E}$ and especially that of $\mathrm{curl}_\Gamma(\pi_A^T\boldsymbol{E})$, let us make the following observations. To begin with, one has $\boldsymbol{E} \in \boldsymbol{H}^{\mathtt{t}}(\Omega)$ for some $\mathtt{t} \in]\frac{1}{2}, 1[$. As a consequence, $\boldsymbol{E}$ has a trace in $\mathbf{H}^{\mathtt{t}-1/2}(\Gamma_A) \subset \mathbf{L}^2(\Gamma_A)$, and in particular, the trace of its tangential components is such that $\pi_A^T\boldsymbol{E} \in \mathbf{PH}_t^{\mathtt{t}-1/2}(\Gamma_A)$. According to Corollary 5.9, one has

$$\|\pi_A^T\boldsymbol{E}\|_{\mathbf{PH}_t^{\mathtt{t}-1/2}(\Gamma_A)} \lesssim \left(\|\boldsymbol{f}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} + \|\mathbf{g}\|_{\pi} + \|\mathbf{g}\|_{\gamma}\right). \tag{6.9}$$

Remark 6.3. Observe that $\mathtt{t} - \frac{1}{2} \in]0, \frac{1}{2}[$. On the other hand, it can happen that the trace $\pi_A^T\boldsymbol{E}$ is more regular, ie. $\pi_A^T\boldsymbol{E} \in \mathbf{PH}_t^{\mathtt{t}''}(\Gamma_A)$ for some $\mathtt{t}'' \in [\frac{1}{2}, 1]$. In this case, one can not bound the norm of the trace $\|\pi_A^T\boldsymbol{E}\|_{\mathbf{PH}_t^{\mathtt{t}''}(\Gamma_A)}$ with respect to the data.

Similarly, one has $\mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{H}^{\mathtt{t}'}(\Omega)$ for some $\mathtt{t}' > \frac{1}{2}$, so $\mathbf{curl}\,\boldsymbol{E}$ has a trace in $\mathbf{H}^{\mathtt{t}'-1/2}(\Gamma_A) \subset \mathbf{L}^2(\Gamma_A)$, and in particular, its normal trace is such that $\gamma_A^n(\mathbf{curl}\,\boldsymbol{E}) \in L^2(\Gamma_A)$. According again to corollary 5.9, one has

$$\|\gamma_A^n(\mathbf{curl}\,\boldsymbol{E})\|_{L^2(\Gamma_A)} \lesssim \{\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{g}\|_{\pi} + \|\mathbf{g}\|_{\gamma}\}.$$

But it is well-known that $\mathrm{curl}_\Gamma(\pi_A^T\boldsymbol{E}) = \mathrm{div}_\Gamma(\gamma_A^T\boldsymbol{E}) = \mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{n}_{|\Gamma_A} = \gamma_A^n(\mathbf{curl}\,\boldsymbol{E})$, so the required regularity on the surface curl part allows one to interpolate $\pi_A^T\boldsymbol{E}$ on the boundary Γ_A by $\Pi_h^{\mathrm{curl}_\Gamma}$, with the estimate (6.8).

Gathering all estimates, we can draw two conclusions. We recall that $\mathtt{r} \in [0, \min(\frac{1}{2}, \mathtt{r}_{\max})[$.
If one wants to achieve an error estimate which includes the continuous dependence with respect to the data, we use (6.9) to find

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}^+(\mathbf{curl},\Omega)} \lesssim h^{\mathtt{r}}\{\|\boldsymbol{f}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} + \|\mathbf{g}\|_{\pi} + \|\mathbf{g}\|_{\gamma}\}. \tag{6.10}$$

Hence the best achievable convergence rate including continuous dependence with respect to the data, which occurs when $\mathtt{r}_{\max} \geq \frac{1}{2}$, is $O(h^{1/2})$.
On the other hand, in the framework of remark 6.3, one finds that

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}^+(\mathbf{curl},\Omega)} \lesssim h^{\min(\mathtt{r}+1/2,\mathtt{t}'')}, \tag{6.11}$$

where $\mathtt{t}'' \in [\frac{1}{2}, 1]$ is such that $\pi_A^T\boldsymbol{E} \in \mathbf{PH}_t^{\mathtt{t}''}(\Gamma_A)$. Hence the best achievable convergence rate, which occurs when $\mathtt{r}_{\max} \geq \frac{1}{2}$ and $\mathtt{t}'' = 1$, is $O(h)$. We note that this last result seems to be an improvement over the classical estimate, see Lemma 5.53 of [31].

7. Numerical illustration. To illustrate the expected convergence rate with a numerical case, we consider a simple benchmark with a manufactured solution. We define Ω as the unit ball: $\Omega = \{\mathbf{x} \in \mathbb{R}^3, |\mathbf{x}| < 1\}$. We assume that $\Gamma_A = \partial\Omega$. The angular frequency ω is set to 1 and the material tensors are respectively defined by

$$\underline{\boldsymbol{\varepsilon}} = \begin{pmatrix} 1+\eta i & & \\ & 1+\eta i & \\ & & -2+\eta i \end{pmatrix}, \qquad \underline{\boldsymbol{\mu}}^{-1} = \underline{\mathbf{I}}, \qquad \alpha = i\omega \tag{7.1}$$

where $\eta > 0$ is a chosen parameter. Let us note that all three coefficients are elliptic (but $\underline{\boldsymbol{\varepsilon}}$ is not Hermitian due to its imaginary part). Moreover, one can notice that $\underline{\boldsymbol{\mu}}^{-1}$, $-\underline{\boldsymbol{\varepsilon}}$ and $-\alpha$ are simultaneously elliptic; hence, this problem is coercive. The range of corresponding ellipticity directions is $\Theta =]\arctan(-1/\eta);\ -\pi/2[$. Different values of η will be tested. When η goes to zero, the ceorcivity deteriorates: at the limit, Θ reduces to $\{-\pi/2\}$, and the coercivity constant goes to 0 as ε_- goes to 0. In fact, when $\eta = 0$, the material becomes hyperbolic, and the problem is most likely ill-posed [15]. We consider a manufactured reference solution which is a plane wave,

$$\boldsymbol{E}_{\text{ref}} = [-1,\ 1,\ 1]^T \exp(i\pi\mathbf{k}\cdot\mathbf{x}), \quad \text{with } \mathbf{k} = \frac{1}{\sqrt{14}}\,[3,\ 2,\ 1]^T. \tag{7.2}$$

The volume source term is chosen accordingly, *i.e.* $\boldsymbol{f} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}_{\text{ref}} - \omega^2\underline{\boldsymbol{\varepsilon}}\boldsymbol{E}_{\text{ref}}$, as well as the right-hand-side term of the boundary conditions.

Numerical simulations are performed with FreeFem++ [23] using unstructured meshes made of tetrahedra and first-order edge finite elements. The discrete variational formulation, recast as a linear system, is then solved by a direct solver. Because the boundary of the meshes (which are polyhedral) does not exactly match the curved border of the spherical domain, the boundary data used in the numerical simulation are evaluated on the sphere and then projected on the surface mesh. This approximation generates an error of the order $O(h)$ [19].

The relative numerical error in $\boldsymbol{H}(\mathbf{curl},\Omega)$-norm is plotted as a function of the mesh size h on Fig. 2. As a reference, the relative error corresponding to the projection of the reference solution on the discrete solution space, which corresponds to the best approximation error according to Céa's lemma, is plotted as well. As the solution $\boldsymbol{E}_{\text{ref}}$ is smooth, it belongs to $\boldsymbol{H}^1(\Omega)$ as well as its curl. In addition, $\pi^T\boldsymbol{E}_{\text{ref}} \in \mathbf{H}^1_t(\partial\Omega)$. Therefore, one expects the error to evolve linearly with the mesh size h. This is what we observe, see Figure 2.

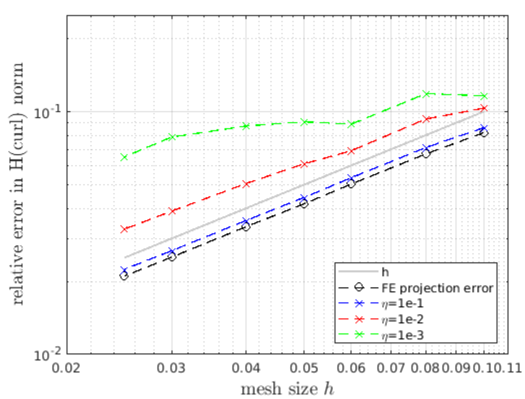

Fig. 2: Convergence of edge finite elements (order 1)

We also observe how the material tensors affect the convergence of the method. We have tested three different values for η: 10^{-1}, 10^{-2} and 10^{-3}. As the reference solution and the meshes are the same through those three cases, the projection error $\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\|\boldsymbol{E}-\boldsymbol{v}_h\|_{\boldsymbol{H}^+(\mathbf{curl};\Omega)}$ corresponding to the projection of the reference solution on the discrete solution space, remains the same. Ideally, the numerical error would be close to this theoretical error. However, we observe that this is not the case. In fact, when η decreases, the error deteriorates. Although the rate of convergence is quite well preserved for $\eta = 10^{-1}, 10^{-2}$, there is an impact on this rate too for $\eta = 10^{-3}$: the error is no more monotone. Also, the error that becomes much larger than the projection error. This can be related to the fact that the coercivity constant of the problem decreases with η, causing the constant C_η hidden in Céa's Lemma (we use the symbol $\lesssim$ in (6.4)) to become larger and larger when η goes to 0. Hence, for a fixed meshsize h, the right-hand side $C_\eta \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{E} - \boldsymbol{v}_h\|_{\boldsymbol{H}^+(\mathbf{curl};\Omega)}$ deteriorates.

8. Conclusion and extensions. In this work, we have derived the mathematical and numerical analysis of the time-harmonic Maxwell problem with Robin condition in elliptic, anisotropic media. This requires first to address the function spaces in which this boundary condition may hold, which is the point of Sections 3 and 4. In Theorem 3.5, we show that the regularity of elements of $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_A) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_A)$ depends only on the geometry of Γ_A (precisely, of its pathological vertices). Thanks to this result, we then provide regularity characterizations for the traces on Γ_A. This regularity depends only on the pathological vertices of Γ_A and of the discontinuity vertices of the coefficient α, if any (or edges if the coefficient is tensor-valued). The variational formulation is then obtained by standard techniques, which however require a few adaptations to deal with elliptic, non-hermitian coefficients. The regularity of the solution is studied via localization and a splitting into regular and singular part. When the form is coercive, we are able to conduct the numerical analysis of the problem and derive the order of convergence. Our results seem to complement the classical one of [31].

There are several interesting directions in which one might extend this work. First of all, the generic results of Section 4 could, in principle, be adapted to piecewise smooth boundaries and piecewise regular scalar coefficients. While our results are quite generic, the techniques used can be adapted in given configurations to get more precise results, as for example in [6]. For tensor-valued coefficients, further investigations could be conducted. Another way of prospect is the numerical analysis of the problem. While here we have done it for coercive forms, which allows the use of Céa's lemma, in more general cases, one would have to prove a uniform discrete inf-sup condition, which seems quite tedious in the general anisotropic plus Robin condition context. Finally, this work provides a suitable framework for the mathematical analysis of decomposed problems and domain decomposition methods making use of impedance conditions; this has been sketched in Chapter 7 of [12].

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REFERENCES

[1] F. Assous, P. Ciarlet Jr, and S. Labrunie. *Mathematical foundations of computational electromagnetism.* Springer, 2018.

[2] H. Barucq and B. Hanouzet. Asymptotic behavior of solutions to Maxwell's system in bounded domains with absorbing Silver–Müller's condition on the exterior boundary. *Asymptotic Analysis*, 15(1):25–40, 1997.

[3] A. Bermudez, P. Gamallo, M. R. Nogueiras, and R. Rodriguez. Approximation of a structural acoustic vibration problem by hexahedral finite elements. *IMA J. Numer. Anal.*, 26:391–421, 2006.

[4] A. Bermudez, R. Rodriguez, and P. Salgado. Numerical solution of eddy current problems in bounded domains using realistic boundary conditions. *Comput. Methods Appl. Mech. Engrg.*, 194:11–426, 2005.

[5] A. Bonito, J.-L. Guermond, and F. Luddens. Regularity of the Maxwell equations in heterogeneous media and Lipschitz domains. *Journal of Mathematical Analysis and applications*, 408(2):498–512, 2013.

[6] A.-S. Bonnet-Ben Dhia, C. Hazard, and F. Monteghetti. Complex-scaling method for the plasmonic resonances of planar subwavelength particles with corners. *Journal of Computational Physics*, 440:110433, 2021.

[7] A. Buffa. Hodge decompositions on the boundary of nonsmooth domains: the multi-connected case. *Mathematical Models and Methods in Applied Sciences*, 11(09):1491–1503, 2001.

[8] A. Buffa and S. H. Christiansen. The electric field integral equation on Lipschitz screens: definitions and numerical approximation. *Numerische Mathematik*, 94(2):229–267, 2003.

[9] A. Buffa and P. Ciarlet Jr. On traces for functional spaces related to Maxwell's equations. Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications. *Mathematical Methods in the Applied Sciences*, 24(1):31–48, 2001.

[10] A. Buffa, M. Costabel, and C. Schwab. Boundary element methods for Maxwell's equations on non-smooth domains. *Numerische Mathematik*, 92(4):679–710, 2002.

[11] A. Buffa, M. Costabel, and D. Sheen. On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. *Journal of Mathematical Analysis and Applications*, 276(2):845–867, 2002.

[12] D. Chicaud. *Analysis of time-harmonic electromagnetic problems in elliptic anisotropic media.* PhD thesis, Institut Polytechnique de Paris, 2021.

[13] D. Chicaud, P. Ciarlet Jr, and A. Modave. Analysis of variational formulations and low-regularity solutions for time-harmonic electromagnetic problems in complex anisotropic media. *SIAM Journal on Mathematical Analysis*, 53(3):2691–2717, 2021.

[14] P. Ciarlet Jr. On the approximation of electromagnetic fields by edge finite elements. Part 1: Sharp interpolation results for low-

regularity fields. *Computers Math. Applic.*, 71:85–104, 2016.
[15] P. Ciarlet Jr and M. Kachanovska. A mathematical study of a hyperbolic metamaterial in free space. *SIAM Journal on Mathematical Analysis*, 54:2216–2250, 2022.
[16] M. Costabel, M. Dauge, and S. Nicaise. Corner singularities and analytic regularity for linear elliptic systems. Part I: Smooth domains. https://hal.archives-ouvertes.fr/hal-00453934/, Feb. 2010.
[17] M. Dauge. *Elliptic boundary value problems on corner domains.* Lecture Notes in Mathematics, **1341**. Springer-Verlag, 1988.
[18] M. Dauge. Singularities of corner problems and problems of corner singularities. In *ESAIM: Proceedings*, volume 6, pages 19–40. EDP Sciences, 1999.
[19] A. Dello Russo and A. Alonso. Finite element approximation of Maxwell eigenproblems on curved Lipschitz polyhedral domains. *Appl. Numer. Math.*, 59:1796–1822, 2009.
[20] B. Després, P. Joly, and J. E. Roberts. A domain decomposition method for the harmonic Maxwell equations. In *Iterative Methods in Linear Algebra*, pages 245–252, 1992.
[21] G. N. Gatica and S. Meddahi. Finite element analysis of a time-harmonic Maxwell problem with an impedance boundary condition. *IMA Journal of Numerical Analysis*, 32(2):534–552, 2012.
[22] P. Grisvard. *Singularities in boundary value problems*, volume 22. Springer, 1992.
[23] F. Hecht. New development in FreeFem++. *Journal of Numerical Mathematics*, 20(3-4):251–265, 2012.
[24] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, 11:237, 2002.
[25] R. Hiptmair, A. Moiola, and I. Perugia. Stability results for the time-harmonic Maxwell equations with impedance boundary conditions. *Mathematical Models and Methods in Applied Sciences*, 21(11):2263–2287, 2011.
[26] A. Kirsch and F. Hettlich. *Mathematical theory of time-harmonic Maxwell's equations.* Springer, 2016.
[27] V. Kondrat'ev and O. Oleinik. Boundary value problems for partial differential equations in non-smooth domains. *Russian Math. Surveys*, 38:1–86, 1983.
[28] V. Kozlov, V. Maz'ya, and J. Rossmann. *Spectral problems associated with corner singularities of solutions to elliptic equations*. Number 85. American Mathematical Society, 2001.
[29] V. Mantic, F. Paris, and J. Berger. Singularities in 2D anisotropic potential problems in multi-material corners Real variable approach. *International Journal of Solids and Structures*, 40:5197–5218, 2003.
[30] D. Mercier. Minimal regularity of the solutions of some transmission problems. *Mathematical Methods in the Applied Sciences*, 26:321–348, 2003.
[31] P. Monk. *Finite element methods for Maxwell's equations.* Oxford University Press, 2003.
[32] J.-C. Nédélec. Mixed finite elements in $\mathbb{R}^3$. *Numer Math.*, 35:315–341, 1980.
[33] J.-C. Nédélec. *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*. Springer, 2001.
[34] J. E. Santos and D. Sheen. On the existence and uniqueness of solutions to Maxwell's equations in bounded domains with application to magnetotellurics. *Mathematical Models and Methods in Applied Sciences*, 10(04):615–628, 2000.