INSTITUT POLYTECHNIQUE DE PARIS

NNT : 2021IPPAE014

Thèse de doctorat

Analysis of time-harmonic electromagnetic problems in elliptic anisotropic media

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à École nationale supérieure de techniques avancées

École doctorale n°574 Mathématiques Hadamard (EDMH)
Spécialité de doctorat: Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, le 7/12/2021, par

Damien Chicaud

Composition du Jury :

Anne-Sophie Bonnet-Ben Dhia Directrice de recherche, ENSTA Paris (Poems)	Examinatrice
Patrick Ciarlet Professeur, ENSTA Paris (Poems)	Directeur de thèse
Xavier Claeys Maître de conférences, Sorbonne Université (LJLL)	Rapporteur
Monique Dauge Directrice de recherche, Université Rennes 1 (IRMAR)	Rapporteure
Bruno Després Professeur, Sorbonne Université (LJLL)	Examinateur
Houssem Haddar Directeur de recherche, Inria (IDEFIX)	Président du jury
Simon Labrunie Maître de conférences, Université de Lorraine (IECL)	Examinateur
Axel Modave Chargé de recherches, ENSTA Paris (Poems)	Co-directeur de thèse
Philippe Pouliguen Responsable innovation, DGA	Invité

Not all those who wander are lost

Remerciements

Espaces

S'il fallait un seul mot pour résumer cette thèse, ce serait sans doute celui-ci. C'est en tous cas, Axel et Patrick, celui qui est certainement revenu le plus fréquemment au cours des (nombreux !) échanges qui ont eu lieu durant ces trois ans. L'un les aime fonctionnels, ce qu'on est parfois en droit de remettre en question ; l'autre les préfère de mémoire, et me pardonnera si je suis ici trop frontal. Vous m'avez, chacun, ouvert à vos espaces (peut-être devrais-je dire vos *domaines*) de recherche, d'exploration ; et nos échanges, à deux ou à trois, de longues heures devant un tableau, ont toujours été riches, et souvent passionnés. Vous avez aussi été disponibles et attentifs pour me permettre de créer petit à petit mon espace, me guidant et m'accompagnant, tout en me laissant infléchir les directions à prendre ou à explorer. Vous m'avez accordé – et régulièrement renouvelé – votre confiance, scientifiquement et non-scientifiquement, et je vous en suis profondément reconnaissant. Vous avez, enfin, supporté ma bonne humeur matinale et mon humour de qualité. Pour tout cela, soyez profondément remerciés.

Traces

Ce serait probablement le deuxième mot à choisir pour qualifier cette thèse. J'aimerais remercier sincèrement Monique Dauge et Xavier Claeys pour l'intérêt et le soin avec lequel ils ont évalué mon travail. Je remercie également Bruno Després, Simon Labrunie, Anne-Sophie Bonnet-Ben Dhia et Houssem Haddar d'être membres du jury. C'est pour moi un honneur de recevoir l'attention de chercheurs sur les traces desquels j'ai tâché d'avancer. J'ose espérer que le voyage vous soit agréable. Merci également à l'ENSTA et la DGA qui ont financé ces travaux, et particulièrement à Philippe Pouliguen qui en a assuré le suivi. Je remercie les anonymes qui, peut-être, liront un jour ce mémoire et pourront y trouver quelque valeur (propre).

Nœuds

Je souhaiterais remercier Pierre Jolivet pour son aide dans le développement du code. Merci à Marcella et Bertrand pour quelques fructueuses discussions, et plus largement à tous ceux, rencontrés ici ou là, qui auront contribué à dénouer des idées à différents moments de la thèse. Merci pour Waves 2019 et pour le CIRM. Je remercie en particulier Émile, Mahran et Christophe, et je demande pardon à ceux que j'oublierais.

Intégration

J'aimerais remercier très chaleureusement l'ensemble des membres de l'UMA pour leur accueil dans ce labo lui aussi chaleureux, bienveillant, et qui cultive avec succès un certain art de vivre. Merci pour les psaumes, les discussions aux pauses café, et l'humour raffiné. Merci aux vieux qui ne sont pas vieux de partager si volontiers leur expérience et leur humanité avec nous autres doctorants. Et puis, en vrac et dans le désordre, merci à Éliane pour les arcanes d'ADUM, à Corinne pour les arcanes tout court, à Nico pour le LaTeX et les dépannages divers ; merci à Frédéric et Sonia pour m'avoir permis d'enseigner dans leurs cours, sans peur et (presque) sans dégâts ; à ceux « de ce bout-ci du couloir » (avec prolongement continu) pour les soirées à Massy ou au Shamrock ; à ceux qui ne jugent pas ceux qui n'ont pas encore fini de manger ; et aussi à tous les autres.

Continuité

J'aimerais adresser un merci plus particulier aux doctorants du labo. Tout d'abord ceux qui m'ont accueilli, avec simplicité et chaleur, en particulier Antoine, Sandrine et Clément ; merci aussi de m'avoir fait partager l'expérience de la colo. (N'oubliez pas de passer par la forêt, c'est plus court ! (J'en profite pour saluer avec beaucoup d'affection tous ceux qui perdront en lisant cette phrase.)) Et puis les suivants : Alex, Amond, Clara, Akram, Meryem, Pierre, Étienne, Laura, Alice, Alan, Cristian, Mathieu, et les petits derniers Thibaut et Quentin. Vous formez une bonne petite équipe. Merci en particulier d'avoir redonné du dynamisme aux échanges entre doctorants, ce qui n'allait forcément pas de soi en 2021. Vous l'avez réussi à grands coups d'apéros et de séminaires : bravo !

Noyau

Les plus attentifs parmi vous auront peut-être remarqué qu'il manque un bureau dans la liste du dessus. C'est tout simplement parce que ce sont des gens de plus grande qualité et pas du tout parce qu'ils m'ont versé des royalties. Merci donc au meilleur bureau, le lieu central de ces trois ans, et à ses occupants : Othmane et ton sens de l'inattendu, Mahran et ton inégalable culture scientifique, JF et nos retours à pieds. Je me souviendrai avec plaisir de nos nombreuses discussions tous les quatre sur une variété de sujets non-lisse et néanmoins assez considérable. Échantillon : Grothendieck, le pape, les inscriptions ADUM, le trajet d'une fourmi dans une boîte de chaussures, le sens du jeûne, la théorie spectrale des opérateurs non auto-adjoints, le gouvernement, les espaces chelou, pourquoi est-on bon en maths (ou pas), Trump, le Dirac est-il une distribution ou une mesure, et j'en oublie sûrement de meilleurs. Vous allez me manquer.

Injections

Un mot qui est étrangement revenu dans le langage courant en 2020-2021. J'aimerais remercier très chaleureusement tous ceux qui, d'une façon ou d'une autre, ont contribué à rendre la vie plus facile pendant cette période. Je remercie en particulier mes parents qui m'ont accueilli très généreusement à plusieurs reprises durant les périodes de confinement, ce dont je garde un beau souvenir. De manière plus générale, vous avez toujours été présents et m'avez toujours soutenu (à défaut, peut-être, de toujours me comprendre), merci pour ça. Un grand merci aussi aux 4 Fantastiques, qui ont installé leurs quartiers (qu'ils soient principaux ou secondaires...) à Malakoff et qui, à une certaine période, se réunissaient à une fréquence quasi-hebdomadaire, ce qui fut également d'un soutien précieux.

Classes

Qu'il me soit également permis de remercier certaines personnes qui, bien qu'elles n'aient pas contribué de manière directe au déroulement de cette thèse, auront tout de même marqué ces trois années de leur présence. À cet effet, j'aimerais commencer par remercier divers groupes issus de l'ENSTA, aux noms souvent rocambolesques et aux intersections généralement non-vides : les Joyeux Lurons, the Quartet, Rose Fuschia aka Moriarty, JC, Point, sans oublier la Troupe des Aventuriers Générateurs d'Entropie. Je salue également ceux qui auront eu l'idée (quelque peu incongrue) de m'accompagner dans l'épopée de la thèse : Marie, Mocia, Gilles, Omar, Émilien et ceux que j'oublie. Alors oui ça fait que certains sont cités plusieurs fois, mais c'est mes remerciements et je fais ce que je veux. Du coup re-merci à Gilles et à Juliette pour la qualité de nos discussions sur tout un tas de sujets.

Intérieur

Différents termes issus du langage mathématique pourraient se prêter à une lecture ignacienne, mais c'est celui d'« intérieur » qui m'a paru correspondre le mieux. Merci donc à Anne, Pauline, Antoine et Clément, ainsi qu'à Nicolas. La période que j'ai passée à l'UMA décrit approximativement celle de notre compagnonnage et, vous le savez, celui-ci a été précieux. Vous avez, bien souvent, été les premiers témoins au long de mes cheminements, parfois sinueux. Petit à petit, nous avons appris à nous connaître, à échanger librement, à nous faire confiance. Nous avons grandi, et nous nous sommes très certainement aidés mutuellement à grandir. C'est un très grand privilège que de pouvoir contempler cela. Merci à vous.

Singularité

D'aucuns pourraient dire qu'en ces trois ans il s'est passé certaines choses « peu banales ». Pour cela, un grand merci aux déclicots et déclicotes avec qui j'ai vécu l'année 2018-2019, ainsi qu'à nos supers accompagnateurs. Là encore, beaucoup de cheminement et de découvertes, mais surtout la joie de parcourir ce chemin avec d'autres. Cette année fut le lieu de belles rencontres, de ressourcement, de paix et de fraternité. Faire l'expérience de cheminer chacun mais ensemble. Tout cela a compté. Nous continuons d'avancer sur nos chemins tous singuliers... Qu'ils puissent s'ouvrir à votre approche, et que le Vent souffle dans votre dos !

Stabilité

Voici à présent le temps de remercier Ceux qui sont tout simplement là, qui l'ont été dès la fondation du monde, et qui n'ont jamais pris la peine de se donner un nom autre que « les meilleurs » ou « la bande » (j'oserais « la famille » si le terme n'avait pas trop perdu de sa valeur). Merci à Hugo pour les si nombreuses connivences ; merci à Iris pour les connaître, et continuer à m'inviter. Merci à Julien pour ta loyauté, et pour avoir, volontairement ou pas, inspiré la forme de ces remerciements. Merci à Samuel, pour être là, ce qui est déjà pas mal ! Merci à Sylvain et Aline pour commencer l'éducation de leur fils avec tant d'ardeur. Merci à Héloïse et Quentin pour leurs présences fugaces, merci à Caro pour la maison toujours ouverte. Merci à Greg pour les nuits longues et par conséquent courtes. Merci à Aude pour savoir respecter le sommeil de ses riverains.

Là aussi, on peut dire qu'il s'en sera passé des choses en trois ans, qui commencent approximativement dans les Pyrénées pour se finir à Fécamp. Des diplômes, quelques-uns. Montrouge, une voisine relou, Malakoff. Un théâtre. Des pizzas, souvent. Un weekend de Pentecôte ; des ecocups. Des guitares, plusieurs ; des guitaristes, plusieurs aussi, mais à des degrés divers ; de la musique, beaucoup. Des réveillons, des lendemains de réveillon (un en particulier), et des lendemains de lendemain de réveillon. Des memes de qualité variable. Des soirées à distance sur skribblio ou Catan. De belles soirées d'été ; des barbecues. Les rives de la Dordogne. Rouen. Des dos-d'âne. Un bébé. D'autres barbecues. Du martini et du cointreau. Une maison dans laquelle on casse tout. Un château avec une table moche dedans. Merci pour tout, et aussi pour tout le reste.

Unicité et Existence

Deux mots pour conclure, qui sont venus de loin. Merci à ceux qui savent quelle en est la valeur.

Abstract | Résumé

Abstract

The numerical simulation of electromagnetic problems in complex physical settings is a trending topic which conveys many scientific and industrial applications, such as the design of optical metamaterials, or the study of cold plasmas. The mathematical and numerical analysis of Maxwell problems is well-known in simple physical contexts, when the material parameters are isotropic. Some results in anisotropic media exist, but they generally tend to focus on the case where the material tensors are real symmetric (or complex Hermitian) definite positive. However, problems in more complex media are not covered by the standard theory. Therefore, new mathematical tools need to be developed to analyse these problems.

This thesis aims at analysing time-harmonic electromagnetic problems for a general class of complex anisotropic material tensors. These are called elliptic materials. We derive an extended functional framework well-suited for these anisotropic problems, generalizing well-known results. We study the well-posedness of Maxwell boundary value problems for Dirichlet, Neumann, and Robin boundary conditions. For the Robin case, the characterization of appropriate function spaces for Robin traces is addressed. The regularity of the solution and its curl is studied, and elements of numerical analysis for edge finite elements are provided. In the perspective of the use of Domain Decomposition Methods (DDM) for accelerated numerical computing, various decomposed formulations are proposed and studied, focusing on their right meaning in terms of function spaces and equivalence with the global problem. These results are complemented with some numerical DDM experimentations in anisotropic media.

Keywords: Maxwell equations, anisotropic media, electromagnetic waves, finite elements, regularity analysis, domain decomposition

Résumé

La simulation numérique de problèmes électromagnétiques dans des configurations physiques complexes est largement utilisée pour de nombreuses applications scientifiques et industrielles, telles que la conception de métamatériaux optiques ou l'étude des plasmas froids. L'analyse mathématique et numérique des problèmes de Maxwell est bien connue dans des contextes physiques simples, où les paramètres du milieu sont isotropes. Des résultats en milieux anistropes existent, mais se limitent généralement au cas des tenseurs réels symétriques (ou complexes hermitiens) définis positifs. Cependant, pour certains milieux plus complexes, les problèmes ne sont pas couverts par la théorie standard. De nouveaux outils mathématiques doivent donc être développés pour analyser ces problèmes.

Dans cette thèse, nous analysons des problèmes électromagnétiques harmoniques en temps pour une classe générale de tenseurs matériels anistropes, appelés elliptiques. Nous développons un cadre fonctionnel étendu adapté à ces problèmes anisotropes, en généralisant les résultats connus. Nous étudions le caractère bien posé de problèmes avec conditions limites de Dirichlet, Neumann ou Robin. Dans le cas Robin, un intérêt particulier est porté à la caractérisation des espaces fonctionnels pour les traces de Robin. Nous étudions la régularité de la solution et de son rotationnel, et donnons des éléments d'analyse numérique. Dans la perspective de l'utilisation de méthodes de décomposition de domaine (DDM) pour une résolution accélérée, nous proposons et étudions différentes formulations décomposées, en nous focalisant sur leurs espaces fonctionnels et leur équivalence avec le problème global. Quelques expérimentations numériques sur la DDM complètent ce travail.

Mots-clés : équations de Maxwell, milieux anisotropes, ondes électromagnétiques, éléments finis, étude de régularité, décomposition de domaine

Contents

Chapter 1

Introduction

1.1 Context and motivations

Electromagnetics is a field encompassing many scientific and industrial applications, which involve configurations and materials of increasing complexity. Among them are the famous metamaterials: they consist in small structures whose assembly can have global properties that greatly differ from those of standard objects. For example, metamaterials could be designed to create materials with negative optical index, or invisibility cloacks (see e.g. [93, 57]). Another popular application field is plasma physics. Plasmas consist in charged gases and are involved in various technological challenges, such as the control of fusion reactions in tokamaks (see e.g. [104, 75, 66]). In these applicative contexts where real experimentations are at best sensitive, costly and potentially harmful, numerical simulations offer a good alternative. This requires simulation methods that are at the same time sufficiently accurate to represent the physical solution, sufficiently efficient to get this representation in a reasonable time, with a reasonable amount of computational resources, and sufficiently robust to deal properly with complex configurations.

In this context, Finite Element Methods (FEM) have shown their accuracy, efficiency and robustness through a wide range of geometries and physical problems. They rely on the mathematical principle of variational formulations, and edge finite elements are particularly well-suited for electromagnetic problems. In isotropic media, their robustness is granted by mathematical arguments: one can prove that the considered problems are well-posed, and estimate the error commited when solving numerically the problem. Besides, when the number of degrees of freedom becomes huge (which is sometimes necessary to get a sufficiently accurate solution), Domain Decomposition Methods (DDM) can be used to reduce computational costs. This is done by splitting a large problem into coupled smaller subproblems, thus allowing parallel resolution or parallel preconditioning.

Unfortunately, the mathematical foundations for the study of anisotropic problems are not as developped as for isotropic ones. Several categories can be distinguished. The best documented class is when the material coefficients are real-valued symmetric (or complex-valued Hermitian) definite positive tensor fields. However, let us point out that neither cloaking metamaterials, nor plasma enter this category. Beyond this class, only limited results are available. In fact, only a few works have addressed other types of anisotropic media. Therefore, the existing mathematical tools for the analysis of Maxwell problems cannot deal with these new materials. New tools are required to analyse mathematically the physical problem and the numerical methods in such configurations.

1.2 Scope and goals

In this work, we aim at developing the mathematical framework for a new, more general class of materials that includes cold plasma, see e.g. [104, 8], and cloaking metamaterials such as in [58, 57]. These materials will be called *elliptic*. This class will be defined in the beginning of Chapter 3. Here, we develop the technical tools that allow us to conduct the analysis of anisotropic Maxwell problems for materials with such coefficients. We focus on three aspects:

- The *well-posedness* of the problem.
  This encompasses the *existence* of the solution, its *uniqueness*, as well as its *continuous dependence on data*, which characterizes the solution's sensitivity to small variations. This is done classically by studying a variational formulation of the problem. However, the mathematical tools allowing this study must be adapted to anisotropic problems.

- The *regularity* of the solution.
  This affects the convergence rate of the numerical edge finite elements method. That is why it is helpful to get *a priori* regularity results on the solution. It also has implications for decomposed problems.

- The analysis of the *decomposed* problem.
  This encompasses several aspects. The first one is the justification and the study of decomposed formulations that are equivalent to the global problem. The second one is a proof of convergence for classical iterative domain decomposition schemes.

In addition, we aim at exploring some more practical questions related to the numerical discretization by standard edge finite elements, as well as to the parallel procedures with standard DDMs.

1.3 Outline

The manuscript is organised as follows. In Chapter 2 we introduce the model equations, and the mathematical framework around it. Function spaces and traces are introduced. The main functional analysis results for Maxwell equations are presented and illustrated on a simple standard problem.

Chapter 3 describes a generalized functional framework designed to deal with anisotropic problems. It extends the tools introduced in Chapter 2 to the broader class of elliptic tensors.

Chapter 4 is an interlude which focuses on surface regularity matters for Robin traces. Several embeddings for traces spaces are shown, and the regularity of Robin traces in various cases is addressed. These results are necessary for the proper analysis of the Robin boundary value problem.

In Chapter 5, we address the well-posedness of Maxwell boundary value problems in anisotropic contexts. We do so for the three main types of boundary conditions: Dirichlet, Neumann, and Robin.

In Chapter 6, the regularity of the solution as well as the solution's curl is addressed. This is done, again, for the three types of problems. We also provide elements of numerical analysis for edge finite elements discretization, which are supported by some numerical experiments with FreeFem++.

Chapter 7 presents different aspects related to domain decomposition in anisotropic contexts. Decomposed problems are written, and their equivalence with the global problem is studied. The convergence of a standard iterative algorithm is investigated. Numerical experimentations with FreeFem++/PETSc are presented in order to investigate the behaviour of standard DDMs for anisotropic complex media.

In the last chapter, general conclusions and perspectives are drawn.

1.4 Main contributions

Below, we draw a list of what we believe are our main original contributions:

- The extension of the functional analysis framework for Maxwell equations to elliptic tensor-valued coefficients (Chapter 3);

- The studies on Robin traces function spaces: the trace spaces embeddings of Section 4.1, and extensions to the cases of heterogeneous scalar coefficient (Sections 4.2.2 and 4.2.3) and tensor-valued coefficient (Section 4.3);

- The well-posedness analysis of Maxwell problems with Dirichlet, Neumann or Robin boundary conditions for broad classes of anisotropic materials (Chapter 5);

- The regularity analysis for the solutions of the same problems (Chapter 6);

- The study of several decomposed formulations for anisotropic problems (Section 7.3);

- Investigations on the theoretical convergence of a classical domain decomposition algorithm for anisotropic problems (Section 7.4);
- Numerical investigations on the behaviour of standard domain decomposition methods for anisotropic problems (Section 7.5).

Some parts of this work have been published in a peer-reviewed journal:

[23] D. Chicaud, P. Ciarlet Jr, and A. Modave. Analysis of variational formulations and low-regularity solutions for time-harmonic electromagnetic problems in complex anisotropic media. *SIAM Journal on Mathematical Analysis*, 53(3):2691–2717, 2021.

Some parts of this work have appeared in a conference proceeding:

[22] D. Chicaud, P. Ciarlet Jr, and A. Modave. Perturbed edge finite element method for the simulation of electromagnetic waves in magnetised plasmas. In *Proceedings of the 14th International Conference on Mathematical and Numerical Aspects of wave Propagation Phenomena (Vienna, Austria)*, pages 434–435, 2019.

This work has been funded by DGA/AID (Direction Générale de l'Armement / Agence de l'Innovation de Défense), scientific field *Ondes acoustiques et radioélectriques*, and by ENSTA Paris. An extension has been granted by LMH (Labex Mathématiques Hadamard).

Chapter 2

Model and mathematical tools for the analysis of Maxwell equations

In this chapter, we present the most important elements of mathematical analysis that are classically required for the study of Maxwell problems. We recall only the main definitions and results. For details, we refer the reader to the monographs of Monk [86], Kirsch and Hettlich [78], and Assous, Ciarlet and Labrunie [7]. We also provide some other references along the way when necessary. Section 2.1 introduces the physical model, equations and boundary conditions. In Section 2.2 we introduce the mathematical framework, function spaces and operators. In Section 2.3 we focus on potentials and fields decompositions. In Section 2.4 we present useful results of functional analysis. We conclude in Section 2.5 by recalling the results that allow to state the well-posedness of a variational formulation and apply them to a classical Maxwell problem.

2.1 The model

In this section, we present the most important aspects of electromagnetics that are necessary to our work. As most of problems arising from physics, electromagnetics are described by a set of PDE (Maxwell equations) posed in the three-dimensional space (or a subspace of it). They are completed with constitutive laws and boundary conditions. From now on, we denote scalar fields by standard fonts; vector fields by bold letters; and tensor fields by underlined bold letters.

2.1.1 Maxwell equations

To begin with, let us recall the four main differential operators that will be useful through this work. All of them are expressed in Cartesian coordinates.

The *gradient*, a vector-valued operator acting on scalar fields: $\nabla v = \begin{pmatrix} \partial_x v \\ \partial_y v \\ \partial_z v \end{pmatrix}$ (sometimes denoted $\mathbf{grad}\, v$).

The *divergence*, a scalar-valued operator acting on vector fields: $\operatorname{div} \mathbf{v} = \partial_x v_x + \partial_y v_y + \partial_z v_z$ (also denoted $\nabla \cdot \mathbf{v}$).

The *curl*, a vector-valued operator acting on vector fields: $\mathbf{curl}\, \mathbf{v} = \begin{pmatrix} \partial_y v_z - \partial_z v_y \\ \partial_z v_x - \partial_x v_z \\ \partial_x v_y - \partial_y v_x \end{pmatrix}$ (also denoted $\nabla \times \mathbf{v}$).

The *Laplacian*, a second-order scalar operator: $\Delta v = \operatorname{div}(\nabla v) = \partial^2_{xx} v + \partial^2_{yy} v + \partial^2_{zz} v$.

With these operators at hand, we are able to write the well-known four Maxwell equations which govern the behaviour of electromagnetic fields:

$$\operatorname{div}\mathbf{D} = \rho \qquad \text{(Gauss law)} \tag{2.1}$$
$$\operatorname{div}\mathbf{B} = 0 \qquad \text{(Gauss law for magnetism)} \tag{2.2}$$
$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \qquad \text{(Faraday law)} \tag{2.3}$$
$$\frac{\partial \mathbf{D}}{\partial t} - \mathbf{curl}\,\mathbf{H} = -\mathbf{J} \qquad \text{(Ampere law)} \tag{2.4}$$

where $\mathbf{D}$ is the electric displacement, $\mathbf{E}$ the electric field, $\mathbf{B}$ the magnetic induction, and $\mathbf{H}$ the magnetic field. ρ is the charge density and $\mathbf{J}$ the current density. Moreover, $\mathbf{J}$ and ρ are related through the charge conservation law:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{J} = 0. \tag{2.5}$$

In the context of our work, $\mathbf{J}$ and ρ, the *sources*, are known data. The unknowns are the electromagnetic fields $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{H}$.

2.1.2 Constitutive laws and material parameters

The Maxwell's equations are completed with constitutive laws

$$\mathbf{D} = \underline{\boldsymbol{\varepsilon}}\mathbf{E} \tag{2.6}$$
$$\mathbf{B} = \underline{\boldsymbol{\mu}}\mathbf{H} \tag{2.7}$$

where $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ are material tensors, characteristic of the material, and called respectively the dielectric tensor and the tensor of magnetic permeability. These relations allow us to reduce to two the number of unknown fields in Maxwell equations, eliminating usually $\mathbf{D}$ and $\mathbf{B}$ (or, more occasionally, $\mathbf{D}$ and $\mathbf{H}$).

The medium is said isotropic when $\underline{\boldsymbol{\varepsilon}} = \varepsilon\underline{\mathbf{I}}$, $\underline{\boldsymbol{\mu}} = \mu\underline{\mathbf{I}}$, where ε and μ are scalar fields. From a physical point of view, this means that the constitutive laws do not depend on the orientation of the medium, which is the case of most usual media. The medium is moreover said homogeneous when ε and μ are constant. This means that the behaviour of the medium is not dependent on space. In isotropic homogeneous media, and without sources, the Maxwell equations induce a wave equation:

$$\partial^2_{tt}\mathbf{E} - (\varepsilon\mu)^{-1}\mathbf{\Delta}\mathbf{E} = \mathbf{0}, \tag{2.8}$$

of propagation speed $c := 1/\sqrt{\varepsilon\mu}$. The main example of isotropic homogeneous medium is vacuum, in which case $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ are fundamental physical constants. The associated propagation speed is nothing but the light velocity, $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$.

In more complex configurations, $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ can be tensor-valued. This expresses anisotropy of the medium, i.e. the fact that the medium interactions with the electromagnetic fields do depend on orientation. As an example, one can point metamaterials, small structures made of several materials that follow a regular pattern. Another example of anisotropic medium is plasma. In the domain of numerical applications, PML can also be regarded as anisotropic media.

2.1.3 Time-harmonic formulation

In the context of time-harmonic problems, we assume that the time dependence of the electromagnetic fields, as well as the sources, is known and periodic, of a given pulsation $\omega > 0$: $A(\mathbf{x}, t) = \Re(A(\mathbf{x})e^{-i\omega t})$. On the other hand, we assume that the material tensors $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ do not vary with time. Under such assumptions, the Maxwell equations rewrite

$$\operatorname{div}\underline{\boldsymbol{\varepsilon}}\mathbf{E} = \rho \tag{2.9}$$
$$\operatorname{div}\underline{\boldsymbol{\mu}}\mathbf{H} = 0 \tag{2.10}$$
$$-i\omega\underline{\boldsymbol{\mu}}\mathbf{H} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \tag{2.11}$$
$$i\omega\underline{\boldsymbol{\varepsilon}}\mathbf{E} + \mathbf{curl}\,\mathbf{H} = \mathbf{J}, \tag{2.12}$$

and the charge conservation law

$$-i\omega\rho + \operatorname{div}\mathbf{J} = 0. \tag{2.13}$$

One can notice that, in this context, equations (2.9) and (2.10) can be deduced from (2.11) and (2.12), taking the divergence and using relation (2.13). Moreover, one can eliminate unknown $\mathbf{H}$ in the two remaining equations, to get the second-order time-harmonic Maxwell equation for the electric field:

$$\mathbf{curl}\,\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E} - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E} = i\omega\mathbf{J} \tag{2.14}$$

which will be our main subject of study through this work. We will generally denote $\mathbf{f}$ the right-hand side of (2.14), $\mathbf{f} = i\omega\mathbf{J}$. Likewise, note that one could write the second-order equation for the magnetic field by eliminating $\mathbf{E}$, to get

$$\mathbf{curl}\,\underline{\boldsymbol{\varepsilon}}^{-1}\,\mathbf{curl}\,\mathbf{H} - \omega^2\underline{\boldsymbol{\mu}}\mathbf{H} = \mathbf{curl}\,\underline{\boldsymbol{\varepsilon}}^{-1}\mathbf{J}. \tag{2.15}$$

Both problems have similar forms.

2.1.4 Boundary conditions

Like any physical problem, equation (2.14) must be completed with appropriate boundary conditions, as soon as the domain of interest is a strict subset of $\mathbb{R}^3$. Let us review the three main ones that will be studied in this work.

Dirichlet conditions

One expects that, along any surface Σ, the tangential components of the electric field are continuous, that is $[\mathbf{E}\times\mathbf{n}]_\Sigma = \mathbf{0}$, where $[\,\cdot\,]_\Sigma$ denotes the jump across Σ, and $\mathbf{n}$ is a unit normal vector field to Σ. If, for example, the domain neighbours a perfectly conducting medium (in which $\mathbf{E} = \mathbf{0}$), there will hold on its boundary Γ

$$\mathbf{E}\times\mathbf{n} = \mathbf{0}, \tag{2.16}$$

which is called a *perfectly conducting condition*. Note that, however, one does not control the normal component of $\mathbf{E}$ at the interface. In particular, even though all the components of $\mathbf{E}$ vanish in a perfect conductor, there does not hold $\mathbf{E}\cdot\mathbf{n} = 0$ on Γ.

More generally, one could impose the value of the electric field's tangential components on Γ:

$$\mathbf{E}\times\mathbf{n} = \mathbf{E}_d\times\mathbf{n}, \quad \text{or} \quad \mathbf{E}\times\mathbf{n} = \mathbf{g}, \tag{2.17}$$

where $\mathbf{E}_d$ is a given electric field outside of Ω, or $\mathbf{g}$ is a given tangential field on Γ. In our work, this type of boundary condition will be referred to as *Dirichlet condition*.

Neumann conditions

On the other hand, along any surface Σ, there also holds $[\mathbf{H}\times\mathbf{n}]_\Sigma = \mathbf{j}_\Sigma$, where $\mathbf{j}_\Sigma$ is the surface current density on Σ. Therefore, one can impose a given surface current at the boundary: $\mathbf{H}\times\mathbf{n} = \tilde{\mathbf{j}}$ on Γ, which also rewrites, in time-harmonic context,

$$\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}\times\mathbf{n} = \mathbf{j}, \tag{2.18}$$

with $\mathbf{j} = i\omega\tilde{\mathbf{j}}$. This condition is sometimes used to represent effects induced by an antenna, see e.g. [8]. In our work, this type of boundary condition will be referred to as *Neumann condition*.

Impedance or Robin conditions

The last type of boudary condition we will consider in this work has to do with radiation and absorption matters. In an isotropic homogeneous medium, one can write the Silver-Müller condition

$$(\mathbf{E} - c\mathbf{B}\times\mathbf{n})\times\mathbf{n} = \mathbf{0}, \tag{2.19}$$

which is an outgoing condition. It models the fact that no wave is incoming from outside the domain of study (cf. [87, 103]). A plane wave which propagates normally to a plane boundary with such a condition would continue its way without being reflected. That is why it is also called a *transparent condition*. However, it is exact only for a plane boundary and for a plane wave with normal incidence. In the other cases, (2.19) is only (but still) an approximate absorbing boundary condition (ABC). It is the lowest-order ABC for Maxwell equations. Refinements are possible, which are not in the scope of this thesis; we refer e.g. to [50]. On the other hand, an inhomogeneous condition is also possible, which then models that a wave is incoming from outside the domain:

$$(\mathbf{E} - c\mathbf{B} \times \mathbf{n}) \times \mathbf{n} = \mathbf{g}. \tag{2.20}$$

The condition can also be written in $(\mathbf{E}, \mathbf{H})$ variables, which gives

$$\left(\mathbf{E} - \sqrt{\frac{\mu}{\varepsilon}}\mathbf{H} \times \mathbf{n}\right) \times \mathbf{n} = \mathbf{g}, \tag{2.21}$$

and, because $Z := \sqrt{\mu/\varepsilon}$ is physically speaking an impedance, it is also called an *impedance condition*. In the time-harmonic regime, one can eliminate $\mathbf{H}$ thanks to Faraday law (2.11), and get

$$-\left(\mu^{-1}\,\mathbf{curl}\,\mathbf{E} \times \mathbf{n}\right) \times \mathbf{n} + \frac{i\omega}{Z}\left(\mathbf{E} \times \mathbf{n}\right) = \frac{i\omega}{Z}\mathbf{g}. \tag{2.22}$$

In the case of non-isotropic media, the proper meaning to give to absorbing or transparent conditions becomes unclear. In this work, we shall consider conditions of the type

$$\mathbf{n} \times \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E} \times \mathbf{n}\right) + \underline{\boldsymbol{\alpha}}\left(\mathbf{E} \times \mathbf{n}\right) = \mathbf{g} \tag{2.23}$$

where $\underline{\boldsymbol{\alpha}}$ can be any chosen tensor. We will refer to them either as *impedance conditions*, or, because they involve linear combinations of the Dirichlet and Neumann traces, as *Robin conditions*.

2.2 Function spaces and operators

In this section, we introduce the functional framework that allows us to study mathematically the Maxwell equations. In the following of our work, Ω shall be a bounded, connected, open subset of $\mathbb{R}^3$, and s.t. at each point $\mathbf{x}$ of its boundary, there exists a Lipschitz-continuous mapping whose graph locally represents the boundary of Ω in a neighbourhood of $\mathbf{x}$, and Ω is locally on one side only of its boundary (the two latter points meaning that Ω is *of Lipschitz boundary*). Such a subset will be called from now on a *Lipschitz domain* of $\mathbb{R}^3$. Moreover, the boundary of Ω will generally be denoted Γ, and $\mathbf{n}$ will denote the unit outward normal to Ω.

We denote by standard fonts the scalar fields and their function spaces, and by bold letters the vector fields and their function spaces; e.g. $\mathbf{H}^1(\Omega) := \left(H^1(\Omega)\right)^3$; $v \in H^1(\Omega)$; $\mathbf{v} \in \mathbf{H}^1(\Omega)$. The tensor fields and their function spaces will be denoted by underlined bold letters, e.g. $\underline{\boldsymbol{\xi}} \in \underline{\mathbf{L}}^\infty(\Omega)$. An index of notations is provided at the end of the manuscript.

2.2.1 Volume function spaces

Distributions spaces

We denote by $\mathcal{D}(\Omega)$ (resp. $\boldsymbol{\mathcal{D}}(\Omega) = (\mathcal{D}(\Omega))^3$) the space of scalar (resp. vector), complex-valued, infinitely differentiable functions of compact support included in Ω.

A linear form T on $\mathcal{D}(\Omega)$ is said continuous if, for all compact set $K \subset \Omega$, there exists $C_K > 0$, $m_K \in \mathbb{N}$ s.t.

$$|T(g)| \leq C_K \sup_{|\alpha| \leq m_K} \sup_{\mathbf{x} \in K} |\partial_\alpha g(\mathbf{x})|, \quad \forall g \in \mathcal{D}(\Omega) \text{ s.t. } \operatorname{supp}(g) \subset K. \tag{2.24}$$

The set of linear continuous forms on $\mathcal{D}(\Omega)$, i.e. the dual space of $\mathcal{D}(\Omega)$, is denoted $\mathcal{D}'(\Omega)$. Similarly, the dual space of $\boldsymbol{\mathcal{D}}(\Omega)$ is denoted $\boldsymbol{\mathcal{D}}'(\Omega)$. They are called distributions spaces. The duality product is denoted $\langle \cdot, \cdot \rangle$ in either scalar or vector case.

One can differentiate objects of $\mathcal{D}'(\Omega)$ in the following sense. Let $T \in \mathcal{D}'(\Omega)$, the j-th partial derivative of T is given by the relation

$$\langle \frac{\partial T}{\partial x_j}, g\rangle = -\langle T, \frac{\partial g}{\partial x_j}\rangle, \quad \forall g \in \mathcal{D}(\Omega). \tag{2.25}$$

The mapping ∂_j defines a linear, continuous operator from $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Omega)$. This is called *differentiation in the sense of distributions*. This way, one can also define operators ∇, div, **curl**, Δ in the sense of distributions.

Lebesgue spaces

For $1 \leq p < \infty$, we introduce the *Lebesgue spaces*

$$L^p(\Omega) := \left\{ v \text{ s.t. } \int_\Omega |v|^p \,\mathrm{d}\mathbf{x} < \infty \right\}, \tag{2.26}$$

which is a Banach space endowed with the norm $\|v\|_{L^p} := \left(\int_\Omega |v|^p \,\mathrm{d}\mathbf{x}\right)^{1/p}$. We also introduce $L^\infty(\Omega)$, the space of essentially bounded functions on Ω, which is also a Banach space endowed with the norm $\|v\|_{L^\infty} := \mathrm{esssup}_{\mathbf{x}\in\Omega}|v(\mathbf{x})|$. For the sake of shortness, we shall omit the subscript $_{(\Omega)}$ in the norms when it is clear that the corresponding domain is Ω.

Among these, a particular space that will play a leading role in the following is $L^2(\Omega)$, the space of square integrable functions on Ω, which is a Hilbert space endowed with the inner product $(u|v) := \int_\Omega u\bar{v} \,\mathrm{d}\mathbf{x}$. All those spaces also have their vector-valued counterparts, and, in the special case $p = 2$, $(\cdot|\cdot)$ will denote either the inner product of $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$.

Standard Sobolev spaces

A notable property of elements of $L^p(\Omega)$, for any $p \in [1, \infty]$, is that they can be considered as distributions, that is, elements of $\mathcal{D}'(\Omega)$. Therefore, differentiating these functions becomes possible in the sense of distributions. This motivates the definition of *Sobolev spaces*: for $m \in \mathbb{N}$,

$$W^{m,p}(\Omega) := \left\{ v \in L^p(\Omega), \forall \alpha \in \mathbb{N}^3 \text{ s.t. } |\alpha| \leq m, \partial_\alpha v \in L^p(\Omega) \right\}, \tag{2.27}$$

where α is a derivative multi-index; that is, all derivatives of v up to order m belong to $L^p(\Omega)$. Such spaces are Banach spaces, endowed with the norm $\|v\|_{W^{m,p}} := \left(\sum_{|\alpha|\leq m} \|\partial_\alpha v\|^p_{L^p}\right)^{1/p}$ if $p < \infty$, and $\|v\|_{W^{m,\infty}} := \max_{|\alpha|\leq m} \|\partial_\alpha v\|_{L^\infty}$.

Once again, the case $p = 2$ plays a special role, as spaces $H^m(\Omega) := W^{m,2}(\Omega)$ are Hilbert spaces, endowed with the inner product

$$(u, v)_{H^m} = \sum_{|\alpha|\leq m} \int_\Omega \partial_\alpha u \, \overline{\partial_\alpha v} \,\mathrm{d}\mathbf{x}. \tag{2.28}$$

It is possible to define Sobolev spaces $H^s(\Omega)$ of fractional order. Let $s = m + \sigma$, with $m \in \mathbb{N}$ and $\sigma \in]0, 1[$. Introduce

$$|v|_{H^\sigma} := \left(\sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|\partial_\alpha v(\mathbf{x}) - \partial_\alpha v(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{3+2\sigma}} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \right)^{1/2}. \tag{2.29}$$

Then $H^s(\Omega) := \{v \in H^m(\Omega) \text{ s.t. } |v|_{H^\sigma} < \infty\}$, and it is a Hilbert space, endowed with the norm $\|v\|_{H^s} := \left(\|v\|^2_{H^m} + |v|^2_{H^\sigma}\right)^{1/2}$ and the associated scalar product. Other (equivalent) definitions are possible, see e.g. [82].

Then, one also introduces the subspace $H^s_0(\Omega)$, for $s \geq 0$, as the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$. Sobolev spaces of negative order are defined by duality: for $s > 0$, $H^{-s}(\Omega)$ is the dual space of $H^s_0(\Omega)$, i.e. the space of *antilinear* continuous forms on $H^s_0(\Omega)$, with $L^2(\Omega)$ taken as the pivot space. They are also Hilbert spaces, endowed with the norm $\|v\|_{H^{-s}} := \sup_{w\in H^s_0(\Omega)\setminus\{0\}} \frac{\langle v, w\rangle_{H^s_0}}{\|w\|_{H^s}}$ (and the associated scalar product). Once again, all those spaces also have their vector-valued counterparts. To conclude, we denote by the susbcript $_{\mathrm{zmv}}$ the subspace of fields of $H^s(\Omega)$ with zero mean value, for $s \leq 0$,

$$H^s_{\mathrm{zmv}}(\Omega) := \{v \in H^s(\Omega),\ (v|1) = 0\}. \tag{2.30}$$

Divergence and curl Sobolev spaces

To finish this section, let us introduce two more Sobolev spaces, which will be of primal interest for the study of electromagnetic fields. These are the spaces related to operators div and **curl**,

$$\mathbf{H}(\mathrm{div},\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega), \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}, \tag{2.31}$$

$$\mathbf{H}(\mathbf{curl},\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega), \mathbf{curl}\,\mathbf{v} \in \mathbf{L}^2(\Omega)\}, \tag{2.32}$$

which are also Hilbert spaces, endowed with their respective inner products $(\mathbf{u},\mathbf{v})_{\mathbf{H}(\mathrm{div})} := (\mathbf{u}|\mathbf{v}) + (\mathrm{div}\,\mathbf{u}|\,\mathrm{div}\,\mathbf{v})$ and $(\mathbf{u},\mathbf{v})_{\mathbf{H}(\mathbf{curl})} := (\mathbf{u}|\mathbf{v}) + (\mathbf{curl}\,\mathbf{u}|\,\mathbf{curl}\,\mathbf{v})$. Note that, while one clearly has

$$\mathbf{H}^1(\Omega) \subset \mathbf{H}(\mathrm{div},\Omega), \quad \mathbf{H}^1(\Omega) \subset \mathbf{H}(\mathbf{curl},\Omega),$$

the converse embeddings are false. Indeed, one simply requires that some linear combination of the derivatives belong to $\mathbf{L}^2(\Omega)$, not the derivatives themselves. In fact, one can even state that, *a priori*,

$$\mathbf{H}(\mathrm{div},\Omega) \cap \mathbf{H}(\mathbf{curl},\Omega) \not\subset \mathbf{H}^1(\Omega),$$

which has strong implications that will be detailed later.

Finally, as for standard Sobolev spaces, we denote with subscript $_0$ the closure of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\mathbf{H}(\mathrm{div},\Omega)$ and $\mathbf{H}(\mathbf{curl},\Omega)$, that is resp. $\mathbf{H}_0(\mathrm{div},\Omega)$ and $\mathbf{H}_0(\mathbf{curl},\Omega)$.

2.2.2 Traces and surface operators

Traces

It is possible to give a meaning to the boundary value of volume fields. This is done thanks to operators called *traces*. We first introduce them for fields of $\mathcal{C}^\infty(\overline{\Omega})$ (resp. $\boldsymbol{\mathcal{C}}^\infty(\overline{\Omega})$). Their extension to Sobolev spaces will be discussed in the next subsection. Recalling that Γ denotes the boundary of Ω and $\mathbf{n}$ the unit outward normal to Ω, let us introduce:

The *trace of scalar fields*, $\gamma : f \mapsto f_{|\Gamma}$ (simply called *trace* when there is no amibiguity).

The *normal trace*, $\gamma^n : \mathbf{f} \mapsto \mathbf{f} \cdot \mathbf{n}_{|\Gamma}$.

The *tangential trace*, $\gamma^T : \mathbf{f} \mapsto \mathbf{f} \times \mathbf{n}_{|\Gamma}$.

The *tangential components trace*, $\pi^T : \mathbf{f} \mapsto \mathbf{n} \times (\mathbf{f} \times \mathbf{n})_{|\Gamma}$.

Surface operators

Let us also introduce some surface operators, defined for surface fields on Γ. Therefore, they are related to tangential traces. For the sake of clarity, let us also give an explicit expression of these operators in Cartesian coordinates, with the convention $\Gamma = \{z = 0\}$ and $\mathbf{n}$ is towards $z > 0$. For the case of a smooth boundary, we refer, e.g., to [105, pp. 61-75].

The *surface gradient*: $\nabla_\Gamma(v_{|\Gamma}) := \pi^T(\nabla v) = \begin{pmatrix} \partial_x v \\ \partial_y v \end{pmatrix}$.

The *surface vector curl*: $\mathbf{curl}_\Gamma(v_{|\Gamma}) := \gamma^T(\nabla v) = \begin{pmatrix} \partial_y v \\ -\partial_x v \end{pmatrix}$.

The *surface divergence*, the dual operator of $-\nabla_\Gamma$: $\mathrm{div}_\Gamma\,\mathbf{v}_{|\Gamma} = \partial_x v_x + \partial_y v_y$.

The *surface scalar curl*, the dual operator of $\mathbf{curl}_\Gamma$: $\mathrm{curl}_\Gamma\,\mathbf{v}_{|\Gamma} = \partial_x v_y - \partial_y v_x$.

The *Laplace-Beltrami operator*: $\Delta_\Gamma v := \mathrm{div}_\Gamma \nabla_\Gamma v = -\mathrm{curl}_\Gamma\,\mathbf{curl}_\Gamma\, v = \partial^2_{xx} v + \partial^2_{yy} v$.

Note that $\mathbf{curl}_\Gamma$ acts on a scalar field and returns a vector one, while curl_Γ acts on a vector field and returns a scalar one. Also, although the definitions above involve volume fields, all these operators can be understood as purely surface operators. Let us also recall that

$$\operatorname{div}_\Gamma \mathbf{curl}_\Gamma = 0 \quad \text{and} \quad \operatorname{curl}_\Gamma \nabla_\Gamma = \mathbf{0}. \tag{2.33}$$

Moreover, if ξ is a surface scalar field, one has

$$\operatorname{div}_\Gamma(\xi \nabla_\Gamma \cdot) = -\operatorname{curl}_\Gamma(\xi \,\mathbf{curl}_\Gamma \cdot). \tag{2.34}$$

2.2.3 Function spaces for traces

In this subsection, we extend the notion of traces to less regular fields, more precisely, elements of particular Sobolev spaces. This is not straightforward. Indeed, elements of Sobolev spaces are defined, as Lebesgue spaces, *almost everywhere*. As a consequence, they do not have a *pointwise* definition, contrarily to fields of $\mathcal{C}^\infty(\overline{\Omega})$. However, when the fields are sufficiently regular, it is possible to extend (in a weaker sense) traces of the field (or of some of its components) to the boundary of the domain.

To that aim, we introduce the Sobolev spaces $H^s(\Gamma)$ as well as their vector-valued counterparts, defined on Γ for $|s| \leq 1$. This is done in a similar way than for volume spaces. In particular, for $0 < s < 1$,

$$\|v\|_{H^s(\Gamma)} := \left(\|v\|^2_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|v(\mathbf{x}) - v(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{2+2s}} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \right)^{1/2}, \tag{2.35}$$

and $\|v\|_{H^{-s}(\Gamma)} := \sup_{w \in H^s(\Gamma)\setminus\{0\}} \dfrac{\langle v, w\rangle_{H^s(\Gamma)}}{\|w\|_{H^s(\Gamma)}}$. Again, other equivalent definitions are possible, cf. [82].

Trace of scalar fields

The first important and classical result is that it is actually possible to give a meaning to $\gamma(v)$ when v belongs to $H^1(\Omega)$.

Theorem 2.2.1. The mapping γ has a unique continuous, surjective extension from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$. Moreover, $H^1_0(\Omega) = \{v \in H^1(\Omega), v_{|\Gamma} = 0\}$.

When it comes to vector valued fields, if $\mathbf{v} \in \mathbf{H}^1(\Omega)$, one can take the scalar trace of each component. However, this is not possible *a priori* for less regular fields of $\mathbf{H}(\operatorname{div}, \Omega)$ or $\mathbf{H}(\mathbf{curl}, \Omega)$. In the following we will see that one can in fact define the normal trace of elements of $\mathbf{H}(\operatorname{div}, \Omega)$, and, likewise, the tangential trace of elements of $\mathbf{H}(\mathbf{curl}, \Omega)$.

Normal trace

The normal trace can be extended in a weak sense to elements of $\mathbf{H}(\operatorname{div}, \Omega)$.

Theorem 2.2.2. The mapping γ^n has a unique continuous extension from $\mathbf{H}(\operatorname{div}, \Omega)$ to $H^{-1/2}(\Gamma)$, which is surjective. Moreover, $\mathbf{H}_0(\operatorname{div}, \Omega) = \{\mathbf{v} \in \mathbf{H}(\operatorname{div}, \Omega), \mathbf{v} \cdot \mathbf{n}_{|\Gamma} = 0\}$.

This comes with the following integration by parts formula.

Theorem 2.2.3. $\forall(\mathbf{u}, v) \in \mathbf{H}(\operatorname{div}, \Omega) \times H^1(\Omega)$,

$$(\mathbf{u}|\nabla v) + (\operatorname{div} \mathbf{u}|v) = \langle \mathbf{u} \cdot \mathbf{n}, v\rangle_{H^{1/2}(\Gamma)}. \tag{2.36}$$

Moreover, a side-product is that, for fields $v \in H^1(\Omega)$ s.t. $\Delta v \in L^2(\Omega)$, there holds $\nabla v \in \mathbf{H}(\operatorname{div}, \Omega)$; so, it is possible to define their normal derivative trace, $\nabla v \cdot \mathbf{n}_{|\Gamma} \in H^{-1/2}(\Gamma)$. The normal derivative is sometimes denoted $\frac{\partial v}{\partial \mathbf{n}}$ or $\partial_\mathbf{n} v$. To finish with that matter, let us introduce a particular type of domain:

Definition 2.2.4. A domain Ω is said *of the* $\mathfrak{A}$*-type* if, for any $\mathbf{x} \in \Omega$, there exists a neighbourhood $\mathcal{V}$ of $\mathbf{x}$ in $\mathbb{R}^3$, and a $\mathcal{C}^2$ diffeomorphism that transforms $\Omega \cap \mathcal{V}$ into one of the following types, where (x_1, x_2, x_3) denote the Cartesian coordinates and $(\rho, \omega) \in \mathbb{R} \times \mathcal{S}^2$ the spherical coordinates in $\mathbb{R}^3$:

1. $[x_1 > 0]$, i.e. $\mathbf{x}$ is a regular point;
2. $[x_1 > 0, x_2 > 0]$, i.e. $\mathbf{x}$ is a point on a salient (outward) edge;
3. $\mathbb{R}^3 \backslash [x_1 \geq 0, x_2 \geq 0]$, i.e. $\mathbf{x}$ is a point on a reentrant (inward) edge;
4. $[\rho > 0, \omega \in \tilde{\Omega}]$, where $\tilde{\Omega}$ is a topologically trivial domain. In particular, if $\partial\tilde{\Omega}$ is smooth, $\mathbf{x}$ is a conical vertex; if $\partial\tilde{\Omega}$ is a made of arcs of great circles, $\mathbf{x}$ is a polyhedral vertex.

In [35], such domains are referred to as (Lipschitz) *corner domains*.

In a domain of the $\mathfrak{A}$-type, one can match normal traces of $\mathbf{H}^1(\Omega)$ vector fields with normal derivative of $H^2(\Omega)$ scalar fields [7, Lemma 3.6.4]:

Proposition 2.2.5. Let Ω be a domain of the $\mathfrak{A}$-type. For any $\mathbf{w} \in \mathbf{H}^1(\Omega)$, there exists $q \in H^2(\Omega)$ s.t.

$$\frac{\partial q}{\partial \mathbf{n}} = \mathbf{w} \cdot \mathbf{n}_{|\Gamma} \quad \text{on } \Gamma, \tag{2.37}$$

with $\|q\|_{H^2} \leq C \|\mathbf{w}\|_{\mathbf{H}^1}$, where $C > 0$ is a constant independent on $\mathbf{w}$.

Tangential and tangential components traces

For elements of $\mathbf{H}(\mathbf{curl}, \Omega)$, it is possible to define in a weak sense the tangential trace, as well as the tangential components trace.

Theorem 2.2.6. The mappings γ^T and π^T have a unique continuous extension from $\mathbf{H}(\mathbf{curl}, \Omega)$ to $\mathbf{H}^{-1/2}(\Gamma)$. Moreover, $\mathbf{H}_0(\mathbf{curl}, \Omega) = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega), \gamma^T \mathbf{v} = 0\} = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega), \pi^T \mathbf{v} = 0\}$.

However, this result does not provide surjectivity, contrarily to the previous Theorems 2.2.1 and 2.2.2. A better understanding of the function spaces related to tangential traces has been provided by Buffa and Ciarlet for piecwise smooth domains [18, 19], and by Buffa, Costabel and Sheen for general domains with Lipschitz boundary [21]. Hereafter we recall their most important results. In the rest of the section we assume that Γ is piecewise smooth. First of all, let us introduce the space of tangential fields of $\mathbf{L}^2(\Gamma)$,

$$\mathbf{L}^2_t(\Gamma) := \{\mathbf{v} \in \mathbf{L}^2(\Gamma), \mathbf{v} \cdot \mathbf{n} = 0\}. \tag{2.38}$$

The inner space of $\mathbf{L}^2_t(\Gamma)$ will generally be denoted $(\cdot, \cdot)_\Gamma$. More generally, for $s > 0$,

$$\mathbf{H}^s_t(\Gamma) := \mathbf{H}^s(\Gamma) \cap \mathbf{L}^2_t(\Gamma). \tag{2.39}$$

We also introduce

$$\mathbf{H}^{1/2}_\perp(\Gamma) := \gamma^T \left(\mathbf{H}^1(\Omega)\right), \tag{2.40}$$

$$\mathbf{H}^{1/2}_\parallel(\Gamma) := \pi^T \left(\mathbf{H}^1(\Omega)\right). \tag{2.41}$$

These spaces are defined in an inner yet equivalent manner in [18]. Their duals are denoted, respectively, $\mathbf{H}^{-1/2}_\perp(\Gamma)$ and $\mathbf{H}^{-1/2}_\parallel(\Gamma)$, where $\mathbf{L}^2_t(\Gamma)$ is taken as the pivot space.

Remark 2.2.7. When Γ is smooth, both spaces $\mathbf{H}^{1/2}_\perp(\Gamma)$ and $\mathbf{H}^{1/2}_\parallel(\Gamma)$ coincide. In fact, there holds

$$\mathbf{H}^{1/2}_\perp(\Gamma) = \mathbf{H}^{1/2}_\parallel(\Gamma) = \mathbf{H}^{1/2}_t(\Gamma),$$

where $\mathbf{H}^{1/2}_t(\Gamma) := \mathbf{H}^{1/2}(\Gamma) \cap \mathbf{L}^2_t(\Gamma)$. However, when Γ is only piecewise smooth, there simply holds

$$\mathbf{H}^{1/2}_t(\Gamma) \subset \mathbf{H}^{1/2}_\perp(\Gamma), \quad \mathbf{H}^{1/2}_t(\Gamma) \subset \mathbf{H}^{1/2}_\parallel(\Gamma).$$

Then, let us introduce

$$\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) := \{\mathbf{v}\in\mathbf{H}_{\parallel}^{-1/2}(\Gamma), \mathrm{div}_\Gamma\,\mathbf{v}\in H^{-1/2}(\Gamma)\}; \tag{2.42}$$

$$\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) := \{\mathbf{v}\in\mathbf{H}_{\perp}^{-1/2}(\Gamma), \mathrm{curl}_\Gamma\,\mathbf{v}\in H^{-1/2}(\Gamma)\}. \tag{2.43}$$

For short, the natural norm of the space $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ is denoted $\|\cdot\|_\gamma$: $\|\mathbf{v}\|_\gamma^2 := \|\mathbf{v}\|^2_{\mathbf{H}_{\parallel}^{-1/2}(\Gamma)} + \|\,\mathrm{div}_\Gamma\,\mathbf{v}\|^2_{H^{-1/2}(\Gamma)}$. The norm of the space $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ is denoted $\|\cdot\|_\pi$. Then, one has the next fundamental results.

Theorem 2.2.8. The mapping γ^T is continuous and surjective from $\mathbf{H}(\mathbf{curl},\Omega)$ to $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$. The mapping π^T is continuous and surjective from $\mathbf{H}(\mathbf{curl},\Omega)$ to $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$.

Moreover, one has a duality result and an integration by parts formula.

Theorem 2.2.9. $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ and $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ are dual spaces. Their duality product is denoted ${}_\gamma\langle\cdot,\cdot\rangle_\pi$. Moreover, one has the following integration by parts formula: $\forall(\mathbf{u},\mathbf{v})\in\mathbf{H}(\mathbf{curl},\Omega)\times\mathbf{H}(\mathbf{curl},\Omega)$,

$$\begin{aligned}(\mathbf{u}|\,\mathbf{curl}\,\mathbf{v}) - (\mathbf{curl}\,\mathbf{u}|\mathbf{v}) &= {}_\gamma\langle\gamma^T\mathbf{u},\pi^T\mathbf{v}\rangle_\pi\\ &= -{}_\pi\langle\pi^T\mathbf{u},\gamma^T\mathbf{v}\rangle_\gamma.\end{aligned} \tag{2.44}$$

To finish with, let us note the following identities.

Proposition 2.2.10. Let $q\in H^1(\Omega)$, there holds, in $H_{\perp}^{-1/2}(\Gamma)$,

$$\nabla_\Gamma\, q_{|\Gamma} = \pi^T(\nabla q). \tag{2.45}$$

Proposition 2.2.11. Let $\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega)$, there holds, in $H^{-1/2}(\Gamma)$,

$$\mathrm{div}_\Gamma(\mathbf{v}\times\mathbf{n})_{|\Gamma} = (\mathbf{curl}\,\mathbf{v})\cdot\mathbf{n}_{|\Gamma}. \tag{2.46}$$

2.3 Extractions of potentials and fields decompositions

The study of electromagnetic fields requires some useful elements of vector analysis that are commonly acknowledged. For example, the fact that one can write a curl-free field as the gradient of a scalar potential; that a divergence-free field can be written as the curl of a vector potential; and more generally, that a vector field $\mathbf{u}$ can be written as

$$\mathbf{u} = \mathbf{curl}\,\mathbf{A} + \nabla\phi, \tag{2.47}$$

with $\mathbf{curl}\,\mathbf{A}$ being the "divergence-free part" of $\mathbf{u}$ and $\nabla\phi$ its "curl-free part". The main goal of this section is to recall the mathematical foundations of such assertions.

2.3.1 Topological matters

Before all, it will be useful to make more precise some assumptions on the domain Ω. Indeed, the domain topology will play an important role in the matter of the following subsections. First, the domain may be of connected boundary or not. If not, we denote $(\Gamma_k)_{0\le k\le K}$ the (maximal) connected components of Γ. Second, the domain may be topologically trivial or not (see [63] for further details):

Definition 2.3.1. A domain Ω is said *topologically trivial* when one can extract a single-valued potential from curl-free smooth fields, that is:

$$\forall\mathbf{v}\in\boldsymbol{\mathcal{C}}^1(\Omega) \text{ s.t. } \mathbf{curl}\,\mathbf{v}=\mathbf{0} \text{ in } \Omega, \quad \text{there exists } p\in\mathcal{C}^0(\Omega) \text{ s.t. } \mathbf{v}=\nabla p \text{ in } \Omega.$$

If not, we assume that the domain satisfies the following condition:

Definition 2.3.2. A domain Ω is said *of genus I* if I is the minimal number s.t. there exists I non-intersecting manifolds $(\Sigma_i)_{1\leq i\leq I}$, called cuts, s.t., letting $\dot\Omega := \Omega\backslash\bigcup_{i=1}^{I}\Sigma_i$,

$$\forall \mathbf{v}\in \boldsymbol{\mathcal{C}}^1(\Omega) \text{ s.t. } \mathbf{curl}\,\mathbf{v}=\mathbf{0} \text{ in } \Omega, \quad \text{there exists } \dot p\in\mathcal{C}^0(\dot\Omega) \text{ s.t. } \mathbf{v}=\nabla\dot p \text{ in } \dot\Omega,$$

Topologically trivial domains are of genus 0. More generally, I is the first Betti number of Ω. For more details on these aspects, we refer to [63].

The following integration by parts formula from [6] will be useful in such domains.

Proposition 2.3.3. In a domain Ω of genus $I>0$, and using the notations of the definition above, there holds the following integration by parts formula:
$\forall\mathbf{v}\in\mathbf{H}_0(\mathrm{div},\Omega),\ \forall\dot q\in H^1(\dot\Omega)$,

$$(\mathbf{v}|\nabla\dot q)_{\dot\Omega}+(\mathrm{div}\,\mathbf{v}|\dot q)_{\dot\Omega}=\sum_{i=1}^{I}\langle\mathbf{v}\cdot\mathbf{n},[\dot q]_{\Sigma_i}\rangle_{H^{1/2}(\Sigma_i)}, \tag{2.48}$$

where $\mathbf{n}$ is a unit normal vector field to Σ_i, and $[\dot q]_{\Sigma_i}$ denotes the jump of $\dot q$ across Σ_i (with sign convention taken according to $\mathbf{n}$).

The notion can be extended to manifolds of $\mathbb{R}^3$. In particular, Γ can be topologically trivial or not, cf. [16].

2.3.2 Extraction of potentials

Here, we recall the main mathematical results concerning extraction of scalar and vector potentials. This subject is widely addressed in the monograph of Girault and Raviart [59], and we recall only the most useful results for our concerns. To begin with, one has the next scalar potential result [59, Th. 2.9].

Theorem 2.3.4 (Scalar potential)**.** Let Ω be a topologically trivial domain. For $\mathbf{v}$ in $\mathbf{L}^2(\Omega)$, there holds

$$\mathbf{curl}\,\mathbf{v}=\mathbf{0} \tag{2.49}$$

if, and only if,

$$\exists p\in H^1_{\mathrm{zmv}}(\Omega) \text{ s.t. } \mathbf{v}=\nabla p. \tag{2.50}$$

The scalar potential p is unique, and there exists a constant $C>0$, independent of $\mathbf{v}$, s.t. $\|p\|_{H^1}\leq C\|\mathbf{v}\|_{\mathbf{L}^2}$.

One has also at hand a vector potential extraction result [59, Th. 3.4].

Theorem 2.3.5 (Vector potential)**.** Let Ω be a domain, and $(\Gamma_k)_{0\leq k\leq K}$ the connected components of its boundary Γ. For $\mathbf{v}$ in $\mathbf{L}^2(\Omega)$, there holds

$$\begin{cases}\mathrm{div}\,\mathbf{v}=0,\\ \langle\mathbf{v}\cdot\mathbf{n},1\rangle_{H^{1/2}(\Gamma_k)}=0,\quad\forall\,0\leq k\leq K,\end{cases} \tag{2.51}$$

if, and only if,

$$\exists\mathbf{w}\in\mathbf{H}^1(\Omega) \text{ s.t. } \mathbf{v}=\mathbf{curl}\,\mathbf{w}, \tag{2.52}$$

with moreover $\mathrm{div}\,\mathbf{w}=0$, and there exists a constant $C>0$, independent of $\mathbf{v}$, s.t. $\|\mathbf{w}\|_{\mathbf{H}^1}\leq C\|\mathbf{v}\|_{\mathbf{L}^2}$.

To finish this subsection, let us also mention an alternate vector potential theorem proved in [7, Th. 3.5.1].

Theorem 2.3.6 (Second vector potential)**.** Let Ω be a domain of genus $I \geq 0$. For $\mathbf{v}$ in $\mathbf{L}^2(\Omega)$, there holds

$$\begin{cases} \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \\ \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_{H^{1/2}(\Sigma_i)} = 0, \quad \forall\, 1 \leq i \leq I, \end{cases} \tag{2.53}$$

if, and only if,

$$\exists \mathbf{w} \in \mathbf{H}_0(\mathbf{curl}, \Omega) \text{ s.t. } \mathbf{v} = \mathbf{curl}\, \mathbf{w}, \tag{2.54}$$

with, additionally, $\operatorname{div} \mathbf{w} = 0$, and, $\forall\, 0 \leq k \leq K$, $\langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_{H^{1/2}(\Gamma_k)} = 0$. Moreover, $\mathbf{w}$ is unique, and there exists a constant $C > 0$, independent of $\mathbf{v}$, s.t. $\|\mathbf{w}\|_{\mathbf{H}(\mathbf{curl})} \leq C \|\mathbf{v}\|_{\mathbf{L}^2}$.

2.3.3 Null spaces of operators

In this subsection, we develop some aspects of the relations underlying the different operators ∇, div, **curl**, and their related spaces. Let us introduce the spaces of vanishing divergence or curl,

$$\mathbf{H}(\operatorname{div} 0, \Omega) := \{\mathbf{v} \in \mathbf{H}(\operatorname{div}, \Omega),\ \operatorname{div} \mathbf{v} = 0\}, \tag{2.55}$$

$$\mathbf{H}(\mathbf{curl}\, 0, \Omega) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega),\ \mathbf{curl}\, \mathbf{v} = 0\}, \tag{2.56}$$

as well as

$$\mathbf{H}_0(\operatorname{div} 0, \Omega) := \mathbf{H}_0(\operatorname{div}, \Omega) \cap \mathbf{H}(\operatorname{div} 0, \Omega), \tag{2.57}$$

$$\mathbf{H}_0(\mathbf{curl}\, 0, \Omega) := \mathbf{H}_0(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathbf{curl}\, 0, \Omega). \tag{2.58}$$

A famous way to represent the relations between all the different operators ∇, div, **curl**, and their related spaces, is the *de Rham complex*:

$$H^1(\Omega) \xrightarrow{\nabla} \mathbf{H}(\mathbf{curl}, \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\operatorname{div}, \Omega) \xrightarrow{\operatorname{div}} L^2(\Omega), \tag{2.59}$$

$$H^1_0(\Omega) \xrightarrow{\nabla} \mathbf{H}_0(\mathbf{curl}, \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}_0(\operatorname{div}, \Omega) \xrightarrow{\operatorname{div}} L^2(\Omega). \tag{2.60}$$

The diagram can be understood as follows: the range of each arrow is contained in the kernel of the following. For example, $\nabla(H^1_0(\Omega)) \subset \mathbf{H}_0(\mathbf{curl}\, 0, \Omega)$. This is sometimes referred to as *exact sequence*, although, from an algebraic point of view, the sequence is in general not exact: $\nabla(H^1_0(\Omega)) \neq \mathbf{H}_0(\mathbf{curl}\, 0, \Omega)$, due to topological concerns. In fact, there holds

$$\mathbf{H}_0(\mathbf{curl}\, 0, \Omega) = \nabla H^1_0(\Omega) \overset{\perp}{\oplus} \mathbf{Z}_N(\Omega), \tag{2.61}$$

with $\mathbf{Z}_N(\Omega) := \mathbf{H}_0(\mathbf{curl}\, 0, \Omega) \cap \mathbf{H}(\operatorname{div} 0, \Omega)$. Likewise, $\mathbf{H}_0(\operatorname{div} 0, \Omega) \neq \mathbf{curl}\, \mathbf{H}_0(\mathbf{curl}, \Omega)$ in general, but

$$\mathbf{H}_0(\operatorname{div} 0, \Omega) = \mathbf{curl}\, \mathbf{H}_0(\mathbf{curl}, \Omega) \overset{\perp}{\oplus} \mathbf{Z}_T(\Omega), \tag{2.62}$$

with $\mathbf{Z}_T(\Omega) := \mathbf{H}(\mathbf{curl}\, 0, \Omega) \cap \mathbf{H}_0(\operatorname{div} 0, \Omega)$.

One can go further in the characterization of the vanishing curl and divergence spaces $\mathbf{Z}_N(\Omega)$ and $\mathbf{Z}_T(\Omega)$, cf. [6]. They are finite dimensional, and their dimensions depend on the topology of Ω. In fact, there holds

$$\mathbf{Z}_N(\Omega) = \nabla Q_N(\Omega), \tag{2.63}$$

$$\mathbf{Z}_T(\Omega) = \widetilde{\nabla Q_T(\dot{\Omega})}, \tag{2.64}$$

where

$$Q_N(\Omega) := \left\{ q \in H^1(\Omega) \mid \Delta q = 0, q_{|\Gamma_0} = 0, \text{ and, for } 1 \leq k \leq K, q_{|\Gamma_k} \text{ constant} \right\}, \tag{2.65}$$

$$Q_T(\dot{\Omega}) := \left\{ \dot{q} \in H^1_{\text{zmv}}(\dot{\Omega}) \mid \operatorname{div} \widetilde{\nabla \dot{q}} = 0 \text{ in } \Omega,\ \widetilde{\nabla \dot{q}} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \text{ and, for } 1 \leq i \leq I, [\dot{q}]_{\Sigma_i} \text{ constant} \right\}, \tag{2.66}$$

and $\tilde{\cdot}$ denotes the continuation to Ω. Hence, $\dim \mathbf{Z}_N(\Omega) = K$, while $\dim \mathbf{Z}_N(\Omega) = I$.

Then, in general, considering a field with vanishing curl and divergence, even with vanishing normal or tangential trace, does not mean the whole field vanishes. This is true only when $\mathbf{Z}_N(\Omega)$ and $\mathbf{Z}_T(\Omega)$ reduce to zero, that is, if Ω is topologically trivial and of connected boundary.

2.3.4 Helmholtz decompositions

We are now in position to address the mathematical meaning behind (2.47). This is done thanks to the so-called *Helmholtz decompositions*. To begin with, let us introduce a few more function spaces:

$$\mathbf{X}_N(\Omega) := \mathbf{H}_0(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega), \tag{2.67}$$

$$\mathbf{X}_T(\Omega) := \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}_0(\mathrm{div}, \Omega); \tag{2.68}$$

and the corresponding spaces with vanishing divergence

$$\mathbf{K}_N(\Omega) := \mathbf{H}_0(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}\, 0, \Omega), \tag{2.69}$$

$$\mathbf{K}_T(\Omega) := \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}_0(\mathrm{div}\, 0, \Omega). \tag{2.70}$$

In these spaces, the subscript $_N$ denotes fields with non-zero normal trace, while the subscript $_T$ denotes fields with non-zero tangential trace. All those spaces are endowed with the norm $(\| \cdot \|_{\mathbf{L}^2} + \| \mathbf{curl} \cdot \|_{\mathbf{L}^2} + \| \mathrm{div} \cdot \|_{L^2})^{1/2}$, which reduces to the $\mathbf{H}(\mathbf{curl}, \Omega)$-norm for spaces with vanishing divergence.

Then, one can write a number of decompositions. The first one involves $\mathbf{H}_0(\mathbf{curl}, \Omega)$, and will be of particular interest for the study of Dirichlet problems. Likewise, the second one is a decomposition of $\mathbf{H}(\mathbf{curl}, \Omega)$, that will be useful when dealing with Neumann problems.

Theorem 2.3.7 (First Helmholtz decomposition)**.** The following decompositions hold:

$$\mathbf{L}^2(\Omega) = \nabla H^1_0(\Omega) \overset{\perp}{\oplus} \mathbf{H}(\mathrm{div}\, 0, \Omega); \tag{2.71}$$

$$\mathbf{H}_0(\mathbf{curl}, \Omega) = \nabla H^1_0(\Omega) \overset{\perp}{\oplus} \mathbf{K}_N(\Omega); \tag{2.72}$$

where orthogonality is taken, respectively, in the sense of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}(\mathbf{curl}, \Omega)$.

Theorem 2.3.8 (Second Helmholtz decomposition)**.** The following decompositions hold:

$$\mathbf{L}^2(\Omega) = \nabla H^1_{\mathrm{zmv}}(\Omega) \overset{\perp}{\oplus} \mathbf{H}_0(\mathrm{div}\, 0, \Omega); \tag{2.73}$$

$$\mathbf{H}(\mathbf{curl}, \Omega) = \nabla H^1_{\mathrm{zmv}}(\Omega) \overset{\perp}{\oplus} \mathbf{K}_T(\Omega); \tag{2.74}$$

where orthogonality is taken, respectively, in the sense of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}(\mathbf{curl}, \Omega)$.

Moreover, one can go further, introducing decompositions of vanishing divergence spaces. In fact, there holds

$$\mathbf{H}(\mathrm{div}\, 0, \Omega) = \nabla Q_N(\Omega) \overset{\perp}{\oplus} \mathbf{curl}\, \mathbf{H}^1(\Omega); \tag{2.75}$$

$$\mathbf{H}_0(\mathrm{div}\, 0, \Omega) = \widetilde{\nabla Q_T(\dot{\Omega})} \overset{\perp}{\oplus} \mathbf{curl}\, \mathbf{H}_0(\mathbf{curl}, \Omega), \tag{2.76}$$

where orthogonality is taken in the sense of $\mathbf{H}(\mathrm{div}, \Omega)$. Because all fields have vanishing divergence, this amounts to $\mathbf{L}^2(\Omega)$-orthogonality. Therefore, the following three-terms decompositions hold:

$$\mathbf{L}^2(\Omega) = \nabla H^1_0(\Omega) \overset{\perp}{\oplus} \nabla Q_N(\Omega) \overset{\perp}{\oplus} \mathbf{curl}\, \mathbf{H}^1(\Omega); \tag{2.77}$$

$$\mathbf{L}^2(\Omega) = \nabla H^1_{\mathrm{zmv}}(\Omega) \overset{\perp}{\oplus} \widetilde{\nabla Q_T(\dot{\Omega})} \overset{\perp}{\oplus} \mathbf{curl}\, \mathbf{H}_0(\mathbf{curl}, \Omega), \tag{2.78}$$

where, in the first decomposition, $\nabla H^1_0(\Omega) \overset{\perp}{\oplus} \nabla Q_N(\Omega)$ is the vanishing curl part, and $\nabla Q_N(\Omega) \overset{\perp}{\oplus} \mathbf{curl}\, \mathbf{H}^1(\Omega)$ is the vanishing divergence part; the same holds for the second one.

Let us introduce a third Helmholtz decomposition, which appears for instance in [86]. It will be useful for the study of the problem with Robin condition. We introduce the space

$$\mathbf{H}^+(\mathbf{curl}, \Omega) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega), \gamma^T \mathbf{v} \in \mathbf{L}^2_t(\Gamma)\} \tag{2.79}$$

which is sometimes denoted $\mathbf{H}^{\text{imp}}(\mathbf{curl}, \Omega)$ in the literature. It is a Hilbert space, endowed with the inner product $(\mathbf{u}, \mathbf{v})_{\mathbf{H}^+(\mathbf{curl})} = (\mathbf{u}, \mathbf{v})_{\mathbf{H}(\mathbf{curl})} + (\gamma^T \mathbf{u}, \gamma^T \mathbf{v})_{\mathbf{L}^2_t(\Gamma)}$. Let us also introduce

$$\mathbf{W}_N(\Omega) := \mathbf{H}^+(\mathbf{curl}, \Omega) \cap \mathbf{H}(\operatorname{div} 0, \Omega). \tag{2.80}$$

Then, one has the following Helmholtz decomposition of $\mathbf{H}^+(\mathbf{curl}, \Omega)$.

Theorem 2.3.9 (Third Helmholtz decomposition)**.** The following Helmholtz decomposition holds:

$$\mathbf{H}^+(\mathbf{curl}, \Omega) = \nabla H^1_0(\Omega) \overset{\perp}{\oplus} \mathbf{W}_N(\Omega), \tag{2.81}$$

where orthogonality is taken in the sense of $\mathbf{H}^+(\mathbf{curl}, \Omega)$.

We finish the subsection by mentioning two surface Helmholtz decompositions (also called *Hodge decompositions*), proved in [19, 21]. Let us assume first that Γ is topologically trivial. Introduce

$$\mathcal{H}(\Gamma) := \left\{ v \in H^1_{\text{zmv}}(\Gamma),\ \Delta_\Gamma v \in H^{-1/2}(\Gamma) \right\}. \tag{2.82}$$

Theorem 2.3.10. If Γ is topologically trivial, the following decompositions hold:

$$\mathbf{H}^{-1/2}_{\parallel}(\operatorname{div}_\Gamma, \Gamma) = \mathbf{curl}_\Gamma(H^{1/2}(\Gamma)) \oplus \nabla_\Gamma(\mathcal{H}(\Gamma)); \tag{2.83}$$

$$\mathbf{H}^{-1/2}_{\perp}(\operatorname{curl}_\Gamma, \Gamma) = \nabla_\Gamma(H^{1/2}(\Gamma)) \oplus \mathbf{curl}_\Gamma(\mathcal{H}(\Gamma)). \tag{2.84}$$

In the more general case of a non-topologically trivial boundary [16], one has to introduce a third space in the decompositions,

$$\mathbb{H} := \left\{ \mathbf{u} \in \mathbf{L}^2_t(\Gamma),\ \operatorname{curl}_\Gamma \mathbf{u} = 0,\ \operatorname{div}_\Gamma \mathbf{u} = 0 \right\}, \tag{2.85}$$

which is finite-dimensional. Then, one has the next three-terms decompositions [16, Theorem 3].

Theorem 2.3.11. More generally, one has the following decompositions:

$$\mathbf{H}^{-1/2}_{\parallel}(\operatorname{div}_\Gamma, \Gamma) = \mathbf{curl}_\Gamma(H^{1/2}(\Gamma)) \oplus \mathbb{H} \oplus \nabla_\Gamma(\mathcal{H}(\Gamma)); \tag{2.86}$$

$$\mathbf{H}^{-1/2}_{\perp}(\operatorname{curl}_\Gamma, \Gamma) = \nabla_\Gamma(H^{1/2}(\Gamma)) \oplus \mathbb{H} \oplus \mathbf{curl}_\Gamma(\mathcal{H}(\Gamma)). \tag{2.87}$$

2.4 Inequalities and embeddings

In this section, we present important functional analysis results for some of the function spaces introduced above. The first ones are Poincaré inequalities, that state a norm equivalence in some subspaces of $H^1(\Omega)$. The next ones are Weber inequalities and compact embeddings that concern the spaces $\mathbf{X}_N(\Omega)$, $\mathbf{X}_T(\Omega)$ and $\mathbf{W}_N(\Omega)$. These results will be helpful for the study of the well-posedness of Maxwell problems.

2.4.1 Poincaré inequalities

To begin with, let us mention a couple of properties of subspaces of $H^1(\Omega)$ that will be useful throughout this work. The first one concerns a norm equivalence in $H^1_0(\Omega)$. It is referred to as *Poincaré inequality*.

Theorem 2.4.1 (Poincaré inequality)**.** There exists a constant $C_P > 0$, dependent only on Ω, s.t.

$$\forall v \in H^1_0(\Omega), \quad \|v\|_{L^2} \leq C_P \|\nabla v\|_{\mathbf{L}^2}. \tag{2.88}$$

Hence, the mapping $v \mapsto \|\nabla v\|_{\mathbf{L}^2}$ defines a norm on $H^1_0(\Omega)$, equivalent to the norm $\|\cdot\|_{H^1}$, and denoted $\|\cdot\|_{H^1_0}$.

By extension, the terminology *Poincaré inequalities* denotes also a class of properties which are similar to Theorem 2.4.1. In particular, it is also valid for fields whose trace vanishes only on a part on the boundary. Let Γ_0 a non-negligible subset of Γ, and the space

$$H^1_{0,\Gamma_0}(\Omega) := \left\{ v \in H^1(\Omega),\ v_{|\Gamma_0} = 0 \right\}. \tag{2.89}$$

In this space, there holds a Poincaré inequality.

Theorem 2.4.2. There exists a constant $C > 0$, dependent only on Ω and Γ_0, s.t.

$$\forall v \in H^1_{0,\Gamma_0}(\Omega), \quad \|v\|_{L^2} \leq C \|\nabla v\|_{\mathbf{L}^2}. \tag{2.90}$$

Hence, the mapping $v \mapsto \|\nabla v\|_{\mathbf{L}^2}$ defines a norm on $H^1_{0,\Gamma_0}(\Omega)$, equivalent to the norm $\|\cdot\|_{H^1}$.

One also has a counterpart to Poincaré inequality for fields with zero mean value, which is generally called *Poincaré-Wirtinger inequality.*

Theorem 2.4.3 (Poincaré-Wirtinger inequality)**.** There exists a constant $C'_P > 0$, dependent only on Ω, s.t.

$$\forall v \in H^1_{\mathrm{zmv}}(\Omega), \quad \|v\|_{L^2} \leq C'_P \|\nabla v\|_{\mathbf{L}^2}. \tag{2.91}$$

Hence, the mapping $v \mapsto \|\nabla v\|_{\mathbf{L}^2}$ defines a norm on $H^1_{\mathrm{zmv}}(\Omega)$, equivalent to the norm $\|\cdot\|_{H^1}$, and denoted $\|\cdot\|_{H^1_{\mathrm{zmv}}}$.

Remark 2.4.4. All those results rely importantly on the assumption that Ω is bounded, and are false, in general, if Ω is unbounded.

2.4.2 Weber inequalities

In the following, we mention useful results that concern the spaces $\mathbf{X}_N(\Omega)$ and $\mathbf{X}_T(\Omega)$. Those results are of two types. The first ones are *Weber inequalities* (sometimes referred to as *Friedrich* or *Gaffney inequalities*), which, somehow as Poincaré inequalities, can be understood as norm equivalence results. The second ones are compact embeddings. All those results are due to Weber [114].

In $\mathbf{X}_N(\Omega)$, it is possible to control the $\mathbf{L}^2(\Omega)$-norm of the fields, somehow in the spirit of Poincaré inequalities.

Theorem 2.4.5 (First Weber inequality)**.** There exists a constant $C_W > 0$ s.t., for all $\mathbf{v}$ in $\mathbf{X}_N(\Omega)$,

$$\|\mathbf{v}\|_{\mathbf{L}^2} \leq C_W \left(\|\,\mathbf{curl}\,\mathbf{v}\|_{\mathbf{L}^2} + \|\operatorname{div}\mathbf{v}\|_{L^2} + \sum_{k=1}^{K} |\langle \mathbf{v}\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Gamma_k)}| \right). \tag{2.92}$$

One has a similar result for elements of $\mathbf{X}_T(\Omega)$.

Theorem 2.4.6 (Second Weber inequality)**.** Let Ω be a domain of genus $I \geq 0$. There exists a constant $C'_W > 0$ s.t., for all $\mathbf{v}$ in $\mathbf{X}_T(\Omega)$,

$$\|\mathbf{v}\|_{\mathbf{L}^2} \leq C'_W \left(\|\,\mathbf{curl}\,\mathbf{v}\|_{\mathbf{L}^2} + \|\operatorname{div}\mathbf{v}\|_{L^2} + \sum_{i=1}^{I} |\langle \mathbf{v}\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Sigma_i)}| \right). \tag{2.93}$$

2.4.3 Compact embeddings

Let us note some compact embedding results. The first one concern the classical Sobolev spaces. It is generally referred to as *compact Sobolev embeddings* and also as *Rellich-Kondrachov* or *Rellich theorem.*

Theorem 2.4.7. Let $s' > s > 0$. Then the embedding of $H^{s'}(\Omega)$ into $H^s(\Omega)$ is compact.

One also has the next compact embedding results, that are related to Weber inequalities and were also proven by Weber [114].

Theorem 2.4.8. The embedding of $\mathbf{X}_N(\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact.

Theorem 2.4.9. The embedding of $\mathbf{X}_T(\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact.

Finally, one also has such a result for $\mathbf{W}_N(\Omega)$ (see [86]).

Theorem 2.4.10. The embedding of $\mathbf{W}_N(\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact.

Remark 2.4.11. Again, all these results rely on the assumption that Ω is bounded, and are false in general when Ω is unbounded.

2.5 Analysis of classical Maxwell problems

To carry out the analysis of PDE problems, one generally rewrites the problem into an equivalent variational formulation. To conclude the chapter, we present the tools that allow us to study the well-posedness of variational formulations, and, in a pedagogical view, we apply them to a classical Maxwell example.

We recall that a bilinear form on a Hilbert space V is said *coercive* on V iff $\exists C > 0,\ \forall v \in V,\ |a(v,v)| \geq C\|v\|_V^2$. The first result that guarantees the well-posedness of such problems is the famous Lax-Milgram lemma.

Theorem 2.5.1 (Lax-Milgram lemma)**.** Let V a Hilbert space, and consider the problem

$$\left|\begin{array}{l}\text{Find } u \in V \text{ s.t., } \forall v \in V,\\ \qquad a(u,v) = \langle f, v\rangle.\end{array}\right. \tag{2.94}$$

If $a(\cdot,\cdot)$ is a coercive, continuous sesquilinear form on V, then the problem (2.94) is well-posed: for all $f \in V'$, there exists a unique $u \in V$ solution to (2.94), which depends continuously on f: $\exists C > 0$, independent of u and f, s.t. $\|u\|_V \leq C\|f\|_{V'}$.

However, in many time-harmonic problems, Lax-Milgram lemma does not apply, because the form is not coercive. In this case, one relies rather on Fredholm alternative.

Theorem 2.5.2 (Fredholm alternative)**.** Given two Hilbert spaces V and H with $V \subset H$, $a(\cdot,\cdot)$ a continuous sesquilinear form on $V \times V$, and $b(\cdot,\cdot)$ a continuous sesquilinear form on $H \times V$, consider the problem

$$\left|\begin{array}{l}\text{Find } u \in V \text{ s.t., } \forall v \in V,\\ a(u,v) + b(u,v) = \langle f, v\rangle.\end{array}\right. \tag{2.95}$$

If $a(\cdot,\cdot)$ is coercive on V and the embedding of V into H is compact, then,

- either, for all f in V', the problem (2.95) admits a unique solution $u \in V$, which depends continuously on f;
- or, the problem (2.95) has solutions if, and only if, f satisfies a finite number n_b of orthogonality conditions;

in this case, the space of solutions is an affine space of dimension n_b. Additionally, the component of the solution which is orthogonal to the corresponding linear vector space, depends continuously on f.

When a formulation can be split with a coercive part and a compact part, as in (2.95), we sometimes say that it enters the *"coercive + compact" framework*. For such problems, the Fredholm alternative applies.

Another consequence of Fredholm alternative is that "existence and uniqueness are equivalent". Thus, if one is able to prove that the solution is unique (using, for example, a unique continuation principle, see [113], [90]), then the problem is well-posed.

A classical Maxwell problem

Let us take as an example the time-harmonic Maxwell equation expressed for the electric field in a homogeneous isotropic medium, completed with a homogeneous Dirichlet condition:

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \omega^2\mathbf{E} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{2.96}$$

where we assume $\mathbf{f} \in \mathbf{L}^2(\Omega)$, and, for simplicity, $\underline{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{\mu}} = \underline{\mathbf{I}}$. We will naturally look for the solution in $\mathbf{H}(\mathbf{curl},\Omega)$. More specifically, because of the boundary condition, one actually has $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl},\Omega)$.

The variational formulation of this problem is obtained by multiplying the volume equation of (2.96) by $\mathbf{F} \in \mathbf{H}_0(\mathbf{curl},\Omega)$, then integrating by parts using formula (2.44). Thus, one gets

$$\left|\begin{array}{l} \text{Find } \mathbf{E} \in \mathbf{H}_0(\mathbf{curl},\Omega) \text{ s.t., } \forall \mathbf{F} \in \mathbf{H}_0(\mathbf{curl},\Omega), \\ \quad (\mathbf{curl}\,\mathbf{E}|\,\mathbf{curl}\,\mathbf{F}) - \omega^2\,(\mathbf{E}|\mathbf{F}) = (\mathbf{f}|\mathbf{F})\,. \end{array}\right. \tag{2.97}$$

Using the Helmholtz decomposition of Theorem 2.3.7, we split $\mathbf{E}$ into two parts: $\mathbf{E} = \nabla p + \tilde{\mathbf{E}}$, with $p \in H_0^1(\Omega)$ and $\tilde{\mathbf{E}} \in \mathbf{K}_N(\Omega)$. Moreover, we can write the two variational formulations satisfied by p and $\tilde{\mathbf{E}}$:

$$\left|\begin{array}{l} \text{Find } p \in H_0^1(\Omega) \text{ s.t., } \forall q \in H_0^1(\Omega), \\ \quad -\omega^2\,(\nabla p|\nabla q) = (\mathbf{f}|\nabla q) \end{array}\right. \tag{2.98}$$

and

$$\left|\begin{array}{l} \text{Find } \tilde{\mathbf{E}} \in \mathbf{K}_N(\Omega) \text{ s.t., } \forall \tilde{\mathbf{F}} \in \mathbf{K}_N(\Omega), \\ (\mathbf{curl}\,\tilde{\mathbf{E}}|\,\mathbf{curl}\,\tilde{\mathbf{F}}) - \omega^2\,(\tilde{\mathbf{E}}|\tilde{\mathbf{F}}) = (\mathbf{f}|\tilde{\mathbf{F}}). \end{array}\right. \tag{2.99}$$

They are obtained by writing $\mathbf{E} = \nabla p + \tilde{\mathbf{E}}$ and taking as test functions ∇q, $q \in H_0^1(\Omega)$ or $\tilde{\mathbf{F}} \in \mathbf{K}_N(\Omega)$, respectively. Conversely, one sums formulations (2.98) and (2.99). Taking advantage of the Helmholtz decomposition (2.72), and posing $\mathbf{E} = \nabla p + \tilde{\mathbf{E}}$, one gets that (2.97) holds for all $\mathbf{F} = \nabla q + \tilde{\mathbf{F}}$, therefore for all $\mathbf{F} \in \mathbf{H}_0(\mathbf{curl},\Omega)$. Hence, the couple of formulations (2.98)-(2.99) is well equivalent to (2.97).

Then, the formulation (2.98) enters the scope of Lax-Milgram lemma (Theorem 2.5.1). Indeed, the corresponding bilinear form is continuous and coercive on $H_0^1(\Omega)$, thanks to Poincaré inequality (Theorem 2.4.1). Therefore, this formulation is well-posed.

Concerning the formulation (2.99), it enters the scope of Fredholm alternative. Indeed, one can split the bilinear form into two parts:

$$a(\mathbf{u},\mathbf{v}) := (\mathbf{curl}\,\mathbf{u}|\,\mathbf{curl}\,\mathbf{v}) + (\mathbf{u}|\mathbf{v})\,, \tag{2.100}$$

which is continuous and coercive on $\mathbf{K}_N(\Omega)$ ($\mathbf{K}_N(\Omega)$ and $\mathbf{H}(\mathbf{curl},\Omega)$ share the same norm), and

$$b(\mathbf{u},\mathbf{v}) := (-\omega^2 - 1)\,(\mathbf{u}|\mathbf{v})\,, \tag{2.101}$$

which is continuous on $\mathbf{L}^2(\Omega) \times \mathbf{K}_N(\Omega)$. As the embedding of $\mathbf{K}_N(\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact (Theorem 2.4.8), one can conclude by Fredholm alternative.

One could proceed similarly with the Neumann or Robin problems. Each boundary condition will lead to a variational formulation posed in a different subspace of $\mathbf{H}(\mathbf{curl},\Omega)$, leading to use slightly different tools adapted to this formulation. Hence, for the Neumann problem, the variational space is the whole space $\mathbf{H}(\mathbf{curl}, \Omega)$. This leads to use the second Helmholtz decomposition of Theorem 2.3.8, the Poincaré-Wirtinger inequality (Theorem 2.4.3) for the first problem and the compact embedding of Theorem 2.4.9 for the second one. For the Robin problem, the variational space would rather be $\mathbf{H}^{+}(\mathbf{curl}, \Omega)$ (one has to check, however, that the boundary condition indeed holds in $\mathbf{L}^2_t(\Gamma)$; this point is discussed in Chapter 4). This leads to use the third Helmholtz decomposition of Theorem 2.3.9 and the compact embedding of Theorem 2.4.10.

Conclusion

We have presented the main tools that are classically used for the study of time-harmonic Maxwell problems. We introduced the necessary function spaces, traces and operators. With this, we are able to write variational formulations for Maxwell problems. We also provided an overview on decompositions, inequalities and compact embeddings that hold for these different spaces. Using these tools, combined with Lax-Milgram lemma and Fredholm alternative, we are able to state the well-posedness of classical Maxwell problems in isotropic media. However, these tools are not sufficient to deal with more complex Maxwell problems, in particular when the material coefficients $\underline{\boldsymbol{\varepsilon}}$, $\underline{\boldsymbol{\mu}}$ are anisotropic. It is then necessary to extend the classical tools presented in this chapter to more complex cases. This is the point of the next chapter.

Chapter 3

Extended tools for the study of anisotropic problems

In this chapter, we generalise to the anisotropic case the tools of the previous chapter that classically arise in the study of Maxwell problems: Helmholtz decompositions, Weber inequalities, and compact embedding results. We extend these results to the more general case of elliptic (possibly non-Hermitian) tensors. Throughout this chapter, we assume that Ω is a Lipschitz domain as introduced in Section 2.2. In Section 3.1 we introduce the ellipticity condition and derive some basic properties. Section 3.2 is devoted to the extension of Helmholtz decompositions, and Section 3.3 to the extension of Weber inequalities and compact embedding results.

3.1 Ellipticity condition

Definition 3.1.1. We say that a tensor field $\underline{\boldsymbol{\xi}}$ is *elliptic* iff $\underline{\boldsymbol{\xi}} \in \underline{\mathbf{L}}^\infty(\Omega)$ and

$$\exists \theta_\xi \in \mathbb{R},\ \exists \xi_- > 0,\quad \text{a.e. in } \Omega,\ \forall \mathbf{z} \in \mathbb{C}^3,\qquad \xi_-|\mathbf{z}|^2 \leq \Re[e^{i\theta_\xi} \cdot \mathbf{z}^* \underline{\boldsymbol{\xi}} \mathbf{z}]. \tag{3.1}$$

Additionally, we will use the notation $\xi_+ := \|\underline{\boldsymbol{\xi}}\|_{\underline{\mathbf{L}}^\infty}$.

The condition can be understood as follows: there exists a "coercivity direction" for $\underline{\boldsymbol{\xi}}$ in the complex plane, given by θ_ξ, while ξ_- gives the "magnitude" of the coercivity. Note that the constants θ_ξ and ξ_- are fixed in the sense that they do not depend on $\mathbf{x}$, but they are not unique. In particular, we denote Θ_ξ the range of admissible directions θ_ξ.

Remark 3.1.2. When the tensor $\underline{\boldsymbol{\xi}}$ is isotropic, i.e. $\underline{\boldsymbol{\xi}}(\mathbf{x}) = \xi(\mathbf{x})\underline{\mathbf{I}}$, the ellipticity condition reduces to

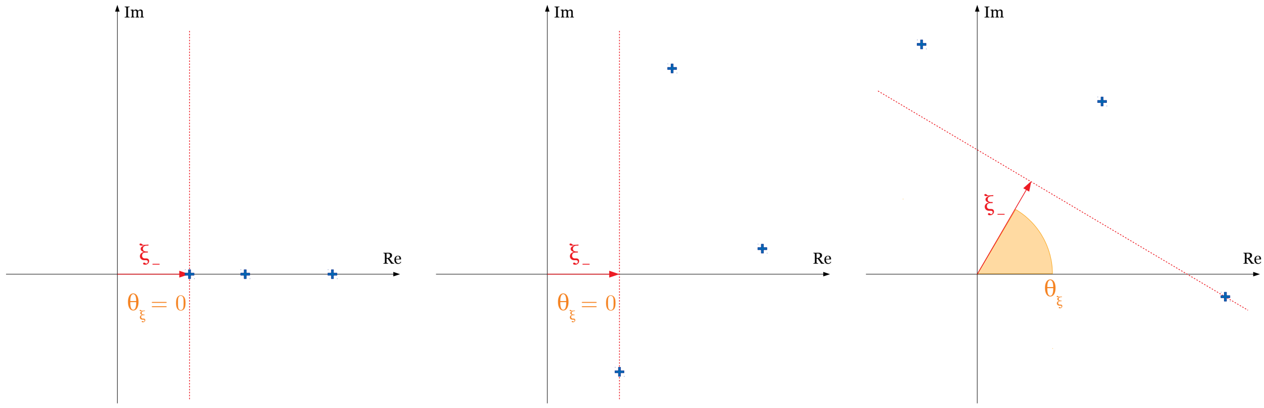
$$\exists \theta_\xi \in \mathbb{R},\ \exists \xi_- > 0,\quad \text{a.e. in } \Omega,\quad \xi_- \leq \Re[e^{i\theta_\xi}\, \xi]. \tag{3.2}$$

Therefore, we shall also use this terminology for scalar fields: a scalar field ξ that satisfies (3.2) is said *elliptic*. A scalar field ξ is elliptic iff $\xi\underline{\mathbf{I}}$ is an elliptic tensor field. In the literature, scalar fields that satisfy (3.2) are also frequently said to be *bounded by below*. Again, θ_ξ is in general not unique, and the range of admissible values for θ_ξ is denoted Θ_ξ.

Remark 3.1.3. Ellipticity can be interpreted in terms of tensor eigenvalues: the condition (3.1) implies that, almost everywhere in Ω, the eigenvalues of $\underline{\boldsymbol{\xi}}(\mathbf{x})$ are contained in a fixed open half-plane of $\mathbb{C}$. However, the converse implication is false in general. For example, in 2D, consider the matrix $\underline{\mathbf{M}} = \begin{pmatrix} 1 & -4 \\ 0 & 2 \end{pmatrix}$. Its eigenvalues are strictly positive, but it is not elliptic: taking $\mathbf{v} = (1,0)^T$ and $\mathbf{w} = (1,1)^T$, one has $\mathbf{v}^*\underline{\mathbf{M}}\mathbf{v} = 1$, but $\mathbf{w}^*\underline{\mathbf{M}}\mathbf{w} = -1$. In fact, it seems that both conditions are equivalent *for normal tensors only.*

Because $\underline{\boldsymbol{\xi}}$ can be non-Hermitian, the mapping $(\mathbf{v}, \mathbf{w}) \mapsto (\underline{\boldsymbol{\xi}}\mathbf{v}|\mathbf{w})$ is, in general, not a scalar product in $\mathbf{L}^2(\Omega)$; orthogonality properties are lateralized, in the sense that $(\underline{\boldsymbol{\xi}}\mathbf{v}|\mathbf{w}) = 0$ is not equivalent to $(\underline{\boldsymbol{\xi}}\mathbf{w}|\mathbf{v}) = 0$.

Let us compare the condition (3.1) to what is usually done in the literature. Most of the results of this chapter are known for Hermitian definite positive tensors (see e.g. [7]). In this case, one has a notion of orthogonality: the mapping $(\underline{\boldsymbol{\xi}} \cdot |\cdot)$ defines a scalar product, which is helpful in many proof techniques. This property makes this case relatively close to the isotropic case. Looking at the eigenvalues of $\underline{\boldsymbol{\xi}}$, they are all real and strictly positive.



(a) Hermitian definite positive tensor (b) tensor with elliptic real part (3.3) (c) elliptic tensor (3.1)

Figure 3.1: Representation of tensors eigenvalues

To the best of our knowledge, few authors address these questions in the case of non-Hermitian tensors. In a recent paper [115], tensors with elliptic real part are considered, i.e.

$$\exists \xi_- > 0, \forall \mathbf{z} \in \mathbb{C}^3, \xi_-|\mathbf{z}|^2 \leq \Re[\mathbf{z}^* \underline{\boldsymbol{\xi}} \mathbf{z}] \quad \text{a.e. in } \Omega. \tag{3.3}$$

This condition is equivalent to the one of [2], where tensors of definite positive Hermitian part are considered. As we noticed, relaxing the Hermitian assumption on $\underline{\boldsymbol{\xi}}$ goes with a loss of orthogonality. With the condition (3.3), the eigenvalues of $\underline{\boldsymbol{\xi}}$ are all contained in the half-plane of $\mathbb{C}$ of positive real part; the imaginary part can be non-zero, contrarily to the Hermitian case. For example, in [4], a model with conductivity is considered. The tensors there are symmetric and definite positive, but complex-valued (therefore not Hermitian). This case is encompassed by the condition (3.3). In these works, authors focus on the Dirichlet problem. In particular, the compact embedding of $\mathbf{X}_N(\underline{\boldsymbol{\xi}};\Omega)$ into $\mathbf{L}^2(\Omega)$ is proven.

Our condition is similar to (3.3), allowing also its rotations in the complex plane (see Fig. 3.1). It is, therefore, more general. As we shall see, it is actually not necessary to assume specifically that the real part is elliptic. In [8], a model from plasma theory is analysed. The authors show that the real part of the eigenvalues do not have a constant sign, so, it does not enter the scope of [115]. However, the imaginary part is elliptic (the authors also show that the tensor is normal). Therefore, this case is covered by our condition, with $\theta_\xi = \pi/2$. This example has motivated us to design the condition (3.1), independently of the work of [115]. Let us also point out that, contrarily to most authors, we treat the three main types of boundary conditions: Dirichlet, Neumann, and Robin.

From now on, $\underline{\boldsymbol{\xi}}$ denotes a tensor that satisfies assumption (3.1). Next, let us state some simple properties that follow from (3.1). First of all, an elliptic tensor has an inverse which is also elliptic.

Proposition 3.1.4. If $\underline{\boldsymbol{\xi}}$ is elliptic, then $\underline{\boldsymbol{\xi}}^{-1}$ is well-defined in $\underline{\mathbf{L}}^\infty(\Omega)$, and is elliptic as well, with $\theta_{\xi^{-1}} = -\theta_\xi$, $\xi_+^{\text{inv}} \leq \xi_-^{-1}$ and $\xi_-^{\text{inv}} = \xi_- \xi_+^{-2}$.

Proof. Let $\mathbf{x}$ in Ω s.t. $\underline{\boldsymbol{\xi}}(\mathbf{x})$ is well-defined. One can first notice that under assumption (3.1), $\underline{\boldsymbol{\xi}}(\mathbf{x})\mathbf{z} = \mathbf{0}$ only if $\mathbf{z} = \mathbf{0}$. Then $\underline{\boldsymbol{\xi}}(\mathbf{x})$ is injective and we can define $\underline{\boldsymbol{\xi}}^{-1}(\mathbf{x})$.

Let $\mathbf{z} \in \mathbb{C}^3$, we pose $\mathbf{y} = \underline{\boldsymbol{\xi}}^{-1}(\mathbf{x})\mathbf{z}$. One has $\xi_-|\mathbf{y}|^2 \leq |\mathbf{y}^* \underline{\boldsymbol{\xi}}(\mathbf{x})\mathbf{y}| \leq |\mathbf{y}||\underline{\boldsymbol{\xi}}(\mathbf{x})\mathbf{y}|$, then $\xi_-|\mathbf{y}| \leq |\underline{\boldsymbol{\xi}}\mathbf{y}|$. We thus have that

$$|\underline{\boldsymbol{\xi}}^{-1}\mathbf{z}| \leq \xi_-^{-1}|\mathbf{z}| \quad \text{a.e. in } \Omega,$$

hence $\underline{\boldsymbol{\xi}}^{-1} \in \underline{\mathbf{L}}^\infty(\Omega)$.

Moreover, it holds that $|\underline{\boldsymbol{\xi}}\mathbf{y}| \leq \xi_+|\mathbf{y}|$ a.e. in Ω, so $\xi_+^{-1}|\mathbf{z}| \leq |\underline{\boldsymbol{\xi}}^{-1}\mathbf{z}|$. Then, $\xi_-|\mathbf{y}|^2 \leq \Re[e^{i\theta_\xi} \cdot \mathbf{y}^*\underline{\boldsymbol{\xi}}\mathbf{y}]$ yields

$$\begin{aligned} \xi_-|\underline{\boldsymbol{\xi}}^{-1}\mathbf{z}|^2 &\leq \Re[e^{i\theta_\xi} \cdot \mathbf{z}^*(\underline{\boldsymbol{\xi}}^{-1})^*\mathbf{z}] \\ \xi_-|\underline{\boldsymbol{\xi}}^{-1}\mathbf{z}|^2 &\leq \Re[e^{-i\theta_\xi} \cdot \mathbf{z}^*\underline{\boldsymbol{\xi}}^{-1}\mathbf{z}] \\ \frac{\xi_-}{\xi_+^2}|\mathbf{z}|^2 &\leq \Re[e^{-i\theta_\xi} \cdot \mathbf{z}^*\underline{\boldsymbol{\xi}}^{-1}\mathbf{z}]. \end{aligned}$$

To sum up, $\underline{\boldsymbol{\xi}}^{-1}$ satisfies an ellipticity condition, with: $\theta_{\xi^{-1}} = -\theta_\xi$, and upper and lower bounds given by $\xi_+^{\text{inv}} \leq \xi_-^{-1}$ and $\xi_-^{\text{inv}} := \xi_-\xi_+^{-2}$. □

We can also note the following property.

Proposition 3.1.5. For any $\mathbf{v} \in \mathbf{L}^2(\Omega)$, one has the following inequalities:

$$\xi_-\|\mathbf{v}\|_{\mathbf{L}^2}^2 \leq \Re\left[e^{i\theta_\xi}\left(\underline{\boldsymbol{\xi}}\mathbf{v}|\mathbf{v}\right)\right] \leq \left|\left(\underline{\boldsymbol{\xi}}\mathbf{v}|\mathbf{v}\right)\right| \leq \xi_+\|\mathbf{v}\|_{\mathbf{L}^2}^2. \tag{3.4}$$

A simple consequence of this is the well-posedness of the associated scalar problems.

Theorem 3.1.6. The Dirichlet problem

$$\left|\begin{aligned} &\text{Find } p \in H_0^1(\Omega) \text{ s.t., } \forall q \in H_0^1(\Omega), \\ &\qquad\qquad \left(\underline{\boldsymbol{\xi}}\nabla p|\nabla q\right) = \ell(q), \end{aligned}\right. \tag{3.5}$$

that is equivalent to

Find $p \in H^1(\Omega)$ s.t.

$$\begin{cases} -\operatorname{div}\underline{\boldsymbol{\xi}}\nabla p = \ell \text{ in } \Omega, \\ \qquad\qquad p = 0 \text{ on } \Gamma, \end{cases} \tag{3.6}$$

is well-posed for all ℓ in $\left(H_0^1(\Omega)\right)' = H^{-1}(\Omega)$, i.e.

$$\exists C > 0,\ \forall \ell \in \left(H_0^1(\Omega)\right)', \exists!\ p \text{ solution to (3.5), with } \|p\|_{H_0^1} \leq C\|\ell\|_{(H_0^1)'}.$$

Theorem 3.1.7. The Neumann problem

$$\left|\begin{aligned} &\text{Find } p \in H_{\text{zmv}}^1(\Omega) \text{ s.t., } \forall q \in H_{\text{zmv}}^1(\Omega), \\ &\qquad\qquad \left(\underline{\boldsymbol{\xi}}\nabla p|\nabla q\right) = \ell(q), \end{aligned}\right. \tag{3.7}$$

that is equivalent, if $\ell \in \mathbf{L}^2(\Omega)$, to

Find $p \in H_{\text{zmv}}^1(\Omega)$ s.t.

$$\begin{cases} -\operatorname{div}\underline{\boldsymbol{\xi}}\nabla p = \ell \text{ in } \Omega, \\ \quad \underline{\boldsymbol{\xi}}\nabla p \cdot \mathbf{n} = 0 \text{ on } \Gamma, \end{cases} \tag{3.8}$$

is well-posed for all ℓ in $\left(H_{\text{zmv}}^1(\Omega)\right)'$, that is

$$\exists C > 0,\ \forall \ell \in \left(H_{\text{zmv}}^1(\Omega)\right)', \exists!\ p \text{ solution to (3.7), with } \|p\|_{H_{\text{zmv}}^1} \leq C\|\ell\|_{(H_{\text{zmv}}^1)'}.$$

Proof. It follows from Lax-Milgram lemma, relations (3.4) and Poincaré (resp. Poincaré-Wirtinger) inequality. □

3.2 Helmholtz decompositions

To begin this section, let us introduce some more function spaces, related to the operator $\operatorname{div}\underline{\boldsymbol{\xi}}\cdot$ instead of just div:

$$\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}},\Omega) := \{\mathbf{v}\in\mathbf{L}^2(\Omega),\ \underline{\boldsymbol{\xi}}\mathbf{v}\in\mathbf{H}(\operatorname{div},\Omega)\}, \tag{3.9}$$
$$\mathbf{H}_0(\operatorname{div}\underline{\boldsymbol{\xi}},\Omega) := \{\mathbf{v}\in\mathbf{L}^2(\Omega),\ \underline{\boldsymbol{\xi}}\mathbf{v}\in\mathbf{H}_0(\operatorname{div},\Omega)\}, \tag{3.10}$$
$$\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega) := \{\mathbf{v}\in\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}},\Omega),\ \operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{v}=0\}, \tag{3.11}$$
$$\mathbf{H}_0(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega) := \mathbf{H}_0(\operatorname{div}\underline{\boldsymbol{\xi}},\Omega)\cap\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega), \tag{3.12}$$

which are Hilbert spaces endowed with the norm $\|\cdot\|_{\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}})} := \left(\|\cdot\|^2_{\mathbf{L}^2}+\|\operatorname{div}\underline{\boldsymbol{\xi}}\cdot\|^2_{L^2}\right)^{1/2}$. We also introduce

$$\mathbf{X}_N(\underline{\boldsymbol{\xi}};\Omega) := \mathbf{H}_0(\mathbf{curl},\Omega)\cap\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}},\Omega), \tag{3.13}$$
$$\mathbf{X}_T(\underline{\boldsymbol{\xi}};\Omega) := \mathbf{H}(\mathbf{curl},\Omega)\cap\mathbf{H}_0(\operatorname{div}\underline{\boldsymbol{\xi}},\Omega), \tag{3.14}$$
$$\mathbf{K}_N(\underline{\boldsymbol{\xi}};\Omega) := \mathbf{H}_0(\mathbf{curl},\Omega)\cap\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega), \tag{3.15}$$
$$\mathbf{K}_T(\underline{\boldsymbol{\xi}};\Omega) := \mathbf{H}(\mathbf{curl},\Omega)\cap\mathbf{H}_0(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega), \tag{3.16}$$
$$\mathbf{W}_N(\underline{\boldsymbol{\xi}};\Omega) := \mathbf{H}^+(\mathbf{curl},\Omega)\cap\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega). \tag{3.17}$$

The spaces $\mathbf{X}_N(\underline{\boldsymbol{\xi}};\Omega)$ and $\mathbf{X}_T(\underline{\boldsymbol{\xi}};\Omega)$ are respectively equipped with the norms $\|\cdot\|_{\mathbf{X}_N(\underline{\boldsymbol{\xi}})}$ and $\|\cdot\|_{\mathbf{X}_T(\underline{\boldsymbol{\xi}})}$, which are both defined as $\left(\|\cdot\|^2_{\mathbf{L}^2}+\|\mathbf{curl}\cdot\|^2_{\mathbf{L}^2}+\|\operatorname{div}\underline{\boldsymbol{\xi}}\cdot\|^2_{L^2}\right)^{1/2}$. In $\mathbf{K}_N(\underline{\boldsymbol{\xi}};\Omega)$ and $\mathbf{K}_T(\underline{\boldsymbol{\xi}};\Omega)$, this reduces to the $\|\cdot\|_{\mathbf{H}(\mathbf{curl})}$-norm. Likewise, the space $\mathbf{W}_N(\underline{\boldsymbol{\xi}};\Omega)$ is equipped with the $\|\cdot\|_{\mathbf{H}^+(\mathbf{curl})}$-norm.

These are natural spaces that arise in the study of anisotropic Maxwell problems. They generalize the standard function spaces $\mathbf{H}(\operatorname{div},\Omega)$, $\mathbf{X}_N(\Omega)$, etc. introduced in Chapter 2. In the following, we extend to these spaces some useful properties of the standard spaces. As a first noticeable consequence of (3.1), one can extend the Helmholtz decompositions 2.3.7 and 2.3.8 into the following results.

Theorem 3.2.1. One has the following first-kind Helmholtz decompositions, whose associated projectors are continuous:

$$\mathbf{L}^2(\Omega) = \nabla H^1_0(\Omega)\oplus\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega); \tag{3.18}$$
$$\mathbf{H}_0(\mathbf{curl},\Omega) = \nabla H^1_0(\Omega)\oplus\mathbf{K}_N(\underline{\boldsymbol{\xi}};\Omega). \tag{3.19}$$

Proof. Let $\mathbf{v}\in\mathbf{L}^2(\Omega)$. The Dirichlet problem

$$\left|\begin{array}{l}\text{Find } p\in H^1_0(\Omega) \text{ s.t., } \forall q\in H^1_0(\Omega),\\ \qquad\left(\underline{\boldsymbol{\xi}}\nabla p|\nabla q\right) = \left(\underline{\boldsymbol{\xi}}\mathbf{v}|\nabla q\right)\end{array}\right.$$

is well-posed by Theorem 3.1.6, and there holds $\|p\|_{H^1_0}\lesssim\|\mathbf{v}\|_{\mathbf{L}^2}$ [1]. We pose $\mathbf{v}_T=\mathbf{v}-\nabla p\in\mathbf{L}^2(\Omega)$. Then $\langle\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{v}_T|q\rangle_{H^1_0} = \left(\underline{\boldsymbol{\xi}}\mathbf{v}_T|\nabla q\right) = 0$, i.e. $\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{v}_T=0$ in $H^{-1}(\Omega)$, and $\mathbf{v}_T\in\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega)$. Moreover, by triangle inequality, one has $\|\mathbf{v}_T\|_{\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}})}=\|\mathbf{v}_T\|_{\mathbf{L}^2}\leq\|\mathbf{v}\|_{\mathbf{L}^2}+\|\nabla p\|_{\mathbf{L}^2}\lesssim\|\mathbf{v}\|_{\mathbf{L}^2}$.

Additionally, the sum is direct: indeed, let $\mathbf{v}\in\nabla H^1_0(\Omega)\cap\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega)$, then $\mathbf{v}=\nabla p$ for a certain $p\in H^1_0(\Omega)$, and $\operatorname{div}\underline{\boldsymbol{\xi}}\nabla p=0$. But the Dirichlet problem is well-posed, so $p=0$.

The second proof is similar, with bounds in $\mathbf{H}(\mathbf{curl},\Omega)$-norm. □

Remark 3.2.2. Contrarily to Theorem 2.3.7, the notion of orthogonality no longer applies, as $(\underline{\boldsymbol{\xi}}\cdot|\cdot)$ is not a scalar product if $\underline{\boldsymbol{\xi}}$ is not Hermitian. Indeed, for $\mathbf{v}\in\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega), q\in H^1_0(\Omega)$, it always holds $\left(\underline{\boldsymbol{\xi}}\mathbf{v}|\nabla q\right)=0$ by integration by parts; however, $\left(\underline{\boldsymbol{\xi}}\nabla q|\mathbf{v}\right)=\left(\nabla q|\underline{\boldsymbol{\xi}}^*\mathbf{v}\right)$ may not vanish. The same remark also apply to the following lemmas and theorems.

[1] Here and in all the following proofs of the chapter, the notation $a\lesssim b$ denotes that there exists a constant $C>0$, *independent of a and b*, s.t. $a\leq Cb$. The constant C depends only on $\underline{\boldsymbol{\xi}}$ and the geometry.

Theorem 3.2.3. One has the following second-kind Helmholtz decompositions, which are continuous:

$$\mathbf{L}^2(\Omega) = \nabla H^1_{\mathrm{zmv}}(\Omega) \oplus \mathbf{H}_0(\operatorname{div} \underline{\boldsymbol{\xi}}0, \Omega); \tag{3.20}$$

$$\mathbf{H}(\mathbf{curl}, \Omega) = \nabla H^1_{\mathrm{zmv}}(\Omega) \oplus \mathbf{K}_T(\underline{\boldsymbol{\xi}}; \Omega). \tag{3.21}$$

Proof. Let $\mathbf{v} \in \mathbf{L}^2(\Omega)$. The Neumann problem

$$\left| \begin{array}{l} \text{Find } p \in H^1_{\mathrm{zmv}}(\Omega) \text{ s.t., } \forall q \in H^1_{\mathrm{zmv}}(\Omega), \\ \qquad\qquad (\underline{\boldsymbol{\xi}}\nabla p|\nabla q) = (\underline{\boldsymbol{\xi}}\mathbf{v}|\nabla q) \end{array} \right.$$

is well-posed by Theorem 3.1.7, and there holds $\|p\|_{H^1_{\mathrm{zmv}}} \lesssim \|\mathbf{v}\|_{\mathbf{L}^2}$. We pose $\mathbf{v}_T = \mathbf{v} - \nabla p \in \mathbf{L}^2(\Omega)$. Noting that the formulation is still valid $\forall q \in H^1(\Omega)$, and taking $q \in H^1_0(\Omega)$, there holds $\langle \operatorname{div} \underline{\boldsymbol{\xi}}\mathbf{v}_T, q\rangle_{H^1_0} = -(\underline{\boldsymbol{\xi}}\mathbf{v}_T|\nabla q) = 0$. Hence $\operatorname{div} \underline{\boldsymbol{\xi}}\mathbf{v}_T = 0$ and $\mathbf{v}_T \in \mathbf{H}(\operatorname{div} \underline{\boldsymbol{\xi}}0, \Omega)$. Moreover, $\forall q \in H^1(\Omega)$, $\langle \underline{\boldsymbol{\xi}}\mathbf{v}_T \cdot \mathbf{n}, q\rangle_{H^{1/2}(\Gamma)} = (\underline{\boldsymbol{\xi}}\mathbf{v}_T|\nabla q) + (\operatorname{div} \underline{\boldsymbol{\xi}}\mathbf{v}_T|q) = 0$. Hence $\mathbf{v}_T \in \mathbf{H}_0(\operatorname{div} \underline{\boldsymbol{\xi}}0, \Omega)$. By triangle inequality, $\|\mathbf{v}_T\|_{\mathbf{L}^2} \lesssim \|\mathbf{v}\|_{\mathbf{L}^2}$.

Additionally, the sum is direct: indeed, let $\mathbf{v} \in \nabla H^1_{\mathrm{zmv}}(\Omega) \cap \mathbf{H}_0(\operatorname{div} \underline{\boldsymbol{\xi}}0, \Omega)$, then $\mathbf{v} = \nabla p$ for a certain $p \in H^1_{\mathrm{zmv}}(\Omega)$, and fulfills $\operatorname{div} \underline{\boldsymbol{\xi}}\nabla p = 0$ and $\underline{\boldsymbol{\xi}}\nabla p \cdot \mathbf{n}_{|\Gamma} = 0$. As the Neumann problem is well-posed, $p = 0$.

The second proof is similar. □

Then, one has a third Helmholtz decomposition for the space $\mathbf{H}^+(\mathbf{curl}, \Omega)$, in the spirit of Theorem 2.3.9.

Theorem 3.2.4. The following Helmholtz decomposition holds:

$$\mathbf{H}^+(\mathbf{curl}, \Omega) = \nabla H^1_0(\Omega) \oplus \mathbf{W}_N(\underline{\boldsymbol{\xi}}; \Omega). \tag{3.22}$$

Proof. Let $\mathbf{v} \in \mathbf{H}^+(\mathbf{curl}, \Omega)$, and $p \in H^1_0(\Omega)$ the unique solution to

$$\left| \begin{array}{l} \text{Find } p \in H^1_0(\Omega) \text{ s.t., } \forall q \in H^1_0(\Omega), \\ \qquad\qquad (\underline{\boldsymbol{\xi}}\nabla p|\nabla q) = (\underline{\boldsymbol{\xi}}\mathbf{v}|\nabla q)\,. \end{array} \right.$$

Then $\mathbf{v}_T := \mathbf{v} - \nabla p \in \mathbf{H}(\mathbf{curl}, \Omega)$, with $\operatorname{div} \underline{\boldsymbol{\xi}}\mathbf{v}_T = 0$ and $\gamma^T\mathbf{v}_T = \gamma^T\mathbf{v} \in \mathbf{L}^2_t(\Gamma)$. Moreover, the sum is direct, because the Dirichlet problem is well-posed. □

One can go further by introducing splittings of spaces $\mathbf{H}(\operatorname{div} \underline{\boldsymbol{\xi}}, \Omega)$ and $\mathbf{H}_0(\operatorname{div} \underline{\boldsymbol{\xi}}, \Omega)$, in the spirit of relations (2.75) and (2.76). They will be useful for the proofs of the next section. These results are dependent of the topology of the domain. We refer to subsection 2.3.1 for the corresponding definitions and notations. For the first one, introduce

$$Q_N(\underline{\boldsymbol{\xi}}; \Omega) := \left\{ q \in H^1(\Omega) \mid \operatorname{div} \underline{\boldsymbol{\xi}}\nabla q = 0, q_{|\Gamma_0} = 0, \text{ and, for } 1 \le k \le K, q_{|\Gamma_k} \text{ constant} \right\}. \tag{3.23}$$

Proposition 3.2.5. The space $Q_N(\underline{\boldsymbol{\xi}}; \Omega)$ is finite dimensional, of dimension K.

Proof. For $1 \le k \le K$, let us introduce the solution $q_k \in Q_N(\underline{\boldsymbol{\xi}}; \Omega)$ to $\operatorname{div} \underline{\boldsymbol{\xi}}\nabla q_k = 0$, $q_{k|\Gamma_0} = 0$, and, $\forall l$, $q_{k|\Gamma_l} = \delta_{kl}$. As the Dirichlet problem is well-posed, q_k exists and is unique. Moreover, the $(q_k)_k$ form a basis of $Q_N(\underline{\boldsymbol{\xi}}; \Omega)$: clearly the family is linearly independent, and, for all $q \in Q_N(\underline{\boldsymbol{\xi}}; \Omega)$, there holds $q = \sum_{k=1}^K q_{|\Gamma_k} q_k$. As a consequence, $\max_k |\cdot_{|\Gamma_k}|$ defines a norm on $Q_N(\underline{\boldsymbol{\xi}}; \Omega)$. □

The next result is the counterpart of decomposition (2.75).

Lemma 3.2.6. One has the following decomposition, which is continuous:

$$\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega) = \nabla Q_N(\underline{\boldsymbol{\xi}},\Omega) \oplus \underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{H}^1(\Omega). \tag{3.24}$$

Proof. Let $\mathbf{v} \in \mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega)$, consider the problem

$$\left|\begin{array}{l}\text{Find } q^\Gamma \in Q_N(\underline{\boldsymbol{\xi}};\Omega) \text{ s.t., } \forall q \in Q_N(\underline{\boldsymbol{\xi}};\Omega),\\ \qquad\qquad (\underline{\boldsymbol{\xi}}\nabla q^\Gamma|\nabla q) = (\underline{\boldsymbol{\xi}}\mathbf{v}|\nabla q)\,.\end{array}\right. \tag{3.25}$$

The problem is well-posed: it is a consequence of Lax-Milgram lemma, along with Poincaré inequality set in $H^1_{0,\Gamma_0}(\Omega)$ and relations (3.4). So, the problem admits a unique solution $q^\Gamma \in Q_N(\underline{\boldsymbol{\xi}};\Omega)$, with $\|\nabla q^\Gamma\|_{\mathbf{L}^2} \lesssim \|\mathbf{v}\|_{\mathbf{L}^2}$. Introducing $\mathbf{z} = \mathbf{v} - \nabla q^\Gamma \in \mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega)$, there holds $\langle\underline{\boldsymbol{\xi}}\mathbf{z}\cdot\mathbf{n},1\rangle_{H^{1/2}(\Gamma_k)} = 0$. Indeed, one has, by integration by parts,

$$\begin{aligned}\langle\underline{\boldsymbol{\xi}}\mathbf{z}\cdot\mathbf{n},1\rangle_{H^{1/2}(\Gamma_k)} &= \langle\underline{\boldsymbol{\xi}}\mathbf{z}\cdot\mathbf{n},q_k\rangle_{H^{1/2}(\Gamma)}\\ &= \big(\underline{\boldsymbol{\xi}}\mathbf{z}|\nabla q_k\big) + \big(\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{z}|q_k\big)\\ &= \big(\underline{\boldsymbol{\xi}}\mathbf{v}|\nabla q_k\big) - \big(\underline{\boldsymbol{\xi}}\nabla q^\Gamma|\nabla q_k\big)\\ &= 0,\end{aligned}$$

the latter by definition of q^Γ. Then, one can apply the Vector Potential Theorem 2.3.5. There exists $\mathbf{w} \in \mathbf{H}^1(\Omega)$ s.t. $\underline{\boldsymbol{\xi}}\mathbf{z} = \mathbf{curl}\,\mathbf{w}$, with $\|\mathbf{w}\|_{\mathbf{H}^1} \lesssim \|\underline{\boldsymbol{\xi}}\mathbf{z}\|_{\mathbf{L}^2} \lesssim \|\mathbf{z}\|_{\mathbf{L}^2}$. So, $\mathbf{v} = \nabla q^\Gamma + \mathbf{z} = \nabla q^\Gamma + \underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{w}$, with $\|\mathbf{w}\|_{\mathbf{H}^1} \lesssim \|\mathbf{v}\|_{\mathbf{L}^2}$ by triangle inequality.

Moreover, the sum is direct. Indeed, let $\mathbf{v} \in \nabla Q_N(\underline{\boldsymbol{\xi}},\Omega) \cap \underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{H}^1(\Omega)$. Then $\mathbf{v} = \nabla q^\Gamma = \underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{w}$ for some $q^\Gamma \in Q_N(\underline{\boldsymbol{\xi}};\Omega), \mathbf{w} \in \mathbf{H}^1(\Omega)$. Thus, for all $q \in Q_N(\underline{\boldsymbol{\xi}};\Omega)$, there holds

$$\begin{aligned}\big(\underline{\boldsymbol{\xi}}\nabla q^\Gamma|\nabla q\big) &= (\mathbf{curl}\,\mathbf{w}|\nabla q)\\ &= {}_\gamma\langle\gamma^T\mathbf{w},\pi^T\nabla q\rangle_\pi\end{aligned}$$

by integration by parts. However, note that $q \in Q_N(\underline{\boldsymbol{\xi}};\Omega)$, so $q_{|\Gamma_k}$ is constant for all k, and $\pi^T\nabla q = 0$. So, for all $q \in Q_N(\underline{\boldsymbol{\xi}};\Omega)$, $\big(\underline{\boldsymbol{\xi}}\nabla q^\Gamma|\nabla q\big) = 0$. As the problem (3.25) is well-posed, $q^\Gamma = 0$ and then $\mathbf{v} = \mathbf{0}$. □

For the splitting of $\mathbf{H}_0(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega)$, we introduce

$$Q_T(\underline{\boldsymbol{\xi}};\dot\Omega) := \left\{\dot q \in H^1_{\mathrm{zmv}}(\dot\Omega) \mid \operatorname{div}\underline{\boldsymbol{\xi}}\widetilde{\nabla}\dot q = 0 \text{ in } \Omega,\ \underline{\boldsymbol{\xi}}\widetilde{\nabla}\dot q\cdot\mathbf{n} = 0 \text{ on } \Gamma,\ \text{and, for } 1 \le i \le I, [\dot q]_{\Sigma_i} \text{ constant}\right\}, \tag{3.26}$$

where $\widetilde{\nabla}\dot q$ denotes the extension of $\nabla\dot q$ to Ω and $[\dot q]_{\Sigma_i}$ denotes the jump of $\dot q$ across Σ_i.

Proposition 3.2.7. The space $Q_T(\underline{\boldsymbol{\xi}};\dot\Omega)$ is finite dimensional, of dimension I.

Proof. For $1 \le i \le I$, one can introduce $\dot q_i$ the unique element of $Q_T(\underline{\boldsymbol{\xi}};\dot\Omega)$ s.t. $[\dot q_i]_{\Sigma_j} = \delta_{ij}$ for $1 \le j \le I$. Then, the $(\dot q_i)_i$ form a basis of $Q_T(\underline{\boldsymbol{\xi}};\dot\Omega)$, as clearly they are linearly independent, and, for all $\dot q \in Q_T(\underline{\boldsymbol{\xi}};\dot\Omega)$, $\dot q = \sum_{i=1}^I [\dot q]_{\Sigma_i}\dot q_i$. Furthermore, $\max_i |[\cdot]_{|\Sigma_i}|$ defines a norm on $Q_T(\underline{\boldsymbol{\xi}};\Omega)$. □

Then, one obtains a decomposition similar to (2.76).

Lemma 3.2.8. One has the following decomposition, which is continuous:

$$\mathbf{H}_0(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega) = \widetilde{\nabla Q_T(\underline{\boldsymbol{\xi}},\dot\Omega)} \oplus \underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{H}_0(\mathbf{curl},\Omega). \tag{3.27}$$

Proof. Let $\mathbf{v} \in \mathbf{H}_0(\operatorname{div} \underline{\boldsymbol{\xi}} 0, \Omega)$, consider the problem

$$\left| \begin{aligned} &\text{Find } \dot{q}^{\Sigma} \in Q_T(\underline{\boldsymbol{\xi}}; \dot{\Omega}) \text{ s.t.}, \forall \dot{q} \in Q_T(\underline{\boldsymbol{\xi}}; \dot{\Omega}), \\ &\qquad (\underline{\boldsymbol{\xi}} \nabla \dot{q}^{\Sigma} | \nabla \dot{q})_{\dot{\Omega}} = (\underline{\boldsymbol{\xi}} \mathbf{v} | \nabla \dot{q})_{\dot{\Omega}} . \end{aligned} \right. \tag{3.28}$$

The problem is well-posed, as a consequence of relations (3.4) and Poincaré-Wirtinger inequality. So, it admits a unique solution $\dot{q}^{\Sigma} \in Q_T(\underline{\boldsymbol{\xi}}; \dot{\Omega})$, with $\|\nabla \dot{q}\|_{\mathbf{L}^2} \lesssim \|\mathbf{v}\|_{\mathbf{L}^2}$. Introducing $\mathbf{z} = \mathbf{v} - \widetilde{\nabla \dot{q}} \in \mathbf{H}_0(\operatorname{div} \underline{\boldsymbol{\xi}} 0, \Omega)$, there holds $\langle \underline{\boldsymbol{\xi}} \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{H^{1/2}(\Sigma_i)} = 0$ for $1 \leq i \leq I$. Indeed, according to the integration by parts formula (2.48),

$$\begin{aligned} \langle \underline{\boldsymbol{\xi}} \mathbf{z} \cdot \mathbf{n}, 1 \rangle_{H^{1/2}(\Sigma_i)} &= \sum_{j=1}^{I} \langle \underline{\boldsymbol{\xi}} \mathbf{z} \cdot \mathbf{n}, [\dot{q}_i]_{\Sigma_j} \rangle_{H^{1/2}(\Sigma_j)} \\ &= (\underline{\boldsymbol{\xi}} \mathbf{z} | \nabla \dot{q}_i)_{\dot{\Omega}} \\ &= (\underline{\boldsymbol{\xi}} \mathbf{v} | \nabla \dot{q}_i)_{\dot{\Omega}} - (\underline{\boldsymbol{\xi}} \nabla \dot{q}^{\Sigma} | \nabla \dot{q}_i)_{\dot{\Omega}} \\ &= 0, \end{aligned}$$

the latter by definition of $\dot{q}^{\Sigma}$. Then, one can invoke the Second Vector Potential Theorem 2.3.6. There exists $\mathbf{w} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ s.t. $\underline{\boldsymbol{\xi}} \mathbf{z} = \mathbf{curl}\, \mathbf{w}$, with $\|\mathbf{w}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{z}\|_{\mathbf{L}^2}$. Thus, $\mathbf{v} = \widetilde{\nabla \dot{q}^{\Sigma}} + \underline{\boldsymbol{\xi}}^{-1} \mathbf{curl}\, \mathbf{w}$, with $\|\mathbf{w}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{v}\|$ by triangle inequality.

Moreover, the sum is direct. Indeed, let $\mathbf{v} \in \widetilde{\nabla Q_T(\underline{\boldsymbol{\xi}}, \dot{\Omega})} \cap \underline{\boldsymbol{\xi}}^{-1} \mathbf{curl}\, \mathbf{H}_0(\mathbf{curl}, \Omega)$. Then $\mathbf{v} = \widetilde{\nabla \dot{q}^{\Sigma}} = \underline{\boldsymbol{\xi}}^{-1} \mathbf{curl}\, \mathbf{w}$ for some $\dot{q}^{\Sigma} \in Q_T(\underline{\boldsymbol{\xi}}; \dot{\Omega}), \mathbf{w} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$. Hence, $\forall \dot{q} \in Q_T(\underline{\boldsymbol{\xi}}; \dot{\Omega})$,

$$(\underline{\boldsymbol{\xi}} \nabla \dot{q}^{\Sigma} | \nabla \dot{q})_{\dot{\Omega}} = (\mathbf{curl}\, \mathbf{w} | \nabla \dot{q})_{\dot{\Omega}} = 0,$$

because $\mathbf{w} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ and $\widetilde{\nabla \dot{q}} \in \mathbf{H}(\mathbf{curl}, \Omega)$. As the problem (3.28) is well-posed, $\dot{q}^{\Sigma} = 0$, and $\mathbf{v} = \mathbf{0}$. □

3.3 Weber inequalities and compact embeddings

The next subsections are dedicated to extending Weber inequalities and compact embedding results for the spaces $\mathbf{X}_N(\underline{\boldsymbol{\xi}}; \Omega)$, $\mathbf{X}_T(\underline{\boldsymbol{\xi}}; \Omega)$ and $\mathbf{W}_N(\underline{\boldsymbol{\xi}}; \Omega)$. These results will be useful to ensure that the variational formulations of the Maxwell problems enter the setting of Fredholm alternative. They depend on the topology of the domain, and we use the notations introduced in subsection 2.3.1.

3.3.1 The space $\mathbf{X}_N(\underline{\boldsymbol{\xi}}; \Omega)$

Let us begin with an extension of the first Weber inequality 2.4.5.

Theorem 3.3.1. Let Ω, be a domain of boundary Γ, and $(\Gamma_k)_{0 \leq k \leq K}$ the maximal connected components of Γ. There exists a constant $C_W > 0$ s.t., for all $\mathbf{y}$ in $\mathbf{X}_N(\underline{\boldsymbol{\xi}}; \Omega)$,

$$\|\mathbf{y}\|_{\mathbf{L}^2} \leq C_W \left(\|\mathbf{curl}\, \mathbf{y}\|_{\mathbf{L}^2} + \|\operatorname{div} \underline{\boldsymbol{\xi}} \mathbf{y}\|_{L^2} + \sum_{k=1}^{K} |\langle \underline{\boldsymbol{\xi}} \mathbf{y} \cdot \mathbf{n}, 1 \rangle_{H^{1/2}(\Gamma_k)}| \right). \tag{3.29}$$

Proof. We proceed by contradiction. Let us assume there exists $(\mathbf{y}_m)$ a sequence of $\mathbf{X}_N(\underline{\boldsymbol{\xi}}; \Omega)$ s.t., $\forall m$, $\|\mathbf{y}_m\|_{\mathbf{L}^2} = 1$, and

$$\|\mathbf{curl}\, \mathbf{y}_m\|_{\mathbf{L}^2} + \|\operatorname{div} \underline{\boldsymbol{\xi}} \mathbf{y}_m\|_{L^2} + \sum_{k=1}^{K} |\langle \underline{\boldsymbol{\xi}} \mathbf{y}_m \cdot \mathbf{n}, 1 \rangle_{H^{1/2}(\Gamma_k)}| \leq \frac{1}{m+1}.$$

The proof makes use of the Helmholtz decomposition (3.19) as well as the decomposition (3.24), so that we split $\mathbf{y}_m$ into three terms.

First, consider the solution to the Dirichlet problem

$$\left|\begin{array}{l}\text{Find } q_m^0 \in H_0^1(\Omega) \text{ s.t., } \forall q \in H_0^1(\Omega),\\ \qquad \left(\underline{\boldsymbol{\xi}}\nabla q_m^0|\nabla q\right) = \left(\underline{\boldsymbol{\xi}}\mathbf{y}_m|\nabla q\right).\end{array}\right. \tag{3.30}$$

By Theorem 3.1.6, this problem is well-posed. Moreover, taking $q = q_m^0$, one gets by integration by parts

$$\left|\left(\underline{\boldsymbol{\xi}}\nabla q_m^0|\nabla q_m^0\right)\right| = \left|\left(-\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m|q_m^0\right)\right| \leq \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2}\|q_m^0\|_{L^2}.$$

Using the relation (3.4) on the left-hand side and Poincaré inequality on the right-hand side, one gets

$$\|\nabla q_m^0\|_{\mathbf{L}^2}^2 \lesssim \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2}\|\nabla q_m^0\|_{\mathbf{L}^2},$$

so

$$\|\nabla q_m^0\|_{\mathbf{L}^2} \lesssim \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2}. \tag{3.31}$$

Hence $\|\nabla q_m^0\|_{\mathbf{L}^2} \longrightarrow 0$.

As a second step, let $\mathbf{x}_m := \mathbf{y}_m - \nabla q_m^0 \in \mathbf{K}_N(\underline{\boldsymbol{\xi}}, \Omega)$ (this is the Helmholtz decomposition (3.19) of $\mathbf{y}_m$). Consider the problem

$$\left|\begin{array}{l}\text{Find } q_m^\Gamma \in Q_N(\underline{\boldsymbol{\xi}};\Omega) \text{ s.t., } \forall q \in Q_N(\underline{\boldsymbol{\xi}};\Omega),\\ \qquad \left(\underline{\boldsymbol{\xi}}\nabla q_m^\Gamma|\nabla q\right) = \left(\underline{\boldsymbol{\xi}}\mathbf{x}_m|\nabla q\right),\end{array}\right. \tag{3.32}$$

with the space $Q_N(\underline{\boldsymbol{\xi}};\Omega)$ defined in (3.23). This problem is also well-posed, following the proof of Lemma 3.2.6. Taking $q = q_m^\Gamma$ and integrating by parts, one has

$$\begin{aligned}\left|\left(\underline{\boldsymbol{\xi}}\nabla q_m^\Gamma|\nabla q_m^\Gamma\right)\right| &= \left|\langle\underline{\boldsymbol{\xi}}\mathbf{x}_m\cdot\mathbf{n}, q_m^\Gamma\rangle_{H^{1/2}(\Gamma)}\right|\\ &= \left|\sum_{k=1}^K\langle\underline{\boldsymbol{\xi}}\mathbf{x}_m\cdot\mathbf{n}, q_m^\Gamma\rangle_{H^{1/2}(\Gamma_k)}\right|\\ &= \left|\sum_{k=1}^K\overline{q_{m|\Gamma_k}^\Gamma}\langle\underline{\boldsymbol{\xi}}\mathbf{x}_m\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Gamma_k)}\right|.\end{aligned}$$

As $Q_N(\underline{\boldsymbol{\xi}};\Omega)$ is a finite-dimensional vector space, all the norms are equivalent, and among them, $\|\nabla\cdot\|_{\mathbf{L}^2}$ and $\max_k|\cdot_{|\Gamma_k}|$ (the latter denoted $\|\cdot\|_{Q_N(\underline{\boldsymbol{\xi}})}$). Then, using additionally relation (3.4), there holds

$$\|q_m^\Gamma\|_{Q_N(\underline{\boldsymbol{\xi}})}^2 \lesssim \left(\|q_m^\Gamma\|_{Q_N(\underline{\boldsymbol{\xi}})}\sum_{k=1}^K\left|\langle\underline{\boldsymbol{\xi}}\mathbf{x}_m\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Gamma_k)}\right|\right).$$

Besides, $\langle\underline{\boldsymbol{\xi}}\mathbf{x}_m\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Gamma_k)} = \langle\underline{\boldsymbol{\xi}}\mathbf{y}_m\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Gamma_k)} - \langle\underline{\boldsymbol{\xi}}\nabla q_m^0\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Gamma_k)}$. Moreover, using the continuity of the normal trace and recalling that $\operatorname{div}\underline{\boldsymbol{\xi}}\nabla q_m^0 = \operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m$ as well as relation (3.31), there holds

$$\begin{aligned}\left|\langle\underline{\boldsymbol{\xi}}\nabla q_m^0\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Gamma_k)}\right| &\lesssim \|\underline{\boldsymbol{\xi}}\nabla q_m^0\cdot\mathbf{n}\|_{H^{-1/2}(\Gamma_k)}\\ &\lesssim \|\nabla q_m^0\|_{\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}})}\\ &\lesssim \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2}.\end{aligned}$$

Therefore,

$$\|q_m^\Gamma\|_{Q_N(\underline{\boldsymbol{\xi}})} \lesssim \left(\sum_{k=1}^K\left|\langle\underline{\boldsymbol{\xi}}\mathbf{y}_m\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Gamma_k)}\right| + \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2}\right), \tag{3.33}$$

and $\|\nabla q_m^\Gamma\|_{\mathbf{L}^2} \longrightarrow 0$.

Thirdly, let $\mathbf{z}_m := \mathbf{x}_m - \nabla q_m^\Gamma = \mathbf{y}_m - \nabla q_m^0 - \nabla q_m^\Gamma$. It belongs to $\mathbf{X}_N(\underline{\boldsymbol{\xi}}, \Omega)$: in fact, $\nabla q_m^\Gamma \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, as $\pi^T(\nabla q_m^\Gamma) = \nabla_\Gamma(q_{m|\Gamma}^\Gamma) = 0$. There holds $\mathbf{curl}\,\mathbf{z}_m = \mathbf{curl}\,\mathbf{y}_m$, $\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{z}_m = 0$, and, moreover, $\langle\underline{\boldsymbol{\xi}}\mathbf{z}_m\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Gamma_k)} = 0$. Indeed, one has by integration by parts

$$\begin{aligned}\langle\underline{\boldsymbol{\xi}}\mathbf{z}_m\cdot\mathbf{n}, 1\rangle_{H^{1/2}(\Gamma_k)} &= \langle\underline{\boldsymbol{\xi}}\mathbf{z}_m\cdot\mathbf{n}, q_k\rangle_{H^{1/2}(\Gamma)}\\ &= \left(\underline{\boldsymbol{\xi}}\mathbf{z}_m|\nabla q_k\right) + \left(\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{z}_m|q_k\right).\end{aligned}$$

However, $\left(\underline{\boldsymbol{\xi}}\mathbf{z}_m|\nabla q_k\right) = \left(\underline{\boldsymbol{\xi}}\mathbf{x}_m|\nabla q_k\right) - \left(\underline{\boldsymbol{\xi}}\nabla q_m^\Gamma|\nabla q_k\right) = 0$ by definition of q_m^Γ; and $\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{z}_m = 0$. Hence $\langle\underline{\boldsymbol{\xi}}\mathbf{z}_m\cdot\mathbf{n},1\rangle_{H^{1/2}(\Gamma_k)} = 0$. This allows us to invoke the vector potential Theorem 2.3.5: there exists $\mathbf{w}_m \in \mathbf{H}^1(\Omega)$ s.t. $\underline{\boldsymbol{\xi}}\mathbf{z}_m = \mathbf{curl}\,\mathbf{w}_m$, and $\|\mathbf{w}_m\|_{\mathbf{H}^1} \lesssim \|\underline{\boldsymbol{\xi}}\mathbf{z}_m\|_{\mathbf{L}^2} \lesssim \|\mathbf{z}_m\|_{\mathbf{L}^2}$ (this is the splitting (3.24) of $\mathbf{x}_m$). Furthermore, we have by integration by parts

$$\left(\mathbf{z}_m|\underline{\boldsymbol{\xi}}\mathbf{z}_m\right) = \left(\mathbf{z}_m|\,\mathbf{curl}\,\mathbf{w}_m\right) = \left(\mathbf{curl}\,\mathbf{z}_m|\mathbf{w}_m\right) = \left(\mathbf{curl}\,\mathbf{y}_m|\mathbf{w}_m\right).$$

Using again relation (3.4), there holds

$$\begin{aligned}\|\mathbf{z}_m\|^2_{\mathbf{L}^2} &\lesssim \|\,\mathbf{curl}\,\mathbf{y}_m\|_{\mathbf{L}^2}\|\mathbf{w}_m\|_{\mathbf{L}^2}\\ &\lesssim \|\,\mathbf{curl}\,\mathbf{y}_m\|_{\mathbf{L}^2}\|\mathbf{z}_m\|_{\mathbf{L}^2},\end{aligned}$$

and so

$$\|\mathbf{z}_m\|_{\mathbf{L}^2} \lesssim \|\,\mathbf{curl}\,\mathbf{y}_m\|_{\mathbf{L}^2}. \tag{3.34}$$

Hence $\|\mathbf{z}_m\|_{\mathbf{L}^2} \longrightarrow 0$.

Finally, as $\mathbf{y}_m = \mathbf{z}_m + \nabla q_m^0 + \nabla q_m^\Gamma$, we have $\|\mathbf{y}_m\|_{\mathbf{L}^2} \longrightarrow 0$, which contradicts $\|\mathbf{y}_m\|_{\mathbf{L}^2} = 1$. □

Furthermore, one can also extend the compact embedding result of Theorem 2.4.8. A similar result has been proven by Alonso and Valli [4].

Theorem 3.3.2. The embedding of $\mathbf{X}_N(\underline{\boldsymbol{\xi}};\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact.

Proof. Let $(\mathbf{y}_m)$ be a bounded sequence of $\mathbf{X}_N(\underline{\boldsymbol{\xi}};\Omega)$. As in the previous proof, we split $\mathbf{y}_m$ into three terms. Introduce $q_m^0 \in H_0^1(\Omega)$, $q_m^\Gamma \in Q_N(\underline{\boldsymbol{\xi}};\Omega)$, and $\mathbf{w}_m \in \mathbf{H}^1(\Omega)$ s.t.

$$\mathbf{y}_m = \nabla q_m^0 + \nabla q_m^\Gamma + \underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{w}_m.$$

Additionally, there holds from the previous proof

$$\begin{aligned}&\|\nabla q_m^0\|_{\mathbf{L}^2} \lesssim \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2};\\ &\|\nabla q_m^\Gamma\|_{\mathbf{L}^2} \lesssim \left(\|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2} + \sum_{k=1}^K \left|\langle\underline{\boldsymbol{\xi}}\mathbf{y}_m\cdot\mathbf{n},1\rangle_{H^{1/2}(\Gamma_k)}\right|\right);\\ &\|\mathbf{w}_m\|_{\mathbf{H}^1} \lesssim \|\mathbf{z}_m\|_{\mathbf{L}^2} \lesssim \|\,\mathbf{curl}\,\mathbf{y}_m\|_{\mathbf{L}^2}.\end{aligned}$$

Let us begin with (q_m^Γ): it is a bounded sequence of the finite-dimensional vector space $Q_N(\underline{\boldsymbol{\xi}};\Omega)$, so it admits a subsequence which converges (in particular in H^1-norm). Besides, (q_m^0) and $(\mathbf{w}_m)$ are bounded sequences of $H^1(\Omega)$ (resp. $\mathbf{H}^1(\Omega)$). Then, by Rellich Theorem, they admit susbsequences (still denoted with the same indices) which converge in $L^2(\Omega)$ (resp. $\mathbf{L}^2(\Omega)$). It remains to prove that the subsequences (∇q_m^0) and $(\mathbf{curl}\,\mathbf{w}_m)$ converge in $\mathbf{L}^2(\Omega)$.

By definition of q_m^0 (see (3.30)), for any q in $H_0^1(\Omega)$, there holds by integration by parts

$$\left(\underline{\boldsymbol{\xi}}\nabla q_m^0|\nabla q\right) = \left(\underline{\boldsymbol{\xi}}\mathbf{y}_m|\nabla q\right) = -\left(\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m|q\right).$$

Using the notation $v_{mn} := v_m - v_n$, one has $\left(\underline{\boldsymbol{\xi}}\nabla q_{mn}^0|\nabla q\right) = \left(\underline{\boldsymbol{\xi}}\mathbf{y}_{mn}|\nabla q\right) = -\left(\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_{mn}|q\right)$. Then, taking $q = q_{mn}^0$,

$$\left|\left(\underline{\boldsymbol{\xi}}\nabla q_{mn}^0|\nabla q_{mn}^0\right)\right| \le \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_{mn}\|_{L^2}\|q_{mn}^0\|_{L^2}.$$

Thanks to relation (3.4),

$$\xi_-\|\nabla q_{mn}^0\|^2_{\mathbf{L}^2} \le 2\sup_m\left(\|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2}\right)\|q_{mn}^0\|_{L^2}.$$

Thus (∇q_m^0) is a Cauchy sequence of $\mathbf{L}^2(\Omega)$, hence converges in this Hilbert space.

We recall that $\mathbf{z}_m = \underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{w}_m \in \mathbf{X}_N(\Omega)$ and $\mathbf{curl}\,\mathbf{z}_m = \mathbf{curl}\,\mathbf{y}_m$ (cf. previous proof). Then, still with the same notations, and by integration by parts,

$$\left(\underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{w}_{mn}|\,\mathbf{curl}\,\mathbf{w}_{mn}\right) = \left(\mathbf{z}_{mn}|\,\mathbf{curl}\,\mathbf{w}_{mn}\right) = \left(\mathbf{curl}\,\mathbf{z}_{mn}|\mathbf{w}_{mn}\right) = \left(\mathbf{curl}\,\mathbf{y}_{mn}|\mathbf{w}_{mn}\right).$$

As $\underline{\boldsymbol{\xi}}^{-1}$ also satisfies an ellipticity condition (Prop. 3.1.4), we get

$$\xi_-^{\text{inv}} \| \mathbf{curl}\, \mathbf{w}_{mn} \|^2_{\mathbf{L}^2} \le \| \mathbf{curl}\, \mathbf{y}_{mn} \|_{\mathbf{L}^2} \| \mathbf{w}_{mn} \|_{\mathbf{L}^2} \le 2 \sup_m \left(\| \mathbf{curl}\, \mathbf{y}_m \|_{\mathbf{L}^2} \right) \| \mathbf{w}_{mn} \|_{\mathbf{L}^2},$$

which proves that $(\mathbf{curl}\, \mathbf{w}_m)$ is a Cauchy, hence converging, sequence of $\mathbf{L}^2(\Omega)$. As $\mathbf{y}_m = \underline{\boldsymbol{\xi}}^{-1}\, \mathbf{curl}\, \mathbf{w}_m + \nabla q_m^0 + \nabla q_m^\Gamma$, we conclude that the subsequence $(\mathbf{y}_m)$ converges in $\mathbf{L}^2(\Omega)$. □

3.3.2 The space $\mathbf{X}_T(\underline{\boldsymbol{\xi}}; \Omega)$

In a similar way, one can extend the properties of the space $\mathbf{X}_T(\Omega)$ to $\mathbf{X}_T(\underline{\boldsymbol{\xi}}; \Omega)$: namely, the second Weber inequality 2.4.6 and the compact embedding 2.4.9.

Theorem 3.3.3. Let Ω be a domain of genus $I \ge 0$, and $(\Sigma_i)_{1 \le i \le I}$ cuts of Ω. There exists a constant $C'_W > 0$ s.t., for all $\mathbf{y}$ in $\mathbf{X}_T(\underline{\boldsymbol{\xi}}; \Omega)$,

$$\| \mathbf{y} \|_{\mathbf{L}^2} \le C'_W \left(\| \mathbf{curl}\, \mathbf{y} \|_{\mathbf{L}^2} + \| \operatorname{div} \underline{\boldsymbol{\xi}} \mathbf{y} \|_{L^2} + \sum_{i=1}^{I} |\langle \underline{\boldsymbol{\xi}} \mathbf{y} \cdot \mathbf{n}, 1 \rangle_{H^{1/2}(\Sigma_i)}| \right). \tag{3.35}$$

Proof. The proof follows a similar structure as in Theorem 3.3.1. By contradiction, we assume there exists $(\mathbf{y}_m)$ a sequence of $\mathbf{X}_T(\underline{\boldsymbol{\xi}}; \Omega)$ s.t., $\forall m$, $\| \mathbf{y}_m \|_{\mathbf{L}^2} = 1$, and

$$\| \mathbf{curl}\, \mathbf{y}_m \|_{\mathbf{L}^2} + \| \operatorname{div} \underline{\boldsymbol{\xi}} \mathbf{y}_m \|_{L^2} + \sum_{i=1}^{I} |\langle \underline{\boldsymbol{\xi}} \mathbf{y}_m \cdot \mathbf{n}, 1 \rangle_{H^{1/2}(\Sigma_i)}| \le \frac{1}{m+1}.$$

We split $\mathbf{y}_m$ into three terms, making use of decompositions (3.21) and (3.27) (recall that $\mathbf{K}_T(\underline{\boldsymbol{\xi}}; \Omega) \subset \mathbf{H}_0(\operatorname{div} \underline{\boldsymbol{\xi}} 0, \Omega)$).

Consider first the solution to the Neumann problem

$$\left| \begin{aligned} &\text{Find } q_m^0 \in H^1_{\text{zmv}}(\Omega) \text{ s.t., } \forall q \in H^1_{\text{zmv}}(\Omega), \\ &\qquad (\underline{\boldsymbol{\xi}} \nabla q_m^0 | \nabla q) = (\underline{\boldsymbol{\xi}} \mathbf{y}_m | \nabla q)\,. \end{aligned} \right. \tag{3.36}$$

The problem is well-posed by Theorem 3.1.7. Taking $q = q_m^0$ and integrating by parts, one gets, as $\mathbf{y}_m \in \mathbf{H}_0(\operatorname{div} \underline{\boldsymbol{\xi}}, \Omega)$:

$$\left| (\underline{\boldsymbol{\xi}} \nabla q_m^0 | \nabla q_m^0) \right| = \left| (-\operatorname{div} \underline{\boldsymbol{\xi}} \mathbf{y}_m | q_m^0) \right| \le \| \operatorname{div} \underline{\boldsymbol{\xi}} \mathbf{y}_m \|_{L^2} \| q_m^0 \|_{L^2}.$$

Using the relation (3.4) on the left-hand side, as well as the Poincaré-Wirtinger inequality on the right-hand side, leads to

$$\| \nabla q_m^0 \|_{\mathbf{L}^2} \lesssim \| \operatorname{div} \underline{\boldsymbol{\xi}} \mathbf{y}_m \|_{L^2}. \tag{3.37}$$

Hence $\| \nabla q_m^0 \|_{\mathbf{L}^2} \longrightarrow 0$.

Secondly, note that $\nabla q_m^0 \in \mathbf{H}_0(\operatorname{div} \underline{\boldsymbol{\xi}}, \Omega)$ (see proof of Th. 3.2.3), and introduce $\mathbf{x}_m := \mathbf{y}_m - \nabla q_m^0 \in \mathbf{K}_T(\underline{\boldsymbol{\xi}}, \Omega)$, with the help of the second-kind Helmholtz decomposition (3.21). Consider the problem

$$\left| \begin{aligned} &\text{Find } \dot{q}_m^\Sigma \in Q_T(\underline{\boldsymbol{\xi}}; \dot{\Omega}) \text{ s.t., } \forall \dot{q} \in Q_T(\underline{\boldsymbol{\xi}}; \dot{\Omega}), \\ &\qquad (\underline{\boldsymbol{\xi}} \nabla \dot{q}_m^\Sigma | \nabla \dot{q})_{\dot{\Omega}} = (\underline{\boldsymbol{\xi}} \mathbf{x}_m | \nabla \dot{q})_{\dot{\Omega}}\,, \end{aligned} \right. \tag{3.38}$$

with the space $Q_T(\underline{\boldsymbol{\xi}}; \dot{\Omega})$ defined in (3.26). This problem is also well-posed, adapting the proof of Theorem 3.1.7 using Poincaré-Wirtinger inequality in $H^1_{\text{zmv}}(\dot{\Omega})$. Taking $\dot{q} = \dot{q}_m^\Sigma$ and using the integration by parts formula (2.48), one has, as $\operatorname{div} \underline{\boldsymbol{\xi}} \mathbf{x}_m = 0$,

$$\begin{aligned} (\underline{\boldsymbol{\xi}} \nabla \dot{q}_m^\Sigma | \nabla \dot{q}_m^\Sigma)_{\dot{\Omega}} &= \sum_{i=1}^{I} \left\langle \underline{\boldsymbol{\xi}} \mathbf{x}_m \cdot \mathbf{n}, [\dot{q}_m^\Sigma]_{\Sigma_i} \right\rangle_{H^{1/2}(\Sigma_i)} \\ &= \sum_{i=1}^{I} \overline{[\dot{q}_m^\Sigma]_{\Sigma_i}} \left\langle \underline{\boldsymbol{\xi}} \mathbf{x}_m \cdot \mathbf{n}, 1 \right\rangle_{H^{1/2}(\Sigma_i)}. \end{aligned}$$

As $Q_T(\underline{\boldsymbol{\xi}};\dot{\Omega})$ is a finite-dimensional vector space, all the norms are equivalent, and among them, $\|\nabla\cdot\|_{\mathbf{L}^2(\dot{\Omega})}$ and $\max_i |[\cdot]_{\Sigma_i}|$. Then, using additionally relation (3.4), there holds

$$\|\dot{q}_m^\Sigma\|^2_{Q_T(\underline{\boldsymbol{\xi}};\dot{\Omega})} \lesssim \left(\|\dot{q}_m^\Sigma\|_{Q_T(\underline{\boldsymbol{\xi}};\dot{\Omega})} \sum_{i=1}^{I} \left|\langle \underline{\boldsymbol{\xi}}\mathbf{x}_m \cdot \mathbf{n}, 1\rangle_{H^{1/2}(\Sigma_i)}\right| \right).$$

Besides, $\langle \underline{\boldsymbol{\xi}}\mathbf{x}_m \cdot \mathbf{n}, 1\rangle_{H^{1/2}(\Sigma_i)} = \langle \underline{\boldsymbol{\xi}}\mathbf{y}_m \cdot \mathbf{n}, 1\rangle_{H^{1/2}(\Sigma_i)} - \langle \underline{\boldsymbol{\xi}}\nabla q_m^0 \cdot \mathbf{n}, 1\rangle_{H^{1/2}(\Sigma_i)}$. Then, there holds, because $\nabla q_m^0 \in \mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}},\Omega)$ and formula (2.48),

$$\begin{aligned}
\left|\langle \underline{\boldsymbol{\xi}}\nabla q_m^0 \cdot \mathbf{n}, 1\rangle_{H^{1/2}(\Sigma_i)}\right| &= \left|\sum_{j=1}^{I} \langle \underline{\boldsymbol{\xi}}\nabla q_m^0 \cdot \mathbf{n}, [\dot{q}_i]_{\Sigma_j}\rangle_{H^{1/2}(\Sigma_j)}\right| \\
&= \left|\left(\underline{\boldsymbol{\xi}}\nabla q_m^0|\nabla\dot{q}_i\right)_{\dot{\Omega}} + \left(\operatorname{div}\underline{\boldsymbol{\xi}}\nabla q_m^0|\dot{q}_i\right)_{\dot{\Omega}}\right| \\
&\lesssim \|\nabla q_m^0\|_{\mathbf{L}^2(\dot{\Omega})}\|\nabla\dot{q}_i\|_{\mathbf{L}^2(\dot{\Omega})} + \|\operatorname{div}\underline{\boldsymbol{\xi}}\nabla q_m^0\|_{L^2(\dot{\Omega})}\|\dot{q}_i\|_{L^2(\dot{\Omega})} \\
&\lesssim \|\nabla q_m^0\|_{\mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}},\Omega)} \\
&\lesssim \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2},
\end{aligned}$$

the latter because of (3.37) and $\operatorname{div}\underline{\boldsymbol{\xi}}\nabla q_m^0 = \operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m$. Hence,

$$\|\dot{q}_m^\Sigma\|_{Q_T(\underline{\boldsymbol{\xi}};\dot{\Omega})} \lesssim \left(\sum_{i=1}^{I} \left|\langle \underline{\boldsymbol{\xi}}\mathbf{y}_m \cdot \mathbf{n}, 1\rangle_{H^{1/2}(\Sigma_i)}\right| + \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2} \right), \tag{3.39}$$

and $\|\widetilde{\nabla\dot{q}_m^\Sigma}\|_{\mathbf{L}^2} = \|\dot{q}_m^\Sigma\|_{Q_T(\underline{\boldsymbol{\xi}};\dot{\Omega})} \longrightarrow 0$.

Thirdly, let $\mathbf{z}_m := \mathbf{x}_m - \widetilde{\nabla\dot{q}_m^\Sigma} = \mathbf{y}_m - \nabla q_m^0 - \widetilde{\nabla\dot{q}_m^\Sigma} \in \mathbf{K}_T(\underline{\boldsymbol{\xi}},\Omega)$. There holds $\operatorname{\mathbf{curl}}\mathbf{z}_m = \operatorname{\mathbf{curl}}\mathbf{y}_m$, $\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{z}_m = 0$, and additionally $\langle \underline{\boldsymbol{\xi}}\mathbf{z}_m \cdot \mathbf{n}, 1\rangle_{H^{1/2}(\Sigma_i)} = 0$. Indeed, one has, by integration by parts,

$$\begin{aligned}
\langle \underline{\boldsymbol{\xi}}\mathbf{z}_m \cdot \mathbf{n}, 1\rangle_{H^{1/2}(\Sigma_i)} &= \sum_{j=1}^{I} \langle \underline{\boldsymbol{\xi}}\mathbf{z}_m \cdot \mathbf{n}, [\dot{q}_i]_{\Sigma_j}\rangle_{H^{1/2}(\Sigma_j)} \\
&= \left(\underline{\boldsymbol{\xi}}\mathbf{z}_m|\nabla\dot{q}_i\right)_{\dot{\Omega}} + \left(\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{z}_m|\dot{q}_i\right)_{\dot{\Omega}},
\end{aligned}$$

with $\left(\underline{\boldsymbol{\xi}}\mathbf{z}_m|\nabla\dot{q}_i\right)_{\dot{\Omega}} = \left(\underline{\boldsymbol{\xi}}\mathbf{x}_m|\nabla\dot{q}_i\right)_{\dot{\Omega}} - \left(\underline{\boldsymbol{\xi}}\nabla\dot{q}_m^\Sigma|\nabla\dot{q}_i\right)_{\dot{\Omega}} = 0$ by definition (3.38) of $\dot{q}_m^\Sigma$; and $\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{z}_m = 0$. Hence, $\langle \underline{\boldsymbol{\xi}}\mathbf{z}_m \cdot \mathbf{n}, 1\rangle_{H^{1/2}(\Sigma_i)} = 0$. One can then invoke the Theorem of vector potential 2.3.6: there exists $\mathbf{w}_m \in \mathbf{X}_N(\Omega)$ s.t. $\underline{\boldsymbol{\xi}}\mathbf{z}_m = \operatorname{\mathbf{curl}}\mathbf{w}_m$, $\operatorname{div}\mathbf{w}_m = 0$, and $\|\mathbf{w}_m\|_{\mathbf{X}_N} \lesssim \|\underline{\boldsymbol{\xi}}\mathbf{z}_m\|_{\mathbf{L}^2} \lesssim \|\mathbf{z}_m\|_{\mathbf{L}^2}$. This is the splitting (3.27) of $\mathbf{x}_m$. Furthermore, we have by integration by parts

$$\left(\mathbf{z}_m|\underline{\boldsymbol{\xi}}\mathbf{z}_m\right) = (\mathbf{z}_m|\operatorname{\mathbf{curl}}\mathbf{w}_m) = (\operatorname{\mathbf{curl}}\mathbf{z}_m|\mathbf{w}_m) = (\operatorname{\mathbf{curl}}\mathbf{y}_m|\mathbf{w}_m).$$

Using again relation (3.4), there holds

$$\|\mathbf{z}_m\|^2_{\mathbf{L}^2} \lesssim \|\operatorname{\mathbf{curl}}\mathbf{y}_m\|_{\mathbf{L}^2}\|\mathbf{w}_m\|_{\mathbf{L}^2} \lesssim \|\operatorname{\mathbf{curl}}\mathbf{y}_m\|_{\mathbf{L}^2}\|\mathbf{z}_m\|_{\mathbf{L}^2}$$

and

$$\|\mathbf{z}_m\|_{\mathbf{L}^2} \lesssim \|\operatorname{\mathbf{curl}}\mathbf{y}_m\|_{\mathbf{L}^2}. \tag{3.40}$$

Hence $\|\mathbf{z}_m\|_{\mathbf{L}^2} \longrightarrow 0$.

Finally, as $\mathbf{y}_m = \mathbf{z}_m + \nabla q_m^0 + \widetilde{\nabla\dot{q}_m^\Sigma}$, we have $\|\mathbf{y}_m\|_{\mathbf{L}^2} \longrightarrow 0$, which contradicts $\|\mathbf{y}_m\|_{\mathbf{L}^2} = 1$. □

We conclude this subsection with the extension of the compact embedding Theorem 2.4.9.

Theorem 3.3.4. The embedding of $\mathbf{X}_T(\underline{\boldsymbol{\xi}};\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact.

Proof. Let $(\mathbf{y}_m)$ be a bounded sequence of $\mathbf{X}_T(\underline{\boldsymbol{\xi}};\Omega)$. As in the previous proof, we split $\mathbf{y}_m$ into three terms, introducing $q_m^0 \in H^1_{\mathrm{zmv}}(\Omega)$, $\dot{q}_m^\Sigma \in Q_T(\underline{\boldsymbol{\xi}};\dot{\Omega})$, and $\mathbf{w}_m \in \mathbf{X}_N(\Omega)$ s.t.

$$\mathbf{y}_m = \nabla q_m^0 + \widetilde{\nabla \dot{q}_m^\Sigma} + \underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{w}_m.$$

Additionally, there holds (see previous proof):

$$\begin{aligned}
&\|\nabla q_m^0\|_{\mathbf{L}^2} \lesssim \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2};\\
&\|\widetilde{\nabla \dot{q}_m^\Sigma}\|_{\mathbf{L}^2} \lesssim \left(\|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2} + \sum_i \left|\langle\underline{\boldsymbol{\xi}}\mathbf{y}_m\cdot\mathbf{n},1\rangle_{H^{1/2}(\Sigma_i)}\right|\right);\\
&\|\mathbf{w}_m\|_{\mathbf{X}_N} \lesssim \|\mathbf{z}_m\|_{\mathbf{L}^2} \lesssim \|\mathbf{curl}\,\mathbf{y}_m\|_{\mathbf{L}^2}.
\end{aligned}$$

Let us begin with $(\dot{q}_m^\Sigma)$: it is a bounded sequence of the finite-dimensional vector space $Q_T(\underline{\boldsymbol{\xi}};\dot{\Omega})$, so it admits a converging subsequence (in particular in H^1-norm). Besides, (q_m^0) is a bounded sequence of $H^1(\Omega)$. Then, by Rellich Theorem, it admits a converging susbsequence (still denoted with the same index) in $L^2(\Omega)$. Similarly, $\mathbf{w}_m$ is a bounded sequence of $\mathbf{X}_N(\Omega)$, so by Theorem 2.4.8 it admits a converging subsequence in $\mathbf{L}^2(\Omega)$. It remains to prove that the subsequences (∇q_m^0) and $(\mathbf{curl}\,\mathbf{w}_m)$ converge in $\mathbf{L}^2(\Omega)$.

Using the notation $v_{mn} := v_m - v_n$, there holds by integration by parts, for any q in $H^1_{\mathrm{zmv}}(\Omega)$,

$$\left(\underline{\boldsymbol{\xi}}\nabla q_{mn}^0|\nabla q\right) = \left(\underline{\boldsymbol{\xi}}\mathbf{y}_{mn}|\nabla q\right) = -\left(\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_{mn}|q\right),$$

by definition of q_m^0 and because $\mathbf{y}_m \in \mathbf{X}_T(\underline{\boldsymbol{\xi}};\Omega)$. Taking $q = q_{mn}^0$, one gets by relation (3.4)

$$\xi_-\|\nabla q_{mn}^0\|^2_{\mathbf{L}^2} \le \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_{mn}\|_{L^2}\|q_{mn}^0\|_{L^2} \le 2\sup_m\left(\|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2}\right)\|q_{mn}^0\|_{L^2}.$$

Thus (∇q_m^0) is a Cauchy sequence of $\mathbf{L}^2(\Omega)$, hence converges in this space.

Furthermore, recalling that

$$\left(\underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{w}_{mn}|\,\mathbf{curl}\,\mathbf{w}_{mn}\right) = \left(\mathbf{z}_{mn}|\,\mathbf{curl}\,\mathbf{w}_{mn}\right) = \left(\mathbf{curl}\,\mathbf{z}_{mn}|\mathbf{w}_{mn}\right) = \left(\mathbf{curl}\,\mathbf{y}_{mn}|\mathbf{w}_{mn}\right)$$

and that $\underline{\boldsymbol{\xi}}^{-1}$ also satisfies an ellipticity condition (Prop. 3.1.4), we get

$$\xi_-^{\mathrm{inv}}\|\mathbf{curl}\,\mathbf{w}_{mn}\|^2_{\mathbf{L}^2} \le \|\mathbf{curl}\,\mathbf{y}_{mn}\|_{\mathbf{L}^2}\|\mathbf{w}_{mn}\|_{\mathbf{L}^2} \le 2\sup_m\left(\|\mathbf{curl}\,\mathbf{y}_m\|_{\mathbf{L}^2}\right)\|\mathbf{w}_{mn}\|_{\mathbf{L}^2},$$

which proves that $(\mathbf{curl}\,\mathbf{w}_m)$ is a Cauchy (hence converging) sequence of $\mathbf{L}^2(\Omega)$. As $\mathbf{y}_m = \underline{\boldsymbol{\xi}}^{-1}\,\mathbf{curl}\,\mathbf{w}_m + \nabla q_m^0 + \widetilde{\nabla \dot{q}_m^\Sigma}$, the subsequence $(\mathbf{y}_m)$ converges in $\mathbf{L}^2(\Omega)$. □

3.3.3 The space $\mathbf{W}_N(\underline{\boldsymbol{\xi}};\Omega)$

To conclude, let us prove the compact embedding of the space $\mathbf{W}_N(\underline{\boldsymbol{\xi}};\Omega)$ into $\mathbf{L}^2(\Omega)$, extending the result of Theorem 2.4.10. A similar result may be found in [100] (in the connected boundary case).

Theorem 3.3.5. The embedding of $\mathbf{W}_N(\underline{\boldsymbol{\xi}};\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact.

Proof. The proof is similar to the one of Theorem 8.1.3 in [7]. Let $(\mathbf{y}_m)$ a bounded sequence of $\mathbf{W}_N(\underline{\boldsymbol{\xi}};\Omega)$. We make use of decompositions (3.22) and (3.24) (recalling that $\mathbf{W}_N(\underline{\boldsymbol{\xi}};\Omega) \subset \mathbf{H}(\operatorname{div}\underline{\boldsymbol{\xi}}0,\Omega)$) to split $\mathbf{y}_m$ into three terms: $\mathbf{y}_m = \mathbf{z}_m + \nabla q_m^0 + \nabla q_m^\Gamma$, where q_m^0 is the unique solution to

$$\left|\begin{array}{l}\text{Find } q_m^0 \in H^1_0(\Omega) \text{ s.t., } \forall q \in H^1_0(\Omega),\\ \qquad\left(\underline{\boldsymbol{\xi}}\nabla q_m^0|\nabla q\right) = \left(\underline{\boldsymbol{\xi}}\mathbf{y}_m|\nabla q\right);\end{array}\right.$$

q_m^Γ the unique solution to

$$\left| \begin{aligned} &\text{Find } q_m^\Gamma \in Q_N(\underline{\boldsymbol{\xi}};\Omega) \text{ s.t., } \forall q \in Q_N(\underline{\boldsymbol{\xi}};\Omega), \\ &\quad \left(\underline{\boldsymbol{\xi}}\nabla q_m^\Gamma|\nabla q\right) = \left(\underline{\boldsymbol{\xi}}(\mathbf{y}_m - \nabla q_m^0)|\nabla q\right); \end{aligned} \right.$$

and $\mathbf{z}_m := \mathbf{y}_m - \nabla q_m^0 - \nabla q_m^\Gamma$. Because those problems are well-posed, the sequences (∇q_m^0) and (∇q_m^Γ), and so $(\mathbf{z}_m)$, are bounded in $\mathbf{L}^2(\Omega)$. The space $Q_N(\underline{\boldsymbol{\xi}};\Omega)$ is of finite dimension, so one can extract a subsequence (denoted with the same index) that converges, in particular in $H^1(\Omega)$-norm. Concerning the sequence (q_m^0) of $H_0^1(\Omega)$, one can extract a subsequence (still denoted with the same index) which converges in $L^2(\Omega)$, by Rellich theorem. Additionally, using the notation $v_{mn} = v_m - v_n$, one has

$$\left(\underline{\boldsymbol{\xi}}\nabla q_{mn}^0|\nabla q_{mn}^0\right) = \left(\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_{mn}|q_{mn}^0\right)$$

hence

$$\|\nabla q_{mn}^0\|_{\mathbf{L}^2}^2 \lesssim \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_{mn}\|_{L^2}\|q_{mn}^0\|_{L^2} \lesssim \sup_m \|\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{y}_m\|_{L^2}\|q_{mn}^0\|_{L^2},$$

and (∇q_{mn}^0) is a Cauchy sequence of $\mathbf{L}^2(\Omega)$, hence converges in this space.

Besides, $\mathbf{z}_m = \mathbf{y}_m - \nabla q_m^0 - \nabla q_m^\Gamma \in \mathbf{H}^+(\mathbf{curl},\Omega)$, with $\mathbf{curl}\,\mathbf{z}_m = \mathbf{curl}\,\mathbf{y}_m$, $\gamma^T\mathbf{z}_m = \gamma^T\mathbf{y}_m$, $\operatorname{div}\underline{\boldsymbol{\xi}}\mathbf{z}_m = 0$ and $\langle\underline{\boldsymbol{\xi}}\mathbf{z}_m\cdot\mathbf{n},1\rangle_{H^{1/2}(\Gamma_k)} = 0\ \forall k$ (for the last item, this is similar to the proof of Theorem 3.3.1). Then, by the vector potential Theorem 2.3.5, there exists $\mathbf{w}_m \in \mathbf{H}^1(\Omega)$ s.t. $\underline{\boldsymbol{\xi}}\mathbf{z}_m = \mathbf{curl}\,\mathbf{w}_m$, and $\|\mathbf{w}_m\|_{\mathbf{H}^1} \lesssim \|\mathbf{z}_m\|_{\mathbf{L}^2}$. This is decomposition (3.24). Furthermore, given $\sigma \in]\frac{1}{2},1[$, thanks to the compact embedding of $\mathbf{H}^1(\Omega)$ into $\mathbf{H}^\sigma(\Omega)$ (see Theorem 2.4.7), one can extract a subsequence of $(\mathbf{w}_m)$ (still denoted with the same index) which converges in $\mathbf{H}^\sigma(\Omega)$. Therefore, $(\mathbf{w}_m)$ converges in $\mathbf{L}^2(\Omega)$, and $(\mathbf{w}_{m|\Gamma})$ converges in $\mathbf{L}^2(\Gamma)$, because the trace mapping is continuous from $\mathbf{H}^\sigma(\Omega)$ to $\mathbf{L}^2(\Gamma)$ (see [7, Th. 2.1.62]). Then,

$$\begin{aligned} (\mathbf{z}_{mn}|\underline{\boldsymbol{\xi}}\mathbf{z}_{mn}) &= (\mathbf{z}_{mn}|\,\mathbf{curl}\,\mathbf{w}_{mn}) \\ &= (\mathbf{curl}\,\mathbf{z}_{mn}|\mathbf{w}_{mn}) + \left(\gamma^T\mathbf{z}_{mn},\pi^T\mathbf{w}_{mn}\right)_\Gamma \\ &= (\mathbf{curl}\,\mathbf{y}_{mn}|\mathbf{w}_{mn}) + \left(\gamma^T\mathbf{y}_{mn},\pi^T\mathbf{w}_{mn}\right)_\Gamma \end{aligned}$$

and

$$\|\mathbf{z}_{mn}\|_{\mathbf{L}^2}^2 \lesssim \sup_m \|\mathbf{y}_m\|_{\mathbf{W}_N(\underline{\boldsymbol{\xi}})}\left(\|\mathbf{w}_{mn}\|_{\mathbf{L}^2} + \|\mathbf{w}_{mn|\Gamma}\|_{\mathbf{L}^2(\Gamma)}\right),$$

which proves that $(\mathbf{z}_m)$ is a Cauchy, hence converging, sequence of $\mathbf{L}^2(\Omega)$. Finally, the whole subsequence $(\mathbf{y}_m)$ converges in $\mathbf{L}^2(\Omega)$. □

Conclusion

We have extended the classical tools for the Maxwell equations, presented in Chapter 2. We do so for a very general class of tensors, that is elliptic tensors, without assuming Hermitian properties. These results will be necessary to carry out the analysis of the variational formulations that arise from Maxwell problems, which is the subject of Chapter 5.

The main results of this chapter are of two kinds. The first ones are the Helmholtz decompositions of Theorems 3.2.1, 3.2.3, and 3.2.4. The second ones are the compact embeddings of Theorems 3.3.2, 3.3.4, and 3.3.5. The reason why we need to derive several different Helmholtz decompositions and compact embeddings lies in boundary conditions. Indeed, different boundary conditions in the Maxwell problem (Dirichlet, Neumann, or Robin) will lead to variational formulations posed in different spaces ($\mathbf{H}_0(\mathbf{curl},\Omega)$, $\mathbf{H}(\mathbf{curl},\Omega)$, or $\mathbf{H}^+(\mathbf{curl},\Omega)$). This is why we have derived the tools appropriate to each of these three different spaces.

Chapter 4

Characterization of function spaces for Robin traces

In this chapter, we shall consider a *generalised Robin* (or *impedance*) boundary condition that reads

$$\pi^T \mathbf{C} + \underline{\boldsymbol{\alpha}}\gamma^T \mathbf{E} = \mathbf{g} \quad \text{on } \Gamma, \tag{4.1}$$

where $\underline{\boldsymbol{\alpha}}$ is a $\mathbb{C}^{2\times 2}$-valued tensor field. This condition is an extension of what is classically referred to as *impedance* or *Robin* condition,

$$\pi^T \mathbf{C} + \alpha\gamma^T \mathbf{E} = \mathbf{g} \quad \text{on } \Gamma, \tag{4.2}$$

where α is a scalar (and generally constant) coefficient. It is inherited from the (truncated) classical Silver-Müller boundary condition [87, 103]. Let us briefly recall why this is so. Assume that, in a neighbourhood of Γ, $\underline{\boldsymbol{\varepsilon}} = \varepsilon\underline{\mathbf{I}}$ and $\underline{\boldsymbol{\mu}} = \mu\underline{\mathbf{I}}$ with $\varepsilon, \mu > 0$. Then the (truncated) Silver-Müller condition writes (see e.g. [7, p. 57])

$$\gamma^T \mathbf{E} + \sqrt{\frac{\mu}{\varepsilon}}\pi^T \mathbf{H} = \mathbf{g} \quad \text{on } \Gamma, \tag{4.3}$$

which is an absorbing condition, and, because of Faraday law in the time-harmonic regime, leads to (4.1) with $\underline{\boldsymbol{\alpha}} = i\omega\sqrt{\varepsilon/\mu}\underline{\mathbf{I}}$ and $\mathbf{C} = \underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E} = (i\omega)^{-1}\mathbf{H}$. When $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ are anisotropic, however, the equivalent to (4.3) becomes less clear. This is why we shall also consider more general conditions in the form of (4.1). The coefficient α or $\underline{\boldsymbol{\alpha}}$ will be called the *impedance coefficient* (which is a language abuse; in the Silver-Müller case, α has more to do with the inverse of an impedance).

Our aim is to clarify the mathematical framework in which the condition (4.1) may hold, for some general classes of coefficients. Indeed, mathematically speaking, one has $\mathbf{E}$, $\mathbf{C} \in \mathbf{H}(\mathbf{curl}, \Omega)$, but the traces $\gamma^T\mathbf{E}$ and $\pi^T\mathbf{C}$ belong to different trace spaces: $\gamma^T\mathbf{E} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, whereas $\pi^T\mathbf{C} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. Therefore, the function space in which (4.1) may hold is unclear. In the literature, it is generally assumed or stated without proper justification (see, e.g., [43, 100, 70, 56]), that the condition (4.2) holds in $\mathbf{L}_t^2(\Gamma)$; and, in particular, that one should look for the solution of the associated time-harmonic Maxwell problem in the space $\mathbf{H}^+(\mathbf{curl}, \Omega) = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega), \gamma^T\mathbf{v} \in \mathbf{L}_t^2(\Gamma)\}$. If one assumes that $\mathbf{g}$ belongs to $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, then both traces $\gamma^T\mathbf{E}$ and $\pi^T\mathbf{C}$ also belong to this function space. Thus, to determine whether the condition (4.2) holds in $\mathbf{L}_t^2(\Gamma)$ reduces to determine whether

$$\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \subset \mathbf{L}_t^2(\Gamma), \tag{4.4}$$

which is a general assertion of functional analysis, independent of the boundary condition. However, this assumption is not always justified. It has been proven in the case of smooth domains by Barucq and Hanouzet [10]. For piecewise smooth domains, however, the result does not always hold. We refer to Theorem 8 of [20], Proposition 4.11 of [17], and sections 5.1.2.1 and 5.1.2.2 in [7]. Some of these results also deal with the case of mixed boundary conditions. To the best of our knowledge, only the case of scalar constant coefficients has been addressed in the literature. Then, the main goal of this chapter is to check whether, under appropriate assumptions on Γ and $\underline{\boldsymbol{\alpha}}$, that will be made more precise later, the boundary condition of (4.1) holds in $\mathbf{L}_t^2(\Gamma)$.

Throughout this chapter, we assume that Γ is either a smooth boundary $\mathcal{C}^2$, or piecewise $\mathcal{C}^2$. When it is piecewise $\mathcal{C}^2$, we assume for simplicity that Ω is a Lipschitz polyhedron, i.e. that the faces of Γ are plane and the edges are straight lines. For more general geometries, we refer to [40]. The chapter is organized as follows. In

Section 4.1 we provide several functional analysis results; in particular, the assertion (4.4) is discussed. In Section 4.2 we address the condition with scalar coefficient, focusing on the smooth or piecewise constant cases. In Section 4.3 we provide a result on the condition with a tensor-valued coefficient.

4.1 General embedding results on surface function spaces

In this section, we find necessary and sufficient conditions for the assertion (4.4) to hold. In fact, we prove a functional analysis result (Theorem 4.1.4) which is actually more precise than statement (4.4). To that aim, we follow the path proposed in sections 5.1.2.1 and 5.1.2.2 of [7]. As a consequence, we are able to state whether the condition (4.2) with a constant coefficient holds in $\mathbf{L}^2_t(\Gamma)$.

Preliminary discussion

We assume for simplicity that Γ is topologically trivial (this assumption, however, is not restrictive; see Remark 4.1.6 below). Let $\mathbf{u} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, we write the surface Helmholtz decompositions of $\mathbf{u}$, first considered as an element of $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, and then as an element of $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ (cf (2.83)-(2.84)):

$$\mathbf{u} = \mathbf{curl}_\Gamma\, \phi^- + \nabla_\Gamma\, \psi^+, \tag{4.5}$$

where $\phi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma)$, $\psi^+ \in \mathcal{H}(\Gamma)$, and

$$\mathbf{u} = \nabla_\Gamma\, \psi^- + \mathbf{curl}_\Gamma\, \phi^+, \tag{4.6}$$

where $\psi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma)$, $\phi^+ \in \mathcal{H}(\Gamma)$. We recall that

$$\mathcal{H}(\Gamma) = \{v \in H^1_{\mathrm{zmv}}(\Gamma),\ \Delta_\Gamma\, v \in H^{-1/2}(\Gamma)\}. \tag{4.7}$$

If either $\phi^- \in H^1(\Gamma)$ or $\psi^- \in H^1(\Gamma)$, then it follows that $\mathbf{u} \in \mathbf{L}^2_t(\Gamma)$, and so the claim (4.4) is proven. Therefore, we investigate whether ϕ^- or ψ^- are actually more regular than $H^{1/2}(\Gamma)$. Subtracting the two decompositions of $\mathbf{u}$, one has

$$\mathbf{curl}_\Gamma \left(\phi^- - \phi^+\right) + \nabla_\Gamma \left(-\psi^- + \psi^+\right) = \mathbf{0} \quad \text{ on } \Gamma. \tag{4.8}$$

We introduce $\phi^{\mathrm{sing}} := \phi^- - \phi^+ \in H^{1/2}_{\mathrm{zmv}}(\Gamma)$ and $\psi^{\mathrm{sing}} := -\psi^- + \psi^+ \in H^{1/2}_{\mathrm{zmv}}(\Gamma)$. Taking respectively the curl_Γ and the div_Γ of equation (4.8), and recalling that $\Delta_\Gamma = -\,\mathrm{curl}_\Gamma\, \mathbf{curl}_\Gamma = \mathrm{div}_\Gamma\, \nabla_\Gamma$, one gets that $\phi^{\mathrm{sing}}, \psi^{\mathrm{sing}}$ are governed by

$$\text{Find } \phi^{\mathrm{sing}}, \psi^{\mathrm{sing}} \in H^{1/2}_{\mathrm{zmv}}(\Gamma) \text{ s.t. } \begin{cases} \Delta_\Gamma\, \phi^{\mathrm{sing}} = 0 \\ \Delta_\Gamma\, \psi^{\mathrm{sing}} = 0 \end{cases} \text{ on } \Gamma. \tag{4.9}$$

Thus the question is actually to determine whether the homogeneous Laplace-Beltrami problem admits *singular solutions*, i.e. solutions in $H^{1/2}(\Gamma) \setminus H^1(\Gamma)$. If not, one would have $\phi^{\mathrm{sing}} \in H^1_{\mathrm{zmv}}(\Gamma)$, hence, because the Laplace-Beltrami problem is well-posed in $H^1_{\mathrm{zmv}}(\Gamma)$, $\phi^{\mathrm{sing}} = 0$.

Study of Laplace-Beltrami problems

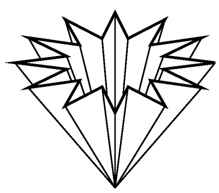


Figure 4.1: Example of pathological vertex

The existence of singular solutions for the Laplace-Beltrami problems (4.9) will depend on the regularity of the surface Γ. To that aim, we introduce the notion of (*semi-*)*pathological vertices* of Γ.

Definition 4.1.1. Let v be a vertex of Γ, that stands at the intersection of K smooth faces denoted Γ_k. Locally, each face Γ_k can be described by polar coordinates (r,θ), with $\theta \in]\theta_k, \theta_{k+1}[$, $\theta_0 = 0$ and $\theta_K = \theta_v$, where θ_v is the sum of all faces angles at the vertex v. If $\theta_v > 4\pi$, the vertex v is said to be *pathological*. In this case, we define $I_v := \max\{q \in \mathbb{N} \mid \theta_v > 4\pi q\} \geq 1$. In the limit case $\theta_v = 4\pi$, the vertex v is said to be *semi-pathological*.

Lemma 4.1.2. The solutions to the problem

$$\text{Find } \phi^{\text{sing}} \in H^{1/2}_{\text{zmv}}(\Gamma) \text{ s.t. } \Delta_\Gamma \phi^{\text{sing}} = 0 \quad \text{on } \Gamma \tag{4.10}$$

are characterized as follows:

- If Γ has no pathological vertex, then $\phi^{\text{sing}} = 0$.
- If Γ has P pathological vertices $(v_p)_{p=1,P}$, then $\phi^{\text{sing}} \in \mathcal{S}$, where $\mathcal{S}$ is a vector space of dimension $2\sum_{p=1}^{P} I_{v_p}$, and is called the space of singularities.

Proof. When Γ is smooth, no solution is in $H^{1/2}(\Gamma) \setminus H^1(\Gamma)$. In this case, the result is a consequence of elliptic regularity. It is obtained by localization (via a finite covering of Γ), and then going back to the parametric plane locally via a smooth mapping, and finally using the standard theory of singularities, see for instance [61, Chapter 2]. In the more general case of a piecewise smooth boundary, one also studies the solution locally. Clearly one can define a finite covering of Γ where all open subsets Γ_A are of one of the three following types: Γ_A is included in one face, Γ_A contains one edge, or Γ_A contains one vertex and the adjacent edges. The proof makes intensive use of results from Grisvard [62].

1. When Γ_A is included in one face. We conclude that $\phi^{\text{sing}}{}_{|\Gamma_A} \in H^1(\Gamma_A)$ as above.
2. When Γ_A contains an edge e_{ij}, at the intersection of two plane faces Γ_i and Γ_j. We denote $\boldsymbol{\tau}_{ij}$ the unit vector tangent to e_{ij}, and, for $k = i,j$, $\boldsymbol{\tau}_k$ the unit vector s.t. $(\boldsymbol{\tau}_{ij}, \boldsymbol{\tau}_k)$ defines a local base on Γ_k. One has $\phi_k := \phi^{\text{sing}}{}_{|\Gamma_k} \in L^2(\Gamma_k)$ with $\Delta_\Gamma \phi_k = 0$ on Γ_k. Then, according to Theorem 1.5.2 in [62], both traces of ϕ_i, ϕ_j belong to $(\widetilde{H}^{1/2}(e_{ij}))'$, and both normal derivatives belong to $(\widetilde{H}^{3/2}(e_{ij}))'$, where $\widetilde{H}^s(e_{ij})$ denotes fields whose continuation by 0 on $\partial\Gamma_A$ belong to $H^s(\partial\Gamma_A)$. Moreover, as $\phi^{\text{sing}}{}_{|\Gamma_A} \in H^{1/2}(\Gamma_A)$ and $\Delta_\Gamma(\phi^{\text{sing}}{}_{|\Gamma_A}) \in L^2(\Gamma_A)$, traces on e_{ij} match, so there holds $\phi_{i|e_{ij}} = \phi_{j|e_{ij}}$, as well as $\partial_{\boldsymbol{\tau}_i}\phi_i = \partial_{\boldsymbol{\tau}_j}\phi_j$ on e_{ij}.

   Then, one maps Γ_A *locally* around e_{ij} into a subset $\Gamma_A^\star$ of the parametric plane. This is done by a piecewise affine, bijective mapping $T : \Gamma_A \to \Gamma_A^\star$. Introducing $\phi_k^\star := \phi_k \circ T^{-1}$ on each $\Gamma_k^\star := T(\Gamma_k)$, and finally $\phi^\star \in L^2(\Gamma_A^\star)$ defined by $\phi^\star_{|\Gamma_k^\star} = \phi_k^\star$ for $k = i,j$, one gets that $\phi^\star \in H^{1/2}(\Gamma_A^\star)$ with $\Delta_\Gamma \phi^\star = 0$ on $\Gamma_A^\star$, thanks to the trace matchings. Therefore, using the result of 1., $\phi^\star \in H^1(\Gamma_A^\star)$. Coming back to $\phi^{\text{sing}}{}_{|\Gamma_A} = T \circ \phi^\star$, it also belongs to $H^1(\Gamma_A)$.
3. When Γ_A contains a vertex v, at the intersection of K smooth faces denoted Γ_k. We denote $e_{k,k+1}$ the edge between Γ_k and Γ_{k+1}, and use the same notations as above. As before, the ϕ_k satisfy $\phi_k \in L^2(\Gamma_k)$ with $\Delta_\Gamma \phi_k = 0$ on Γ_k, and trace matchings: $\phi_k = \phi_{k+1}$ as well as $\partial_{\boldsymbol{\tau}_k}\phi_k = \partial_{\boldsymbol{\tau}_{k+1}}\phi_{k+1}$ on $e_{k,k+1}$ in a weak sense. Thus, outside any neighbourhood of the vertex v, we conclude thanks to points 1. and 2. that ϕ^{sing} is of H^1-regularity.

   In a neighbourhood Γ_v of the vertex, we follow Kondratiev's theory [79, 39]. Using polar coordinates (r,θ) in each Γ_k, one finds (see [62, §2.3]) that the solution to the problem locally belongs to $\text{span}_{\lambda\in\Lambda}(r^\lambda \varphi_\lambda(\theta))$, where $(\varphi_\lambda)_\lambda$ are eigenfunctions of the operator $\varphi \mapsto -\varphi''$ on $[0,\theta_v]$, i.e. $\varphi_\lambda(\theta) = \exp(\pm i\lambda\theta)$; moreover, the admissible values of λ are constrained by periodic boundary conditions (trace matchings on $e_{K,1}$). Thus, a basis of solutions is *locally* given by

   $$\phi_\lambda^\pm(r,\theta) = r^\lambda e^{\pm i\lambda\theta}, \quad \lambda \in \Lambda := \frac{2\pi}{\theta_v}\mathbb{Z}, \tag{4.11}$$

   where the value of λ is prescribed by the periodicity condition.

   The regularity of each $\phi_\lambda^\pm$ is given by Theorem 1.2.18 in [62]: one has that, for $s \in]0,1[$,

   $$\phi_\lambda^\pm \in H^s \iff \lambda > s - 1; \tag{4.12}$$

in particular, $\phi_\lambda^\pm \in H^{1/2} \iff \lambda > -\frac{1}{2}$. On the other hand, $\phi_\lambda^\pm \in H^1 \iff \lambda \geq 0$ ($\lambda = 0$ gives the constant solution). So, (nonzero) local singular solutions exist if, and only if, there exists $\lambda \in \Lambda$ s.t. $-\frac{1}{2} < \lambda < 0$. That is when $-\frac{1}{2} < -\frac{2\pi}{\theta_v}$, i.e. $\theta_v > 4\pi$; in other words, when v is *pathological*.

To reconstruct a global solution of (4.10) starting from $\phi_\lambda^\pm$, we introduce a cut-off function $\chi \in \mathcal{C}^\infty(\Gamma)$ whose value is 1 in Γ_v and 0 on $\partial\Gamma_A$, and s.t. $\Delta_\Gamma(\chi\phi_\lambda^\pm) \in H^{-1}(\Gamma)$. Then, one solves the (well-posed) problem

$$\text{Find } w_\lambda^\pm \in H^1_{\text{zmv}}(\Gamma) \text{ s.t. } \Delta_\Gamma w_\lambda^\pm = \Delta_\Gamma(\chi\phi_\lambda^\pm) \quad \text{in } \Gamma. \tag{4.13}$$

Introducing $s_{v,\lambda}^\pm := w_\lambda^\pm - \chi\phi_\lambda^\pm \in H^{1/2}(\Gamma)$, we have found $2I_v$ independent singular solutions that do not vanish at the neighbourhood of the vertex v. We proceed similarly with the other vertices. Outside the neighbourhood of the vertices, there are no singular solutions because of points 1. and 2. (we also refer to Lemma 2.3.4 in [62]).

One concludes that there exist singular solutions of (4.10) as soon as Γ has at least one pathological vertex. More precisely, there exist a basis of $2\sum_{p=1}^P I_{v_p}$ (independent) singular solutions $s_{v,\lambda}^\pm \in H^{1/2}(\Gamma) \setminus H^1(\Gamma)$ arising from the different pathological vertices v_p. On the other hand, in the absence of pathological vertices, we conclude that there are no singular solutions in $H^{1/2}(\Gamma) \setminus H^1(\Gamma)$. Thus, we have obtained and characterized all the singular solutions to the homogenous Laplace-Beltrami problem, by spanning all singular solutions for each pathological vertex of Γ. □

In what follows, we need to introduce surface Sobolev spaces for more regular exponents than $|s| \leq 1$. When Ω is a polyhedron, we define (cf. [7]), for $0 < s < \frac{1}{2}$,

$$H^{1+s}(\Gamma) := \left\{ v \in H^1(\Gamma),\ \nabla_\Gamma v \in \mathbf{H}_t^s(\Gamma_i),\ \forall i \right\}, \tag{4.14}$$

where the (Γ_i) are the faces of Γ.

Lemma 4.1.3. There exists $s_{\max} \in]0, \frac{1}{2}]$ s.t., $\forall s \in [0, s_{\max}[$,

$$\mathcal{H}(\Gamma) \subset H^{1+s}_{\text{zmv}}(\Gamma). \tag{4.15}$$

This embedding is continuous, and the value of $s_{\max}$ depends only on geometry:

- If Γ has pathological vertices, then $s_{\max} = \min_{p=1,P} \left(\dfrac{2\pi}{\theta_{v_p}}\right) < \dfrac{1}{2}$;
- If Γ has no pathological vertex, then $s_{\max} = \frac{1}{2}$;
- If, in addition, Γ has no semi-pathological vertex, then (4.15) also holds for $s = \frac{1}{2}$.

Proof. Let $\varphi \in \mathcal{H}(\Gamma)$: there holds $\varphi \in H^1_{\text{zmv}}(\Gamma)$, and $g := \Delta_\Gamma \varphi \in H^{-1/2}(\Gamma)$. Therefore, φ can be interpreted as the (regular) solution of a Laplace-Beltrami problem with data in $H^{-1/2}(\Gamma)$. The extra H^{1+s}-regularity of φ is then limited to $s \in [0, \frac{1}{2}]$, the value $\frac{1}{2}$ coming from the Shift Theorem for the Laplace-Beltrami operator with data in $H^{-1/2}$ (see eg. [88], §5.4.1). However, this extra-regularity is also driven by the regularity at the vertices v of Γ (if any). As before, the solutions are locally given at a vertex v by

$$\phi_\lambda^\pm(r,\theta) = r^\lambda e^{\pm i\lambda\theta}, \quad \lambda \in \Lambda := \frac{2\pi}{\theta_v}\mathbb{Z}, \tag{4.16}$$

with now $\lambda \geq 0$ because $\varphi \in H^1(\Gamma)$; and even $\lambda > 0$, because $\lambda = 0$ stands for the constant solution. So, the minimal exponent of local regularity at a vertex v is $s = \frac{2\pi}{\theta_v}$. Hence, if there are pathological vertices, (4.15) holds with $s_{\max} = \min_{p=1,P}\left(\frac{2\pi}{\theta_{v_p}}\right) < \frac{1}{2}$. While, if there is no pathological vertex, the regularity is now limited to $\frac{1}{2}$ because of the Shift Theorem: we conclude that, if moreover there is no semi-pathological vertex, the embedding (4.15) holds for $s = \frac{1}{2}$. In the limit case of semi-pathological vertices, one concludes that (4.15) holds for all $s < \frac{1}{2}$. Finally, in all of the above, one has continuous dependence of the embeddings, because the shift theorem also ensures that $\|\varphi\|_{H^{1+s}(\Gamma)} \lesssim \|g\|_{H^{-1/2}(\Gamma)} \lesssim \|\varphi\|_{\mathcal{H}(\Gamma)}$. [1] □

[1] Here and in the rest of the chapter, the notation $a \lesssim b$ denotes that there exists a constant $C > 0$, *independent of* a *and* b, s.t. $a \leq Cb$. The constant C depends only on the geometry, and on α when this is meaningful.

Main result

With this, one can derive the embedding results below, that generalize the result of [7, Remark 5.1.5].

Theorem 4.1.4. If Γ has pathological vertices, then, $\exists s_{\max} \in]0, \frac{1}{2}[$, $\forall s \in [0, s_{\max}[$,

$$\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \subset \mathbf{H}_t^s(\Gamma) \oplus \nabla_\Gamma \mathcal{S}, \tag{4.17}$$

$$\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \subset \mathbf{H}_t^s(\Gamma) \oplus \mathbf{curl}_\Gamma \mathcal{S}, \tag{4.18}$$

where $\mathcal{S}$ is the finite dimensional vector subspace of singularities.
If Γ has no pathological vertex, then, $\forall s \in [0, \frac{1}{2}[$,

$$\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \subset \mathbf{H}_t^s(\Gamma). \tag{4.19}$$

If, in addition, Γ has no semi-pathological vertex, then (4.19) also holds for $s = \frac{1}{2}$.
Moreover, all those embeddings are continuous.

Proof. As in the beginning of the section, let us consider $\mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, with the surface Helmholtz decompositions of $\mathbf{u}$,

$$\mathbf{u} = \mathbf{curl}_\Gamma \phi^- + \nabla_\Gamma \psi^+, \tag{4.20}$$

$$\mathbf{u} = \nabla_\Gamma \psi^- + \mathbf{curl}_\Gamma \phi^+. \tag{4.21}$$

Taking the curl_Γ of (4.20) and the div_Γ of (4.21), one gets that $\phi^-, \psi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma)$ are governed by

$$\Delta_\Gamma \phi^- = -\,\mathrm{curl}_\Gamma \mathbf{u}, \tag{4.22}$$

$$\Delta_\Gamma \psi^- = \mathrm{div}_\Gamma \mathbf{u}. \tag{4.23}$$

Likewise, $\phi^+, \psi^+ \in H^1_{\mathrm{zmv}}(\Gamma)$ are governed by

$$\Delta_\Gamma \phi^+ = -\,\mathrm{curl}_\Gamma \mathbf{u}, \tag{4.24}$$

$$\Delta_\Gamma \psi^+ = \mathrm{div}_\Gamma \mathbf{u}. \tag{4.25}$$

Moreover, ϕ^+, ψ^+ are actually the unique solutions to the problems (4.24) and (4.25), because the Laplace-Beltrami problem is well-posed in $H^1_{\mathrm{zmv}}(\Gamma)$.

The regularity of ϕ^+, ψ^+ enters the scope of Lemma 4.1.3.

- If Γ has pathological vertices $(v_p)_{p=1,P}$, we find that $\phi^+, \psi^+ \in H^{1+s}(\Gamma)$ for all $s < s_{\max}$ with $s_{\max} := \min_{p=1,P} \left(\frac{2\pi}{\theta_{v_p}}\right) \in]0, \frac{1}{2}[$.
- If Γ has no pathological vertex, we find that $\phi^+, \psi^+ \in H^{1+s}(\Gamma)$ for all $s < \frac{1}{2}$.
- If in addition Γ has no semi-pathological vertex, $\phi^+, \psi^+ \in H^{3/2}(\Gamma)$.

Moreover, for admissible s, one also has that $\|\phi^+\|_{H^{1+s}(\Gamma)} \lesssim \|\,\mathrm{curl}_\Gamma \mathbf{u}\|_{H^{-1/2}(\Gamma)}$, and $\|\psi^+\|_{H^{1+s}(\Gamma)} \lesssim \|\,\mathrm{div}_\Gamma \mathbf{u}\|_{H^{-1/2}(\Gamma)}$.

To analyse the regularity of ϕ^- and ψ^-, we recall that $\phi^{\mathrm{sing}} = \phi^- - \phi^+$ and $\psi^{\mathrm{sing}} = \psi^- - \psi^+$ are solutions of the problem (4.9). These solutions are characterized by Lemma 4.1.2.

- If Γ has pathological vertices, then $\phi^- = \phi^{\mathrm{sing}} - \phi^+$, where ϕ^{sing} is a singular solution which belongs to $\mathcal{S}$ (Lemma 4.1.2). Besides, $\phi^+, \psi^+ \in H^{1+s}(\Gamma)$ for $s < s_{\max}$ because of Lemma 4.1.3. So, thanks to decomposition (4.20), we conclude that, for all $s \in [0, s_{\max}[$, $\mathbf{u}$ belongs to $\mathbf{H}_t^s(\Gamma) + \mathbf{curl}_\Gamma \mathcal{S}$. This is embedding (4.18). Similarly, there also holds $\psi^- = \psi^{\mathrm{sing}} + \psi^+$ with $\psi^{\mathrm{sing}} \in \mathcal{S}$ and $\phi^+, \psi^+ \in H^{1+s}(\Gamma)$ for $s < s_{\max}$. Using now decomposition (4.21), we conclude that for all $s \in [0, s_{\max}[$, $\mathbf{u}$ belongs to $\mathbf{H}_t^s(\Gamma) + \nabla_\Gamma \mathcal{S}$. This is embedding (4.17).

- If Γ has no pathological vertex, then $\phi^{\text{sing}} = 0$, $\psi^{\text{sing}} = 0$, because of Lemma 4.1.2. Therefore, $\phi^- = \phi^+$, $\psi^- = \psi^+$. Moreover, according to Lemma 4.1.3, both fields $\phi^+, \psi^+ \in H^{1+s}(\Gamma)$ for all $s < 1/2$. Hence we conclude from (4.5) that $\mathbf{u}$ belongs to $\mathbf{H}_t^s(\Gamma)$ for all $s < 1/2$. This is embedding (4.19).
- If moreover Γ has no semi-pathological vertex, then (4.19) also holds for $s = \frac{1}{2}$.

Moreover, all results come with stability estimates on ϕ^+ and ψ^+. Indeed, $\| \mathbf{curl}_\Gamma \phi^+ + \nabla_\Gamma \psi^+ \|_{\mathbf{H}_t^s(\Gamma)} \lesssim \| \operatorname{div}_\Gamma \mathbf{u} \|_{H^{-1/2}(\Gamma)} + \| \operatorname{curl}_\Gamma \mathbf{u} \|_{H^{-1/2}(\Gamma)}$. So, if $\mathbf{u} \in \mathbf{H}_t^s(\Gamma)$,

$$\|\mathbf{u}\|_{\mathbf{H}_t^s(\Gamma)} \lesssim \|\mathbf{u}\|_\gamma + \|\mathbf{u}\|_\pi \tag{4.26}$$

which proves that embedding (4.19) is continuous. If $\mathbf{u}$ has a singular part, one writes $\mathbf{u} = \nabla_\Gamma \psi^{\text{sing}} + \nabla_\Gamma \psi^+ + \mathbf{curl}_\Gamma \phi^+$ so, by triangle inequality, there holds $\| \nabla_\Gamma \psi^{\text{sing}} \|_{H^{-1/2}(\Gamma)} \lesssim \|\mathbf{u}\|_\gamma + \|\mathbf{u}\|_\pi$. Because $\mathcal{S}$ is finite dimensional, all norms are equivalent (in particular, $\| \cdot \|_{H^{1/2}(\Gamma)}$ and $\| \nabla_\Gamma \cdot \|_{H^{-1/2}(\Gamma)}$), and we have obtained a stability estimate for the singular part. Hence embedding (4.17) is continuous. One can do the same for embedding (4.18). □

Remark 4.1.5. In fact, there holds

$$\mathbf{H}_t^s(\Gamma) \oplus \nabla_\Gamma \mathcal{S} = \mathbf{H}_t^s(\Gamma) \oplus \mathbf{curl}_\Gamma \mathcal{S}. \tag{4.27}$$

Indeed, looking at the singularities $\phi_\lambda^\pm(r,\theta) = r^\lambda e^{\pm i\lambda\theta}$, at a vertex v, one can observe that the $\mathbf{curl}_\Gamma$ of one is the ∇_Γ of the other.

The result obtained in Theorem 4.1.4 is stronger than the conjecture (4.4), provided that Γ is without pathological vertices. As a matter of fact, a by-product of the result (4.19) is that the embedding in $\mathbf{L}_t^2(\Gamma)$ holds, and moreover that it is *compact*. On the other hand, the embedding (4.4) *does not hold* if Γ has pathological vertices. However, the singularities that appear in (4.17)-(4.18) belong to a finite-dimensional vector space and, once they are taken into account, the remaining "regular" function space $\mathbf{H}_t^s(\Gamma)$ (for some $s > 0$) still embeds compactly in $\mathbf{L}_t^2(\Gamma)$.

Remark 4.1.6. The proof can be adapted to non-topologically trivial boundaries. In this case [16], one has extra-terms in the decompositions (4.6) and (4.5):

$$\mathbf{u} = \mathbf{curl}_\Gamma \phi^- + \nabla_\Gamma \psi^+ + \mathbf{h}_1, \tag{4.28}$$

$$\mathbf{u} = \nabla_\Gamma \psi^- + \mathbf{curl}_\Gamma \phi^+ + \mathbf{h}_2, \tag{4.29}$$

where $\mathbf{h}_1$, $\mathbf{h}_2$ belong to the space

$$\mathbb{H} := \left\{ \mathbf{u} \in \mathbf{L}_t^2(\Gamma), \operatorname{curl}_\Gamma \mathbf{u} = 0, \operatorname{div}_\Gamma \mathbf{u} = 0 \right\} \tag{4.30}$$

(see Proposition 2.3.11). However, these terms vanish when taking the $\operatorname{curl}_\Gamma$ or the $\operatorname{div}_\Gamma$ of the decompositions. So, this has no impact on the above lines about the regularity of ϕ^-, ψ^-, nor ϕ^+, ψ^+. Moreover, $\mathbb{H} \subset \mathbf{L}_t^2(\Gamma)$, so the embedding

$$\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \cap \mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma, \Gamma) \subset \mathbf{L}_t^2(\Gamma)$$

occurs under the same conditions than in Theorem 4.1.4. Finally, as $\mathbb{H}$ is finite-dimensional (see [16]), it also embeds compactly into $\mathbf{L}_t^2(\Gamma)$.

4.2 Regularity of Robin traces with scalar impedance coefficient

In this section, we consider the case of scalar, but heterogeneous impedance coefficient. The coefficient will be assumed either regular or piecewise constant. Our aim is to determine whether, in these cases, the Robin boundary condition may hold in $\mathbf{L}_t^2(\Gamma)$.

4.2.1 Constant coefficient

To begin, let us give the result for the classical impendance condition with scalar constant coefficient

$$\pi^T \mathbf{C} + \alpha \gamma^T \mathbf{E} = \mathbf{g} \quad \text{on } \Gamma, \tag{4.31}$$

with $\alpha \in \mathbb{C} \setminus \{0\}$, and $\mathbf{g} \in \mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \cap \mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$.

Theorem 4.2.1. If Γ has pathological vertices, $\exists s_{\max} \in]0, \frac{1}{2}[$ s.t., $\forall s \in [0, s_{\max}[$,

$$\gamma^T \mathbf{E} \in \mathbf{H}^s_t(\Gamma) \oplus \mathbf{curl}_\Gamma \mathcal{S}; \tag{4.32}$$
$$\pi^T \mathbf{C} \in \mathbf{H}^s_t(\Gamma) \oplus \nabla_\Gamma \mathcal{S}, \tag{4.33}$$

(the same also holding for $\mathbf{H}^s_t(\Gamma) \oplus \mathbf{curl}_\Gamma \mathcal{S}$). If Γ has no pathological vertex, then, $\forall s \in [0, \frac{1}{2}[$,

$$\gamma^T \mathbf{E} \in \mathbf{H}^s_t(\Gamma); \tag{4.34}$$
$$\pi^T \mathbf{C} \in \mathbf{H}^s_t(\Gamma). \tag{4.35}$$

If additionally Γ has no semi-pathological vertex, then (4.34)-(4.35) also hold for $s = \frac{1}{2}$.

Proof. Because $\gamma^T \mathbf{E} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, $\pi^T \mathbf{C} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\mathbf{g} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, and α is constant, one has in fact that each term of (4.31) belongs to $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. Then, the result is a consequence of Theorem 4.1.4. □

4.2.2 Smooth coefficient

At present, let us consider some more general classes of coefficients. In this subsection, we consider

$$\pi^T \mathbf{C} + \alpha \gamma^T \mathbf{E} = \mathbf{g} \quad \text{on } \Gamma, \tag{4.36}$$

with $\mathbf{g} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\alpha \in W^{2,\infty}(\Gamma)$ a smooth, elliptic impedance coefficient. We recall that α is elliptic iff

$$\exists \theta_\alpha \in \mathbb{R},\ \exists \alpha_- > 0,\ \text{ a.e. in } \Gamma, \quad \alpha_- \leq \Re[e^{i\theta_\alpha} \alpha]. \tag{4.37}$$

Contrarily to the previous subsection, one cannot straightforwardly apply Theorem 4.1.4. In fact, because α is now heterogeneous, one cannot simply write that $\gamma^T \mathbf{E}, \pi^T \mathbf{C} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. However, one gets the same result.

Theorem 4.2.2. Assume $\alpha \in W^{2,\infty}(\Gamma)$ and is elliptic.
If Γ has pathological vertices, $\exists s_{\max} \in]0, \frac{1}{2}[$, $\forall s \in [0, s_{\max}[$,

$$\gamma^T \mathbf{E} \in \mathbf{H}^s_t(\Gamma) \oplus \mathbf{curl}_\Gamma \mathcal{S}; \tag{4.38}$$
$$\pi^T \mathbf{C} \in \mathbf{H}^s_t(\Gamma) \oplus \nabla_\Gamma \mathcal{S}, \tag{4.39}$$

where $\mathcal{S}$ is the finite dimensional space of singularities introduced in Lemma 4.1.2, and $s_{\max}$ is the regularity exponent defined in Lemma 4.1.3.
If Γ has no pathological vertex, then, $\forall s \in [0, \frac{1}{2}[$,

$$\gamma^T \mathbf{E} \in \mathbf{H}^s_t(\Gamma) \quad \text{and} \quad \pi^T \mathbf{C} \in \mathbf{H}^s_t(\Gamma), \tag{4.40}$$

with moreover

$$\|\gamma^T \mathbf{E}\|_{\mathbf{H}^s_t(\Gamma)} \lesssim \|\mathbf{g}\|_\pi + \|\pi^T \mathbf{C}\|_\pi + \|\gamma^T \mathbf{E}\|_\gamma; \tag{4.41}$$
$$\|\pi^T \mathbf{C}\|_{\mathbf{H}^s_t(\Gamma)} \lesssim \|\mathbf{g}\|_\gamma + \|\pi^T \mathbf{C}\|_\pi + \|\gamma^T \mathbf{E}\|_\gamma. \tag{4.42}$$

If additionally Γ has no semi-pathological vertex, then (4.40) and (4.41)-(4.42) also hold for $s = \frac{1}{2}$.

Proof. We recall that $\gamma^T \mathbf{E} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\pi^T(\underline{\boldsymbol{\mu}}^{-1}\, \mathbf{curl}\ \mathbf{E}) \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. Let us write the surface

Helmholtz decompositions in these spaces (cf (2.83)-(2.84)):

$$\gamma^T \mathbf{E} = \mathbf{curl}_\Gamma\, \phi^- + \nabla_\Gamma\, \psi^+, \qquad \phi^- \in H^{1/2}_{\text{zmv}}(\Gamma), \quad \psi^+ \in \mathcal{H}(\Gamma); \tag{4.43}$$

$$\pi^T \mathbf{C} = \nabla_\Gamma\, \psi^- + \mathbf{curl}_\Gamma\, \phi^+, \qquad \psi^- \in H^{1/2}_{\text{zmv}}(\Gamma), \quad \phi^+ \in \mathcal{H}(\Gamma). \tag{4.44}$$

Just as in the beginning of Section 4.1, our goal is to determine whether ϕ^-, ψ^- actually belong to $H^1(\Gamma)$, so that one has indeed $\gamma^T \mathbf{E} \in \mathbf{L}^2_t(\Gamma)$ and $\pi^T \left(\underline{\boldsymbol{\mu}}^{-1}\, \mathbf{curl}\ \mathbf{E}\right) \in \mathbf{L}^2_t(\Gamma)$. To that aim, we will investigate the scalar problems governing ϕ^- and ψ^-.

Let us begin with ϕ^-. We note that $\text{curl}_\Gamma\, \mathbf{g}$ and $\text{curl}_\Gamma\, \pi^T \mathbf{C}$ both belong to $H^{-1/2}(\Gamma)$. Taking the curl_Γ of (4.36), and using the decompositions (4.43) and (4.44), one gets the problem governing ϕ^-:

$$\text{curl}_\Gamma \left(\alpha\ \ \mathbf{curl}_\Gamma\, \phi^-\right) = \text{curl}_\Gamma\, \mathbf{g} + \Delta_\Gamma\, \phi^+ - \text{curl}_\Gamma \left(\alpha\ \ \nabla_\Gamma\, \psi^+\right). \tag{4.45}$$

As α is a scalar field, we take advantage of the Leibniz formula

$$-\,\text{curl}_\Gamma(\alpha\, \mathbf{curl}_\Gamma\, v) = \alpha\, \Delta_\Gamma\, v + \mathbf{curl}_\Gamma\, \alpha \cdot \mathbf{curl}_\Gamma\, v. \tag{4.46}$$

Combining it with the above formula, there holds

$$\Delta_\Gamma\, \phi^- = -\alpha^{-1} \left(\text{curl}_\Gamma\, \mathbf{g} + \Delta_\Gamma\, \phi^+ - \text{curl}_\Gamma \left(\alpha\, \nabla_\Gamma\, \psi^+\right) + \mathbf{curl}_\Gamma\, \alpha \cdot \mathbf{curl}_\Gamma\, \phi^-\right), \tag{4.47}$$

whose right-hand side belongs to $(H^1_{\text{zmv}}(\Gamma))'$. Thanks to the Leibniz formula, we are now in a situation that involves only the operator Δ_Γ in the left-hand side, as in the proof of Theorem 4.1.4. Then, we split ϕ^- into a regular part ϕ^{reg} and a singular part ϕ^{sing}. $\phi^{\text{reg}} \in H^1_{\text{zmv}}(\Gamma)$ is the unique solution to

$$\Delta_\Gamma\, \phi^{\text{reg}} = -\alpha^{-1} \left(\text{curl}_\Gamma\, \mathbf{g} + \Delta_\Gamma\, \phi^+ - \text{curl}_\Gamma \left(\alpha\, \nabla_\Gamma\, \psi^+\right) + \mathbf{curl}_\Gamma\, \alpha \cdot \mathbf{curl}_\Gamma\, \phi^-\right), \tag{4.48}$$

while $\phi^{\text{sing}} \in H^{1/2}_{\text{zmv}}(\Gamma)$ is governed by

$$\Delta_\Gamma\, \phi^{\text{sing}} = 0. \tag{4.49}$$

The singular part is characterized by Lemma 4.1.2. If Γ has pathological vertices, ϕ^{sing} is a singular solution that belongs to the finite-dimensional space $\mathcal{S}$. Otherwise, $\phi^{\text{sing}} = 0$.

The extra-regularity of ϕ^{reg} is driven, as in the proof of Lemma 4.1.3, by the regularity of the right-hand side and by the regularity at the vertices. For the right-hand side, one has $\text{curl}_\Gamma\, \mathbf{g} \in H^{-1/2}(\Gamma)$, as well as $\Delta_\Gamma\, \phi^+$; and the extra-term $\nabla_\Gamma\, \alpha \cdot \nabla_\Gamma\, \phi^- \in H^{-1/2}(\Gamma)$, because $\nabla_\Gamma\, \alpha \in \mathbf{W}^{1,\infty}(\Gamma)$ and $\nabla_\Gamma\, \phi^- \in \mathbf{H}^{-1/2}(\Gamma)$; finally, $\alpha^{-1} \in W^{1,\infty}(\Gamma)$ (because $\alpha \in W^{1,\infty}(\Gamma)$ and is elliptic). The last term to study is $\text{curl}_\Gamma\, (\alpha\, \nabla_\Gamma\, \psi^+)$.

- If Γ has pathological vertices, there holds, because of Lemma 4.1.3, $\phi^+, \psi^+ \in H^{1+s}(\Gamma)$ for $s < s_{\max} < \frac{1}{2}$. Then, the right-hand side in (4.48) belongs to $H^{s-1}(\Gamma)$ for all $s < s_{\max}$. So, following the arguments in the proof of Lemma 4.1.3, one has also $\phi^{\text{reg}} \in H^{1+s}(\Gamma)$ for all $s < s_{\max}$ ($s_{\max}$ is the upper bound that comes both from the regularity at vertices, and from the regularity of the right-hand side, because of ψ^+). Finally, $\phi^- = \phi^{\text{reg}} + \phi^{\text{sing}} \in H^{1+s}(\Gamma) \oplus \mathcal{S}$.
- On the other hand, if Γ has no pathological vertex, then, from Lemma 4.1.3, we know that $\phi^+, \psi^+ \in H^{1+s}(\Gamma)$, for all $s \in]0, \frac{1}{2}[$. Therefore, as moreover $\alpha \in W^{1,\infty}(\Gamma)$, the whole right-hand side in (4.48) belongs to $H^{s-1}(\Gamma)$. Again because of the arguments of Lemma 4.1.3, we find that $\phi^{\text{reg}} \in H^{1+s}(\Gamma)$. Hence, $\gamma^T \mathbf{E} \in \mathbf{H}^s_t(\Gamma)$.
- If additionally Γ has no semi-pathological vertex, the above argument also holds for $s = \frac{1}{2}$.

Moreover, one has $\Delta_\Gamma\, \phi^+ = \text{curl}_\Gamma\, \pi^T \mathbf{C}$, so $\|\phi^+\|_{H^{1+s}(\Gamma)} \lesssim \|\pi^T \mathbf{C}\|_\pi$. Similarly, $\|\psi^+\|_{H^{1+s}(\Gamma)} \lesssim \|\gamma^T \mathbf{E}\|_\gamma$. Then, $\|\phi^{\text{reg}}\|_{H^{1+s}(\Gamma)} \lesssim \|\mathbf{g}\|_\pi + \|\pi^T \mathbf{C}\|_\pi + \|\gamma^T \mathbf{E}\|_\gamma$. Thus, in the absence of singular part, one finds

$$\|\gamma^T \mathbf{E}\|_{\mathbf{H}^s_t(\Gamma)} \lesssim \|\mathbf{g}\|_\pi + \|\pi^T \mathbf{C}\|_\pi + \|\gamma^T \mathbf{E}\|_\gamma. \tag{4.50}$$

This gives the result for $\gamma^T \mathbf{E}$.

To deal with ψ^-, we multiply (4.36) by $\alpha^{-1} \in W^{1,\infty}(\Gamma)$ (because $\alpha \in W^{1,\infty}(\Gamma)$ and is elliptic) and take the div_Γ, which gives

$$\text{div}_\Gamma(\alpha^{-1}\, \nabla_\Gamma\, \psi^-) = \text{div}_\Gamma(\alpha^{-1} \mathbf{g}) - \text{div}_\Gamma \left(\alpha^{-1}\, \mathbf{curl}_\Gamma\, \phi^+\right) - \Delta_\Gamma\, \psi^+. \tag{4.51}$$

Using once again the Leibniz formula, one gets

$$\Delta_\Gamma \psi^- = \alpha \left(\operatorname{div}_\Gamma(\alpha^{-1}\mathbf{g}) - \operatorname{div}_\Gamma\left(\alpha^{-1}\,\mathbf{curl}_\Gamma\,\phi^+\right) - \Delta_\Gamma \psi^+ - \nabla_\Gamma \alpha^{-1}\cdot\nabla_\Gamma \psi^-\right), \tag{4.52}$$

with $\nabla_\Gamma \alpha^{-1} = \alpha^{-2}\nabla_\Gamma \alpha \in \mathbf{W}^{1,\infty}(\Gamma)$.

Then, we proceed as above. We split ψ^- into a regular part and a singular part, $\psi^- = \psi^{\mathrm{reg}} + \psi^{\mathrm{sing}}$, where $\psi^{\mathrm{reg}} \in H^1_{\mathrm{zmv}}(\Gamma)$ is the unique *regular* solution of (4.52), and $\psi^{\mathrm{sing}} \in H^{1/2}_{\mathrm{zmv}}(\Gamma)$ satisfies $\Delta_\Gamma \psi^{\mathrm{sing}} = 0$.

- If Γ has pathological vertices, ψ^{sing} is a singularity that belongs to $\mathcal{S}$. Moreover, there holds $\phi^+, \psi^+ \in H^{1+s}(\Gamma)$ for $s < s_{\max}$, and $\psi^{\mathrm{reg}} \in H^{1+s}(\Gamma)$ (same proof as above). We conclude that $\psi^- \in H^{1+s}(\Gamma)\oplus\mathcal{S}$, for $s < s_{\max}$.
- If Γ has no pathological vertex, $\psi^{\mathrm{sing}} = 0$. Moreover, the whole right-hand side of (4.51) now belongs to $H^{s-1}(\Gamma)$ for all $s \in]0,\frac{1}{2}[$. Using once again the same arguments as above, we conclude that $\psi^- \in H^{1+s}(\Gamma)$, and that $\pi^T\mathbf{C} \in \mathbf{H}^s_t(\Gamma)$.
- If additionally Γ has no semi-pathological vertex, the above argument also holds for $s = \frac{1}{2}$.

Moreover, one also has an estimate for ψ^{reg}: $\|\psi^{\mathrm{reg}}\|_{H^{1+s}(\Gamma)} \lesssim \|\mathbf{g}\|_\gamma + \|\pi^T\mathbf{C}\|_\pi + \|\gamma^T\mathbf{E}\|_\gamma$. Therefore, in the absence of singular part, one gets

$$\|\pi^T\mathbf{C}\|_{\mathbf{H}^s_t(\Gamma)} \lesssim \|\mathbf{g}\|_\gamma + \|\pi^T\mathbf{C}\|_\pi + \|\gamma^T\mathbf{E}\|_\gamma \tag{4.53}$$

which gives the result for $\pi^T\mathbf{C}$. □

Remark 4.2.3. Even when there is a singular part, we have shown that the regular part depends continuously on $\|\mathbf{g}\|_\gamma + \|\mathbf{g}\|_\pi$, $\|\pi^T\mathbf{C}\|_\pi$, and $\|\gamma^T\mathbf{E}\|_\gamma$.

Remark 4.2.4. An alternate and simpler proof is possible if one assumes that α admits a smooth lifting $\tilde{\alpha} \in W^{1,\infty}(\Omega)$. In this case, $\tilde{\mathbf{E}} := \tilde{\alpha}\mathbf{E} \in \mathbf{H}(\mathbf{curl},\Omega)$, and there holds $\alpha\gamma^T\mathbf{E} = \gamma^T(\tilde{\alpha}\mathbf{E})$. So, the Robin condition (4.36) rewrites

$$\pi^T\mathbf{C} + \gamma^T\tilde{\mathbf{E}} = \mathbf{g} \quad \text{ on } \Gamma, \tag{4.54}$$

and the regularity for $\pi^T\mathbf{C}$ and $\gamma^T\tilde{\mathbf{E}}$ is obtained by Theorem 4.1.4. Then, one recovers the regularity of $\gamma^T\mathbf{E} = \alpha^{-1}\gamma^T\tilde{\mathbf{E}}$, because $\alpha \in W^{1,\infty}(\Gamma)$ and is elliptic.

Remark 4.2.5. As we saw above, when α is not constant, one has to study two different problems: the one governing ϕ^-, that gives the regularity of $\gamma^T\mathbf{E}$, and the one governing ψ^-, that gives the regularity of $\pi^T\mathbf{C}$. Both problems are not symmetric: it is more natural to begin with ϕ^-, which is governed by (4.45). Then, one always has two alternate manners to deal with the second problem.

The first way is to take the $\operatorname{div}_\Gamma$ of (4.36) directly, to get

$$\Delta_\Gamma \psi^- = \operatorname{div}_\Gamma \mathbf{g} - \operatorname{div}_\Gamma(\alpha\,\mathbf{curl}_\Gamma\,\phi^-) - \operatorname{div}_\Gamma(\alpha\,\nabla_\Gamma \psi^+), \tag{4.55}$$

where we can now take advantange of the known regularity of ϕ^-. Also, note that this problem is in some sense simpler than the one on ϕ^-, because it simply involves the standard Laplace-Beltrami operator instead of operator $\operatorname{curl}_\Gamma(\alpha\,\mathbf{curl}_\Gamma\,\cdot)$. So, one can deal with it using directly the results of Section 4.1. However, doing so may be uncomfortable when ϕ^- has a singular part, because then a singular part arises in the right-hand side of (4.55).

The second way is to multiply (4.36) by α^{-1} (which is legitimate because α is elliptic), and consider the alternate condition

$$\alpha^{-1}\pi^T\mathbf{C} + \gamma^T\mathbf{E} = \alpha^{-1}\mathbf{g} \quad \text{ on } \Gamma, \tag{4.56}$$

as we did in the proof. With this condition, the situation is reversed compared to (4.36). The first natural problem to deal with becomes the one governing ψ^-, which is obtained taking the $\operatorname{div}_\Gamma$ of (4.56):

$$\operatorname{div}_\Gamma(\alpha^{-1}\,\nabla_\Gamma \psi^-) = \operatorname{div}_\Gamma(\alpha^{-1}\mathbf{g}) - \operatorname{div}_\Gamma\left(\alpha^{-1}\,\mathbf{curl}_\Gamma\,\phi^+\right) - \Delta_\Gamma \psi^+. \tag{4.57}$$

In this case, one has to study the singularities of operator $\operatorname{div}_\Gamma(\alpha^{-1}\,\nabla_\Gamma\,\cdot)$. This is the way we have proceeded in the proof of Theorem 4.2.2. If one wants to pursue the reasoning in the paradigm of the alternate condition (4.56), one can take the $\operatorname{curl}_\Gamma$ and get the second problem

$$\Delta_\Gamma \phi^- = -\operatorname{curl}_\Gamma(\alpha^{-1}\mathbf{g}) + \operatorname{curl}_\Gamma(\alpha^{-1}\,\nabla_\Gamma \psi^-) + \operatorname{curl}_\Gamma(\alpha^{-1}\,\mathbf{curl}_\Gamma\,\phi^+), \tag{4.58}$$

which is the problem governing ϕ^-, in which we can plug the now known regularity of ψ^-, and which involves simply operator Δ_Γ.

4.2.3 Piecewise constant coefficient

In this subsection, we assume that α is an elliptic piecewise constant field on Γ. Then, one has to deal with the singularities arising from the discontinuity lines and vertices of the coefficient. This is done in the spirit of Lemma 4.1.2. The (plane) regions where α is constant are denoted Γ_k and called *coefficient faces*. They will play the same role as the faces of Γ in Lemma 4.1.2. They are naturally plane, because included in faces of Γ. Similarly, one will have to deal with *coefficient edges* (lines where two *coefficient faces* meet), and with *coefficient vertices* (points where three or more *coefficient faces* meet). Again for simplicity, we assume that the coefficient edges are straight lines.

As before, the condition reads

$$\pi^T \mathbf{C} + \alpha\gamma^T \mathbf{E} = \mathbf{g} \quad \text{on } \Gamma. \tag{4.59}$$

As in the previous subsection, we write the surface Helmholtz decompositions (2.83)-(2.84) of $\gamma^T\mathbf{E}$ and $\pi^T\mathbf{C}$, to get

$$\left(\nabla_\Gamma \psi^- + \mathbf{curl}_\Gamma \phi^+\right) + \alpha\left(\mathbf{curl}_\Gamma \phi^- + \nabla_\Gamma \psi^+\right) = \mathbf{g} \tag{4.60}$$

where $\phi^-, \psi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma)$, and $\phi^+, \psi^+ \in \mathcal{H}(\Gamma)$. Then, taking the curl_Γ of (4.60), $\phi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma)$ is governed by

$$\mathrm{curl}_\Gamma\left(\alpha\ \mathbf{curl}_\Gamma \phi^-\right) = \mathrm{curl}_\Gamma \mathbf{g} + \Delta_\Gamma \phi^+ - \mathrm{curl}_\Gamma\left(\alpha\ \nabla_\Gamma \psi^+\right). \tag{4.61}$$

So, one has to study the singular solutions of the problem

$$\text{Find } \phi^{\mathrm{sing}} \in H^{1/2}_{\mathrm{zmv}}(\Gamma) \text{ s.t. } \mathrm{curl}_\Gamma\left(\alpha\ \mathbf{curl}_\Gamma \phi^{\mathrm{sing}}\right) = 0. \tag{4.62}$$

This is done, as in Lemma 4.1.2, by localizing in a face, on an edge, or around a coefficient vertex v.

Lemma 4.2.6. Assume α is elliptic. Let v a coefficient vertex, and the eigenproblem

$$\left|\begin{aligned}&\text{Find } (\lambda,\varphi) \in \mathbb{C}\times H^1_{\mathrm{per}}([0,\theta_v]) \text{ s.t. } ,\forall\psi \in H^1_{\mathrm{per}}([0,\theta_v]),\\ &\qquad\int_0^{\theta_v} \alpha\varphi'\bar{\psi}'\mathrm{d}\theta - \lambda^2\int_0^{\theta_v}\alpha\varphi\bar{\psi}\mathrm{d}\theta = 0,\end{aligned}\right. \tag{4.63}$$

where the subscript $_{\mathrm{per}}$ stands for the periodic fields. The solution to the problem

$$\text{Find } \phi^{\mathrm{sing}} \in H^{1/2}_{\mathrm{zmv}}(\Gamma) \text{ s.t. } \mathrm{curl}_\Gamma\left(\alpha\ \mathbf{curl}_\Gamma \phi^{\mathrm{sing}}\right) = 0 \text{ on } \Gamma \tag{4.64}$$

is 0 if, for all coefficient vertices, there is no λ solution to (4.63) s.t. $\Re(\lambda) \in]-\frac{1}{2},0[$; otherwise, it belongs to a finite dimensional space of singularities $\mathcal{S}_1$.

Proof. As in Lemma 4.1.2, we study the regularity locally in a neighbourhood, denoted Γ_A, of each point of Γ, and this domain can be of two types: Γ_A is included in one coefficient face; or Γ_A contains either one coefficient edge, or one coefficient vertex and the adjacent coefficient edges. Indeed, the edge type can be seen as a special case of the vertex type by picking an arbitrary point v on the edge: it corresponds to a vertex type with two faces, each of them with a face angle equal to π (so $\theta_v = 2\pi$).

1. It is included in a *coefficient face* Γ_i. In this case, α is constant in Γ_A, and there holds $\phi^{\mathrm{sing}} \in H^1(\Gamma_A)$ as in Lemma 4.1.2.

2. It contains a *coefficient vertex* at the intersection of K coefficient faces Γ_k, possibly with $K = 2$ and both face angles equal to π to cover also the *coefficient edge* as a particular case. We follow again Kondratiev theory, and look for non-constant solutions that write *locally* $\phi_\lambda(r,\theta) = r^\lambda\varphi_\lambda(\theta)$, where $\varphi_{\lambda|\Gamma_k}(\theta) = c_k e^{\pm i\lambda\theta}$ for some $\lambda \in \mathbb{C}$ and a coefficient $c_k \in \mathbb{C}$. The explicit form of $\varphi_{\lambda|\Gamma_k}$ is a consequence of the fact that α depends only on the angular coordinate θ, and that α is constant on each Γ_k. Then, one can notice that ϕ_λ satisfies $2K$ linear compatibility conditions corresponding to the (weak) continuity of the Dirichlet and Neumann traces at each coefficient edge. The existence of a non-trivial solution to this set of $2K$ equations (this is with at least one non-zero coefficient) leads to a dispersion equation governing λ. For the particular

case of a *coefficient edge*, one finds by direct computations on the 4 equations that $\lambda \in \mathbb{Z}$, and in addition $\lambda > 0$ because $\varphi_\lambda \in H^{1/2}(\Gamma_A)$. Hence $\phi^{\mathrm{sing}}{}_{|\Gamma_A} \in H^1(\Gamma_A)$.

For the case of an actual coefficient vertex, going back to the condition $\mathrm{curl}_\Gamma(\alpha\,\mathbf{curl}_\Gamma\,\phi^{\mathrm{sing}}) = 0$ leads to the eigenproblem governing $\varphi \in H^1_{\mathrm{per}}([0,\theta_v])$:

$$\forall \psi \in H^1_{\mathrm{per}}([0,\theta_v]), \qquad \int_0^{\theta_v} \alpha\varphi'\bar{\psi}'\mathrm{d}\theta - \lambda^2 \int_0^{\theta_v} \alpha\varphi\bar{\psi}\mathrm{d}\theta = 0. \tag{4.65}$$

We recall that the regularity of ϕ_λ is determined only by λ (cf. again Th. 1.2.18 in [62]): for $0 < s < 1$,

$$\phi_\lambda \in H^s(\Gamma_A) \Longleftrightarrow \Re(\lambda) > s - 1.$$

Moreover, one can note that $\lambda = 0$ leads to the constant (thus not singular) solution; and that $\lambda \in i\mathbb{R}\setminus\{0\}$ is not an eigenvalue (in this case, the sesquilinear form in problem (4.63) is coercive, because α is elliptic). Therefore, the problem (4.64) admits singular solutions if, and only if, there are solutions to (4.63) s.t. $\Re(\lambda) \in \,]-\frac{1}{2}, 0[$ (As in Lemma 4.1.2, a local singularity ϕ_λ can be continued on Γ by the means of a cut-off function).

There remain to prove that the singular solutions belong to a finite dimensional vector space. First, even if there are eigenvalues leading to singular solutions, one can always state that they are isolated. Indeed, introducing the operator the operator $\mathcal{L}_\lambda$ defined on $H^1_{\mathrm{per}}([0,\theta_v])$ by:

$$\forall \varphi, \psi \in H^1_{\mathrm{per}}([0,\theta_v]),\ (\mathcal{L}_\lambda\varphi, \psi)_{H^1_{\mathrm{per}}([0,\theta_v])} := \int_0^{\theta_v} \alpha\varphi'\bar{\psi}'\mathrm{d}\theta - \lambda^2 \int_0^{\theta_v} \alpha\varphi\bar{\psi}\mathrm{d}\theta, \tag{4.66}$$

it is clear that $\mathcal{L}_\lambda - \mathcal{L}_\mu$ is a compact operator $\forall\lambda,\mu$, thanks to Rellich theorem, and that $\mathcal{L}_\mu$ is an isomorphism for all $\mu \in i\mathbb{R}\setminus\{0\}$. Hence, $\mathcal{L}_\lambda = \mathcal{L}_i + (\mathcal{L}_\lambda - \mathcal{L}_i)$ is a Fredholm operator. As moreover the family of operators $(\mathcal{L}_\lambda)$ is analytic w.r.t. λ, one can apply the analytic Fredholm Theorem (see, e.g., [80, Th. 1.1.1]): the spectrum of $\mathcal{L}$ consists only of isolated eigenvalues, which are of finite multiplicities, and do not have accumulation points.

Let us show that there is a finite number of λ in the strip $\left\{\lambda \in \mathbb{C},\ \Re(\lambda) \in \,]-\frac{1}{2}, 0[\right\}$. Let $\lambda = a + ib$, one has

$$\mathcal{L}_\lambda(\varphi,\varphi) = \int_0^{\theta_v} \left[\alpha|\varphi'|^2 + \left(\alpha(b^2 - a^2) - 2\alpha iab\right)|\varphi|^2\right]\mathrm{d}\theta.$$

On one hand, there holds

$$\Re\left[e^{i\theta_\alpha}\int_0^{\theta_v} \alpha(b^2-a^2)|\varphi|^2\right] \geq \alpha_-(b^2-a^2)\|\varphi\|^2_{L^2}$$

as soon as $|b| > |a|$. On the other hand,

$$\left|-2iab\int_0^{\theta_v} \alpha|\varphi|^2\mathrm{d}\theta\right| \leq 2|a||b|\alpha_+\|\varphi\|^2_{L^2}.$$

We conclude that $\mathcal{L}_\lambda$ is coercive as soon as

$$|\Im(\lambda)| > |\Re(\lambda)| \quad \text{and} \quad \alpha_-|\Im(\lambda)|^2 \geq 2|\Re(\lambda)||\Im(\lambda)|\alpha_+ + \alpha_-|\Re(\lambda)|^2, \tag{4.67}$$

where the second condition is simply a polynomial of second order on $|\Im(\lambda)|$ (as a function of $|\Re(\lambda)|$). So, clearly, for imaginary parts large enough, $\mathcal{L}_\lambda$ is an isomorphism. Therefore, the conditions $\Re(\lambda) \in \,]-\frac{1}{2}, 0[$ and the negation of (4.67) define a bounded region of $\mathbb{C}$. We conclude thanks to the analytic Fredholm theorem that there is a finite number of eigenvalues inside this region. Out of it, in the strip $\Re(\lambda) \in \,]-\frac{1}{2}, 0[$, there are no eigenvalues. We conclude that the space of singular solutions, i.e. solutions of (4.64) in $H^{1/2}(\Gamma)\setminus H^1(\Gamma)$, is finite dimensional. □

Remark 4.2.7. Alternatively, it is possible to estimate the regularity exponent in a given configuration by solving numerically the eigenproblems (4.63) at each coefficient vertex, see e.g. [15].

Having studied the singularities of operator $\mathrm{curl}_\Gamma(\alpha\,\mathbf{curl}_\Gamma\,\cdot)$, we are now in position to state our result.

Theorem 4.2.8. Assume that Γ has no pathological vertex.
If α is elliptic and s.t., for all its coefficient vertices, the eigenproblem (4.63) is s.t. there is no λ s.t. $\Re(\lambda) \in]-\frac{1}{2}, 0[$, then, there exists $s_{\max} \in]0, \frac{1}{2}[$ s.t., $\forall s \in [0, s_{\max}[$,

$$\gamma^T \mathbf{E} \in \mathbf{H}_t^s(\Gamma) \quad \text{and} \quad \pi^T \mathbf{C} \in \mathbf{H}_t^s(\Gamma), \tag{4.68}$$

with moreover

$$\|\gamma^T \mathbf{E}\|_{\mathbf{H}_t^s(\Gamma)} \lesssim \|\mathbf{g}\|_\pi + \|\pi^T \mathbf{C}\|_\pi + \|\gamma^T \mathbf{E}\|_\gamma; \tag{4.69}$$
$$\|\pi^T \mathbf{C}\|_{\mathbf{H}_t^s(\Gamma)} \lesssim \|\mathbf{g}\|_\gamma + \|\pi^T \mathbf{C}\|_\pi + \|\gamma^T \mathbf{E}\|_\gamma. \tag{4.70}$$

Else, there exists $s_{\max} \in]0, \frac{1}{2}[$ s.t., $\forall s \in [0, s_{\max}[$,

$$\gamma^T \mathbf{E} \in \mathbf{H}_t^s(\Gamma) \oplus \mathbf{curl}_\Gamma \, \mathcal{S}_1; \tag{4.71}$$
$$\pi^T \mathbf{C} \in \mathbf{H}_t^s(\Gamma) \oplus \nabla_\Gamma \, \mathcal{S}_2, \tag{4.72}$$

where $\mathcal{S}_1, \mathcal{S}_2$ are finite dimensional spaces of sigularities.

Proof. Let us assume first that the hypotheses of the theorem are satisfied. If, for all coefficient vertices, there is no λ of real part in $]-\frac{1}{2}, 0[$, then there is no singular solution to (4.64), and there holds $\phi^- \in H^1(\Gamma)$. Moreover, one has actually extra-regularity for ϕ^-. This extra-regularity is driven, on one hand, by the regularity of the right-hand side in (4.61). As $\psi^+ \in H^{3/2}(\Gamma)$ (because of Lemma 4.1.3), and α is piecewise constant, there holds $\alpha \nabla_\Gamma \psi^+ \in \mathbf{H}^s(\Gamma)$ for all $s < \frac{1}{2}$, (with $\|\alpha \nabla_\Gamma \psi^+\|_{\mathbf{H}^s(\Gamma)} \lesssim \|\psi^+\|_{H^{1+s}(\Gamma)}$); thus the right-hand side of (4.61) belongs to H^{s-1}. On the other hand, it is also driven by the lowest (strictly) positive eigenvalue of (4.63) (since $\lambda \notin i\mathbb{R}$, see the proof of Lemma 4.2.6). Therefore, there holds $\phi^- \in H^{1+s}(\Gamma)$ for $s < s_{\max}$, with $s_{\max} = \min(\frac{1}{2}, \min \Re(\lambda))$. Thus, $\gamma^T \mathbf{E} \in \mathbf{H}_t^s(\Gamma)$. The estimates are obtained as in Theorem 4.2.2.

To get the result for $\pi^T \mathbf{C}$, we follow the first way described in Remark 4.2.5. Taking the div_Γ of (4.59), $\psi^- \in H^{1/2}_{\mathrm{zmv}}(\Gamma)$ is governed by

$$\Delta_\Gamma \psi^- = \mathrm{div}_\Gamma \, \mathbf{g} - \mathrm{div}_\Gamma \left(\alpha \; \mathbf{curl}_\Gamma \, \phi^-\right) - \mathrm{div}_\Gamma \left(\alpha \; \nabla_\Gamma \psi^+\right). \tag{4.73}$$

This problem involves simply the standard Laplace-Beltrami operator, so, since Γ has no pathological vertex, there are no singular solutions (Lemma 4.1.2). Besides, the right-hand side of (4.73) is meaningful in $H^{s-1}(\Gamma)$, because $\psi^+ \in H^{3/2}(\Gamma)$ and $\phi^- \in H^{1+s}(\Gamma)$. We conclude that $\psi^- \in H^{1+s}(\Gamma)$, and $\pi^T \mathbf{C} \in \mathbf{H}_t^s(\Gamma)$, with estimates obtained as in Theorem 4.2.2.

On the other hand, if there are singular solutions to (4.64), then $\phi^- \in H^1(\Gamma) \oplus \mathcal{S}_1$, where $\mathcal{S}_1$ is the space of solutions of (4.64) in $H^{1/2}(\Gamma) \backslash H^1(\Gamma)$. Moreover, this corresponds also to the space of solutions in $H^{1/2}(\Gamma) \backslash H^{1+s}(\Gamma)$, for $s > 0$ smaller than the lowest strictly positive value of $\Re(\lambda)$. Therefore, $\phi^- \in H^{1+s}(\Gamma) \oplus \mathcal{S}_1$, where $\mathcal{S}_1$ is the finite-dimensional subspace of singularities. To deal with ψ^-, we proceed as in the second way of Remark 4.2.5. We multiply (4.59) by α^{-1} (α is elliptic) and take its div_Γ, which gives

$$\mathrm{div}_\Gamma(\alpha^{-1} \nabla \psi^-) = \mathrm{div}_\Gamma(\alpha^{-1} \mathbf{g}) - \mathrm{div}_\Gamma \left(\alpha^{-1} \, \mathbf{curl}_\Gamma \, \phi^+\right) - \Delta_\Gamma \psi^+. \tag{4.74}$$

One has then to study the singularities of operator $\mathrm{div}_\Gamma(\alpha^{-1} \nabla_\Gamma \cdot)$. This is done in the same manner as for operator $\mathrm{curl}_\Gamma(\alpha \, \mathbf{curl}_\Gamma \cdot)$. One concludes that $\psi^- \in H^{1+s}(\Gamma) \oplus \mathcal{S}_2$, where $\mathcal{S}_2$ is the finite-dimensional of singularities associated to operator $\mathrm{div}_\Gamma(\alpha^{-1} \nabla_\Gamma \cdot)$. □

Remark 4.2.9. The proof can be adapted to more complex geometries. In particular, in the case where Γ has pathological vertices, one has to take in consideration both types of singularities: those arising from the *geometry* at pathological vertices (Lemma 4.1.2); and those arising from the *coefficient* vertices (Lemma 4.2.6).

4.3 Investigation on Robin traces with a tensor-valued coefficient

In this section, we consider the more general boundary condition

$$\pi^T \mathbf{C} + \underline{\alpha} \gamma^T \mathbf{E} = \mathbf{g} \quad \text{on } \Gamma, \tag{4.75}$$

where the impedance coefficient is tensor-valued. In the following, we assume that $\underline{\boldsymbol{\alpha}} \in \underline{\mathbf{PW}}^{2,\infty}(\Gamma)$ (the set of piecewise $\underline{\mathbf{W}}^{2,\infty}$ fields), and is elliptic, i.e.:

$$\exists\theta_\alpha \in \mathbb{R},\ \exists\alpha_- > 0,\quad \text{a.e. in } \Gamma,\ \forall \mathbf{z} \in \mathbb{C}^2,\qquad \alpha_-|\mathbf{z}|^2 \leq \Re[e^{i\theta_\alpha} \cdot \mathbf{z}^* \underline{\boldsymbol{\alpha}} \mathbf{z}]. \tag{4.76}$$

Similarly to the previous sections, our goal is to determine whether this boundary condition may be meaningful in $\mathbf{L}_t^2(\Gamma)$, assuming as before that $\mathbf{g} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$.

Proposition 4.3.1. Assume $\underline{\boldsymbol{\alpha}} \in \underline{\mathbf{PW}}^{2,\infty}(\Gamma)$ and is elliptic. If Γ has no pathological vertices, and if $\underline{\boldsymbol{\alpha}}$ is s.t. the problem

$$\text{Find } \phi^{\text{sing}} \in H^{1/2}_{\text{zmv}}(\Gamma) \text{ s.t. } \mathrm{curl}_\Gamma(\underline{\boldsymbol{\alpha}}\, \mathbf{curl}_\Gamma\, \phi) = 0 \text{ on } \Gamma$$

has no singular solution in $H^{1/2}(\Gamma) \setminus H^1(\Gamma)$, then

$$\gamma^T \mathbf{E} \in \mathbf{L}_t^2(\Gamma) \quad \text{and} \quad \pi^T \mathbf{C} \in \mathbf{L}_t^2(\Gamma).$$

Proof. We recall that $\gamma^T \mathbf{E} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\pi^T \mathbf{C} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. As before, we write their surface Helmholtz decompositions to get

$$\left(\nabla_\Gamma\, \psi^- + \mathbf{curl}_\Gamma\, \phi^+\right) + \underline{\boldsymbol{\alpha}} \left(\mathbf{curl}_\Gamma\, \phi^- + \nabla_\Gamma\, \psi^+\right) = \mathbf{g}, \tag{4.77}$$

where $\phi^-, \psi^- \in H^{1/2}_{\text{zmv}}(\Gamma)$, and $\phi^+, \psi^+ \in \mathcal{H}(\Gamma)$. Just as in the previous sections, our goal is to determine whether ϕ^-, ψ^- actually belong to $H^1(\Gamma)$, so that one has indeed $\gamma^T \mathbf{E} \in \mathbf{L}_t^2(\Gamma)$ and $\pi^T \left(\underline{\boldsymbol{\mu}}^{-1}\, \mathbf{curl}\, \mathbf{E}\right) \in \mathbf{L}_t^2(\Gamma)$. To that aim, we investigate the scalar problems governing ϕ^- and ψ^-.

Let us begin with ϕ^-. Indeed, we note that $\mathrm{curl}_\Gamma\, \mathbf{g}$ and $\mathrm{curl}_\Gamma\, \pi^T \mathbf{C} \in H^{-1/2}(\Gamma)$. Thus, one can take the curl_Γ of (4.77). One gets the problem governing ϕ^-:

$$\mathrm{curl}_\Gamma \left(\underline{\boldsymbol{\alpha}}\ \ \mathbf{curl}_\Gamma\, \phi^-\right) = \mathrm{curl}_\Gamma\, \mathbf{g} + \Delta_\Gamma\, \phi^+ - \mathrm{curl}_\Gamma \left(\underline{\boldsymbol{\alpha}}\ \ \nabla_\Gamma\, \psi^+\right). \tag{4.78}$$

Hence, let us split ϕ^- into a regular and a singular part. Indeed, because $\underline{\boldsymbol{\alpha}}$ is elliptic and the right-hand side belongs to $(H^1_{\text{zmv}}(\Gamma))'$, the problem (4.78) is well-posed in $H^1_{\text{zmv}}(\Gamma)$, so it admits a unique *regular* solution in $H^1_{\text{zmv}}(\Gamma)$, denoted ϕ^{reg}. On the other hand, $\phi^{\text{sing}} := \phi^- - \phi^{\text{reg}}$ is a *singular* solution to the corresponding homogeneous problem

$$\text{Find } \phi^{\text{sing}} \in H^{1/2}_{\text{zmv}}(\Gamma) \text{ s.t. } \mathrm{curl}_\Gamma(\underline{\boldsymbol{\alpha}}\ \ \mathbf{curl}_\Gamma\, \phi^{\text{sing}}) = 0 \quad \text{on } \Gamma. \tag{4.79}$$

By assumption, there exist no singular solutions to this problem. So, one can conclude that $\phi^- = \phi^{\text{reg}} \in H^1(\Gamma)$. This shows that $\gamma^T \mathbf{E} \in \mathbf{L}_t^2(\Gamma)$.

In a second step, we study ψ^-, following the first way described in Remark 4.2.5. Indeed, we now know that $\mathrm{div}_\Gamma(\underline{\boldsymbol{\alpha}} \gamma^T \mathbf{E}) \in H^{-1}(\Gamma)$, so we can take the div_Γ of (4.77) to write the problem governing ψ^-:

$$\Delta_\Gamma\, \psi^- = \mathrm{div}_\Gamma\, \mathbf{g} - \mathrm{div}_\Gamma \left(\underline{\boldsymbol{\alpha}}\ \ \mathbf{curl}_\Gamma\, \phi^-\right) - \mathrm{div}_\Gamma \left(\underline{\boldsymbol{\alpha}}\ \ \nabla_\Gamma\, \psi^+\right), \tag{4.80}$$

where we now benefit of the extra-regularity result of ϕ^-. As before, let us split ψ^- into two parts: $\psi^{\text{reg}} \in H^1_{\text{zmv}}(\Gamma)$ is the only *regular* solution to (4.80) (the problem is well-posed in $H^1_{\text{zmv}}(\Gamma)$). On the other hand, $\psi^{\text{sing}} := \psi^- - \psi^{\text{reg}}$ is solution to

$$\text{Find } \psi^{\text{sing}} \in H^{1/2}_{\text{zmv}}(\Gamma) \text{ s.t. } \Delta_\Gamma\, \psi^{\text{sing}} = 0 \quad \text{on } \Gamma. \tag{4.81}$$

Here, we recover a standard Laplace-Beltrami problem, which does not involve $\underline{\boldsymbol{\alpha}}$, contrarily to (4.78). Therefore, one can apply directly the results of the first section, in particular the first item of Lemma 4.1.2. As, by assumption, Γ has no pathological vertices, we conclude that $\psi^{\text{sing}} = 0$ and $\psi^- \in H^1(\Gamma)$. This shows that $\pi^T \mathbf{C} \in \mathbf{L}_t^2(\Gamma)$. Moreover, as Γ has no pathological vertices, $\mathbf{g}$ also belongs to $\mathbf{L}_t^2(\Gamma)$ by Theorem 4.1.4. So, finally, the whole boundary condition holds in $\mathbf{L}_t^2(\Gamma)$. □

Remark 4.3.2. As in the previous section, one could also study the extra-regularity of ϕ^- and ψ^-, to determine whether $\gamma^T \mathbf{E}$, $\pi^T \mathbf{C}$ belong to $\mathbf{H}_t^s(\Gamma)$ for a certain $s > 0$. This extra-regularity will be driven by the eigenvalues of operator $\mathrm{curl}_\Gamma(\underline{\boldsymbol{\alpha}}\, \mathbf{curl}_\Gamma\, \cdot)$ in addition to the ones of operator Δ_Γ.

Remark 4.3.3. One could also consider the alternate Robin condition

$$\underline{\boldsymbol{\beta}}\pi^T \mathbf{C} + \gamma^T \mathbf{E} = \tilde{\mathbf{g}} \quad \text{on } \Gamma. \tag{4.82}$$

This yields the same results, except the two steps of the proof are inverted. Indeed, in this case, it is meaningful to take first the div_Γ of (4.82). This leads to consider first the problem governing ψ^-; this time, it involves the operator $\mathrm{div}_\Gamma(\underline{\boldsymbol{\beta}}\,\nabla_\Gamma\,\cdot\,)$. If this problem has no singular solution, one concludes that $\pi^T\mathbf{C} \in \mathbf{L}^2_t(\Gamma)$. It is then meaningful to take the curl_Γ of (4.82). This gives the problem governing ϕ^-, which involves only the standard Laplace-Beltrami operator. From that we conclude that $\gamma^T\mathbf{E} \in \mathbf{L}^2_t(\Gamma)$ (see also Remark 4.2.5).
It is therefore transparent to multiply the boundary condition by $\underline{\boldsymbol{\alpha}}^{-1}$ (so that $\underline{\boldsymbol{\beta}} = \underline{\boldsymbol{\alpha}}^{-1}$, with a right-hand side $\tilde{\mathbf{g}} = \underline{\boldsymbol{\alpha}}^{-1}\mathbf{g}$). In the tensor-valued case, note that there holds $\mathrm{div}_\Gamma\left((\det\underline{\boldsymbol{\alpha}})\ \underline{\boldsymbol{\alpha}}^{-1}\,\nabla_\Gamma\, v\right) = -\,\mathrm{curl}_\Gamma(\underline{\boldsymbol{\alpha}}\,\mathbf{curl}_\Gamma\, v)$.

Conclusion

The results of this chapter allow us to state whether the Robin boundary condition may hold in $\mathbf{L}^2_t(\Gamma)$, so that both Dirichlet and Neumann traces belong to this space. This depends both on the impedance coefficient α or $\underline{\boldsymbol{\alpha}}$ and on the geometry of Γ.

When the coefficient is scalar and constant, the proposition depends only on the geometry. In fact, we have proven a far stronger result than simply stating whether the boundary condition holds in $\mathbf{L}^2_t(\Gamma)$ or not. This is summarized in the imbeddings of Theorem 4.1.4. These are purely functional analysis results: they state the relation between the space $\mathbf{H}^{-1/2}_{\parallel}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}_\Gamma, \Gamma)$ and spaces of Sobolev regularity. A consequence of this result is that, when the imbedding (4.19) holds, the Robin condition with constant coefficient holds in $\mathbf{H}^s(\Gamma)$ for a $s > 0$ (hence, it also holds in $\mathbf{L}^2_t(\Gamma)$). This result is strongly related to the study of solutions to the Laplace-Beltrami problem.

Then, we have also considered the case of a Robin condition with heterogeneous coefficient α or $\underline{\boldsymbol{\alpha}}$. This leads to deal to the operator $\mathrm{curl}_\Gamma(\underline{\boldsymbol{\alpha}}\,\mathbf{curl}_\Gamma\,\cdot)$, instead of Δ_Γ. However, when the coefficient is scalar and smooth enough, one can come back to Δ_Γ thanks to the Leibniz formula. Thus, one obtains similar results as in the constant case. In particular, the singularities space is the same. When α is piecewise constant, one really has to study the singularities of $\mathrm{curl}_\Gamma(\alpha\,\mathbf{curl}_\Gamma\,\cdot)$. We are able to prove that the singularities space is again finite-dimensional, and that there holds extra-regularity for the regular part. More precise results can be obtained for given configurations; we refer e.g. to [15]. When $\underline{\boldsymbol{\alpha}}$ is tensor-valued, we provide only a merely abstract result. Once again, one would have to study the singularities of operator $\mathrm{curl}_\Gamma(\underline{\boldsymbol{\alpha}}\,\mathbf{curl}_\Gamma\,\cdot)$ in a given configuration to get a more precise result.

Chapter 5

Variational formulations and well-posedness of Maxwell boundary value problems

In this chapter, we derive $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming variational formulations for the time-harmonic Maxwell equation completed with Dirichlet, Neumann, or Robin boundary condition, and study their well-posedness. Throughout this chapter, Ω is a Lipschitz domain, as defined in the beginning of Section 2.2. The angular frequency $\omega > 0$ is fixed. We consider a general class of material tensors: we assume $\underline{\boldsymbol{\mu}}, \underline{\boldsymbol{\varepsilon}} \in \underline{\mathbf{L}}^\infty(\Omega)$ and satisfy an ellipticity condition as defined in (3.1). The corresponding ellipticity directions may be different: $\theta_\varepsilon \neq \theta_\mu$ in general.

In the isotropic case (that is, when $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ are scalars), the study of such problems is well-known. We refer, among others, to the monographs of Monk [86] and Assous, Ciarlet and Labrunie [7]. Let us note that variational formulations other than $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming can be considered. The formulation can be *augmented* by adding a divergence term, see e.g. [36, 24]; these are also called *regularized* formulations. Mixed formulations, augmented or not, can also be considered, see e.g. [8, 49]. When it comes to anisotropic media, the most documented cases are when the tensors are Hermitian definite positive, see e.g. [69, 14, 25]. Up to our knowledge, only a few works address the case of non-Hermitian material tensors. A case with complex-valued symmetric tensors have been studied in [4]. The authors show that the problem enters Fredholm alternative and prove the uniqueness of the solution. In [66], various variational formulations have been derived for particular tensors coming from plasma theory. The authors show that in this case, the associated form is coercive. Very recently [115], tensors with elliptic real part have been considered, as perturbations of Hermitian tensors, It is proven that the problem enters Fredholm alternative. Moreover, all these works generally focus on the Dirichlet problem.

Here, we provide a well-posedness analysis for elliptic tensors: this class of materials is more general than the materials considered in the previously cited works. Moreover, we address the three types of boundary conditions, which is generally not the case in the previous works. In our work, well-posedness is to be understood in Fredholm sense. Indeed, coerciveness does not hold in general, so one has to rely on Fredholm Alternative. This study is achieved thanks to the extended tools for anisotropic tensors developed in Chapter 3: Helmholtz decompositions and compact embeddings.

We derive the variational formulations and study their well-posedness successively for the three types of problems: Dirichlet is addressed in Section 5.1, Neumann in Section 5.2 and finally Robin in Section 5.3. We point out that for the latter one, results from Chapter 4 are also required.

5.1 The Dirichlet problem

In this section, we consider the time-harmonic Maxwell problem completed with a non-homogeneous Dirichlet boundary condition on Γ:

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{E}\times\mathbf{n} = \mathbf{g} & \text{on } \Gamma, \end{cases} \tag{5.1}$$

where we assume $\mathbf{f} \in \mathbf{L}^2(\Omega)$, and $\mathbf{g} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. We shall seek $\mathbf{E}$ in $\mathbf{H}(\mathbf{curl}, \Omega)$.

Before deriving the variational formulation of problem (5.1), let us reduce the problem to a homogeneous form, as it is usual when one deals with inhomogeneous Dirichlet problems. To that aim, we introduce a lifting of the boundary data, $\mathbf{E}_d \in \mathbf{H}(\mathbf{curl}, \Omega)$, s.t. $\mathbf{g} = \mathbf{E}_d \times \mathbf{n}$ on Γ. The new unknown $\mathbf{E}_0 := \mathbf{E} - \mathbf{E}_d$ satisfies the homogeneous problem

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_0) - \omega^2 \underline{\boldsymbol{\varepsilon}}\mathbf{E}_0 = \mathbf{f} - \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_d) + \omega^2 \underline{\boldsymbol{\varepsilon}}\mathbf{E}_d & \text{in } \Omega, \\ \mathbf{E}_0 \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma. \end{cases} \tag{5.2}$$

Therefore, we shall seek $\mathbf{E}_0$ in $\mathbf{H}_0(\mathbf{curl}, \Omega)$. Then, one can write the variational formulation of this problem.

Theorem 5.1.1. The problem (5.2) is equivalent to the variational formulation

$$\left| \begin{array}{l} \text{Find } \mathbf{E}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega) \text{ s.t., } \forall \mathbf{F} \in \mathbf{H}_0(\mathbf{curl}, \Omega), \\ \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_0 | \,\mathbf{curl}\,\mathbf{F}\right) - \omega^2 \left(\underline{\boldsymbol{\varepsilon}}\mathbf{E}_0 | \mathbf{F}\right) = \left(\mathbf{f} + \omega^2 \underline{\boldsymbol{\varepsilon}}\mathbf{E}_d | \mathbf{F}\right) - \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_d | \,\mathbf{curl}\,\mathbf{F}\right). \end{array} \right. \tag{5.3}$$

Proof. The variational formulation is obtained by standard techniques. We multiply the first line of (5.2) by $\mathbf{F} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ and integrate by parts. Conversely, one takes $\mathbf{F} \in \boldsymbol{\mathcal{D}}(\Omega)$ and differentiates in $\boldsymbol{\mathcal{D}}'(\Omega)$, to conclude that $\mathbf{E}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ satisfies the volume equation of (5.2) in the sense of distributions. □

Remark 5.1.2. With $\mathbf{E}_0$ governed by (5.3), the total field $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_d$ satisfies both the volume equation and boundary condition of (5.1). Alternatively, $\mathbf{E}$ is governed by the variational formulation

$$\left| \begin{array}{l} \text{Find } \mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega) \text{ s.t.} \\ \forall \mathbf{F} \in \mathbf{H}_0(\mathbf{curl}, \Omega), \quad \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E} | \,\mathbf{curl}\,\mathbf{F}\right) - \omega^2 \left(\underline{\boldsymbol{\varepsilon}}\mathbf{E} | \mathbf{F}\right) = \ell_{\mathrm{D}}(\mathbf{F}), \\ \mathbf{E} \times \mathbf{n} = \mathbf{g} \text{ on } \Gamma, \end{array} \right. \tag{5.4}$$

where $\ell_{\mathrm{D}} : \mathbf{F} \mapsto (\mathbf{f}|\mathbf{F})$ belongs to $(\mathbf{H}_0(\mathbf{curl}, \Omega))'$. However, this formulation cannot be used directly to study the well-posedness of the problem, as the unknown $\mathbf{E}$ and the test function $\mathbf{F}$ do not belong to the same function space. So, for the analysis, one relies rather on the formulation (5.3). On the other hand, when it comes to the numerical resolution, one would rather use the formulation (5.4) to solve the non-homogeneous Dirichlet problem numerically.

In the following, we denote $\ell_{\mathrm{D},0}$ the antilinear continuous form on $\mathbf{H}_0(\mathbf{curl}, \Omega)$ defined by the right-hand side of (5.3), depending on the data $\mathbf{f}$ and $\mathbf{E}_d$:

$$\ell_{\mathrm{D},0} : \mathbf{F} \mapsto \left(\mathbf{f} + \omega^2 \underline{\boldsymbol{\varepsilon}}\mathbf{E}_d | \mathbf{F}\right) - \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_d | \,\mathbf{curl}\,\mathbf{F}\right). \tag{5.5}$$

In addition, we observe that

$$\|\ell_{\mathrm{D},0}\|_{(\mathbf{H}_0(\mathbf{curl}))'} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{E}_d\|_{\mathbf{H}(\mathbf{curl})}.\text{[1]} \tag{5.6}$$

Well-posedness

The well-posedness analysis of the formulation (5.3) is done in two steps. First, we shall rewrite it into two equivalent problems, by the means of the (first-kind) Helmholtz decomposition derived in Theorem 3.2.1. Then, we shall see that both of these problems are well-posed.

Lemma 5.1.3. The formulation (5.3) is equivalent to set $\mathbf{E}_0 = \nabla p + \tilde{\mathbf{E}}$, with $p \in H_0^1(\Omega)$ and $\tilde{\mathbf{E}} \in \mathbf{K}_N(\underline{\boldsymbol{\varepsilon}}; \Omega)$, respectively governed by

$$\left| \begin{array}{l} \text{Find } p \in H_0^1(\Omega) \text{ s.t., } \forall q \in H_0^1(\Omega), \\ \qquad -\omega^2 \left(\underline{\boldsymbol{\varepsilon}}\nabla p | \nabla q\right) = \ell_{\mathrm{D},0}(\nabla q) \end{array} \right. \tag{5.7}$$

[1]Here and in the rest of the chapter, the notation $a \lesssim b$ denotes that there exists a constant $C > 0$, *independent of* a *and* b, s.t. $a \le Cb$. The constant C depends only on the geometry, the frequency ω, and the coefficients $\underline{\boldsymbol{\mu}}$, $\underline{\boldsymbol{\varepsilon}}$ (and α).

and

$$\left| \begin{array}{l} \text{Find } \tilde{\mathbf{E}} \in \mathbf{K}_N(\underline{\varepsilon};\Omega) \text{ s.t., } \forall \tilde{\mathbf{F}} \in \mathbf{K}_N(\underline{\varepsilon};\Omega), \\ \left(\underline{\boldsymbol{\mu}}^{-1} \,\mathbf{curl}\, \tilde{\mathbf{E}} | \,\mathbf{curl}\, \tilde{\mathbf{F}}\right) - \omega^2 \left(\underline{\varepsilon}\tilde{\mathbf{E}}|\tilde{\mathbf{F}}\right) = \omega^2 \left(\underline{\varepsilon}\nabla p|\tilde{\mathbf{F}}\right) + \ell_{\mathrm{D},0}(\tilde{\mathbf{F}}). \end{array} \right. \tag{5.8}$$

Proof. Direct. Let us introduce the (first-kind) Helmholtz decomposition (3.19) of $\mathbf{E}_0$: $\mathbf{E}_0 = \nabla p + \tilde{\mathbf{E}}$, with $p \in H_0^1(\Omega)$ and $\tilde{\mathbf{E}} \in \mathbf{K}_N(\underline{\varepsilon};\Omega)$. Taking $\mathbf{F} = \nabla q$ for any $q \in H_0^1(\Omega)$ in (5.3), we get

$$-\omega^2 \left(\underline{\varepsilon}(\nabla p + \tilde{\mathbf{E}})|\nabla q\right) = \ell_{\mathrm{D},0}(\nabla q)$$

and, since $\tilde{\mathbf{E}} \in \mathbf{K}_N(\underline{\varepsilon};\Omega)$, it holds that, $\forall q \in H_0^1(\Omega)$, $\left(\underline{\varepsilon}\tilde{\mathbf{E}}|\nabla q\right) = 0$, so

$$-\omega^2 \left(\underline{\varepsilon}\nabla p|\nabla q\right) = \ell_{\mathrm{D},0}(\nabla q).$$

On the other hand, for $\tilde{\mathbf{E}} = \mathbf{E}_0 - \nabla p \in \mathbf{K}_N(\underline{\varepsilon};\Omega)$, one has

$$\left(\underline{\boldsymbol{\mu}}^{-1} \,\mathbf{curl}\, \tilde{\mathbf{E}} | \,\mathbf{curl}\, \mathbf{F}\right) - \omega^2 \left(\underline{\varepsilon}\tilde{\mathbf{E}}|\mathbf{F}\right) = \omega^2 \left(\underline{\varepsilon}\nabla p|\mathbf{F}\right) + \ell_{\mathrm{D},0}(\mathbf{F})$$

for any $\mathbf{F} \in \mathbf{H}_0(\mathbf{curl},\Omega)$, hence in particular for any $\tilde{\mathbf{F}} \in \mathbf{K}_N(\underline{\varepsilon};\Omega)$.

Reverse. By summation of (5.7) and (5.8), one has, $\forall q \in H_0^1(\Omega), \forall \tilde{\mathbf{F}} \in \mathbf{K}_N(\underline{\varepsilon};\Omega)$:

$$\left(\underline{\boldsymbol{\mu}}^{-1} \,\mathbf{curl}\, \tilde{\mathbf{E}} | \,\mathbf{curl}\, \tilde{\mathbf{F}}\right) - \omega^2 \left(\underline{\varepsilon}\tilde{\mathbf{E}}|\tilde{\mathbf{F}}\right) - \omega^2 \left(\underline{\varepsilon}\nabla p|\nabla q\right) = \omega^2 \left(\underline{\varepsilon}\nabla p|\tilde{\mathbf{F}}\right) + \ell_{\mathrm{D},0}(\tilde{\mathbf{F}}) + \ell_{\mathrm{D},0}(\nabla q).$$

One can add the null terms $\left(\underline{\boldsymbol{\mu}}^{-1} \,\mathbf{curl}\, \nabla p | \,\mathbf{curl}\, \tilde{\mathbf{F}}\right)$, $\left(\underline{\boldsymbol{\mu}}^{-1} \,\mathbf{curl}(\tilde{\mathbf{E}} + \nabla p) | \,\mathbf{curl}\, \nabla q\right)$ and $-\omega^2 \left(\underline{\varepsilon}\tilde{\mathbf{E}}|\nabla q\right)$, and pose $\mathbf{E}_0 := \tilde{\mathbf{E}} + \nabla p \in \mathbf{H}_0(\mathbf{curl},\Omega)$ to get, after simple rearrangements:

$$\left(\underline{\boldsymbol{\mu}}^{-1} \,\mathbf{curl}\, \mathbf{E}_0 | \,\mathbf{curl}(\tilde{\mathbf{F}} + \nabla q)\right) - \omega^2 \left(\underline{\varepsilon}\mathbf{E}_0|\tilde{\mathbf{F}} + \nabla q\right) = \ell_{\mathrm{D},0}(\tilde{\mathbf{F}} + \nabla q).$$

Finally, as $\tilde{\mathbf{F}}$ and q span respectively $\mathbf{K}_N(\underline{\varepsilon};\Omega)$ and $H_0^1(\Omega)$, the sum $\tilde{\mathbf{F}} + \nabla q$ spans the whole space $\mathbf{H}_0(\mathbf{curl},\Omega)$, thanks to the (first-kind) Helmholtz decomposition (3.19). □

Remark 5.1.4. The term $\omega^2 \left(\underline{\varepsilon}\nabla p|\tilde{\mathbf{F}}\right)$ in formulation (5.8) vanishes automatically *only* if $\underline{\varepsilon}$ is a Hermitian tensor field.

Now, let us study the well-posedness of variational formulations (5.7) and (5.8). For the first one, the result is quite straightforward.

Lemma 5.1.5. The formulation (5.7) is well-posed, and there holds $\|p\|_{H_0^1} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{E}_d\|_{\mathbf{L}^2}$.

Proof. It is an immediate consequence of Theorem 3.1.6. Since $\ell_{\mathrm{D},0} \in (\mathbf{H}_0(\mathbf{curl},\Omega))'$, and ∇ is a continuous mapping from $H_0^1(\Omega)$ to $\mathbf{H}_0(\mathbf{curl},\Omega)$, one has $\ell_{\mathrm{D},0} \circ \nabla \in H^{-1}(\Omega)$, with $\ell_{\mathrm{D},0}(\nabla q) = \left(\mathbf{f} + \omega^2 \underline{\varepsilon}\mathbf{E}_d|\nabla q\right)$. □

To study formulation (5.8), we use the compact embedding of $\mathbf{K}_N(\underline{\varepsilon};\Omega)$ into $\mathbf{L}^2(\Omega)$ (Theorem 3.3.2). The formulation then enters Fredholm alternative (see Theorem 2.5.2).

Lemma 5.1.6. The formulation (5.8) enters Fredholm alternative, and there holds the following:

- either the problem admits a unique solution $\tilde{\mathbf{E}}$ in $\mathbf{K}_N(\underline{\varepsilon};\Omega)$, which depends continuously on the data $\mathbf{f}$ and $\mathbf{E}_d$:
$$\|\tilde{\mathbf{E}}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{E}_d\|_{\mathbf{H}(\mathbf{curl})};$$
- or, the problem has solutions if, and only if, $\mathbf{f}$ and $\mathbf{E}_d$ satisfies a finite number of compatibility conditions; in this case, the space of solutions is an affine space of finite dimension. Additionally, the component of the solution which is orthogonal (in the sense of the $\mathbf{H}_0(\mathbf{curl},\Omega)$ inner product) to the corresponding linear vector space, depends continuously on the data $\mathbf{f}$ and $\mathbf{E}_d$.

Proof. Let us split the left-hand side of (5.8) in two terms. We introduce two sesquilinear forms defined on $\mathbf{H}_0(\mathbf{curl},\Omega)$, namely

$$a : (\mathbf{u},\mathbf{v}) \mapsto \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{u}|\,\mathbf{curl}\,\mathbf{v}\right) + \mathrm{e}^{\mathrm{i}\theta_\mu}\,(\mathbf{u}|\mathbf{v})$$

and

$$b : (\mathbf{u},\mathbf{v}) \mapsto -\omega^2\,(\underline{\varepsilon}\mathbf{u}|\mathbf{v}) - \mathrm{e}^{\mathrm{i}\theta_\mu}\,(\mathbf{u}|\mathbf{v})\,,$$

where we recall that $\underline{\boldsymbol{\mu}}^{-1}$ satisfies assumption (3.1), and use the notations of Proposition 3.1.4.

We recall that $\mathbf{K}_N(\underline{\varepsilon};\Omega)$ is equipped with the same norm as $\mathbf{H}(\mathbf{curl},\Omega)$. The form a is clearly continuous on $\mathbf{K}_N(\underline{\varepsilon};\Omega)$: $|a(\mathbf{u},\mathbf{v})| \lesssim \|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})}\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl})}$. It is also coercive on $\mathbf{K}_N(\underline{\varepsilon};\Omega)$, indeed:

$$\begin{aligned}
|a(\mathbf{v},\mathbf{v})| &= |\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\,\mathbf{curl}\,\mathbf{v}\right) + \mathrm{e}^{\mathrm{i}\theta_\mu}\,(\mathbf{v}|\mathbf{v})\,| \\
&\geq \Re\left[\mathrm{e}^{-\mathrm{i}\theta_\mu}\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\,\mathbf{curl}\,\mathbf{v}\right) + (\mathbf{v}|\mathbf{v})\right] \\
&\geq \mu_-^{\mathrm{inv}}\|\,\mathbf{curl}\,\mathbf{v}\|^2_{\mathbf{L}^2} + \|\mathbf{v}\|^2_{\mathbf{L}^2} \\
&\geq \min(\mu_-^{\mathrm{inv}},1)\,\|\mathbf{v}\|^2_{\mathbf{H}(\mathbf{curl})}.
\end{aligned}$$

Besides, $b(\mathbf{u},\mathbf{v}) \lesssim \|\mathbf{u}\|_{\mathbf{L}^2}\|\mathbf{v}\|_{\mathbf{L}^2} \lesssim \|\mathbf{u}\|_{\mathbf{L}^2}\|\mathbf{v}\|_{\mathbf{K}_N(\underline{\varepsilon})}$, so the form b is continuous on $\mathbf{L}^2(\Omega)\times\mathbf{K}_N(\underline{\varepsilon};\Omega)$. Finally, the embedding of $\mathbf{K}_N(\underline{\varepsilon};\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact by Theorem 3.3.2. Hence the formulation (5.8) enters the coercive + compact framework, and the Fredholm alternative (Theorem 2.5.2) holds. The bound proceeds from the bounds on $\ell_{\mathrm{D},0}$ (5.6) and on p (Lemma 5.1.5). □

Grouping the previous results, we are able to conclude on the well-posedness of problem (5.3), then of problem (5.1).

Theorem 5.1.7. The problem (5.3) with unknown $\mathbf{E}_0$ enters Fredholm alternative:

- either the problem admits a unique solution $\mathbf{E}_0$ in $\mathbf{H}_0(\mathbf{curl},\Omega)$, which depends continuously on the data $\mathbf{f}$ and $\mathbf{E}_d$: $\mathbf{K}_N(\underline{\varepsilon};\Omega)$, which depends continuously on the data $\mathbf{f}$ and $\mathbf{E}_d$:

$$\|\mathbf{E}_0\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{E}_d\|_{\mathbf{H}(\mathbf{curl})}; \tag{5.9}$$

- or, the problem has solutions if, and only if, $\mathbf{f}$ and $\mathbf{E}_d$ satisfy a finite number of compatibility conditions; in this case, the space of solutions is an affine space of finite dimension. Additionally, the component of the solution which is orthogonal (in the sense of the $\mathbf{H}_0(\mathbf{curl},\Omega)$ inner product) to the corresponding linear vector space, depends continuously on the data $\mathbf{f}$ and $\mathbf{E}_d$.

Moreover, each alternative occurs simultaneously for formulation (5.8) and formulation (5.3).

Corollary 5.1.8. If the problem (5.3) is well-posed, then the problem (5.1) is well-posed as well, and

$$\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{g}\|_\gamma. \tag{5.10}$$

Proof. The problems (5.3) and (5.1) are equivalent. In particular, the solution $\mathbf{E}$ reconstructed from problem (5.3) is independent of the choice of the lifting $\mathbf{E}_d$. Indeed, let $\mathbf{E}_d^1, \mathbf{E}_d^2 \in \mathbf{H}(\mathbf{curl},\Omega)$ two liftings of $\mathbf{g}$. Then, we introduce $\mathbf{E}_0^1$ and $\mathbf{E}_0^2 \in \mathbf{H}_0(\mathbf{curl},\Omega)$ the corresponding solutions to problem (5.3), and define $\mathbf{E}^1 := \mathbf{E}_0^1 + \mathbf{E}_d^1$, $\mathbf{E}^2 := \mathbf{E}_0^2 + \mathbf{E}_d^2$. Then, $\mathbf{E}^1$ and $\mathbf{E}^2$ satisfy the same boundary value problem (5.1). In particular, $\mathbf{E}^1 - \mathbf{E}^2 \in \mathbf{H}_0(\mathbf{curl},\Omega)$ is solution to the homogeneous Dirichlet problem with null data. As this problem is well-posed, $\mathbf{E}^1 = \mathbf{E}^2$.

The bound is obtained taking the infimum on the liftings $\mathbf{E}_d$:

$$\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \inf_{\mathbf{E}_d}\|\mathbf{E}_d\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{g}\|_\gamma,$$

where we recall that $\|\cdot\|_\gamma$ denotes the norm of $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$. This grants the continuous dependence of $\mathbf{E}$ w.r.t. data $\mathbf{f}$ and $\mathbf{g}$. □

In what preceeds, we have simply assumed that the coefficients $\underline{\varepsilon}$ and $\underline{\boldsymbol{\mu}}$ are elliptic, independently from each other: in particular, their ellipticity directions can be independent. It is possible to go further if one assumes moreover interplay between the coefficients $\underline{\varepsilon}$ and $\underline{\boldsymbol{\mu}}$. To that aim, we introduce the notion of *simultaneous ellipticity*: we say that $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\varepsilon}$ are simultaneously elliptic iff $\Theta_{\mu^{-1}} \cap \Theta_{-\varepsilon} \neq \emptyset$: that is, there exists $\theta \in \mathbb{R}$, $\mu^{\text{inv}}_{-,\theta}, \varepsilon_{-,\theta} > 0$ s.t.

$$\Re[e^{i\theta}(\mathbf{v}^*\underline{\boldsymbol{\mu}}^{-1}\mathbf{v})] \geq \mu^{\text{inv}}_{-,\theta}|\mathbf{v}|^2 \quad \text{and} \quad \Re[e^{i\theta}(-\mathbf{v}^*\underline{\varepsilon}\mathbf{v})] \geq \varepsilon_{-,\theta}|\mathbf{v}|^2, \tag{5.11}$$

simultaneously. This means that $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\varepsilon}$ share a common ellipticity direction. Note that $\mu^{\text{inv}}_{-,\theta}, \varepsilon_{-,\theta}$ can be different from $\mu^{\text{inv}}_{-}, \varepsilon_{-}$: it depends on the relative positions of θ, θ_μ and θ_ε.

In this case, the bilinear form associated to the problem is coercive. Indeed, there holds

$$\begin{aligned}\Re\left[e^{i\theta}\left(\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\,\mathbf{curl}\,\mathbf{v}\right) - \omega^2\left(\underline{\varepsilon}\mathbf{v}|\mathbf{v}\right)\right)\right] &\geq \mu^{\text{inv}}_{-,\theta}\|\,\mathbf{curl}\,\mathbf{v}\|^2 + \omega^2\varepsilon_{-,\theta}\|\mathbf{v}\|^2 \\ &\geq \min(\mu^{\text{inv}}_{-,\theta},\, \omega^2\varepsilon_{-,\theta})\|\mathbf{v}\|^2_{\mathbf{H}(\mathbf{curl})}.\end{aligned}$$

Theorem 5.1.9. If $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\varepsilon}$ are simultaneously elliptic, then the Dirichlet problem is well-posed: there exists a unique solution $\mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega)$ to (5.1), with moreover

$$\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{g}\|_\gamma. \tag{5.12}$$

5.2 The Neumann problem

In this section, the problem is completed with a Neumann boundary condition:

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\underline{\varepsilon}\mathbf{E} = \mathbf{f} & \text{in } \Omega, \\ \underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E} \times \mathbf{n} = \mathbf{j} & \text{on } \Gamma, \end{cases} \tag{5.13}$$

where $\mathbf{j}$ is a boundary data which can be interpreted as a surface current. We assume that $\mathbf{f} \in \mathbf{L}^2(\Omega)$, and $\mathbf{j} \in \mathbf{H}^{-1/2}_{\parallel}(\text{div}_\Gamma, \Gamma)$. The natural space to look for $\mathbf{E}$ is $\mathbf{H}(\mathbf{curl}, \Omega)$.

Using standard techniques, we write the variational formulation of the Neumann problem (5.13).

Theorem 5.2.1. The problem (5.13) is equivalent to the variational formulation

$$\left|\begin{array}{l}\text{Find } \mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega) \text{ s.t., } \forall \mathbf{F} \in \mathbf{H}(\mathbf{curl}, \Omega), \\ \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}|\,\mathbf{curl}\,\mathbf{F}\right) - \omega^2\left(\underline{\varepsilon}\mathbf{E}|\mathbf{F}\right) = (\mathbf{f}|\mathbf{F}) + {}_\gamma\langle \mathbf{j}, \pi^T\mathbf{F}\rangle_\pi.\end{array}\right. \tag{5.14}$$

Proof. We multiply the volume equation of (5.13) by a test function $\mathbf{F} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and integrate by parts using formula (2.44), to get

$$\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}|\,\mathbf{curl}\,\mathbf{F}\right) - \omega^2\left(\underline{\varepsilon}\mathbf{E}|\mathbf{F}\right) - {}_\gamma\langle\gamma^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}), \pi^T\mathbf{F}\rangle_\pi = (\mathbf{f}|\mathbf{F})\,. \tag{5.15}$$

Then, using the boundary condition of (5.13) gives the variational formulation (5.14).

Conversely, one takes $\mathbf{F} \in \boldsymbol{\mathcal{D}}(\Omega)$ in (5.14) and differentiates in $\boldsymbol{\mathcal{D}}'(\Omega)$ to conclude that the volume Maxwell equation

$$\mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\underline{\varepsilon}\mathbf{E} = \mathbf{f}$$

holds in $\boldsymbol{\mathcal{D}}'(\Omega)$, and in $\mathbf{L}^2(\Omega)$. Multipliying anew this equation by $\mathbf{F} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and integrating by parts, one recovers equation (5.15). Substrating this to (5.14), one concludes that ${}_\gamma\langle\gamma^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}), \pi^T\mathbf{F}\rangle_\pi = {}_\gamma\langle\mathbf{j}, \pi^T\mathbf{F}\rangle_\pi$, for all $\mathbf{F} \in \mathbf{H}(\mathbf{curl}, \Omega)$; i.e. that $\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E} \times \mathbf{n} = \mathbf{j}$ holds in $\mathbf{H}^{-1/2}_{\parallel}(\text{div}_\Gamma, \Gamma)$, because the mapping π^T is surjective from $\mathbf{H}(\mathbf{curl}, \Omega)$ to $\mathbf{H}^{-1/2}_{\perp}(\text{curl}_\Gamma, \Gamma) = \left(\mathbf{H}^{-1/2}_{\parallel}(\text{div}_\Gamma, \Gamma)\right)'$. $\square$

In the following, we denote ℓ_{N} the antilinear continuous form on $\mathbf{H}(\mathbf{curl}, \Omega)$ defined by the right-hand side, depending on the data $\mathbf{f}$ and $\mathbf{j}$:

$$\ell_{\mathrm{N}} : \mathbf{F} \mapsto (\mathbf{f}|\mathbf{F}) + {}_{\gamma}\langle \mathbf{j}, \pi^T \mathbf{F}\rangle_{\pi}, \tag{5.16}$$

and we note that

$$\|\ell_{\mathrm{N}}\|_{(\mathbf{H}(\mathbf{curl}))'} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{j}\|_{\gamma}. \tag{5.17}$$

Well-posedness

Here, the variational formulation is posed in the space $\mathbf{H}(\mathbf{curl}, \Omega)$. The analysis follows the same steps as for the Dirichlet problem, using the appropriate tools related to the space $\mathbf{H}(\mathbf{curl}, \Omega)$. In particular, we recast the formulation (5.14) using the second-kind Helmholtz decomposition (Theorem 3.2.3).

Lemma 5.2.2. The formulation (5.14) is equivalent to set $\mathbf{E} = \nabla p + \tilde{\mathbf{E}}$, with $p \in H^1_{\mathrm{zmv}}(\Omega)$ and $\tilde{\mathbf{E}} \in \mathbf{K}_T(\underline{\varepsilon}; \Omega)$, respectively governed by

$$\left| \begin{array}{l} \text{Find } p \in H^1_{\mathrm{zmv}}(\Omega) \text{ s.t., } \forall q \in H^1_{\mathrm{zmv}}(\Omega), \\ \qquad -\omega^2 \left(\underline{\varepsilon}\nabla p|\nabla q\right) = \ell_{\mathrm{N}}(\nabla q) \end{array} \right. \tag{5.18}$$

and

$$\left| \begin{array}{l} \text{Find } \tilde{\mathbf{E}} \in \mathbf{K}_T(\underline{\varepsilon}; \Omega) \text{ s.t., } \forall \tilde{\mathbf{F}} \in \mathbf{K}_T(\underline{\varepsilon}; \Omega), \\ \left(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \tilde{\mathbf{E}} | \,\mathbf{curl}\, \tilde{\mathbf{F}}\right) - \omega^2 \left(\underline{\varepsilon}\tilde{\mathbf{E}}|\tilde{\mathbf{F}}\right) = \omega^2 \left(\underline{\varepsilon}\nabla p|\tilde{\mathbf{F}}\right) + \ell_{\mathrm{N}}(\tilde{\mathbf{F}}). \end{array} \right. \tag{5.19}$$

Proof. Direct. Let us introduce the (second-kind Helmholtz) decomposition (3.21) of $\mathbf{E}$: $\mathbf{E} = \nabla p + \tilde{\mathbf{E}}$, with $p \in H^1_{\mathrm{zmv}}(\Omega)$ and $\tilde{\mathbf{E}} \in \mathbf{K}_T(\underline{\varepsilon}; \Omega)$. Taking $\mathbf{F} = \nabla q$ for any $q \in H^1_{\mathrm{zmv}}(\Omega)$ yields

$$-\omega^2 \left(\underline{\varepsilon}(\tilde{\mathbf{E}} + \nabla p)|\nabla q\right) = \ell_{\mathrm{N}}(\nabla q),$$

which gives (5.18), as $\tilde{\mathbf{E}} \in \mathbf{H}_0(\mathrm{div}\, \underline{\varepsilon}0)$. On the other hand, there holds

$$\left(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \tilde{\mathbf{E}} | \,\mathbf{curl}\, \mathbf{F}\right) - \omega^2 \left(\underline{\varepsilon}\tilde{\mathbf{E}}|\mathbf{F}\right) = \omega^2 \left(\underline{\varepsilon}\nabla p|\mathbf{F}\right) + \ell_{\mathrm{N}}(\mathbf{F})$$

for any $\mathbf{F} \in \mathbf{H}(\mathbf{curl}, \Omega)$, hence for any $\tilde{\mathbf{F}} \in \mathbf{K}_T(\underline{\varepsilon}; \Omega)$.

Reverse. We sum equations (5.18) and (5.19) and introduce $\mathbf{E} := \tilde{\mathbf{E}} + \nabla p \in \mathbf{H}(\mathbf{curl}, \Omega)$. Adding the complementary vanishing terms, one gets, after rearrangements:

$$\left(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E} | \,\mathbf{curl}(\tilde{\mathbf{F}} + \nabla q)\right) - \omega^2 \left(\underline{\varepsilon}\mathbf{E}|(\tilde{\mathbf{F}} + \nabla q)\right) = \ell_{\mathrm{N}}(\tilde{\mathbf{F}} + \nabla q).$$

As q and $\tilde{\mathbf{F}}$ span respectively $H^1_{\mathrm{zmv}}(\Omega)$ and $\mathbf{K}_T(\underline{\varepsilon}; \Omega)$, we conclude by the (second-kind) Helmholtz decomposition (3.21) that the sum $\tilde{\mathbf{F}} + \nabla q$ spans the whole space $\mathbf{H}(\mathbf{curl}, \Omega)$; hence the result. □

Then, the analysis proceeds as for the Dirichlet problem. For that reason, we present directly the final result. The study of the problem (5.18) is quite easy. For the problem (5.19), we rely on the compact embedding of $\mathbf{K}_T(\underline{\varepsilon}; \Omega)$ into $\mathbf{L}^2(\Omega)$ (Theorem 3.3.4) to show that it enters Fredholm alternative.

Theorem 5.2.3. The formulation (5.14) enters Fredholm alternative, and the following conclusions hold:

- either the problem admits a unique solution $\mathbf{E}$ in $\mathbf{H}(\mathbf{curl}, \Omega)$, which depends continuously on the data $\mathbf{f}$ and $\mathbf{j}$:
$$\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{j}\|_{\gamma}; \tag{5.20}$$
- or, the problem has solutions if, and only if, $\mathbf{f}$ and $\mathbf{j}$ satisfy a finite number of compatibility conditions; in this case, the space of solutions is an affine space of finite dimension. Additionally, the component of

the solution which is orthogonal (in the sense of the $\mathbf{H}(\mathbf{curl},\Omega)$ inner product) to the corresponding linear vector space, depends continuously on the data $\mathbf{f}$ and $\mathbf{j}$.

Proof. Let us study sucessively the problems (5.18) and (5.19).

1. The formulation (5.18) is well-posed by Theorem 3.1.7. In fact, the form ℓ_{N} is continuous on $\mathbf{H}(\mathbf{curl},\Omega)$ and the mapping ∇ is continuous from $H^1_{\mathrm{zmv}}(\Omega)$ to $\mathbf{H}(\mathbf{curl},\Omega)$. Moreover, $\|p\|_{H^1_{\mathrm{zmv}}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{j}\|_\gamma$.

2. Besides, the formulation (5.19) fits the coercive + compact framework. Indeed, as in the proof of Lemma 5.1.6, one can split the left-hand side in two parts. The term

$$a(\mathbf{u},\mathbf{v}) := \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{u}|\,\mathbf{curl}\,\mathbf{v}\right) + e^{i\theta_\mu}\,(\mathbf{u}|\mathbf{v})$$

is continuous, and coercive, on $\mathbf{H}(\mathbf{curl},\Omega)$. Indeed, for the latter, there holds

$$\begin{aligned}|a(\mathbf{v},\mathbf{v})| &\geq \Re\left[\mathrm{e}^{-\mathrm{i}\theta_\mu}\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\,\mathbf{curl}\,\mathbf{v}\right) + (\mathbf{v}|\mathbf{v})\right]\\ &\geq \min(\mu_-^{\mathrm{inv}},1)\|\mathbf{v}\|^2_{\mathbf{H}(\mathbf{curl})}.\end{aligned}$$

Therefore, it also continuous and coercive on $\mathbf{K}_T(\underline{\boldsymbol{\varepsilon}};\Omega)$, because this space is equipped with the same norm as $\mathbf{H}(\mathbf{curl},\Omega)$. The remaining term

$$b(\mathbf{u},\mathbf{v}) := -\omega^2\,(\underline{\boldsymbol{\varepsilon}}\mathbf{u}|\mathbf{v}) - \mathrm{e}^{i\theta_\mu}\,(\mathbf{u}|\mathbf{v})$$

is continuous on $\mathbf{L}^2(\Omega)\times\mathbf{K}_T(\underline{\boldsymbol{\varepsilon}};\Omega)$, and the embedding of $\mathbf{K}_T(\underline{\boldsymbol{\varepsilon}};\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact by Theorem 3.3.4. Hence the conclusions of Fredholm alternative apply to formulation (5.19), with the bound $\|\tilde{\mathbf{E}}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{j}\|_\gamma + \|p\|_{H^1_{\mathrm{zmv}}}$.

Finally, the same holds for formulation (5.14), with the bound $\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{j}\|_\gamma$ obtained by triangle inequality. ☐

As in the Dirichlet problem, one can get a stronger result assuming that the coefficients $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\varepsilon}}$ are simultaneously elliptic. Indeed, in this case, the problem is actually coercive.

Theorem 5.2.4. If $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\varepsilon}}$ are simultaneously elliptic, then there exists a unique solution $\mathbf{E}\in\mathbf{H}(\mathbf{curl},\Omega)$ to (5.13), with moreover

$$\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{j}\|_\gamma. \tag{5.21}$$

5.3 The Robin problem

In this section, we consider the Maxwell problem completed with a generalised impedance (Robin) boundary condition:

$$\begin{cases}\mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E} = \mathbf{f} & \text{in } \Omega,\\ \pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}) + \alpha\gamma^T\mathbf{E} = \mathbf{g} & \text{on } \Gamma,\end{cases} \tag{5.22}$$

where we assume $\mathbf{f}\in\mathbf{L}^2(\Omega)$, and $\mathbf{g}\in\mathbf{L}^2_t(\Gamma)$. Here, we look for $\mathbf{E}$ *a priori* in the space $\mathbf{H}(\mathbf{curl},\Omega)$. However, in the literature, the variational space for this kind of problem is frequently set to $\mathbf{H}^+(\mathbf{curl},\Omega) = \{\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega), \gamma^T\mathbf{v}\in\mathbf{L}^2_t(\Gamma)\}$, assuming that the boundary condition holds in $\mathbf{L}^2_t(\Gamma)$.

As we saw in Chapter 4, this assumption is in fact not trivial, because both traces $\gamma^T\mathbf{E}$ and $\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E})$ belong *a priori* to different trace spaces: $\gamma^T\mathbf{E}\in\mathbf{H}^{-1/2}_\parallel(\mathrm{div}_\Gamma,\Gamma)$, whereas $\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E})\in\mathbf{H}^{-1/2}_\perp(\mathrm{curl}_\Gamma,\Gamma)$. To ensure that one can legitimately look for $\mathbf{E}$ in $\mathbf{H}^+(\mathbf{curl},\Omega)$, we make some additional assumptions: $\mathbf{g}\in\mathbf{H}^{-1/2}_\parallel(\mathrm{div}_\Gamma,\Gamma)\cap\mathbf{H}^{-1/2}_\perp(\mathrm{curl}_\Gamma,\Gamma)$, the boundary Γ of the domain has no pathological vertices (see Definition 4.1.1), and $\alpha\in L^\infty(\Gamma)$ is an elliptic scalar field (cf. (4.76)) s.t. operator $\mathrm{curl}_\Gamma(\alpha\,\mathbf{curl}_\Gamma\,\cdot)$ has no singular solution. This is the case, for example, if $\alpha\in W^{2,\infty}(\Gamma)$. In this case, the conclusions of Theorem 4.2.2 apply, and each term of the boundary condition of (5.22) holds (at least) in $\mathbf{L}^2_t(\Gamma)$. Other configurations are possible, see e.g. Theorem 4.2.8.

With these assumptions at hand, we ensure that a solution $\mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega)$ to (5.22) indeed belongs to the space $\mathbf{H}^+(\mathbf{curl}, \Omega) = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega), \gamma^T \mathbf{v} \in \mathbf{L}^2_t(\Gamma)\}$. In the following, we derive an equivalent variational formulation to (5.22) that is set in $\mathbf{H}^+(\mathbf{curl}, \Omega)$. In fact, it seems that $\mathbf{H}^+(\mathbf{curl}, \Omega)$ is the appropriate variational space to deal with the problem (5.22), just as $\mathbf{H}_0(\mathbf{curl}, \Omega)$ is for the Dirichlet problem or $\mathbf{H}(\mathbf{curl}, \Omega)$ is for the Neumann problem.

Besides, we also assume that $\alpha \in L^\infty(\Gamma)$ is elliptic, and, furthermore, that $\Theta_{\mu^{-1}} \cap \Theta_{-\alpha} \neq \emptyset$, i.e. that $\underline{\boldsymbol{\mu}}^{-1}$ and $-\alpha$ are *simultaneously elliptic*. In other words, there exists $\theta \in \mathbb{R}$, $\mu^{\mathrm{inv}}_{-,\theta}, \alpha_{-,\theta} > 0$ s.t.

$$\Re[e^{i\theta}(\mathbf{v}^* \underline{\boldsymbol{\mu}}^{-1} \mathbf{v})] \geq \mu^{\mathrm{inv}}_{-,\theta} |\mathbf{v}|^2 \quad \text{and} \quad \Re[e^{i\theta}(-\mathbf{v}^* \alpha \mathbf{v})] \geq \alpha_{-,\theta} |\mathbf{v}|^2$$

simultaneously ($\underline{\boldsymbol{\mu}}^{-1}$ and $-\alpha$ share a common ellipticity direction). Note that $\mu^{\mathrm{inv}}_{-,\theta}, \alpha_{-,\theta}$ can be different from $\mu^{\mathrm{inv}}_{-}, \alpha_{-}$.

We recall that $\mathbf{H}^+(\mathbf{curl}, \Omega)$ is a Hilbert space when equipped with the graph norm, and we denote $(\cdot, \cdot)_\Gamma$ the scalar product in $\mathbf{L}^2_t(\Gamma)$. Then, let us derive the variational formulation of (5.22).

Theorem 5.3.1. The problem (5.22) is equivalent to the variational formulation

$$\left| \begin{array}{l} \text{Find } \mathbf{E} \in \mathbf{H}^+(\mathbf{curl}, \Omega) \text{ s.t., } \forall \mathbf{F} \in \mathbf{H}^+(\mathbf{curl}, \Omega), \\ \left(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E} | \mathbf{curl}\, \mathbf{F}\right) - \omega^2 \left(\underline{\boldsymbol{\varepsilon}} \mathbf{E} | \mathbf{F}\right) - \left(\alpha \gamma^T \mathbf{E}, \gamma^T \mathbf{F}\right)_\Gamma = (\mathbf{f} | \mathbf{F}) - \left(\mathbf{g}, \gamma^T \mathbf{F}\right)_\Gamma . \end{array} \right. \tag{5.23}$$

Proof. Direct. Let us multiply the volume equation of (5.22) by a test function $\mathbf{F} \in \mathbf{H}^+(\mathbf{curl}, \Omega)$ and integrate by parts:

$$\left(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E} | \mathbf{curl}\, \mathbf{F}\right) - \omega^2 \left(\underline{\boldsymbol{\varepsilon}} \mathbf{E} | \mathbf{F}\right) + {}_\pi\langle \pi^T (\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}), \gamma^T \mathbf{F} \rangle_\gamma = (\mathbf{f} | \mathbf{F}) \tag{5.24}$$

Using the boundary condition, and the fact that it holds in $\mathbf{L}^2_t(\Gamma)$, one gets that

$$\begin{aligned} {}_\pi\langle \pi^T (\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}), \gamma^T \mathbf{F} \rangle_\gamma &= {}_\pi\langle \mathbf{g} - \alpha \gamma^T \mathbf{E}, \gamma^T \mathbf{F} \rangle_\gamma \\ &= \left(\mathbf{g} - \alpha \gamma^T \mathbf{E}, \gamma^T \mathbf{F}\right)_\Gamma \end{aligned}$$

and then (5.23) holds.

Reverse. Taking $\mathbf{F} \in \boldsymbol{\mathcal{D}}(\Omega)$ in (5.23) and differentiating in $\boldsymbol{\mathcal{D}}'(\Omega)$, we find $\mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}) - \omega^2 \underline{\boldsymbol{\varepsilon}} \mathbf{E} = \mathbf{f}$ in $\boldsymbol{\mathcal{D}}'(\Omega)$, hence in $\mathbf{L}^2(\Omega)$. Once again, we multiply by a test function $\mathbf{F} \in \mathbf{H}^+(\mathbf{curl}, \Omega)$ and integrate by parts to recover (5.24). Subtracting it to (5.23), we get

$${}_\pi\langle \pi^T (\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}), \gamma^T \mathbf{F} \rangle_\gamma - \left(\mathbf{g}, \gamma^T \mathbf{F}\right)_\Gamma = -\left(\alpha \gamma^T \mathbf{E}, \gamma^T \mathbf{F}\right)_\Gamma ,$$

and, because $\mathbf{g} \in \mathbf{H}^{-1/2}_\perp(\mathrm{curl}_\Gamma, \Gamma)$,

$${}_\pi\langle \pi^T (\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}), \gamma^T \mathbf{F} \rangle_\gamma - {}_\pi\langle \mathbf{g}, \gamma^T \mathbf{F} \rangle_\gamma = -\left(\alpha \gamma^T \mathbf{E}, \gamma^T \mathbf{F}\right)_\Gamma .$$

Moreover, this holds for all $\mathbf{F} \in \boldsymbol{\mathcal{C}}^\infty(\overline{\Omega}) \subset \mathbf{H}^+(\mathbf{curl}, \Omega)$. As $\boldsymbol{\mathcal{C}}^\infty(\overline{\Omega})$ is dense in $\mathbf{H}(\mathbf{curl}, \Omega)$, and γ^T is surjective from $\mathbf{H}(\mathbf{curl}, \Omega)$ to $\mathbf{H}^{-1/2}_\parallel(\mathrm{div}_\Gamma, \Gamma)$, we have that $\gamma^T(\boldsymbol{\mathcal{C}}^\infty(\overline{\Omega}))$ is dense in $\mathbf{H}^{-1/2}_\parallel(\mathrm{div}_\Gamma, \Gamma)$. Therefore, we conclude by density that

$$\pi^T (\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}) - \mathbf{g} = -\alpha \gamma^T \mathbf{E}$$

holds in $\mathbf{H}^{-1/2}_\perp(\mathrm{curl}_\Gamma, \Gamma) = (\mathbf{H}^{-1/2}_\parallel(\mathrm{div}_\Gamma, \Gamma))'$. □

In the following, we introduce ℓ_R the antilinear continuous form on $\mathbf{H}^+(\mathbf{curl}, \Omega)$ defined by the right-hand side,

$$\ell_\mathrm{R} : \mathbf{F} \mapsto (\mathbf{f} | \mathbf{F}) - \left(\mathbf{g}, \gamma^T \mathbf{F}\right)_\Gamma , \tag{5.25}$$

with

$$\|\ell_\mathrm{R}\|_{(\mathbf{H}^+(\mathbf{curl}))'} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{g}\|_{\mathbf{L}^2_t(\Gamma)} . \tag{5.26}$$

Well-posedness

The rest of the analysis follows the same steps as in the previous sections. First, we rely on the Helmholtz decomposition of $\mathbf{H}^+(\mathbf{curl}, \Omega)$ (Theorem 3.2.4) to recast the variational formulation (5.23).

Lemma 5.3.2. The formulation (5.23) can be equivalently recast as: set $\mathbf{E} = \nabla p + \tilde{\mathbf{E}}$, with $p \in H_0^1(\Omega)$ and $\tilde{\mathbf{E}} \in \mathbf{W}_N(\underline{\varepsilon}; \Omega)$, respectively governed by

$$\left| \begin{array}{l} \text{Find } p \in H_0^1(\Omega) \text{ s.t., } \forall q \in H_0^1(\Omega), \\ \qquad -\omega^2 \left(\underline{\varepsilon} \nabla p | \nabla q\right) = (\mathbf{f} | \nabla q) \end{array} \right. \tag{5.27}$$

and

$$\left| \begin{array}{l} \text{Find } \tilde{\mathbf{E}} \in \mathbf{W}_N(\underline{\varepsilon}; \Omega) \text{ s.t., } \forall \tilde{\mathbf{F}} \in \mathbf{W}_N(\underline{\varepsilon}; \Omega), \\ \left(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \tilde{\mathbf{E}} | \mathbf{curl}\, \tilde{\mathbf{F}}\right) - \omega^2 \left(\underline{\varepsilon} \tilde{\mathbf{E}} | \tilde{\mathbf{F}}\right) - \left(\alpha \gamma^T \tilde{\mathbf{E}}, \gamma^T \tilde{\mathbf{F}}\right)_\Gamma = (\mathbf{f} | \tilde{\mathbf{F}}) - \left(\mathbf{g}, \gamma^T \tilde{\mathbf{F}}\right)_\Gamma + \omega^2 \left(\underline{\varepsilon} \nabla p | \tilde{\mathbf{F}}\right). \end{array} \right. \tag{5.28}$$

Proof. Direct. Taking $\mathbf{F} = \nabla q$ for any $q \in H_0^1(\Omega)$ in (5.23) yields $-\omega^2 \left(\underline{\varepsilon}(\nabla p + \tilde{\mathbf{E}}) | \nabla q\right) = (\mathbf{f} | \nabla q)$. Because $\operatorname{div} \underline{\varepsilon} \tilde{\mathbf{E}} = 0$, one gets (5.27). Then, for $\tilde{\mathbf{E}} = \mathbf{E} - \nabla p \in \mathbf{W}_N(\underline{\varepsilon}, \Omega)$, there holds

$$\left(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \tilde{\mathbf{E}} | \mathbf{curl}\, \mathbf{F}\right) - \omega^2 \left(\underline{\varepsilon}(\nabla p + \tilde{\mathbf{E}}) | \mathbf{F}\right) - \left(\alpha \gamma^T \tilde{\mathbf{E}}, \gamma^T \mathbf{F}\right)_\Gamma = (\mathbf{f} | \mathbf{F}) - \left(\mathbf{g}, \gamma^T \mathbf{F}\right)_\Gamma$$

for any $\mathbf{F} \in \mathbf{H}^+(\mathbf{curl}, \Omega)$, hence in particular for any $\tilde{\mathbf{F}} \in \mathbf{W}_N(\underline{\varepsilon}; \Omega)$.

Reverse. We sum (5.27) and (5.28), pose $\mathbf{E} = \nabla p + \tilde{\mathbf{E}} \in \mathbf{H}^+(\mathbf{curl}, \Omega)$. Adding the null terms $\left(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \nabla p | \mathbf{curl}\, \tilde{\mathbf{F}}\right)$, $\left(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E} | \mathbf{curl}\, \nabla q\right)$, $-\omega^2 \left(\underline{\varepsilon} \tilde{\mathbf{E}} | \nabla q\right)$, $-\left(\alpha \gamma^T \nabla p, \gamma^T \tilde{\mathbf{F}}\right)_\Gamma$, $-\left(\alpha \gamma^T \mathbf{E}, \gamma^T \nabla q\right)_\Gamma$, and $\left(\mathbf{g}, \gamma^T \nabla q\right)_\Gamma$, one gets

$$\left(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E} | \mathbf{curl}(\nabla q + \tilde{\mathbf{F}})\right) - \omega^2 \left(\underline{\varepsilon} \mathbf{E} | \nabla q + \tilde{\mathbf{F}}\right) - \left(\alpha \gamma^T \mathbf{E}, \gamma^T (\nabla q + \tilde{\mathbf{F}})\right)_\Gamma = \left(\mathbf{f} | \nabla q + \tilde{\mathbf{F}}\right) - \left(\mathbf{g}, \gamma^T (\nabla q + \tilde{\mathbf{F}})\right)_\Gamma.$$

As q and $\tilde{\mathbf{F}}$ span respectively $H_0^1(\Omega)$ and $\mathbf{W}_N(\underline{\varepsilon}; \Omega)$, the sum $\nabla q + \tilde{\mathbf{F}}$ spans the whole space $\mathbf{H}^+(\mathbf{curl}, \Omega)$, according to Theorem 3.2.4. Hence (5.23) holds. □

Then, the analysis proceeds as in the previous sections. Here, we rely on the compact embedding of $\mathbf{W}_N(\underline{\varepsilon}; \Omega)$ into $\mathbf{L}^2(\Omega)$ (Theorem 3.3.5). One more difference is that one needs an additional assumption on $\underline{\boldsymbol{\mu}}$ and α to conclude.

Theorem 5.3.3. If $\underline{\boldsymbol{\mu}}^{-1}$ and $-\alpha$ are simultaneously elliptic, then the problem (5.23) enters Fredholm alternative, and

- either the problem admits a unique solution $\mathbf{E}$ in $\mathbf{H}(\mathbf{curl}, \Omega)$, which depends continuously on the data $\mathbf{f}$ and $\mathbf{g}$:
$$\|\mathbf{E}\|_{\mathbf{H}^+(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{g}\|_{\mathbf{L}^2_t(\Gamma)}; \tag{5.29}$$
- or, the problem has solutions if, and only if, $\mathbf{f}$ and $\mathbf{g}$ satisfy a finite number of compatibility conditions; in this case, the space of solutions is an affine space of finite dimension. Additionally, the component of the solution which is orthogonal (in the sense of the $\mathbf{H}^+(\mathbf{curl}, \Omega)$ inner product) to the corresponding linear vector space, depends continuously on the data $\mathbf{f}$ and $\mathbf{g}$.

Proof. We study successively the formulations (5.27) and (5.28).

1. The formulation (5.27) is clearly well-posed by Lax-Milgram lemma.
2. For the formulation (5.28),

   by assumption, we have, for $\theta \in \Theta_{\mu^{-1}} \cap \Theta_{-\alpha}$,

$$\Re[e^{i\theta}(\mathbf{v}^* \underline{\boldsymbol{\mu}}^{-1} \mathbf{v})] \geq \mu_{-,\theta}^{\text{inv}} |\mathbf{v}|^2 \quad \text{and} \quad \Re[e^{i\theta}(-\mathbf{v}^* \alpha \mathbf{v})] \geq \alpha_{-,\theta} |\mathbf{v}|^2$$

simultaneously. Then, let us introduce the sesquilinear form on $\mathbf{W}_N(\underline{\varepsilon};\Omega)$

$$a(\mathbf{u},\mathbf{v}) := \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{u}|\,\mathbf{curl}\,\mathbf{v}\right) + e^{-i\theta}\,(\mathbf{u}|\mathbf{v}) - \left(\alpha\gamma^T\mathbf{u},\gamma^T\mathbf{v}\right)_\Gamma .$$

It is continuous and coercive on $\mathbf{W}_N(\underline{\varepsilon};\Omega)$: indeed, one checks that $|a(\mathbf{u},\mathbf{v})| \lesssim \|\mathbf{u}\|_{\mathbf{H}^+(\mathbf{curl},\Omega)}\|\mathbf{v}\|_{\mathbf{H}^+(\mathbf{curl},\Omega)}$, and that

$$\begin{aligned}|a(\mathbf{v},\mathbf{v})| &\geq \Re[e^{i\theta}a(\mathbf{v},\mathbf{v})]\\ &\geq \mu^{\mathrm{inv}}_{-,\theta}\|\,\mathbf{curl}\,\mathbf{v}\|^2_{\mathbf{L}^2} + \|\mathbf{v}\|^2_{\mathbf{L}^2} + \alpha_{-,\theta}\|\gamma^T\mathbf{v}\|^2_{\mathbf{L}^2_t(\Gamma)},\\ &\geq \min(\mu^{\mathrm{inv}}_{-,\theta},1,\alpha_{-,\theta})\,\|\mathbf{v}\|^2_{\mathbf{H}^+(\mathbf{curl},\Omega)},\end{aligned}$$

with $\mathbf{W}_N(\underline{\varepsilon};\Omega)$ and $\mathbf{H}^+(\mathbf{curl},\Omega)$ sharing the same norm. Introducing the complementary form

$$b(\mathbf{u},\mathbf{v}) := -\omega^2(\underline{\varepsilon}\mathbf{u}|\mathbf{v}) + e^{-i\theta}(\mathbf{u}|\mathbf{v}), \tag{5.30}$$

it is continuous on $\mathbf{L}^2(\Omega)\times\mathbf{W}_N(\underline{\varepsilon};\Omega)$. Moreover, the embedding of $\mathbf{W}_N(\underline{\varepsilon};\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact by Theorem 3.3.5. Hence the formulation (5.28) enters the coercive + compact framework, and the conclusions of Fredholm alternative apply to problem (5.28).

Grouping both results, one has the result for problem (5.23). □

Remark 5.3.4. Note that one needs "compatibility" in some sense between $\underline{\boldsymbol{\mu}}$ and α (simultaneous ellipticity), but, on the other hand, no condition on $\underline{\varepsilon}$ is required.

Remark 5.3.5. One can proceed similarly if $\underline{\boldsymbol{\alpha}}$ is a tensor-valued coefficient, provided it allows the boundary condition to hold in $\mathbf{L}^2_t(\Gamma)$. If $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\alpha}}$ are simultaneously elliptic, then the problem (5.23) enters Fredholm alternative. The proof is the same.

In what preceeds, we have assumed interplay only between $\underline{\boldsymbol{\mu}}$ and α, in order to get the Fredholm character of the problem. Again, one can go further if one assumes interplay between all three coefficients $\underline{\varepsilon}$, $\underline{\boldsymbol{\mu}}$ and α. If $\underline{\boldsymbol{\mu}}^{-1}$, $-\underline{\varepsilon}$ and $-\alpha$ are simultaneously elliptic, i.e. if $\Theta_{\mu^{-1}}\cap\Theta_{-\varepsilon}\cap\Theta_{-\alpha}\neq\emptyset$, then there exists $\theta\in\mathbb{R}$, $\mu^{\mathrm{inv}}_{-,\theta}$, $\varepsilon_{-,\theta}$, $\alpha_{-,\theta}>0$ s.t.

$$\Re[e^{i\theta}(\mathbf{v}^*\underline{\boldsymbol{\mu}}^{-1}\mathbf{v})] \geq \mu^{\mathrm{inv}}_{-,\theta}|\mathbf{v}|^2,\quad \Re[e^{i\theta}(-\mathbf{v}^*\underline{\varepsilon}\mathbf{v})] \geq \varepsilon_{-,\theta}|\mathbf{v}|^2,\quad \Re[e^{i\theta}(-\mathbf{v}^*\alpha\mathbf{v})] \geq \alpha_{-,\theta}|\mathbf{v}|^2, \tag{5.31}$$

simultaneously. As a consequence, the bilinear form associated to the Robin problem is coercive. Indeed,

$$\begin{aligned}\Re\left[e^{i\theta}\left(\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\,\mathbf{curl}\,\mathbf{v}\right) - \omega^2\left(\underline{\varepsilon}\mathbf{v}|\mathbf{v}\right) - \left(\alpha\gamma^T\mathbf{E},\gamma^T\mathbf{F}\right)_\Gamma\right)\right] &\geq \mu^{\mathrm{inv}}_{-,\theta}\|\,\mathbf{curl}\,\mathbf{v}\|^2 + \omega^2\varepsilon_{-,\theta}\|\mathbf{v}\|^2 + \alpha_{-,\theta}\|\gamma^T\mathbf{v}\|^2_{\mathbf{L}^2_t(\Gamma)}\\ &\geq \min(\mu^{\mathrm{inv}}_{-,\theta},\,\omega^2\varepsilon_{-,\theta},\,\alpha_{-,\theta})\|\mathbf{v}\|^2_{\mathbf{H}^+(\mathbf{curl})}.\end{aligned}$$

Again, this is also valid for a tensor-valued $\underline{\boldsymbol{\alpha}}$.

Theorem 5.3.6. If $\underline{\boldsymbol{\mu}}^{-1}$, $-\underline{\varepsilon}$ and $-\alpha$ (or $-\underline{\boldsymbol{\alpha}}$) are simultaneously elliptic, i.e. $\Theta_{\mu^{-1}}\cap\Theta_{-\varepsilon}\cap\Theta_{-\alpha}\neq\emptyset$, then there exists a unique solution $\mathbf{E}\in\mathbf{H}^+(\mathbf{curl},\Omega)$ to the Robin problem (5.22), with moreover

$$\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{g}\|_{\mathbf{L}^2_t(\Gamma)}. \tag{5.32}$$

The magnetic problem

Alternatively, one could choose to solve the second-order time-harmonic magnetic problem, eliminating $\mathbf{E}$ instead of $\mathbf{H}$. The equation reads (see (2.15))

$$\mathbf{curl}\,\underline{\varepsilon}^{-1}\,\mathbf{curl}\,\mathbf{H} - \omega^2\underline{\boldsymbol{\mu}}\mathbf{H} = \tilde{\mathbf{f}}, \tag{5.33}$$

with $\tilde{\mathbf{f}} = \mathbf{curl}\,\underline{\varepsilon}^{-1}\mathbf{J}$. In other words, the roles of $\underline{\varepsilon}$ and $\underline{\boldsymbol{\mu}}$ are permuted compared to the electric problem. Moreover, the Dirichlet and Neumann conditions are also permuted: a Neumann condition for the electric field is a Dirichlet condition for the magnetic field, and vice-versa. Therefore, the Dirichlet and Neumann magnetic problems are well-posed under the same conditions than the electric ones.

Considering the Robin condition, expressing it in terms of the magnetic field, one gets

$$\pi^T(\underline{\varepsilon}^{-1}\,\mathbf{curl}\,\mathbf{H}) - \alpha^{-1}\omega^2\gamma^T\mathbf{H} = \tilde{\mathbf{g}}, \tag{5.34}$$

where $\tilde{\mathbf{g}} = \pi^T(\underline{\varepsilon}^{-1}\mathbf{J}) + i\omega\alpha^{-1}(\mathbf{g}\times\mathbf{n})$.

Remark 5.3.7. Here, we specifically assume that the impedance coefficient is scalar-valued. When it is tensor-valued, the computations are a bit more complex.

The variational formulation of the problem (5.33)-(5.34) reads:

$$\left|\begin{array}{l}\text{Find } \mathbf{H}\in\mathbf{H}^+(\mathbf{curl},\Omega) \text{ s.t., } \forall\mathbf{F}\in\mathbf{H}^+(\mathbf{curl},\Omega),\\ \left(\underline{\varepsilon}^{-1}\,\mathbf{curl}\,\mathbf{H}|\,\mathbf{curl}\,\mathbf{F}\right) - \omega^2\left(\underline{\boldsymbol{\mu}}\mathbf{H}|\mathbf{F}\right) + \left(\omega^2\alpha^{-1}\gamma^T\mathbf{H},\gamma^T\mathbf{F}\right)_\Gamma = \left(\tilde{\mathbf{f}}|\mathbf{F}\right) - \left(\tilde{\mathbf{g}},\gamma^T\mathbf{F}\right)_\Gamma.\end{array}\right. \tag{5.35}$$

Then, one can notice a curious thing. To get the Fredholm character of problem (5.35), one needs apparently a different condition than for the electric problem (5.23), although both problems are supposed to be equivalent. In particular, the coefficients involved are not the same: here, one requires simultaneous ellipticity between $\underline{\varepsilon}^{-1}$ and α^{-1}, independently of $\underline{\boldsymbol{\mu}}$, while for the electric problem we required simultaneous ellipticity between $\underline{\boldsymbol{\mu}}^{-1}$ and $-\alpha$, independently of $\underline{\varepsilon}$.

Besides, in order to get the coercivity of the magnetic problem (5.35), one has to assume interplay between all three coefficients: namely, that $\underline{\varepsilon}^{-1}$, $-\underline{\boldsymbol{\mu}}$ and α^{-1} are simultaneously elliptic (i.e. $\Theta_{\varepsilon^{-1}}\cap\Theta_{-\mu}\cap\Theta_{\alpha^{-1}}\neq\emptyset$). Compare to the electric problem: to go from the electric to the magnetic problem boils down to change the triplet of coefficients $(\underline{\boldsymbol{\mu}}^{-1}, -\underline{\varepsilon}, -\alpha)$ into $(\underline{\varepsilon}^{-1}, -\underline{\boldsymbol{\mu}}, \alpha^{-1})$, that is, all coefficients are changed by the transformation $\xi\mapsto-\xi^{-1}$. Thus, note that both conditions $\Theta_{\mu^{-1}}\cap\Theta_{-\varepsilon}\cap\Theta_{-\alpha}\neq\emptyset$ and $\Theta_{\varepsilon^{-1}}\cap\Theta_{-\mu}\cap\Theta_{\alpha^{-1}}\neq\emptyset$ are equivalent. Therefore, both electric and magnetic problems are coercive under the same conditions.

Conclusion

We have studied time-harmonic Maxwell problems for a wide class of material tensors: we only assume that $\underline{\varepsilon}$ and $\underline{\boldsymbol{\mu}}$ are elliptic (Definition 3.1.1). In this context, we have derived and studied $\mathbf{H}(\mathbf{curl},\Omega)$-conforming variational formulations for the time-harmonic Maxwell problem with the three main types of boundary conditions: Dirichlet (Theorem 5.1.7), Neumann (Theorem 5.2.3), and Robin (Theorem 5.3.3).

We prove that well-posedness in Fredholm sense holds in very general contexts. However, we do not provide explicit compatibility conditions to ensure that the problems have unique solutions. To conclude, one should establish the uniqueness of the solution. This is usually done thanks to a unique continuation principle. However, up to our knowledge, it seems that the unique continuation principle has been established only when $\underline{\varepsilon}, \underline{\boldsymbol{\mu}}$ are real symmetric [113, 90]. We also refer to [9]. Alternatively, one can go further and prove the coercivity of the formulations assuming moreover the simultaneous ellipticity of the coefficients. This is the case in some physical contexts such as plasma, see for example [8]. In this case, the problem is truly well-posed.

One could also consider the case of mixed boundary conditions, for example Dirichlet on one part of the boundary and Neumann on the other. We refer to the work of Fernandes and Gilardi [55] who treated it in the case of real symmetric material tensors.

Chapter 6

Analysis of the regularity of electromagnetic fields

In this chapter, we study the regularity of the solutions of the problems presented in the previous chapter, i.e. the time-harmonic Maxwell equation completed with Dirichlet, Neumann, or Robin boundary condition. We focus on the regularity in the Sobolev scale, in the cases where the data are more regular than assumed in Chapter 5. We are interested in the regularity of both the solution itself and of its curl. This is motivated by numerical analysis considerations: indeed, when discretizing the variational formulations with $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming edge finite elements, the order of convergence of the method is driven by both the regularity of the solution itself and of its curl. We specifically assume throughout this chapter that Γ is of class $\mathcal{C}^2$. As in the previous chapter, $\underline{\boldsymbol{\varepsilon}}$, $\underline{\boldsymbol{\mu}}$ shall be elliptic tensors, with ellipticity directions that may be different. Additional assumptions on the regularity of $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ will be made in the different sections.

The study of the regularity of electromagnetic fields originates in the work of Birman and Solomyak [12]. The regularity of electromagnetic fields has been studied in [37] for piecewise constant isotropic media, in [77], [14] and [25] for piecewise smooth symmetric definite positive tensors. A few works address the case of non-Hermitian tensors. For tensors with elliptic real part, $\mathbf{H}^1(\Omega)$-regularity [2] and Hölder regularity [1] results have been established. Very recently, similar results have been obtained in [112] for real elliptic tensors. All these works focus on the Dirichlet problem.

The analysis relies mainly on two elements: decompositions into regular and singular parts, and regularity results from the theory of elliptic problems. For the Robin problem, it also relies on the considerations of Chapter 4. We treat successively the three boundary conditions: the Dirichlet problem is addressed in Section 6.1, Neumann in Section 6.2 and Robin in Section 6.3. In Section 6.4, we provide elements of numerical analysis and a numerical illustration.

6.1 Regularity in the Dirichlet problem

Let us recall that the Dirichlet problem reads:

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{E}\times\mathbf{n} = \mathbf{g} & \text{on } \Gamma, \end{cases} \tag{6.1}$$

where we assume, as in Section 5.1, that $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. Therefore, $\mathbf{g}$ admits a lifting $\mathbf{E}_d \in \mathbf{H}(\mathbf{curl}, \Omega)$ s.t. $\mathbf{g} = \mathbf{E}_d \times \mathbf{n}$ on Γ, and one can rewrite the problem with unknown $\mathbf{E}_0 := \mathbf{E} - \mathbf{E}_d \in \mathbf{H}_0(\mathbf{curl}, \Omega)$.

Let us make here some further assumptions. In this section, we specifically assume that $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\mu}} \in \underline{\boldsymbol{\mathcal{C}}}^1(\overline{\Omega})$. Concerning the data, we assume that $\mathbf{E}_d \in \mathbf{H}^r(\Omega)$ and $\mathbf{curl}\,\mathbf{E}_d \in \mathbf{H}^{r'}(\Omega)$, and that $\mathrm{div}\,\mathbf{f} \in H^{s-1}(\Omega)$, for given r, r', s in $[0,1] \setminus \{\frac{1}{2}\}$. The lowest-regular case, $r = r' = s = 0$, corresponds to the hypotheses of Section 5.1. The highest-regular case corresponds to $\mathbf{E}_d, \mathbf{curl}\,\mathbf{E}_d \in \mathbf{H}^1(\Omega)$, and $\mathbf{f} \in \mathbf{H}(\mathrm{div}, \Omega)$. Then, we investigate the extra-regularity of $\mathbf{E}$ and $\mathbf{curl}\,\mathbf{E}$, depending on r, r' and s.

6.1.1 Extra-regularity of the solution

We focus first on the regularity of the solution itself. Let us recall a theorem of continuous splitting fields of $\mathbf{H}_0(\mathbf{curl}, \Omega)$ into a regular and a singular part [69, Lemma 2.4] (see also [7, Th. 3.6.7]):

Theorem 6.1.1. Let $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$. There exists $\mathbf{u}^{\text{reg}} \in \mathbf{H}^1(\Omega)$ and $\phi \in H^1_0(\Omega)$, s.t.

$$\mathbf{u} = \mathbf{u}^{\text{reg}} + \nabla\phi \quad \text{ in } \Omega; \tag{6.2}$$

additionally, there exists a constant $C > 0$, independent of $\mathbf{u}$, s.t.

$$\|\mathbf{u}^{\text{reg}}\|_{\mathbf{H}^1} + \|\phi\|_{H^1_0} \leq C\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})}. \tag{6.3}$$

Remark 6.1.2. In fact, one can show that $\mathbf{u}^{\text{reg}} \in \mathbf{H}^1_0(\Omega)$.

Remark 6.1.3. The result holds for any domain of Lipschitz boundary.

With Theorem 6.1.1 at hand, let us introduce the splitting of $\mathbf{E}_0$:

$$\mathbf{E}_0 = \mathbf{E}^{\text{reg}} + \nabla\phi_E \tag{6.4}$$

where $\mathbf{E}^{\text{reg}} \in \mathbf{H}^1(\Omega)$, $\phi_E \in H^1_0(\Omega)$, and

$$\|\mathbf{E}^{\text{reg}}\|_{\mathbf{H}^1} + \|\phi_E\|_{H^1_0} \lesssim \|\mathbf{E}_0\|_{\mathbf{H}(\mathbf{curl})}.^{1} \tag{6.5}$$

Therefore, the regularity of $\mathbf{E}$ is driven by the regularity of its singular part, $\nabla\phi_E$. In the following, our aim is thus to study the extra-regularity of the field ϕ_E. We recall that $\mathbf{E}_0$ satisfies (taking $\mathbf{F} = \nabla q$ in (5.3)), $\forall q \in H^1_0(\Omega)$, $-\omega^2 \left(\underline{\boldsymbol{\varepsilon}}\mathbf{E}_0|\nabla q\right) = \left(\mathbf{f} + \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E}_d|\nabla q\right)$. Then, ϕ_E is governed by the following Dirichlet problem:

$$\left|\begin{array}{l}\text{Find } \phi_E \in H^1_0(\Omega) \text{ s.t., } \forall q \in H^1_0(\Omega),\\ \omega^2 \left(\underline{\boldsymbol{\varepsilon}}\nabla\phi_E|\nabla q\right) = \left(\operatorname{div}\mathbf{f} + \omega^2 \operatorname{div}\underline{\boldsymbol{\varepsilon}}\mathbf{E}_d + \omega^2 \operatorname{div}\underline{\boldsymbol{\varepsilon}}\mathbf{E}^{\text{reg}}|q\right).\end{array}\right. \tag{6.6}$$

The regularity of the solutions of elliptic problems has been widely studied. Here, we rely on the result of [38, Th. 3.4.5] for Dirichlet problems, recalled hereafter. For problems set in a domain with a smooth boundary, with a smooth coefficient, one generally has a shift of 2 from the regularity of the data to the regularity solution; that is why they are sometimes referred to as *(regularity) shift results.*

Theorem 6.1.4 (Shift Theorem)**.** Let Ω be a bounded domain of boundary Γ, ℓ in $\left(H^1_0(\Omega)\right)'$, and p governed by

$$\left|\begin{array}{l}\text{Find } p \in H^1_0(\Omega) \text{ s.t., } \forall q \in H^1_0(\Omega),\\ \qquad\qquad \left(\underline{\boldsymbol{\xi}}\nabla p|\nabla q\right) = \ell(q).\end{array}\right. \tag{6.7}$$

If the tensor coefficient $\underline{\boldsymbol{\xi}}$ is elliptic, then the problem (6.7) is well-posed; if additionally $\underline{\boldsymbol{\xi}} \in \underline{\boldsymbol{\mathcal{C}}}^1(\overline{\Omega})$ and Γ is of class $\mathcal{C}^2$, then, $\forall\sigma \in [1, 2] \setminus \{\frac{3}{2}\}$,

$$\ell \in \left(H^{2-\sigma}_0(\Omega)\right)' \implies p \in H^\sigma(\Omega); \tag{6.8}$$

additionally, there exists $C_\sigma > 0$ s.t., $\forall\ell \in \left(H^{2-\sigma}_0(\Omega)\right)'$,

$$\|p\|_{H^\sigma} \leq C_\sigma\|\ell\|_{\left(H^{2-\sigma}_0\right)'}. \tag{6.9}$$

Consequently, one can derive the following regularity result for the field $\mathbf{E}$.

[1] As in the previous chapter, the notation $a \lesssim b$ denotes that there exists a constant $C > 0$, *independent of* a *and* b, s.t. $a \leq Cb$. The constant C depends only on the geometry, the frequency ω, and the coefficients $\underline{\boldsymbol{\mu}}$, $\underline{\boldsymbol{\varepsilon}}$ (and α).

Theorem 6.1.5. Let $\mathbf{E}$ governed by (6.1), split as $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_d$ with $\mathbf{E}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$. If Γ is of class $\mathcal{C}^2$, $\underline{\varepsilon} \in \underline{\mathcal{C}}^1(\overline{\Omega})$, $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\operatorname{div} \mathbf{f} \in H^{s-1}(\Omega)$, and $\mathbf{E}_d \in \mathbf{H}^r(\Omega)$, with r, s in $[0,1] \setminus \{\frac{1}{2}\}$, then

$$\mathbf{E} \in \mathbf{H}^{\min(s,r)}(\Omega), \tag{6.10}$$

with, moreover,

$$\|\mathbf{E}\|_{\mathbf{H}^{\min(s,r)}} \lesssim \|\mathbf{E}_0\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{div} \mathbf{f}\|_{H^{s-1}} + \|\mathbf{E}_d\|_{\mathbf{H}^r}. \tag{6.11}$$

Remark 6.1.6. No regularity assumption on $\underline{\mu}$ is required here.

Proof. We split $\mathbf{E}_0$ as in (6.4). Let us apply Theorem 6.1.4 to the problem (6.6) governing ϕ_E. We introduce the form ℓ defined by the right-hand side,

$$\ell : q \mapsto \left(\operatorname{div} \mathbf{f} + \omega^2 \operatorname{div} \underline{\varepsilon} \mathbf{E}_d + \omega^2 \operatorname{div} \underline{\varepsilon} \mathbf{E}^{\mathrm{reg}} | q\right).$$

Consider each term: one has $\operatorname{div} \mathbf{f} \in H^{s-1}(\Omega) = \left(H_0^{1-s}(\Omega)\right)'$; as $\underline{\varepsilon} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$, there holds $\underline{\varepsilon}\mathbf{E}^{\mathrm{reg}} \in \mathbf{H}^1(\Omega)$ and $\operatorname{div} \underline{\varepsilon}\mathbf{E}^{\mathrm{reg}} \in L^2(\Omega)$; similarly, $\underline{\varepsilon}\mathbf{E}_d \in \mathbf{H}^r(\Omega)$, so $\operatorname{div} \underline{\varepsilon}\mathbf{E}_d \in H^{r-1}(\Omega) = \left(H_0^{1-r}(\Omega)\right)'$, because $r \neq \frac{1}{2}$. It follows that ℓ defines a continuous form on $H_0^{\max(1-s,1-r)}(\Omega)$. In other words, $\ell \in H^{\min(s-1,r-1)}(\Omega) = \left(H_0^{\max(1-s,1-r)}(\Omega)\right)'$. In addition, one has the bound

$$\begin{aligned}
\|\ell\|_{H^{\min(s,r)-1}} &\lesssim \|\operatorname{div} \mathbf{f}\|_{H^{s-1}} + \|\operatorname{div} \underline{\varepsilon}\mathbf{E}_d\|_{H^{r-1}} + \|\operatorname{div} \underline{\varepsilon}\mathbf{E}^{\mathrm{reg}}\|_{\mathbf{L}^2} \\
&\lesssim \|\operatorname{div} \mathbf{f}\|_{H^{s-1}} + \|\mathbf{E}_d\|_{\mathbf{H}^r} + \|\mathbf{E}^{\mathrm{reg}}\|_{\mathbf{H}^1} \\
&\lesssim \|\operatorname{div} \mathbf{f}\|_{H^{s-1}} + \|\mathbf{E}_d\|_{\mathbf{H}^r} + \|\mathbf{E}_0\|_{\mathbf{H}(\mathbf{curl})},
\end{aligned}$$

the latter because of (6.5). We conclude by the Shift Theorem 6.1.4 that $\phi_E \in H^{\min(s,r)+1}(\Omega)$, with

$$\|\phi_E\|_{H^{\min(s,r)+1}} \lesssim \|\operatorname{div} \mathbf{f}\|_{H^{s-1}} + \|\mathbf{E}_d\|_{\mathbf{H}^r} + \|\mathbf{E}_0\|_{\mathbf{H}(\mathbf{curl})}.$$

Hence $\mathbf{E} = \mathbf{E}^{\mathrm{reg}} + \nabla\phi_E + \mathbf{E}_d \in \mathbf{H}^{\min(s,r)}(\Omega)$, with the bound

$$\begin{aligned}
\|\mathbf{E}\|_{\mathbf{H}^{\min(s,r)}} &\lesssim \|\mathbf{E}^{\mathrm{reg}}\|_{\mathbf{H}^1} + \|\nabla\phi_E\|_{\mathbf{H}^{\min(s,r)}} + \|\mathbf{E}_d\|_{\mathbf{H}^r} \\
&\lesssim \|\mathbf{E}_0\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{div} \mathbf{f}\|_{H^{s-1}} + \|\mathbf{E}_d\|_{\mathbf{H}^r},
\end{aligned}$$

where we used (6.5) and the bound on $\|\phi_E\|_{H^{\min(s,r)+1}}$ to conclude. ☐

Assuming further that the Dirichlet problem is well-posed, one finds that $\mathbf{E}$ in its extra-regularity norm depends continuously on the data in their appropriate regularity norms only.

Corollary 6.1.7. If additionally the problem (5.3) is well-posed, then

$$\|\mathbf{E}\|_{\mathbf{H}^{\min(s,r)}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\operatorname{div} \mathbf{f}\|_{H^{s-1}} + \|\mathbf{E}_d\|_{\mathbf{H}^r} + \|\mathbf{curl}\, \mathbf{E}_d\|_{\mathbf{L}^2}. \tag{6.12}$$

Proof. It results of combining Theorem 6.1.5 with the bound of Theorem 5.1.7. ☐

6.1.2 Extra-regularity of the solution's curl

To study the regularity of the solution's curl, we rather focus on the field $\mathbf{C} := \underline{\mu}^{-1} \mathbf{curl}\, \mathbf{E} \in \mathbf{L}^2(\Omega)$. One can note that, from a physical point of view, this field is very much related to the magnetic field. Indeed, in the time-harmonic regime, one has $\underline{\mu}^{-1} \mathbf{curl}\, \mathbf{E} = i\omega\mathbf{H}$, where $\mathbf{H}$ is the magnetic field. Moreover, one has $\mathbf{curl}\, \mathbf{C} = \mathbf{f} + \omega^2 \underline{\varepsilon}\mathbf{E} \in \mathbf{L}^2(\Omega)$, hence $\mathbf{C} \in \mathbf{H}(\mathbf{curl}, \Omega)$ with the bound

$$\|\mathbf{C}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{L}^2}, \tag{6.13}$$

because $\underline{\mu}$ is elliptic.

The ingredients introduced in the previous subsection will have their counterparts in this one. First, one has a theorem analogous to Theorem 6.1.1, of continuous splitting into a regular part and a singular part, for fields of $\mathbf{H}(\mathbf{curl}, \Omega)$ [7, Th. 3.6.7] (for a similar result, see also [69, Lemma 2.4]):

Theorem 6.1.8. Let Ω be a domain of the $\mathfrak{A}$-type (see Definition 2.2.4), and $\mathbf{u}$ in $\mathbf{H}(\mathbf{curl}, \Omega)$. There exists $\mathbf{u}^{\mathrm{reg}}$ in $\mathbf{H}^1(\Omega)$ and ϕ in $H^1_{\mathrm{zmv}}(\Omega)$, s.t.

$$\mathbf{u} = \mathbf{u}^{\mathrm{reg}} + \nabla\phi \quad \text{in } \Omega, \tag{6.14}$$

with additionally $\mathbf{u}^{\mathrm{reg}} \cdot \mathbf{n} = 0$ on Γ, and there exists a constant $C > 0$, independent of $\mathbf{u}$, s.t.

$$\|\mathbf{u}^{\mathrm{reg}}\|_{\mathbf{H}^1} + \|\phi\|_{H^1_{\mathrm{zmv}}} \leq C\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl})}. \tag{6.15}$$

Remark 6.1.9. As Ω is a domain with boundary of class $\mathcal{C}^2$, it is automatically of the $\mathfrak{A}$-type.

Then, let us introduce the splitting of $\mathbf{C}$ by Theorem 6.1.8:

$$\mathbf{C} = \mathbf{C}^{\mathrm{reg}} + \nabla\phi_C, \tag{6.16}$$

where $\mathbf{C}^{\mathrm{reg}} \in \mathbf{H}^1(\Omega)$, $\phi_C \in H^1_{\mathrm{zmv}}(\Omega)$, $\underline{\boldsymbol{\mu}}\mathbf{C}^{\mathrm{reg}} \cdot \mathbf{n} = 0$ and

$$\|\mathbf{C}^{\mathrm{reg}}\|_{\mathbf{H}^1} + \|\phi_C\|_{H^1_{\mathrm{zmv}}} \lesssim \|\mathbf{C}\|_{\mathbf{H}(\mathbf{curl})}. \tag{6.17}$$

Observing that $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_d$ with $\mathbf{E}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, there holds, for all q in $H^1_{\mathrm{zmv}}(\Omega)$, $\left(\underline{\boldsymbol{\mu}}\mathbf{C}|\nabla q\right) = (\mathbf{curl}\,\mathbf{E}|\nabla q) = (\mathbf{curl}\,\mathbf{E}_d|\nabla q)$. Thus, ϕ_C is governed by the following Neumann problem:

$$\left|\begin{array}{l}\text{Find } \phi_C \in H^1_{\mathrm{zmv}}(\Omega) \text{ s.t., } \forall q \in H^1_{\mathrm{zmv}}(\Omega),\\ \left(\underline{\boldsymbol{\mu}}\nabla\phi_C|\nabla q\right) = \left(\mathbf{curl}\,\mathbf{E}_d - \underline{\boldsymbol{\mu}}\mathbf{C}^{\mathrm{reg}}|\nabla q\right).\end{array}\right. \tag{6.18}$$

Once again, one may determine the regularity of ϕ_C by the means of the shift regularity results for elliptic problems in smooth domains with smooth coefficient, here in the Neumann case. We rely on the result of [38, Th. 3.4.5], recalled hereafter:

Theorem 6.1.10 (Shift Theorem)**.** Let Ω be a bounded domain of boundary Γ, ℓ in $\left(H^1_{\mathrm{zmv}}(\Omega)\right)'$, and p governed by

$$\left|\begin{array}{l}\text{Find } p \in H^1_{\mathrm{zmv}}(\Omega) \text{ s.t., } \forall q \in H^1_{\mathrm{zmv}}(\Omega),\\ \qquad\qquad \left(\underline{\boldsymbol{\xi}}\nabla p|\nabla q\right) = \ell(q).\end{array}\right. \tag{6.19}$$

If the tensor coefficient $\underline{\boldsymbol{\xi}}$ is elliptic, then the problem (6.19) is well-posed; if additionally $\underline{\boldsymbol{\xi}} \in \underline{\boldsymbol{\mathcal{C}}}^1(\overline{\Omega})$ and Γ is of class $\mathcal{C}^2$, then,

(i) $\forall\sigma \in [1, \frac{3}{2}[$,

$$\ell \in \left(H^{2-\sigma}_{\mathrm{zmv}}(\Omega)\right)' \implies p \in H^\sigma(\Omega), \tag{6.20}$$

and there exists $C_\sigma > 0$ s.t., $\forall\ell \in \left(H^{2-\sigma}_{\mathrm{zmv}}(\Omega)\right)'$,

$$\|p\|_{H^\sigma} \leq C_\sigma\|\ell\|_{\left(H^{2-\sigma}_{\mathrm{zmv}}\right)'}. \tag{6.21}$$

(ii) If there exists $\sigma \in]\frac{3}{2}, 2]$ s.t. ℓ writes $\ell(q) = (f|q) + \langle g, q\rangle_{H^{1/2}(\Gamma)}$, with $f \in L^2(\Omega)$ and $g \in H^{\sigma-3/2}(\Gamma)$, then

$$p \in H^\sigma(\Omega) \tag{6.22}$$

and there exists $C_\sigma > 0$, independent of f and g, s.t.

$$\|p\|_{H^\sigma} \leq C_\sigma\left(\|f\|_{L^2} + \|g\|_{H^{\sigma-3/2}(\Gamma)}\right). \tag{6.23}$$

Remark 6.1.11. This theorem is slightly more subtle than its equivalent for the Dirichlet problem, Theorem 6.1.4. Indeed, in the case (i), the theorem can be understood in a variational manner, just as in Theorem 6.1.4. In fact, one can note that $\mathbf{H}^{\sigma-1}(\Omega)$ identifies with $\mathbf{H}^{\sigma-1}_0(\Omega)$. However, because of this, the normal derivative $\nabla p \cdot \mathbf{n}_{|\Gamma}$ will be meaningless in general. So, the boundary value problem solved by p is unclear.

On the other hand, in the case (ii), the normal derivative $\nabla p \cdot \mathbf{n}_{|\Gamma}$ is meaningful in $H^{\sigma-3/2}(\Gamma)$ (because $\nabla p \in \mathbf{H}^{\sigma-1}(\Omega)$ and $\sigma - 1 > 1/2$). Then, p solves the boundary value problem

$$\begin{cases} -\operatorname{div} \underline{\xi} \nabla p = f & \text{in } \Omega, \\ \nabla p \cdot \mathbf{n} = g & \text{on } \Gamma. \end{cases} \tag{6.24}$$

In this case, the proof of the result relies on local analysis arguments; see [38] for details. Moreover, the result (ii) also holds if one only assumes $f \in H^{\sigma-2}(\Omega)$.

Applying this result to problem (6.18), one gets the regularity of ϕ_C, then deduces the regularity of $\mathbf{C}$ and $\mathbf{curl}\,\mathbf{E}$.

Theorem 6.1.12. Let $\mathbf{E}$ governed by (6.1), split as $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_d$ with $\mathbf{E}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$. If Γ is of class $\mathcal{C}^2$, $\underline{\boldsymbol{\mu}} \in \underline{\mathcal{C}}^1(\overline{\Omega})$, and $\mathbf{curl}\,\mathbf{E}_d \in \mathbf{H}^{r'}(\Omega)$ with r' in $[0, 1] \setminus \{\frac{1}{2}\}$, then

$$\mathbf{curl}\,\mathbf{E} \in \mathbf{H}^{r'}(\Omega), \tag{6.25}$$

with

$$\|\,\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{r'}} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{L}^2} + \|\,\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'}}. \tag{6.26}$$

Remark 6.1.13. No regularity assumption on $\underline{\boldsymbol{\varepsilon}}$ (other than $\underline{\boldsymbol{\varepsilon}} \in \underline{\mathbf{L}}^\infty(\Omega)$) is required here.

Proof. We want to apply Theorem 6.1.10 to the problem (6.18) governing ϕ_C. Let us introduce the antilinear continuous form on $H^1_{\text{zmv}}(\Omega)$ defined by the right-hand side,

$$\ell : q \mapsto \left(\mathbf{curl}\,\mathbf{E}_d - \underline{\boldsymbol{\mu}}\mathbf{C}^{\text{reg}} | \nabla q\right).$$

To determine the regularity of ϕ_C, one wants to determine whether the form ℓ belongs to $\left(H^{2-\sigma}_{\text{zmv}}(\Omega)\right)'$ for $\sigma \in [1, \frac{3}{2}[$ as large as possible, in order to apply Theorem 6.1.10 (i).

If $r' < \frac{1}{2}$, then $\mathbf{H}^{r'}(\Omega)$ identifies with $\mathbf{H}^{r'}_0(\Omega)$, the dual space of $\mathbf{H}^{-r'}(\Omega)$. Hence the product $(\mathbf{curl}\,\mathbf{E}_d | \nabla q)$ is meaningful as soon as $q \in H^{1-r'}_{\text{zmv}}(\Omega)$, because $\nabla q \in \mathbf{H}^{-r'}(\Omega)$; the same holds for the term $\left(\underline{\boldsymbol{\mu}}\mathbf{C}^{\text{reg}} | \nabla q\right)$. This means that ℓ belongs to $\left(H^{1-r'}_{\text{zmv}}(\Omega)\right)'$, and the Shift Theorem 6.1.10 (i), with $\sigma = 1 + r'$, ensures that $\phi_C \in H^{1+r'}(\Omega)$, with the bound

$$\|\phi_C\|_{H^{1+r'}} \lesssim \|\,\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'}} + \|\underline{\boldsymbol{\mu}}\mathbf{C}^{\text{reg}}\|_{\mathbf{H}^1} \lesssim \|\,\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'}} + \|\mathbf{C}^{\text{reg}}\|_{\mathbf{H}^1}.$$

On the other hand, if $r' > \frac{1}{2}$, then $\mathbf{H}^{r'}_0(\Omega)$ does not identify with $\mathbf{H}^{r'}(\Omega)$, and, as soon as $\mathbf{curl}\,\mathbf{E}_d \cdot \mathbf{n}_{|\Gamma} \neq 0$, the product $(\mathbf{curl}\,\mathbf{E}_d | \nabla q)$ can be meaningless if one has only $q \in H^{1-r'}_{\text{zmv}}(\Omega)$ (see Remark 6.1.11). However, $\mathbf{curl}\,\mathbf{E}_d \cdot \mathbf{n}_{|\Gamma}$ now makes sense in $H^{r'-1/2}(\Gamma)$, and, as $\underline{\boldsymbol{\mu}}\mathbf{C}^{\text{reg}} \cdot \mathbf{n} = 0$, ℓ rewrites by integrations by parts

$$\ell(q) = \left(\operatorname{div} \underline{\boldsymbol{\mu}}\mathbf{C}^{\text{reg}} | q\right) + \langle \mathbf{curl}\,\mathbf{E}_d \cdot \mathbf{n}, q\rangle_{H^{1/2}(\Gamma)}.$$

Here, the limiting regularity is not the volume one, as $\operatorname{div} \underline{\boldsymbol{\mu}}\mathbf{C}^{\text{reg}} \in L^2(\Omega)$, but the one of the boundary data: $\mathbf{curl}\,\mathbf{E}_d \cdot \mathbf{n}_{|\Gamma} \in H^{r'-1/2}(\Gamma)$. Hence, ℓ satisfies the assumptions of the Shift Theorem 6.1.10 (ii) with $\sigma = r' + 1$, and we conclude that $\phi_C \in H^{1+r'}(\Omega)$, with the bound

$$\begin{aligned} \|\phi_C\|_{H^{1+r'}} &\lesssim \|\,\mathbf{curl}\,\mathbf{E}_d \cdot \mathbf{n}\|_{H^{r'-1/2}(\Gamma)} + \|\operatorname{div} \underline{\boldsymbol{\mu}}\mathbf{C}^{\text{reg}}\|_{\mathbf{L}^2} \\ &\lesssim \|\,\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'-1/2}(\Gamma)} + \|\operatorname{div} \underline{\boldsymbol{\mu}}\mathbf{C}^{\text{reg}}\|_{\mathbf{L}^2} \\ &\lesssim \|\,\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'}} + \|\mathbf{C}^{\text{reg}}\|_{\mathbf{H}^1}. \end{aligned}$$

Finally, $\phi_C \in H^{1+r'}(\Omega)$ in all the considered cases (and with the same upper bound), so that $\mathbf{C} = \mathbf{C}^{\text{reg}} + \nabla\phi_C \in \mathbf{H}^{r'}(\Omega)$, with the bound

$$\begin{aligned} \|\mathbf{C}\|_{\mathbf{H}^{r'}} &\lesssim \|\mathbf{C}^{\text{reg}}\|_{\mathbf{H}^1} + \|\nabla\phi_C\|_{\mathbf{H}^{r'}} \\ &\lesssim \|\mathbf{C}^{\text{reg}}\|_{\mathbf{H}^1} + \|\,\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'}} \\ &\lesssim \|\mathbf{C}\|_{\mathbf{H}(\mathbf{curl})} + \|\,\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'}}, \end{aligned}$$

the latter because of (6.17). As $\mathbf{curl}\,\mathbf{E} = \underline{\boldsymbol{\mu}}\mathbf{C}$ and $\underline{\boldsymbol{\mu}} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$, one has also $\mathbf{curl}\,\mathbf{E} \in \mathbf{H}^{r'}(\Omega)$, with moreover $\|\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{r'}} \lesssim \|\mathbf{C}\|_{\mathbf{H}^{r'}}$. With the help of (6.13), one finally gets that

$$\begin{aligned} \|\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{r'}} &\lesssim \|\mathbf{C}\|_{\mathbf{H}^{r'}} \\ &\lesssim \|\mathbf{C}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'}} \\ &\lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'}} \end{aligned}$$

which concludes the proof. □

Just as in the first subsection, one can go further if the Dirichlet problem is additionally well-posed. It ensures the continuous dependence of the regularity of $\mathbf{curl}\,\mathbf{E}$ w.r.t. the regularity of the data.

Corollary 6.1.14. If additionally the problem (5.3) is well-posed, then

$$\|\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{r'}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{E}_d\|_{\mathbf{L}^2} + \|\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'}}. \tag{6.27}$$

Proof. In the bound of Theorem 6.1.12, one recalls that $\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{E}_0\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{E}_d\|_{\mathbf{H}(\mathbf{curl})}$, then concludes using the bound of Theorem 5.1.7. □

Summary

The last theorem sums up the regularity results of this section.

Theorem 6.1.15. Let $\mathbf{E}$ governed by (6.1), split as $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_d$ with $\mathbf{E}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$. If Γ is of class $\mathcal{C}^2$, $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\mu}} \in \underline{\boldsymbol{\mathcal{C}}}^1(\overline{\Omega})$, $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\operatorname{div}\mathbf{f} \in H^{s-1}(\Omega)$, $\mathbf{E}_d \in \mathbf{H}^r(\Omega)$, and $\mathbf{curl}\,\mathbf{E}_d \in \mathbf{H}^{r'}(\Omega)$ with r, r', s in $[0,1] \setminus \{\frac{1}{2}\}$, then

$$\mathbf{E} \in \mathbf{H}^{\min(s,r)}(\Omega) \quad \text{and} \quad \mathbf{curl}\,\mathbf{E} \in \mathbf{H}^{r'}(\Omega). \tag{6.28}$$

Additionally, if the problem is well-posed,

$$\|\mathbf{E}\|_{\mathbf{H}^{\min(s,r)}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\operatorname{div}\mathbf{f}\|_{H^{s-1}} + \|\mathbf{E}_d\|_{\mathbf{H}^r} + \|\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{L}^2}; \tag{6.29}$$

$$\|\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{r'}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{E}_d\|_{\mathbf{L}^2} + \|\mathbf{curl}\,\mathbf{E}_d\|_{\mathbf{H}^{r'}}. \tag{6.30}$$

In the lower regularity case, one recovers the stability estimate (5.9). In the higher regularity case, one finds that $\mathbf{E}, \mathbf{curl}\,\mathbf{E} \in \mathbf{H}^1(\Omega)$, with continuous dependence w.r.t. $\mathbf{f} \in \mathbf{H}(\operatorname{div}, \Omega)$, $\mathbf{E}_d \in \mathbf{H}^1(\Omega)$ and $\mathbf{curl}\,\mathbf{E}_d \in \mathbf{H}^1(\Omega)$.

6.2 Regularity in the Neumann problem

Let us recall the Neumann problem:

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E} = \mathbf{f} & \text{in } \Omega, \\ \underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E} \times \mathbf{n} = \mathbf{j} & \text{on } \Gamma, \end{cases} \tag{6.31}$$

where we assume, as in Section 5.2, $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{j} \in \mathbf{H}_{\parallel}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$. Then, $\mathbf{j}$ is the tangential trace of a field $\mathbf{C}_d \in \mathbf{H}(\mathbf{curl}, \Omega)$, i.e. $\mathbf{j} = \mathbf{C}_d \times \mathbf{n}$ on Γ, with $\|\mathbf{j}\|_\gamma \lesssim \|\mathbf{C}_d\|_{\mathbf{H}(\mathbf{curl})}$.

Let us make some further assumptions. In this section, we assume that $\underline{\boldsymbol{\varepsilon}} \in \underline{\mathbf{W}}^{1,\infty}$ and $\underline{\boldsymbol{\mu}} \in \underline{\boldsymbol{\mathcal{C}}}^1(\overline{\Omega})$. For the data, we assume that $\mathbf{f} \in \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}^s(\Omega)$ for a given $s \in [0,1]$, and that $\mathbf{C}_d \in \mathbf{H}^{r'}(\Omega)$ and is s.t. $\mathbf{curl}\,\mathbf{C}_d \in \mathbf{H}^r(\Omega)$ for given $r', r \in [0,1] \setminus \{\frac{1}{2}\}$. The tools introduced in the previous sections will be reused. For presentation purposes, we begin here with the regularity of the curl, which relies on the same arguments as in Section 6.1.1.

6.2.1 Extra-regularity of the solution's curl

Rather than working directly with $\mathbf{curl}\,\mathbf{E}$, let us introduce $\mathbf{C} := \underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E} \in \mathbf{L}^2(\Omega)$. There holds $\mathbf{curl}\,\mathbf{C} = \mathbf{f} + \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E} \in \mathbf{L}^2(\Omega)$, hence $\mathbf{C} \in \mathbf{H}(\mathbf{curl}, \Omega)$, with the bound

$$\|\mathbf{C}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{L}^2}. \tag{6.32}$$

Moreover, $\mathbf{C} \times \mathbf{n} = \mathbf{C}_d \times \mathbf{n}$. Letting $\mathbf{C}_0 := \mathbf{C} - \mathbf{C}_d \in \mathbf{H}_0(\mathbf{curl}, \Omega)$, we introduce the splitting of $\mathbf{C}_0$ by Theorem 6.1.1:

$$\mathbf{C}_0 = \mathbf{C}^{\mathrm{reg}} + \nabla\phi_C, \tag{6.33}$$

where $\mathbf{C}^{\mathrm{reg}} \in \mathbf{H}^1(\Omega)$, $\phi_C \in H^1_0(\Omega)$, and

$$\|\mathbf{C}^{\mathrm{reg}}\|_{\mathbf{H}^1} + \|\phi_C\|_{H^1_0} \lesssim \|\mathbf{C}_0\|_{\mathbf{H}(\mathbf{curl})}. \tag{6.34}$$

As $\operatorname{div}\underline{\boldsymbol{\mu}}\mathbf{C} = 0$, ϕ_C is governed by the Dirichlet problem

$$\left|\begin{array}{l}\text{Find } \phi_C \in H^1_0(\Omega) \text{ s.t., } \forall q \in H^1_0(\Omega),\\ \left(\underline{\boldsymbol{\mu}}\nabla\phi_C|\nabla q\right) = \left(\operatorname{div}\underline{\boldsymbol{\mu}}\mathbf{C}_d + \operatorname{div}\underline{\boldsymbol{\mu}}\mathbf{C}^{\mathrm{reg}}|q\right).\end{array}\right. \tag{6.35}$$

As in section 6.1.1, one can apply the Shift Theorem 6.1.4 for the Dirichlet problem to get the regularity of ϕ_C, then of $\mathbf{C}$ and finally of $\mathbf{curl}\,\mathbf{E}$.

Theorem 6.2.1. Let $\mathbf{E}$ governed by (6.31). If Γ is of class $\mathcal{C}^2$, $\underline{\boldsymbol{\mu}} \in \underline{\boldsymbol{\mathcal{C}}}^1(\overline{\Omega})$, and $\mathbf{C}_d \in \mathbf{H}^{r'}(\Omega)$ with $r' \in [0,1]\backslash\{\frac{1}{2}\}$, then

$$\mathbf{curl}\,\mathbf{E} \in \mathbf{H}^{r'}(\Omega), \tag{6.36}$$

and

$$\|\,\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{r'}} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{C}_d\|_{\mathbf{H}^{r'}} + \|\,\mathbf{curl}\,\mathbf{C}_d\|_{\mathbf{L}^2} + \|\mathbf{f}\|_{\mathbf{L}^2}. \tag{6.37}$$

Remark 6.2.2. No regularity assumption on $\underline{\boldsymbol{\varepsilon}}$ (other than $\underline{\boldsymbol{\varepsilon}} \in \underline{\mathbf{L}}^\infty(\Omega)$) is required here.

Proof. The proof is as in Theorem 6.1.5: we apply the Shift Theorem 6.1.4 to the Dirichlet problem (6.35) governing ϕ_C. Let us introduce ℓ the right-hand side of (6.35),

$$\ell : q \mapsto \left(\operatorname{div}\underline{\boldsymbol{\mu}}\mathbf{C}_d + \operatorname{div}\underline{\boldsymbol{\mu}}\mathbf{C}^{\mathrm{reg}}|q\right).$$

As $\underline{\boldsymbol{\mu}} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$, there holds $\operatorname{div}\underline{\boldsymbol{\mu}}\mathbf{C}^{\mathrm{reg}} \in L^2(\Omega)$ and $\operatorname{div}\underline{\boldsymbol{\mu}}\mathbf{C}_d \in H^{r'-1}(\Omega) = \left(H^{1-r'}_0(\Omega)\right)'$, the latter because $r' \neq \frac{1}{2}$. Then $\ell \in \left(H^{1-r'}_0(\Omega)\right)'$. Hence, by the Shift Theorem 6.1.4, we conclude that $\phi_C \in H^{1+r'}(\Omega)$, with

$$\begin{aligned}\|\phi_C\|_{H^{1+r'}} &\lesssim \|\operatorname{div}\underline{\boldsymbol{\mu}}\mathbf{C}_d\|_{H^{r'-1}} + \|\operatorname{div}\underline{\boldsymbol{\mu}}\mathbf{C}^{\mathrm{reg}}\|_{\mathbf{L}^2}\\ &\lesssim \|\mathbf{C}_d\|_{\mathbf{H}^{r'}} + \|\mathbf{C}^{\mathrm{reg}}\|_{\mathbf{H}^1}.\end{aligned}$$

Therefore, $\mathbf{C} = \mathbf{C}^{\mathrm{reg}} + \nabla\phi_C + \mathbf{C}_d \in \mathbf{H}^{r'}(\Omega)$, with

$$\begin{aligned}\|\mathbf{C}\|_{\mathbf{H}^{r'}} &\lesssim \|\mathbf{C}_d\|_{\mathbf{H}^{r'}} + \|\mathbf{C}^{\mathrm{reg}}\|_{\mathbf{H}^1} + \|\phi_C\|_{H^{1+r'}}\\ &\lesssim \|\mathbf{C}_d\|_{\mathbf{H}^{r'}} + \|\mathbf{C}^{\mathrm{reg}}\|_{\mathbf{H}^1}.\end{aligned}$$

Finally, as $\underline{\boldsymbol{\mu}} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$, one also has $\mathbf{curl}\,\mathbf{E} = \underline{\boldsymbol{\mu}}\mathbf{C} \in \mathbf{H}^{r'}(\Omega)$, with

$$\begin{aligned}\|\,\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{r'}} &\lesssim \|\mathbf{C}\|_{\mathbf{H}^{r'}}\\ &\lesssim \|\mathbf{C}_d\|_{\mathbf{H}^{r'}} + \|\mathbf{C}^{\mathrm{reg}}\|_{\mathbf{H}^1}\\ &\lesssim \|\mathbf{C}_d\|_{\mathbf{H}^{r'}} + \|\mathbf{C}_0\|_{\mathbf{H}(\mathbf{curl})}\\ &\lesssim \|\mathbf{C}_d\|_{\mathbf{H}^{r'}} + \|\,\mathbf{curl}\,\mathbf{C}_d\|_{\mathbf{L}^2} + \|\mathbf{C}\|_{\mathbf{H}(\mathbf{curl})}\\ &\lesssim \|\mathbf{C}_d\|_{\mathbf{H}^{r'}} + \|\,\mathbf{curl}\,\mathbf{C}_d\|_{\mathbf{L}^2} + \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{L}^2},\end{aligned}$$

using sucessively the bound (6.34) on $\mathbf{C}^{\mathrm{reg}}$, the triangle inequality on $\mathbf{C}_0$, and the bound (6.32) on $\mathbf{C}$ to conclude. □

Corollary 6.2.3. If additionally the problem is well-posed, then

$$\|\operatorname{\mathbf{curl}} \mathbf{E}\|_{\mathbf{H}^{r'}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\operatorname{\mathbf{curl}} \mathbf{C}_d\|_{\mathbf{L}^2} + \|\mathbf{C}_d\|_{\mathbf{H}^{r'}}. \tag{6.38}$$

Proof. It results from combining the bounds of Theorem 6.2.1 and Theorem 5.2.3. □

6.2.2 Extra-regularity of the solution

To estimate the regularity of the solution itself, we follow the same approach as in section 6.1.2. However, rather than working with $\mathbf{E}$ directly, let us introduce $\mathbf{G} := \operatorname{\mathbf{curl}} \mathbf{C} = \mathbf{f} + \omega^2 \underline{\varepsilon} \mathbf{E} \in \mathbf{L}^2(\Omega)$. As $\underline{\varepsilon} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$ and $\mathbf{f} \in \mathbf{H}(\mathbf{curl}, \Omega)$, one has $\operatorname{\mathbf{curl}} \mathbf{G} = \operatorname{\mathbf{curl}} \mathbf{f} + \omega^2 \operatorname{\mathbf{curl}} \underline{\varepsilon} \mathbf{E} \in \mathbf{L}^2(\Omega)$, hence $\mathbf{G} \in \mathbf{H}(\mathbf{curl}, \Omega)$, with the bound

$$\|\mathbf{G}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{H}(\mathbf{curl})}. \tag{6.39}$$

By Theorem 6.1.8, one can introduce the splitting of $\mathbf{G}$:

$$\mathbf{G} = \mathbf{G}^{\mathrm{reg}} + \nabla \phi_G, \tag{6.40}$$

where $\mathbf{G}^{\mathrm{reg}} \in \mathbf{H}^1(\Omega)$ with $\mathbf{G}^{\mathrm{reg}} \cdot \mathbf{n} = 0$, $\phi_G \in H^1_{\mathrm{zmv}}(\Omega)$, and

$$\|\mathbf{G}^{\mathrm{reg}}\|_{\mathbf{H}^1} + \|\phi_G\|_{H^1_{\mathrm{zmv}}} \lesssim \|\mathbf{G}\|_{\mathbf{H}(\mathbf{curl})}. \tag{6.41}$$

Additionally, there holds, $\forall q \in H^1_{\mathrm{zmv}}(\Omega)$, $(\mathbf{G}|\nabla q) = (\operatorname{\mathbf{curl}} \mathbf{C}|\nabla q) = (\operatorname{\mathbf{curl}} \mathbf{C}_d|\nabla q)$, as $\mathbf{C}_0 \in \mathbf{H}_0(\mathbf{curl}, \Omega)$. Then ϕ_G is governed by the Neumann problem

$$\left| \begin{array}{l} \text{Find } \phi_G \in H^1_{\mathrm{zmv}}(\Omega) \text{ s.t., } \forall q \in H^1_{\mathrm{zmv}}(\Omega), \\ \qquad (\nabla \phi_G|\nabla q) = (\operatorname{\mathbf{curl}} \mathbf{C}_d - \mathbf{G}^{\mathrm{reg}}|\nabla q)\,. \end{array} \right. \tag{6.42}$$

As in section 6.1.2, the regularity of ϕ_G is given by the Shift Theorem 6.1.10 for the Neumann problem. Then, one can conclude on the regularity of $\mathbf{G}$ and $\mathbf{E}$.

Theorem 6.2.4. Let $\mathbf{E}$ governed by (6.31). If Γ is of class $\mathcal{C}^2$, $\underline{\varepsilon} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$, $\mathbf{f} \in \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}^s(\Omega)$ with $s \in [0, 1]$, and $\operatorname{\mathbf{curl}} \mathbf{C}_d \in \mathbf{H}^r(\Omega)$ with $r \in [0, 1] \setminus \{\frac{1}{2}\}$, then

$$\mathbf{E} \in \mathbf{H}^{\min(r,s)}(\Omega), \tag{6.43}$$

and

$$\|\mathbf{E}\|_{\mathbf{H}^{\min(r,s)}} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{\mathbf{curl}} \mathbf{C}_d\|_{\mathbf{H}^r} + \|\mathbf{f}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{H}^s}. \tag{6.44}$$

Remark 6.2.5. No regularity assumption on $\underline{\boldsymbol{\mu}}$ is required here.

Proof. The proof is as in Theorem 6.1.12: we want to apply the Shift Theorem 6.1.10 to the problem (6.42) governing ϕ_G. Let us introduce the antilinear continuous form on $H^1_{\mathrm{zmv}}(\Omega)$,

$$\ell : q \mapsto (\operatorname{\mathbf{curl}} \mathbf{C}_d - \mathbf{G}^{\mathrm{reg}}|\nabla q)\,.$$

To determine the regularity of ϕ_G, one wants to determine whether the form ℓ belongs to $\left(H^{2-\sigma}_{\mathrm{zmv}}(\Omega)\right)'$ for $\sigma \in [1, \frac{3}{2}[$ as large as possible.

If $r < \frac{1}{2}$, then $\mathbf{H}^r(\Omega)$ identifies with $\mathbf{H}^r_0(\Omega)$, the dual space of $\mathbf{H}^{-r}(\Omega)$. Hence the product $(\operatorname{\mathbf{curl}} \mathbf{C}_d - \mathbf{G}^{\mathrm{reg}}|\nabla q)$ is meaningful as soon as $q \in H^{1-r}_{\mathrm{zmv}}(\Omega)$. This means that $\ell \in \left(H^{1-r}_{\mathrm{zmv}}(\Omega)\right)'$, and the Shift Theorem 6.1.10 (i) ensures that $\phi_G \in H^{1+r}(\Omega)$, with

$$\|\phi_G\|_{H^{1+r}} \lesssim \|\operatorname{\mathbf{curl}} \mathbf{C}_d\|_{\mathbf{H}^r} + \|\mathbf{G}^{\mathrm{reg}}\|_{\mathbf{H}^1}.$$

On the other hand, if $r > \frac{1}{2}$, the previous argument is not valid, as $\mathbf{H}^r_0(\Omega)$ does not identify with $\mathbf{H}^r(\Omega)$ (see Remark 6.1.11). However, $\operatorname{\mathbf{curl}} \mathbf{C}_d \cdot \mathbf{n}$ makes sense in $H^{r-1/2}(\Gamma)$, and, as $\mathbf{G}^{\mathrm{reg}} \cdot \mathbf{n} = 0$, ℓ rewrites by integrations by parts

$$\ell(q) = (\operatorname{div} \mathbf{G}^{\mathrm{reg}}|q) + \langle \operatorname{\mathbf{curl}} \mathbf{C}_d \cdot \mathbf{n}, q\rangle_{H^{1/2}(\Gamma)}.$$

As $\operatorname{div} \mathbf{G}^{\text{reg}} \in L^2(\Omega)$, ℓ satisfies the assumptions of the Shift Theorem 6.1.10 (ii), with $\sigma = 1 + r$, and we conclude that $\phi_G \in H^{1+r}(\Omega)$, with the same bound as above.

Finally, $\phi_G \in H^{1+r}(\Omega)$ in all the considered cases, and $\mathbf{G} = \mathbf{G}^{\text{reg}} + \nabla\phi_G \in \mathbf{H}^r(\Omega)$, with

$$\begin{aligned}\|\mathbf{G}\|_{\mathbf{H}^r} &\lesssim \|\mathbf{G}^{\text{reg}}\|_{\mathbf{H}^1} + \|\phi_G\|_{H^{1+r}}\\ &\lesssim \|\mathbf{G}^{\text{reg}}\|_{\mathbf{H}^1} + \|\operatorname{\mathbf{curl}} \mathbf{C}_d\|_{\mathbf{H}^r}\\ &\lesssim \|\mathbf{G}\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{\mathbf{curl}} \mathbf{C}_d\|_{\mathbf{H}^r}\end{aligned}$$

using the bound (6.41) on $\mathbf{G}^{\text{reg}}$. In addition, we note that $\underline{\boldsymbol{\varepsilon}}^{-1} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$, because $\underline{\boldsymbol{\varepsilon}}^{-1} \in \underline{\mathbf{L}}^{\infty}(\Omega)$ (Proposition 3.1.4) and $\underline{\boldsymbol{\varepsilon}} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$. Recalling that $\mathbf{E} = \omega^{-2}\underline{\boldsymbol{\varepsilon}}^{-1}(\mathbf{G} - \mathbf{f})$, with $\mathbf{f} \in \mathbf{H}^s(\Omega)$, there holds $\mathbf{E} \in \mathbf{H}^{\min(r,s)}(\Omega)$, with the bound

$$\begin{aligned}\|\mathbf{E}\|_{\mathbf{H}^{\min(r,s)}} &\lesssim \|\mathbf{f}\|_{\mathbf{H}^s} + \|\mathbf{G}\|_{\mathbf{H}^r}\\ &\lesssim \|\mathbf{f}\|_{\mathbf{H}^s} + \|\mathbf{G}\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{\mathbf{curl}} \mathbf{C}_d\|_{\mathbf{H}^r}\\ &\lesssim \|\mathbf{f}\|_{\mathbf{H}^s} + \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{\mathbf{curl}} \mathbf{C}_d\|_{\mathbf{H}^r}\end{aligned}$$

coming from triangle inequality and the bound (6.39) on $\mathbf{G}$. □

Corollary 6.2.6. If additionally the problem is well-posed, then

$$\|\mathbf{E}\|_{\mathbf{H}^{\min(r,s)}} \lesssim \|\operatorname{\mathbf{curl}} \mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{f}\|_{\mathbf{H}^s} + \|\mathbf{C}_d\|_{\mathbf{L}^2} + \|\operatorname{\mathbf{curl}} \mathbf{C}_d\|_{\mathbf{H}^r}. \tag{6.45}$$

Proof. It results from combining the bounds of Theorem 6.2.4 and Theorem 5.2.3. □

Summary

To conclude, we sum up the regularity results of this section.

Theorem 6.2.7. Let $\mathbf{E}$ governed by (6.31). If Γ is of class $\mathcal{C}^2$, $\underline{\boldsymbol{\mu}} \in \underline{\boldsymbol{\mathcal{C}}}^1(\overline{\Omega})$, $\underline{\boldsymbol{\varepsilon}} \in \underline{\mathbf{W}}^{1,\infty}(\Omega)$, $\mathbf{f} \in \mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}^s(\Omega)$ with s in $[0, 1]$, $\mathbf{C}_d \in \mathbf{H}^{r'}(\Omega)$, and $\operatorname{\mathbf{curl}} \mathbf{C}_d \in \mathbf{H}^r(\Omega)$ with r', r in $[0, 1] \setminus \frac{1}{2}$, then

$$\mathbf{E} \in \mathbf{H}^{\min(r,s)}(\Omega) \quad \text{and} \quad \operatorname{\mathbf{curl}} \mathbf{E} \in \mathbf{H}^{r'}(\Omega), \tag{6.46}$$

with additionally, if the problem is well-posed,

$$\|\mathbf{E}\|_{\mathbf{H}^{\min(r,s)}} \lesssim \|\operatorname{\mathbf{curl}} \mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{f}\|_{\mathbf{H}^s} + \|\mathbf{C}_d\|_{\mathbf{L}^2} + \|\operatorname{\mathbf{curl}} \mathbf{C}_d\|_{\mathbf{H}^r}; \tag{6.47}$$

$$\|\operatorname{\mathbf{curl}} \mathbf{E}\|_{\mathbf{H}^{r'}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\operatorname{\mathbf{curl}} \mathbf{C}_d\|_{\mathbf{L}^2} + \|\mathbf{C}_d\|_{\mathbf{H}^{r'}}. \tag{6.48}$$

In the lower regularity case, one recovers something similar to the stability estimate (5.20), but with unnecessary extra-term $\|\operatorname{\mathbf{curl}} \mathbf{f}\|_{\mathbf{L}^2}$. In the higher regularity case, one finds that $\mathbf{E}, \operatorname{\mathbf{curl}} \mathbf{E} \in \mathbf{H}^1(\Omega)$, with continuous dependence w.r.t. $\mathbf{f}$, $\mathbf{C}_d$ and $\operatorname{\mathbf{curl}} \mathbf{C}_d \in \mathbf{H}^1(\Omega)$.

6.3 Regularity in the Robin problem

The Robin problem reads:

$$\begin{cases}\operatorname{\mathbf{curl}}(\underline{\boldsymbol{\mu}}^{-1} \operatorname{\mathbf{curl}} \mathbf{E}) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E} = \mathbf{f} & \text{in } \Omega,\\ \pi^T(\underline{\boldsymbol{\mu}}^{-1} \operatorname{\mathbf{curl}} \mathbf{E}) + \alpha\gamma^T\mathbf{E} = \mathbf{g} & \text{on } \Gamma,\end{cases} \tag{6.49}$$

As in Section 5.3, we assume that α is elliptic, that $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}_{\parallel}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \cap \mathbf{H}_{\perp}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$.

Here, we assume furthermore that $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\mu}} \in \underline{\boldsymbol{\mathcal{C}}}^1(\overline{\Omega})$, and $\operatorname{div} \mathbf{f} \in H^{s-1}(\Omega)$ for a given s in $[0, 1] \setminus \{\frac{1}{2}\}$. We assume moreover that α is s.t. $\gamma^T\mathbf{E}, \pi^T(\underline{\boldsymbol{\mu}}^{-1} \operatorname{\mathbf{curl}} \mathbf{E}) \in \mathbf{H}^r(\Gamma)$ for some $r > 0$ (see the results of Chapter 4). Because Γ is of class $\mathcal{C}^2$, clearly it has no (semi-)pathological vertex; moreover, no extra-assumption on $\mathbf{g}$ is necessary, because the regularity of $\mathbf{g}$ is entirely driven by the geometry of Γ: by Theorem 4.1.4, $\mathbf{g} \in \mathbf{H}_t^{1/2}(\Gamma)$. The extra-regularity of $\mathbf{E}$ and $\mathbf{curl}\,\mathbf{E}$ is obtained by reusing the results established in the previous sections.

Theorem 6.3.1. Let $\mathbf{E}$ governed by (6.49). If Γ is of class $\mathcal{C}^2$, $\underline{\varepsilon}, \underline{\mu} \in \underline{\mathcal{C}}^1(\overline{\Omega})$, $\mathbf{f} \in \mathbf{L}^2(\Omega)$ s.t. $\operatorname{div}\mathbf{f} \in H^{s-1}(\Omega)$ with s in $[0,1] \setminus \{\frac{1}{2}\}$, $\mathbf{g} \in \mathbf{H}_{\parallel}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \cap \mathbf{H}_{\perp}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$, and if α is elliptic and s.t. $\exists r \in]0, \frac{1}{2}]$, $\gamma^T\mathbf{E}$, $\pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}) \in \mathbf{H}_t^r(\Gamma)$, then, one has

$$\mathbf{E} \in \mathbf{H}^{\min(r+1/2,s)}(\Omega) \quad \text{and} \quad \mathbf{curl}\,\mathbf{E} \in \mathbf{H}^{r+1/2}(\Omega) \tag{6.50}$$

with the bounds

$$\|\mathbf{E}\|_{\mathbf{H}^{\min(r+1/2,s)}} \lesssim \|\operatorname{div}\mathbf{f}\|_{H^{s-1}} + \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\gamma^T\mathbf{E}\|_{\mathbf{H}_t^r(\Gamma)} \tag{6.51}$$

$$\|\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{r+1/2}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E})\|_{\mathbf{H}_t^r(\Gamma)}. \tag{6.52}$$

Proof. By assumption, there holds $\gamma^T\mathbf{E}, \pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}) \in \mathbf{H}_t^r(\Gamma)$ for $r > 0$. Therefore, $\gamma^T\mathbf{E}$ admits a lifting $\mathbf{E}_d \in \mathbf{H}^{r+1/2}(\Omega)$, s.t. $\gamma^T\mathbf{E} = \gamma^T\mathbf{E}_d$ on Γ, and $\|\mathbf{E}_d\|_{\mathbf{H}^{r+1/2}} \lesssim \|\gamma^T\mathbf{E}\|_{\mathbf{H}_t^r(\Gamma)}$. Then, $\mathbf{E}$ is governed by the Dirichlet problem

$$\begin{cases} \mathbf{curl}(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\underline{\varepsilon}\mathbf{E} = \mathbf{f} & \text{in } \Omega, \\ \gamma^T\mathbf{E} = \gamma^T\mathbf{E}_d & \text{on } \Gamma. \end{cases}$$

We use the regularity result of Theorem 6.1.5 to conclude that $\mathbf{E} \in \mathbf{H}^{\min(s,r+1/2)}(\Omega)$. Moreover, there holds

$$\begin{aligned} \|\mathbf{E}\|_{\mathbf{H}^{\min(s,r+1/2)}} &\lesssim \|\mathbf{E} - \mathbf{E}_d\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{div}\mathbf{f}\|_{H^{s-1}} + \|\mathbf{E}_d\|_{\mathbf{H}^{r+1/2}} \\ &\lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{div}\mathbf{f}\|_{H^{s-1}} + \|\mathbf{E}_d\|_{\mathbf{H}^{r+1/2}} + \|\mathbf{E}_d\|_{\mathbf{H}(\mathbf{curl})} \\ &\lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{div}\mathbf{f}\|_{H^{s-1}} + \|\gamma^T\mathbf{E}\|_{\mathbf{H}_t^r(\Gamma)} + \|\gamma^T\mathbf{E}\|_{\gamma} \\ &\lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{div}\mathbf{f}\|_{H^{s-1}} + \|\gamma^T\mathbf{E}\|_{\mathbf{H}_t^r(\Gamma)}. \end{aligned}$$

We proceed similarly for the curl of the solution. Because $\pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}) \in \mathbf{H}_t^r(\Gamma)$, it admits a lifting $\mathbf{C}_d \in \mathbf{H}^{r+1/2}(\Omega)$ s.t. $\pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}) = \pi^T\mathbf{C}_d$ on Γ, with $\|\mathbf{C}_d\|_{\mathbf{H}^{r+1/2}} \lesssim \|\pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E})\|_{\mathbf{H}_t^r(\Gamma)}$. Then, $\mathbf{E}$ is also governed by the Neumann problem

$$\begin{cases} \mathbf{curl}(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\underline{\varepsilon}\mathbf{E} = \mathbf{f} & \text{in } \Omega, \\ \pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}) = \pi^T\mathbf{C}_d & \text{on } \Gamma. \end{cases}$$

Using here the regularity result of Theorem 6.2.1, we conclude that $\mathbf{curl}\,\mathbf{E} \in \mathbf{H}^{r+1/2}(\Omega)$, with moreover

$$\begin{aligned} \|\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{r+1/2}} &\lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{C}_d\|_{\mathbf{H}^{r+1/2}} + \|\mathbf{C}_d\|_{\mathbf{H}(\mathbf{curl})} \\ &\lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{L}^2} + \|\pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E})\|_{\mathbf{H}_t^r(\Gamma)} + \|\pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E})\|_{\pi} \\ &\lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{L}^2} + \|\pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E})\|_{\mathbf{H}_t^r(\Gamma)} + \|\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} \\ &\lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{f}\|_{\mathbf{L}^2} + \|\pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E})\|_{\mathbf{H}_t^r(\Gamma)}, \end{aligned}$$

the latter because $\mathbf{curl}(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}) = \omega^2\underline{\varepsilon}\mathbf{E} + \mathbf{f}$. □

Remark 6.3.2. The result also holds for a tensor-valued coefficient $\underline{\boldsymbol{\alpha}}$ that allows the boundary condition to hold in $\mathbf{H}_t^r(\Gamma)$; the proof is the same.

To get a result with continuous dependence w.r.t. the data only, one needs to make some additional hypotheses.

Corollary 6.3.3. Let all the assumptions of Theorem 6.3.1 be satisfied, and assume moreover that the problem (6.49) is well-posed. If α is a piecewise constant coefficient s.t. there are no singularities at coefficient vertices, then $\exists r \in]0, \frac{1}{2}]$,

$$\|\mathbf{E}\|_{\mathbf{H}^{\min(r+1/2,s)}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\operatorname{div}\mathbf{f}\|_{H^{s-1}} + \|\mathbf{g}\|_\pi + \|\mathbf{g}\|_\gamma; \tag{6.53}$$
$$\|\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{r+1/2}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{g}\|_\gamma + \|\mathbf{g}\|_\pi. \tag{6.54}$$

If $\alpha \in W^{2,\infty}(\Gamma)$, then (6.53)-(6.54) also hold for $r = \frac{1}{2}$.

Proof. One takes advantage of stability estimates for the Dirichlet and Neumann traces obtained in Chapter 4. If α is piecewise constant with no singularities at coefficient vertices, one has the bounds

$$\|\gamma^T\mathbf{E}\|_{\mathbf{H}_t^r(\Gamma)} \lesssim \|\mathbf{g}\|_\pi + \|\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E})\|_\pi + \|\gamma^T\mathbf{E}\|_\gamma; \tag{6.55}$$
$$\|\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E})\|_{\mathbf{H}_t^r(\Gamma)} \lesssim \|\mathbf{g}\|_\gamma + \|\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E})\|_\pi + \|\gamma^T\mathbf{E}\|_\gamma, \tag{6.56}$$

that follow from Theorem 4.2.8, $\forall r < s_{\max}$ with $s_{\max}$ defined in Theorem 4.2.8. If α is smooth, (6.55)-(6.56) also hold for $r = \frac{1}{2}$ by Theorem 4.2.2 (there are no singularities, because Γ is of class $\mathcal{C}^2$). The result is then obtained by combining the bounds (6.55) and (6.56) with the ones of Theorem 6.3.1. One concludes using the fact that $\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\mathbf{g}\|_{\mathbf{L}_t^2(\Gamma)}$ (Theorem 5.3.3) and that $\|\mathbf{g}\|_{\mathbf{L}_t^2(\Gamma)} \lesssim \|\mathbf{g}\|_\gamma + \|\mathbf{g}\|_\pi$ (Theorem 4.1.4). □

Alternative approach

Alternatively, one can obtain regularity and continuous dependence w.r.t. the data only up to exponent 1/2 if $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\mu}}$ are isotropic in a neighbourhood of Γ, i.e. $\underline{\boldsymbol{\varepsilon}} = \varepsilon\underline{\mathbf{I}}$, $\underline{\boldsymbol{\mu}} = \mu\underline{\mathbf{I}}$. This result is weaker than the previous one, but it calls arguments of a different kind, so we present it in pedagogical views.

Remark 6.3.4. The isotropy assumption is reasonable: in many applications, impedance conditions are imposed on artificial boundaries to truncate the computational domain, and the artificial boundary can, most of the time, be placed in an isotropic media (air or vacuum for example).

Proposition 6.3.5. Let all the assumptions of Theorem 6.3.1 be satisfied. If moreover $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\mu}}$ are isotropic in a neighbourhood of Γ, and if the problem (6.49) is well-posed, then one has the bound

$$\|\mathbf{E}\|_{\mathbf{H}^{\min(s,1/2)}} \lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\operatorname{div}\mathbf{f}\|_{H^{s-1}} + \|\mathbf{g}\|_{\mathbf{L}_t^2(\Gamma)}. \tag{6.57}$$

Proof. We proceed by localization. Let Ω_B a neighbourhood of Γ in Ω s.t. $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\mu}}$ are isotropic in Ω_B. We introduce a cut-off function $\chi \in \mathcal{C}^\infty(\overline{\Omega})$ whose value is 1 in a (smaller) neighbourhood of Γ, and support is included in Ω_B. Therefore, there holds $\mathbf{E} = \chi\mathbf{E} + (1-\chi)\mathbf{E}$. Let us consider each term separately.

We note that $(1-\chi)\mathbf{E}$ satisfies a homogeneous Dirichlet problem, as $(1-\chi)\mathbf{E} = \mathbf{0}$ in a neighbourhood of Γ. Thus, because of Theorem 6.1.5, one has $(1-\chi)\mathbf{E} \in \mathbf{H}^s(\Omega)$, with

$$\|(1-\chi)\mathbf{E}\|_{\mathbf{H}^s} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})} + \|\operatorname{div}\mathbf{f}\|_{H^{s-1}}.$$

For $\chi\mathbf{E} \in \mathbf{H}^+(\mathbf{curl}, \Omega)$, we use the Helmholtz decomposition of Theorem 3.2.4: $\chi\mathbf{E} = \nabla p + \tilde{\mathbf{E}}$, where $p \in H_0^1(\Omega_B)$ and $\tilde{\mathbf{E}} \in \mathbf{W}_N(\varepsilon; \Omega_B)$. The potential p is governed by

$$\left|\begin{array}{l}\text{Find } p \in H_0^1(\Omega_B) \text{ s.t., } \forall q \in H_0^1(\Omega_B),\\ \qquad -\omega^2\,(\varepsilon\nabla p|\nabla q) = (\mathbf{f}|\nabla q).\end{array}\right.$$

Using Shift Theorem 6.1.4, there holds $p \in H^{1+s}(\Omega_B)$, with $\|p\|_{H^{1+s}(\Omega_B)} \lesssim \|\operatorname{div}\mathbf{f}\|_{H^{s-1}(\Omega)}$. For the field $\tilde{\mathbf{E}}$, one has $\tilde{\mathbf{E}} \in \mathbf{H}^+(\mathbf{curl}, \Omega_B)$, and, moreover, $\operatorname{div}\varepsilon\tilde{\mathbf{E}} = 0$. Because ε is scalar in Ω_B, there holds $\nabla\varepsilon\cdot\tilde{\mathbf{E}} + \varepsilon\operatorname{div}\tilde{\mathbf{E}} = 0$, so $\operatorname{div}\tilde{\mathbf{E}} \in L^2(\Omega_B)$, with (because $\varepsilon \in W^{1,\infty}(\Omega)$ and is elliptic)

$$\|\operatorname{div}\tilde{\mathbf{E}}\|_{L^2(\Omega_B)} \lesssim \|\tilde{\mathbf{E}}\|_{\mathbf{L}^2(\Omega_B)}.$$

Then, using a result due to Costabel [34], there holds $\tilde{\mathbf{E}} \in \mathbf{H}^{1/2}(\Omega_B)$, with

$$\begin{aligned}\|\tilde{\mathbf{E}}\|_{\mathbf{H}^{1/2}(\Omega_B)} &\lesssim \|\tilde{\mathbf{E}}\|_{\mathbf{H}^+(\mathbf{curl},\Omega_B)} + \|\operatorname{div}\tilde{\mathbf{E}}\|_{L^2(\Omega_B)}\\ &\lesssim \|\tilde{\mathbf{E}}\|_{\mathbf{H}^+(\mathbf{curl},\Omega_B)}\\ &\lesssim \|\mathbf{E}\|_{\mathbf{H}^+(\mathbf{curl},\Omega_B)} + \|p\|_{H^1_0(\Omega_B)}.\end{aligned}$$

So, $\chi\mathbf{E} \in \mathbf{H}^{\min(s,1/2)}(\Omega_B)$, with

$$\|\chi\mathbf{E}\|_{\mathbf{H}^{\min(s,1/2)}(\Omega_B)} \lesssim \|\mathbf{E}\|_{\mathbf{H}^+(\mathbf{curl},\Omega_B)} + \|\operatorname{div}\mathbf{f}\|_{H^{s-1}(\Omega)}.$$

Finally, and assuming moreover that the problem (6.49) is well-posed, there holds

$$\begin{aligned}\|\mathbf{E}\|_{\mathbf{H}^{\min(s,1/2)}} &\lesssim \|\chi\mathbf{E}\|_{\mathbf{H}^{\min(s,1/2)}(\Omega_B)} + \|(1-\chi)\mathbf{E}\|_{\mathbf{H}^s(\Omega)}\\ &\lesssim \|\mathbf{f}\|_{\mathbf{L}^2} + \|\operatorname{div}\mathbf{f}\|_{H^{s-1}} + \|\mathbf{g}\|_{\mathbf{L}^2_t(\Gamma)}.\end{aligned}$$

One can proceed similarly with the curl of the solution. □

Remark 6.3.6. Moreover, one can also use the localization technique to deal separately with different boundary conditions on different connected components of the boundary. This is the case, for example, when one considers the scattering of an object surrounded by vacuum. On the object boundary, one may have Dirichlet or Neumann conditions; on the exterior (artificial) boundary, one imposes an impedance condition. More generally, if one has mixed boundary conditions, but each type of boundary condition corresponds to a distinct connected component of the boundary, then all results of this chapter apply.

6.4 $\mathbf{H}(\mathbf{curl},\Omega)$-conforming finite element discretization

Edge finite element methods are natural candidates for the numerical solution of electromagnetic problems. Since these methods lead to $\mathbf{H}(\mathbf{curl})$-conforming approximations, some features of the numerical solutions can be rather easily studied by leveraging the results obtained for the exact problems. While the comprehensive numerical analysis of the approximate problems is out of the scope of this work, this section aims at giving a few numerical illustrations for the considered problems. After introducing a standard edge finite element discretization and basic results, we derive an *a priori* error estimate, which is obtained by using the regularity estimates. Elementary numerical results are then proposed to illustrate the expected convergence rate of the method.

6.4.1 Discretization and *a priori* error estimate

We consider a shape regular familly of meshes $(\mathcal{T}_h)_h$ for the domain Ω. For the sake of simplicity, we assume that the domain Ω is a Lipschitz polyhedron. Each mesh $\mathcal{T}_h$ is made up of closed non-overlapping tetrahedra, generically denoted by K, and is indexed by $h := \max_K h_K$, where h_K is the diameter of K. Denoting by ρ_K the diameter of the largest ball inscribed in K, we assume that there exists a shape regularity parameter $\varsigma > 0$ such that for all h, for all $K \in \mathcal{T}_h$, it holds $h_K \leq \varsigma\rho_K$.

Finite dimensional subspaces $(\mathbf{V}_h)_h$ of $\mathbf{H}(\mathbf{curl},\Omega)$ are defined by using the so-called Nédélec's first family of edge finite elements. Elements of degree 1 are considered. One has

$$\mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{H}(\mathbf{curl},\Omega), \mathbf{v}_{h|K} \in \mathcal{R}_1(K),\ \forall K \in \mathcal{T}_h\}, \tag{6.58}$$

where $\mathcal{R}_1(K)$ is the six-dimensional vector space of polynomials on K

$$\mathcal{R}_1(K) := \{\mathbf{v} \in \boldsymbol{P}_1(K)\ :\ \mathbf{v}(\mathbf{x}) = \boldsymbol{a} + \boldsymbol{b}\times\mathbf{x},\ \boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^3\}. \tag{6.59}$$

In this space, the fields are approximated by piecewise order 1 polynomials. Their curl is approximated by piecewise constant functions. The subspaces verify the approximability property (see e.g. [86, Lemma 7.10])

$$\lim_{h\to 0}\left(\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\mathbf{v}-\mathbf{v}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)}\right) = 0,\ \forall\mathbf{v} \in \mathbf{H}(\mathbf{curl},\Omega). \tag{6.60}$$

We also introduce the closed subspaces $(\mathbf{V}_h^0)_h$ with $\mathbf{V}_h^0 := \mathbf{V}_h \cap \mathbf{H}_0(\mathbf{curl},\Omega)$, which also verify the approximability property in $\mathbf{H}_0(\mathbf{curl},\Omega)$.

Using the standard Galerkin approach, the variational formulation of the approximate problem is obtained by seeking the solution in $\mathbf{V}_h$ with test functions in $\mathbf{V}_h^0$ or $\mathbf{V}_h$ for the Dirichlet and Neumann cases, respectively. Therefore, the discrete Dirichlet problem reads

$$\left|\begin{array}{l}\text{Find } \mathbf{E}_h \in \mathbf{V}_h \text{ s.t.}\\ \forall \mathbf{F}_h \in \mathbf{V}_h^0, \quad a(\mathbf{E}_h,\mathbf{F}_h) = \ell_{\mathrm{D}}(\mathbf{F}_h),\\ \mathbf{E}_h \times \mathbf{n} = \mathbf{g}_h \text{ on } \Gamma,\end{array}\right. \tag{6.61}$$

and the discrete Neumann problem reads

$$\left|\begin{array}{l}\text{Find } \mathbf{E}_h \in \mathbf{V}_h \text{ s.t., } \forall \mathbf{F}_h \in \mathbf{V}_h,\\ a(\mathbf{E}_h,\mathbf{F}_h) = \ell_{\mathrm{N}}(\mathbf{F}_h),\end{array}\right. \tag{6.62}$$

with the sesquilinear form

$$a : (\mathbf{u},\mathbf{v}) \mapsto (\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{u}|\,\mathbf{curl}\,\mathbf{v}) - \omega^2(\underline{\varepsilon}\mathbf{u}|\mathbf{v}) \tag{6.63}$$

defined on $\mathbf{H}(\mathbf{curl},\Omega)$. The linear forms ℓ_{D} and ℓ_{N} are defined in sections 5.1 and 5.2, respectively. The right-hand-side term $\mathbf{g}_h$ is the projection of $\mathbf{g}$ onto $\gamma^T\mathbf{V}_h$. For simplicity, in the remaining, we assume that the integrals are computed exactly.

As a first result, we derive a sharp error estimate for the interpolation of the solutions of both problems onto the finite element space. Let π_h denote the classical interpolation operator from $\mathbf{H}_0(\mathbf{curl},\Omega)$ onto $\mathbf{V}_h^0$, resp. from $\mathbf{H}(\mathbf{curl},\Omega)$ onto $\mathbf{V}_h$. One has the following interpolation error estimate, cf. [11].

Theorem 6.4.1. Let $\sigma \in]1/2,1]$ and $\sigma' \in]0,1]$. For all $\mathbf{v} \in \left\{\mathbf{v} \in \mathbf{H}^\sigma(\Omega),\ \mathbf{curl}\,\mathbf{v} \in \mathbf{H}^{\sigma'}(\Omega)\right\}$, it holds that

$$\|\mathbf{v} - \pi_h\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \lesssim h^{\min(\sigma,\sigma')}\left(\|\mathbf{v}\|_{\mathbf{H}^\sigma} + \|\,\mathbf{curl}\,\mathbf{v}\|_{\mathbf{H}^{\sigma'}}\right). \tag{6.64}$$

In this result, $\sigma > 1/2$ is assumed for simplicity. A similar result can be obtained for $\sigma \in]0,1]$ with the help of the combined interpolation operator (see Section 4.2 in [25]), but this result is more involved. Indeed, the norm of the gradient part of the decomposition of $\mathbf{v}$ (given in Theorems 6.1.1 or 6.1.8) then appears in the right-hand side of (6.64), in addition to both terms already there. Nevertheless, since the gradient part is bounded by the norm of the data (see again Theorems 6.1.1 or 6.1.8), the same conclusion stands in this general case. Then, observe that one can replace the field $\mathbf{v}$ with $\mathbf{E}$ in equation (6.64). Using Theorems 6.1.15 and 6.2.7, the norms $\|\mathbf{E}\|_{\mathbf{H}^\sigma}$ and $\|\,\mathbf{curl}\,\mathbf{E}\|_{\mathbf{H}^{\sigma'}}$ are bounded by the norms on the data, and the exponents become $\sigma = \min(s,r)$ and $\sigma' = r'$, where s, r, r' are the extra-regularity exponents for the data. Injecting the regularity estimates in equation (6.64) then gives

$$\|\mathbf{E} - \pi_h\mathbf{E}\|_{\mathbf{H}(\mathbf{curl},\Omega_h)} \lesssim h^{\min(s,r,r')}, \tag{6.65}$$

where the bounds on the exponents are defined in Theorems 6.1.15 and 6.2.7 for the Dirichlet and Neumann cases, respectively.

In order to derive an *a priori* error estimate for both problems, one has to bound the error between the numerical solution and the exact solution with the interpolation error. For a problem with a coercive sesquilinear form, it is known that an *a priori* error estimate for the numerical solution is obtained thanks to Céa's lemma (see e.g. [53]).

Lemma 6.4.2 (Céa)**.** When the sesquilinear form $a(\cdot,\cdot)$ is coercive, it holds that

$$\exists C > 0,\ \forall h,\ \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le C \inf_{\mathbf{w}_h \in \mathbf{V}_h} \|\mathbf{E} - \mathbf{w}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)}. \tag{6.66}$$

Using $\mathbf{w}_h = \pi_h\mathbf{E}$ and the estimates (6.65) and (6.66), one has the final result.

Theorem 6.4.3. When the problem is coercive, there holds

$$\|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)} \lesssim h^{\min(s,r,r')}, \tag{6.67}$$

where the exponents depend only on the regularity of the data.

Let us highlight that the regularity results have been obtained for a boundary of class $\mathcal{C}^2$, while the interpolation error estimates are for Lipschitz polyhedral domains. The error resulting from this geometric approximation can be studied thanks to the framework introduced by Dello Russo and Alonso [41]. Following Section 8 there, one obtains additional terms in the right-hand side of (6.66), which are asymptotically all in the order of $O(h)$.

On the other hand, to obtain a similar estimate for a problem with a non-coercive sesquilinear form, one has to prove a uniform discrete inf-sup condition and to combine it with a generalised Céa's lemma. In our case, when $a(\cdot,\cdot)$ is not coercive, deriving a uniform discrete inf-sup condition requires tedious developments. We refer for instance to [64] and [26] for analyses in slightly different contexts. Provided that such a result is available, the estimate (6.67) holds.

For the Robin problem, we refer to the monograph of Monk [86]. By Lemma 7.10 there, $\mathbf{V}_h$ also satisfy the approximability property in $\mathbf{H}^+(\mathbf{curl},\Omega)$. Moreover, following Lemma 5.53 in the same reference, one gets that

$$\|\mathbf{E} - \pi_h\mathbf{E}\|_{\mathbf{H}^+(\mathbf{curl},\Omega_h)} \lesssim h^{s-1/2} \tag{6.68}$$

if $\mathbf{E},\ \mathbf{curl}\,\mathbf{E} \in \mathbf{H}^s(\Omega)$. Combined with Céa's Lemma, this gives

$$\|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{H}^+(\mathbf{curl},\Omega)} \lesssim h^{s-1/2}. \tag{6.69}$$

The author points that this result is probably not optimal, as there is a loss of 1/2 order in the convergence rate.

6.4.2 Numerical illustration

To illustrate the expected convergence rate with a numerical case, we consider a simple benchmark with a manufactured solution. Let a spherical domain of unit radius centred at the origin, $\Omega = \{\mathbf{x} \in \mathbb{R}^3, \|\mathbf{x}\| < 1\}$, the angular frequency $\omega = 1$ and the material tensors

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} 1+\eta i & & \\ & 1+\eta i & \\ & & -2+\eta i \end{pmatrix}, \qquad \underline{\underline{\mu}}^{-1} = \underline{\underline{\mathbf{I}}}, \tag{6.70}$$

where $\eta \in \mathbb{R}$ is a chosen parameter. Let us note that both tensors are elliptic, but $\underline{\underline{\varepsilon}}$ is not Hermitian due to its imaginary part. Moreover, one can notice that the eigenvalues of $\underline{\underline{\mu}}^{-1}$ and $-\underline{\underline{\varepsilon}}$ are contained in the same open complex half-plane; hence, this problem is coercive (see also Lemma 2.3 of [102]). Different values of η will be tested. When η goes to zero, the (best) ceorcivity constant deteriorates. In fact, when $\eta = 0$, the material becomes hyperbolic, and the problem is most likely ill-posed. We consider a manufactured reference solution which is a plane wave,

$$\mathbf{E}_{\text{ref}} = [-1,\ 1,\ 1]^T \exp(i\pi\mathbf{k}\cdot\mathbf{x}), \quad \text{with } \mathbf{k} = \frac{1}{\sqrt{14}}[3,\ 2,\ 1]^T. \tag{6.71}$$

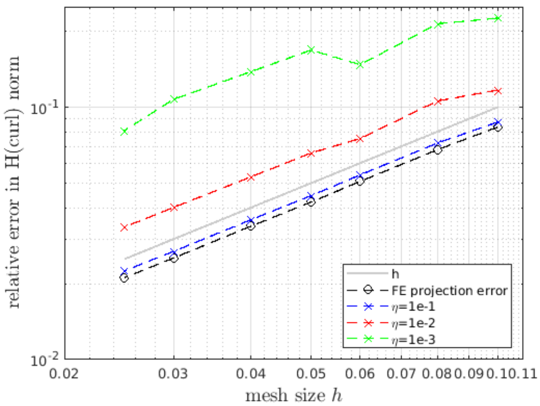
The volume source term is chosen accordingly, *i.e.* $\mathbf{f} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_{\text{ref}} - \omega^2\underline{\underline{\varepsilon}}\mathbf{E}_{\text{ref}}$, as well as the right-hand-side term of the boundary conditions.

Numerical simulations are performed with FreeFem++ [68] using unstructured meshes made of tetrahedra and first-degree edge finite elements. The mesh sizes and number of degrees of freedom are summarized on Table 6.1. The problem is discretized using Nédélec first family of edge finite elements described above. The linear system is then solved by a direct solver. Because the boundary of the meshes (which are polyhedral) does not exactly match the curved border of the spherical domain, the boundary data used in the numerical simulation are evaluated on the sphere and then projected on the surface mesh. It has been proven that this geometric approximation introduces a geometric error of the order $O(h)$ [41].

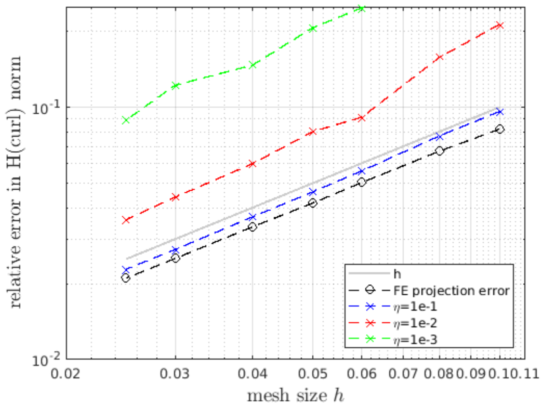
Mesh size	0.1	0.08	0.06	0.05	0.04	0.03	0.025
Degrees of freedom	3372	6543	14604	24589	46544	107537	181298

Table 6.1: Parameters of the different meshes

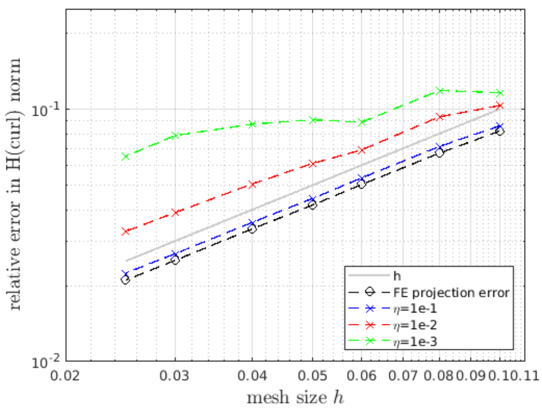
The relative numerical error in $\mathbf{H}(\mathbf{curl})$-norm is plotted as a function of the mesh size h on Fig. 6.1 for Dirichlet, Neumann and Robin cases (with $\alpha = i\pi$ for the Robin case). As a reference, the relative error corresponding to the projection of the reference solution on the discrete solution space, which corresponds to the best approximation error according to Céa's lemma, is plotted as well. As the solution $\mathbf{E}_{\text{ref}}$ is smooth, it belongs to $\mathbf{H}^1(\Omega)$ as well as its curl. Therefore, one expects the error to evolve linearly with the mesh size h, at least for Dirichlet and Neumann problems. The results reported on Fig. 6.1 show that the convergence behaves effectively like $O(h)$ for both problems. The same convergence rate is observed for the Robin problem. This tends to confirm that the estimate (6.69) coming from [86] is indeed suboptimal.



(a) Dirichlet



(b) Neumann



(c) Robin

Figure 6.1: Convergence of edge finite elements (order 1)

We also observe how the material tensors affect the convergence of the method. We have tested three different values for η: 10^{-1}, 10^{-2} and 10^{-3}. As the reference solution and the meshes are the same through those three cases, the relative error corresponding to the projection of the reference solution on the discrete solution space, which corresponds to the best approximation error according to Céa lemma, is the same. Ideally, the numerical error would be close to this theoretical error. However, we observe that this is not the case. In fact, when η decreases, the error becomes well larger. The rate of convergence is quite well preserved for $\eta = 10^{-2}$, but, for $\eta = 10^{-3}$, there is an impact on the convergence too: the error is no more monotone. This seems to indicate that we are maybe still in a pre-asymptotic regime. There is a large component of the error that is not the theoretical FE error. This can be related to the fact that the coercivity constant of the problem decreases with η, causing the constant

C in Céa's Lemma 6.4.2 to become huge.

Conclusion

We have studied the regularity of the solutions to the Dirichlet, Neumann, or Robin time-harmonic Maxwell problems, as well as of their curl. This regularity depends on the geometry of the domain, on the smoothness of the parameters, and on the regularity of the data.

In the most optimistic case, that is, when $\operatorname{div}\mathbf{f} \in L^2(\Omega)$ (or $\mathbf{f} \in \mathbf{H}^1(\Omega)$ for the Neumann case), and when the traces have liftings that belong to $\mathbf{H}^1(\Omega)$ as well as their curl, one finds that

$$\mathbf{E} \in \mathbf{H}^1(\Omega) \quad \text{and} \quad \mathbf{curl}\,\mathbf{E} \in \mathbf{H}^1(\Omega). \tag{6.72}$$

In less optimistic cases, one may have regularity exponents in $]0,1[$ for the solution and its curl. These regularity exponents determine the order of convergence of the edge finite elements methods if one wants to solve the problem numerically.

Moreover, for the Dirichlet and Neumann problems, we have also proven the continuous dependence of the solution and its curl in these norms w.r.t. the data. For the Robin problem, the regularity of the solution as well the continuous dependence is a bit more subtle, and appears to be closely related to the regularity of traces, which has been studied in Chapter 4.

With respect to the geometry, we have assumed that Γ is of class $\mathcal{C}^2$. However, this assumption could be relaxed. In fact, shift theorems also exist for other types of domains, e.g. convex domains. In non-convex domains, an extra-treatment of reentrant edges and corners is necessary; we refer to [35] for regularity studies in polyhedral domains. For shift theorems in settings with non-smooth domains and Hermitian tensors, we refer e.g. to [65]. In a similar manner, one could also consider settings with piecewise smooth coefficients as in [27].

Our work also allows to consider problems in which different boundary conditions hold on different connected components of the boundary, such as e.g. scattering problems with impedance condition. In such cases, the results of this chapter may be extended by localization techniques (see Remark 6.3.6). For truly mixed boundary conditions, i.e. when different boundary conditions hold on different parts of the boundary that are connected, one has to find appropriate shift theorems; we refer to the work of Jochmann [76, 77].

The regularity of the solution and its curl drive the convergence order of edge finite element discretization, for Dirichlet and Neumann problems. This is done by standard numerical analysis arguments when the form is coercive. This is illustrated by some simple numerical experiments: we recover the expected convergence rate for the edge finite element method. We also observe the impact of the material parameters on the accuracy of the results. It deteriorates when the coervity constant of the problem decreases. For the Robin problem, our result gives a gap of $1/2$ between the regularity exponents and the convergence order of the method; this gap is not observed in the numerical experiments. This call to further developments; in fact, we believe that the theoretical result could be enhanced. For non-coercive problems, a uniform discrete inf-sup condition has to be proven; we refer to [64].

Chapter 7

Analysis of Domain Decomposition for anisotropic Maxwell problems

Generally speaking, Domain Decomposition Methods (DDM) refer to methods that consist in solving a global problem making use of smaller problems associated to subdomains of the original domain. They are designed to allow parallel solving procedures and parallel preconditioning, in order to reduce the computational cost of the solution. These methods rely on the bet that it is less costly to solve several times small problems than to solve once one huge problem. In the case of parallel solvers, the PDE problem is reformulated as coupled PDE local problems.

In this chapter, we focus on DDM for electromagnetic problems in anisotropic media. To the best of our knowledge, only a few contributions have already tackled this topic. Here, we explore this topic in three different directions, focusing mainly on considerations that are related to PDEs analysis. We study various decomposed formulations which may be used in DDMs, and analyse their equivalence to the global problem. We investigate the convergence of a classical iterative procedure designed at the continuous level. Finally, we explore the influence of impedance condition for anisotropic problems on numerical experiments.

Section 7.1 is devoted to the introduction of general concepts. Section 7.2 presents the state of the art. In Section 7.3, we study several decomposed formulations for anisotropic Maxwell problems, focusing on functional analysis aspects and equivalence with the global problem. In Section 7.4, we investigate the convergence of an iterative DD procedure. Section 7.5 is devoted to numerical experiments.

7.1 Domain partition and general notions

To begin with, let us introduce the geometric concepts and notations that arise from the domain partitioning.

Partitioning classes

We shall consider a decomposition of the domain Ω into subdomains (Ω_i), s.t. $\overline{\Omega} = \bigcup_{i=1}^{N_d} \overline{\Omega}_i$. This decomposition can be *overlapping* or not. If the decomposition is overlapping, there exists regions where two (or more) subdomains intersect, then called *overlaps*. In the following, we shall mainly focus on non-overlapping decompositions. In this case, the subdomains (Ω_i) form a true partition of the original domain Ω, s.t. $\Omega_i \cap \Omega_j = \emptyset$, $\forall i \neq j$. A global field can be defined by simply putting together all fields defined on each subdomain. The boundary of each subdomain Ω_i can be made of several parts. One part may (or may not) coincide with the actual boundary of the global domain Ω. In this case, we denote it $\Gamma_i := \Gamma \cap \partial\Omega_i$, and call it the *exterior boundary* of Ω_i. On the other hand, the other parts of the boundary of Ω_i are artificial borders caused by the domain partition. We call them the *interfaces*. For $i \neq j$, we denote $\Sigma_{ij} := \mathrm{int}(\partial\Omega_i \cap \partial\Omega_j)$ the interface between Ω_i and Ω_j if its Hausdorff dimension is 2. Otherwise, we use the convention $\Sigma_{ij} = \emptyset$. The reunion of all interfaces is called the *skeleton* of the decomposition, and denoted $\Sigma := \bigcup_{i,j\neq i} \Sigma_{ij}$.

The partition of the domain can be realized in different manners, allowing cross edges/points or not. Let us distinguish two types of them: *boundary cross edges/points*, where an interface intersects the exterior boundary, and *interior cross edges/points*, where two interfaces intersect each other. Extra-difficulties can arise from these geometries. We regroup them in three categories of growing size:

1. The decomposition is s.t. none of the interfaces intersect each other, nor intersect the exterior boundary (for example, if the subdomains are like the layers of an onion).

2. The decomposition is s.t. none of the interfaces intersect each other, but intersections between interfaces and the exterior boundary are allowed (for example, if the subdomains are like the slices of a cake). This is a decomposition with *boundary* cross edges/points, but without *interior* cross edges/points.

3. The most general case, in which interfaces can intersect each other and intersect the exterior border as well. This is a decomposition with both *boundary* and *interior* cross edges/points.

Some methods are designed only for decompositions of type 1 or types 1 and 2. In our work, we allow all three types of decompositions.

PDE and algebraic considerations

The problem decomposition can be done at the continuous level or at the discrete level. In a large amount of contributions, the splitting is done at the discrete level. In this case, the global problem is a huge algebraic system to solve, leading to considerations that are more related to numerical linear algebra. The linear system has a block structure, each block corresponding to a subdomain: the system is mainly block-diagonal, with extra-diagonal parts corresponding to the coupling between subdomains at the interfaces or overlapping regions. The main question is then how to solve efficiently this huge system, preferably in a parallel manner. This is generally done through an iterative procedure. In a solver approach, one designs directly an iterative DDM procedure that will converge to the solution of the global problem. In a preconditioning approach, one preconditions the system taking advantage of the decomposition. The preconditioned system can then be solved using standard iterative solvers (block Jacobi, GMRES, ...). However, this discrete point of view is not the one in the core of our work.

In this work, we introduce the splitting at the PDE level, and focus on considerations that are more related to PDEs analysis. This allows us to take advantage of the inner properties of the problem. Introducing a splitting leaves you with local problems that are *a priori* incomplete, because of the extra-boundaries introduced between subdomains (the *interfaces*) with no corresponding boundary conditions. From an analytic point of view, the boundary conditions to be set on the interfaces (then called *interface* or *transmission conditions*) play a critical role to ensure that the decomposed problem is equivalent to the global problem, i.e., that it is possible to reconstruct the global solution from the solutions of local problems. In DDMs, the decomposed problem is used to define parallel solving procedures or parallel preconditioners. In the first case, one solves a sequence of local problems, whose solutions are expected to converge to the global solution. This is for example the case of the original Schwarz algorithm. Then, one has to ensure that each local problem is well-posed, and that the sequence of solutions given by the iterative procedure will indeed converge to the global solution. In the preconditioning case, one simply has to ensure that each of the local problems is well-posed.

7.2 State of the art of Domain Decomposition for Maxwell problems

As domain decomposition involves a lot of different aspects (PDE problems, numeric linear algebra, etc...), one can find a broad range of approaches in the literature. For a general overview, we refer to the monographs of Quarteroni and Valli [97], Toselli and Widlund [110], Mathew [85] and Dolean, Jolivet and Nataf [47]. In these works, the reference study case is the Laplace equation. However, dealing with other types of problems often bring extra-difficulties. Therefore, let us review the main works that have been done in domain decomposition for Maxwell equations.

For definite (or *static*) Maxwell problems, that are of type

$$\mathbf{curl}\,\eta_2\,\mathbf{curl}\,\mathbf{u} + \eta_1\mathbf{u} = \mathbf{f}, \tag{7.1}$$

with $\eta_1, \eta_2 > 0$, various domain decomposition methods have been studied by Toselli and collaborators. For such problems, the PhD thesis of Toselli [106] is a good starting point. Overlapping Schwarz preconditioning methods have been proposed in [71, 107], for scalar constant coefficients only. Convergence and conditioning results are provided. Some improvements were provided by [92]. There, the standard Schwarz method is used, with Dirichlet conditions at the boundaries of subdomains: only the values of the field on the interfaces are exchanged between subdomains. On the other hand, substructuring methods (that are algorithms with different levels of acceleration, such as Schur methods) have also been proposed, first in 2D by [109, 111, 98] and later by [44]. 3D problems have been considered in [73] with scalar coefficients, and with symmetric definite positive coefficients in [108]. In all these works, the splitting is done at the discrete level, and the focus is set on numeric linear algebra considerations. In [72], a preconditioner for static Maxwell problems that makes use of decomposition into regular and singular part has been proposed.

When it comes to time-harmonic problems, the Maxwell problem becomes indefinite (the variational formulation is no longer coercive), which poses extra-difficulty. An additive Schwarz preconditioner has been proposed in [60]. In [13], a two-level preconditioner is proposed, and dependance with respect to the wavenumber is discussed. Most of the time, convergence analysis is provided only in the case of two subdomains with plane interface. However, ordinary Schwarz methods have trouble converging in this case, as was explained by Ernst and Gander [54], even for (scalar) Helmholtz problems. For that reason, when dealing with time-harmonic problems, it is necessary to design more elaborate methods than classical iterative methods by improving the coupling between subdomains, for example using more elaborate transmission conditions than just Dirichlet.

The first use of transmission conditions for time-harmonic Maxwell problems is attributed to Després, Joly and Roberts [42, 43]. There, a method with impedance transmission conditions, exchanging Robin traces instead of just Dirichlet, is proposed. This choice of transmission condition is related to the Silver-Müller condition, which is the zeroth-order absorbing boundary condition for electromagnetic waves. An iterative procedure is proposed that makes use of impedance transmission conditions linking the solution at one iteration to the next. The convergence at the continuous level is proven whatever the number of subdomains. Discretization using mixed finite elements is also discussed. Offsprings of the Schwarz method that make use of this type of transmission condition (or more elaborate ones) are generally called *optimized Schwarz methods*.

A significant step for the understanding of domain decomposition for time-harmonic Maxwell problems at the PDE level has also been made by Alonso and Valli [3]. In that work, only the low-frequency case with absorption is considered, s.t. the variational formulation of the problem is coercive. Yet, a decomposed problem is written at the continuous level, and equivalence with the global problem is proven. Moreover, an iterative procedure is proposed both at the continuous level and at the discrete level, and a proof of convergence (for 2 domains) is provided.

Later, a number of variations of impedance conditions have been proposed, most of them inspired by non-reflecting boundary conditions techniques. Collino et al. proposed an integral transmission operator and studied several of its approximations [32, 33]. A variety of second-order conditions have been studied by Alonso and Gerardo [5], with comparable works in [45] and [96]. A more general second-order condition was then proposed by Peng, Rawat and Lee [99]. Edge and corner treatment has further been proposed in [94, 95]. The overlapping counterpart was proposed in [46]. More elaborate transmission conditions using Padé approximations were used in [52, 84]. PML-based transmission conditions were introduced in [101].

Let us also note that some authors have studied the first-order Maxwell system, discretized by discontinuous Galerkin methods. Optimized Schwarz techniques, using absorbing boundary conditions of order 0 [48] or 2 [45, 51], have been proposed. Similar results have been obtained with hybridizable discontinuous Galerkin discretizations in [81, 67].

Alternatively, some methods specifically enforce continuity condition at the interfaces. Saddle-points formulations have been proposed, first for static Maxwell problems [74]. For time-harmonic problems, it has been proposed in the context of plasma physics in [8].

It is worth noting that most of these works focus on isotropic (and often homogeneous) problems. Up to our knowledge, only a very few authors addressed domain decomposition for anisotropic Maxwell problems. This is the case of Toselli [108] for coercive problems with real symmetric tensors; and of Back et al. [8] for a plasma problem, which is also coercive – but not Hermitian.

7.3 Decomposed formulations for anistropic problems

Let us consider the time-harmonic Maxwell equation in the domain Ω, completed (for simplicity) with a Robin boundary condition on Γ. The global problem reads

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E} = \mathbf{f} & \text{in } \Omega, \\ \pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}) + \alpha\gamma^T\mathbf{E} = \mathbf{g} & \text{on } \Gamma. \end{cases} \tag{7.2}$$

where we assume that Γ has no pathological vertices, $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{g} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \cap \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, $\underline{\boldsymbol{\varepsilon}}$, $\underline{\boldsymbol{\mu}}$ are elliptic, and the parameter α is elliptic and s.t. the boundary condition holds in $\mathbf{L}_t^2(\Gamma)$. Under these hypotheses, the boundary condition of (7.2) holds in $\mathbf{L}_t^2(\Gamma)$, and one can derive an equivalent variational formulation to the global problem (7.2) in $\mathbf{H}^+(\mathbf{curl}, \Omega)$. This has been done in Section 5.3. The variational formulation reads:

$$\left|\begin{array}{l} \text{Find } \mathbf{E} \in \mathbf{H}^+(\mathbf{curl}, \Omega) \text{ s.t., } \forall \mathbf{F} \in \mathbf{H}^+(\mathbf{curl}, \Omega), \\ \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E} \middle| \,\mathbf{curl}\,\mathbf{F}\right) - \omega^2\left(\underline{\boldsymbol{\varepsilon}}\mathbf{E}|\mathbf{F}\right) - \left(\alpha\gamma^T\mathbf{E}, \gamma^T\mathbf{F}\right)_\Gamma = (\mathbf{f}|\mathbf{F}) - \left(\mathbf{g}, \gamma^T\mathbf{F}\right)_\Gamma. \end{array}\right. \tag{7.3}$$

Remark 7.3.1. In this work, we focus on the problem with impedance boundary condition, because our aim is to derive a DDM based on impedance transmission conditions. Thus, having the same type of exterior boundary condition makes the analysis easier. However, one could consider other types of exterior boundary conditions; in this case, note that if the DDM is not onion-like, the local problems involve mixed boundary conditions, which require a special attention.

7.3.1 First decomposed formulations

Let us introduce a non-overlapping decomposition $(\Omega_i)_i$ of Ω, as defined before. For a given field $\mathbf{v}$ defined on Ω, we use the index i to denote its restriction on the subdomain Ω_i:

$$\mathbf{v}_i := \mathbf{v}_{|\Omega_i}. \tag{7.4}$$

We also denote $\mathbf{n}_i$ the *outward* unit normal vector to the domain Ω_i. Thus, on the interface Σ_{ij}, note that there holds $\mathbf{n}_i = -\mathbf{n}_j$. Because of this, we also have to make the difference between the traces γ_i^T and γ_j^T coming from one side or another, which then have opposite sign: $\gamma_i^T = -\gamma_j^T$. On the contrary, $\pi_i^T = \pi_j^T$, because of the double cross product.

A priori, we shall look for $\mathbf{E}$ solution to the decomposed problem in the space

$$\mathbf{PH}_{\mathrm{Rob}}(\mathbf{curl}, \Omega) := \left\{\mathbf{v} \in \mathbf{L}^2(\Omega),\ \forall i,\ \mathbf{v}_i \in \mathbf{H}(\mathbf{curl}, \Omega_i),\ \gamma_i^T\mathbf{v}_i \in \mathbf{L}_t^2(\Gamma_i)\right\}, \tag{7.5}$$

which is the subset of piecewise $\mathbf{H}(\mathbf{curl})$ fields that is well-suited to the exterior boundary condition of (7.2). Then, at the most elementary level, the decomposed problem reads as follows:

Find $\mathbf{E} \in \mathbf{PH}_{\mathrm{Rob}}(\mathbf{curl}, \Omega)$ s.t., $\forall i, j$,

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E}_i = \mathbf{f}_i & \text{in } \Omega_i, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_i^T\mathbf{E}_i = \mathbf{g}_i & \text{on } \Gamma_i, \\ \gamma_i^T(\mathbf{E}_i - \mathbf{E}_j) = \mathbf{0} & \text{on } \Sigma_{ij}, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}(\mathbf{E}_i - \mathbf{E}_j)) = \mathbf{0} & \text{on } \Sigma_{ij}, \end{cases} \tag{7.6}$$

where $\mathbf{f}_i := \mathbf{f}_{|\Omega_i}$ and $\mathbf{g}_i := \mathbf{g}_{|\Gamma_i}$.

Remark 7.3.2. Similarly, the natural spaces for the Dirichlet or Neumann decomposed problems would be

$$\mathbf{PH}_{\mathrm{Dir}}(\mathbf{curl}, \Omega) := \left\{\mathbf{v} \in \mathbf{L}^2(\Omega),\ \forall i,\ \mathbf{v}_i \in \mathbf{H}(\mathbf{curl}, \Omega_i),\ \gamma_i^T\mathbf{v}_i = \mathbf{0} \text{ on } \Gamma_i\right\}; \tag{7.7}$$

$$\mathbf{PH}_{\mathrm{Neu}}(\mathbf{curl}, \Omega) := \left\{\mathbf{v} \in \mathbf{L}^2(\Omega),\ \forall i,\ \mathbf{v}_i \in \mathbf{H}(\mathbf{curl}, \Omega_i)\right\}. \tag{7.8}$$

The decomposed formulation (7.6) can be adapted to other boundary conditions straightforwardly.

Lemma 7.3.3. Let $\mathbf{E}$ solution to the global problem (7.2). Then $\mathbf{E}$ satisfies (7.6).

Proof. First of all, $\mathbf{E} \in \mathbf{H}^+(\mathbf{curl}, \Omega)$, so clearly for all i, $\mathbf{E}_i \in \mathbf{H}(\mathbf{curl}, \Omega_i)$, and $\gamma_i^T \mathbf{E}_i \in \mathbf{L}_t^2(\Gamma_i)$. Therefore, $\mathbf{E} \in \mathbf{PH}_{\mathrm{Rob}}(\mathbf{curl}, \Omega)$. As $\mathbf{E}$ is solution to the global problem (7.2), it satisfies the volume equation on each subdomain. Similarly, it satisfies the boundary condition on each part of the exterior border. Moreover, $\mathbf{E}$ has also to satisfy continuity conditions at the interfaces. Indeed, because $\mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega)$ *globally*, the Dirichlet trace $\gamma^T \mathbf{E}$ is continuous over all interfaces. Similarly, because of the volume equation of (7.2), there holds $\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega)$ globally. Hence, the Neumann trace $\pi^T(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E})$ is also continuous over any interface. To conclude, $\mathbf{E}$ satisfies (7.6). □

The system (7.6) can be understood as follows. First of all, the equation must be satisfied on each subdomain; similarly, the boundary condition must be satisfied on each part of the exterior boundary. Moreover, the continuity of the Dirichlet traces (*i.e.* $\gamma^T \mathbf{E}$) and Neumann traces (*i.e.* $\pi^T(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E})$) is enforced. The continuity of the Dirichlet trace ensures that the global solution belongs to the right function space, that is $\mathbf{H}(\mathbf{curl}, \Omega)$. It can be understood as an essential condition. The continuity of the Neumann trace, on the other hand, is more like a natural condition: it ensures that the equation will be satisfied globally (in the sense of distributions) on Ω, not just on each subdomain.

In the following, we shall want to take advantage of the regularity of the global solution. Therefore, we assume in the remaining of this work that the following hypothesis holds:

H.0 The problem (7.2) is s.t. $\exists\, s > \frac{1}{2}$, $\mathbf{E}$, $\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E} \in \mathbf{H}^s(\Omega)$.

This grants that the traces $\gamma^T \mathbf{E}$, $\pi^T(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E})$ belong to $\mathbf{L}_t^2(\Sigma_{ij})$ for any interface. This will be useful in the following of our work: in particular, it grants that one can split surface duality products on the different parts of the subdomains boundaries.

The validity of hypothesis H.0 can be discussed with the help of results in the spirit of Chapter 6, especially Theorem 6.3.1 which holds for domains of $\mathcal{C}^2$ boundary. If the domain is not $\mathcal{C}^2$, regularity results in the same spirit may also be obtained, e.g. if the domain is convex. More precisely, if the Shift Theorem 6.1.4 for Dirichlet problems holds in Ω, then Theorems 6.1.5, 6.2.1 and finally 6.3.1 also hold. If the domain is not convex, the regularity of the solution is also driven by behaviour at re-entrant edges and corners, see e.g. [35].

Then, let us introduce the space

$$\mathbf{PH}^+(\mathbf{curl}, \Omega) := \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega),\ \forall i,\ \mathbf{v}_i \in \mathbf{H}^+(\mathbf{curl}, \Omega_i) \right\}, \tag{7.9}$$

equipped with the norm $\|\mathbf{v}\|_{\mathbf{PH}^+(\mathbf{curl},\Omega)} = \sum_{i=1}^{N_d} \|\mathbf{v}_i\|_{\mathbf{H}^+(\mathbf{curl},\Omega_i)}$. We emphasize that, compared to $\mathbf{PH}_{\mathrm{Rob}}(\mathbf{curl}, \Omega)$, we impose not only that $\gamma_i^T \mathbf{v}_{|\Gamma_i} \in \mathbf{L}_t^2(\Gamma_i)$ on the exterior borders, but that $\gamma_i^T \mathbf{v}_i \in \mathbf{L}_t^2(\partial\Omega_i)$ on all subdomain boundaries; in particular, one has $\mathbf{L}_t^2$-regularity also on the interfaces. Then, one can also write a decomposed problem in the space $\mathbf{PH}^+(\mathbf{curl}, \Omega)$:

Find $\mathbf{E} \in \mathbf{PH}^+(\mathbf{curl}, \Omega)$ s.t., $\forall i, j$,

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}_i) - \omega^2 \underline{\boldsymbol{\varepsilon}} \mathbf{E}_i = \mathbf{f}_i & \text{in } \Omega_i, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}_i) + \alpha \gamma_i^T \mathbf{E}_i = \mathbf{g}_i & \text{on } \Gamma_i, \\ \gamma_i^T(\mathbf{E}_i - \mathbf{E}_j) = \mathbf{0} & \text{on } \Sigma_{ij}, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}(\mathbf{E}_i - \mathbf{E}_j)) = \mathbf{0} & \text{on } \Sigma_{ij}. \end{cases} \tag{7.10}$$

Lemma 7.3.4. Let $\mathbf{E}$ solution to the global problem (7.2) s.t. hypothesis H.0 is fulfilled. Then $\mathbf{E}$ satisfies (7.10).

Proof. First of all, $\mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega)$, so clearly $\mathbf{E}_i \in \mathbf{H}(\mathbf{curl}, \Omega_i)$ for all i. Moreover, thanks to hypothesis H.0, there holds $\mathbf{E} \in \mathbf{H}^s(\Omega)$ for some $s > \frac{1}{2}$. Since $\mathbf{H}_t^{s-1/2}(\Sigma_{ij}) \subset \mathbf{L}_t^2(\Sigma_{ij})$, this implies that the Dirichlet (and Neumann) traces satisfy

$$\gamma_i^T \mathbf{E}_i,\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}_i) \in \mathbf{L}_t^2(\Sigma_{ij}), \quad \forall i, j. \tag{7.11}$$

Because $\mathbf{E} \in \mathbf{H}^+(\mathbf{curl}, \Omega)$, the same holds for exterior borders. Therefore, one has indeed $\mathbf{E} \in \mathbf{PH}^+(\mathbf{curl}, \Omega)$. The rest of the proof is as in Lemma 7.3.3. □

7.3.2 A decomposed formulation with impedance conditions

Another way to write the decomposed problem is to combine both interface conditions into new ones, now involving linear combinations of both traces. To that aim, we extend α from Γ to $\Gamma \cup \Sigma$ in such a manner that $\alpha \in L^\infty(\Gamma \cup \Sigma)$ and is elliptic on $\Gamma \cup \Sigma$. Then, it is possible to write a new decomposed problem, with impedance transmission conditions:

Find $\mathbf{E} \in \mathbf{PH}^+(\mathbf{curl}, \Omega)$ s.t., $\forall i, j$,

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) - \omega^2 \underline{\varepsilon}\mathbf{E}_i = \mathbf{f}_i & \text{in } \Omega_i, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_i^T \mathbf{E}_i = \mathbf{g}_i & \text{on } \Gamma_i, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_i^T \mathbf{E}_i = \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) + \alpha\gamma_i^T \mathbf{E}_j & \text{on } \Sigma_{ij}. \end{cases} \tag{7.12}$$

Remark 7.3.5. The key argument here is that the last line of (7.12) in fact hides two interface conditions. Indeed, permuting i and j, one gets, because of the normal convention, the additional condition

$$\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) - \alpha\gamma_i^T \mathbf{E}_i = \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) - \alpha\gamma_i^T \mathbf{E}_j \quad \text{on } \Sigma_{ij}. \tag{7.13}$$

Therefore, we have indeed changed the two interface conditions of (7.10) into two other (independent) ones. As a result, both systems are equivalent.

Lemma 7.3.6. The systems (7.10) and (7.12) are equivalent: $\mathbf{E} \in \mathbf{PH}^+(\mathbf{curl}, \Omega)$ is solution of (7.10) iff it is solution of (7.12).

Proof. Let $\mathbf{E}$ solution of (7.10). Both interface conditions hold, in particular, in $\mathbf{L}^2_t(\Sigma_{ij})$. Thus, one can make linear combinations of them ($\alpha \in L^\infty(\Sigma_{ij})$), which leads to the last line of (7.12) holding in $\mathbf{L}^2_t(\Sigma_{ij})$.

Conversely, assume $\mathbf{E}$ is solution of (7.12), with interface conditions holding in $\mathbf{L}^2_t(\Sigma_{ij})$. Recall that one has in fact two conditions on Σ_{ij}:

$$\begin{cases} \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_i^T \mathbf{E}_i = \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) + \alpha\gamma_i^T \mathbf{E}_j, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) - \alpha\gamma_i^T \mathbf{E}_i = \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) - \alpha\gamma_i^T \mathbf{E}_j, \end{cases} \tag{7.14}$$

both holding in $\mathbf{L}^2_t(\Sigma_{ij})$. Thus, one can take the sum or the difference, to recover the interface conditions of (7.10). □

One can then write a variational formulation of this decomposed problem.

Lemma 7.3.7. The decomposed problem (7.12) is equivalent to the following variational formulation:

$$\left| \begin{aligned} &\text{Find } \mathbf{E} \in \mathbf{PH}^+(\mathbf{curl}, \Omega) \text{ s.t., } \forall \mathbf{F} \in \mathbf{PH}^+(\mathbf{curl}, \Omega), \\ &\sum_i \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i | \,\mathbf{curl}\,\mathbf{F}_i\right)_{\Omega_i} - \omega^2 \left(\underline{\varepsilon}\mathbf{E}_i | \mathbf{F}_i\right)_{\Omega_i} - \left(\alpha\gamma_i^T \mathbf{E}_i, \gamma_i^T \mathbf{F}_i\right)_{\Gamma_i} \\ &\quad + \sum_i \sum_{j \neq i} \left(\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\ \mathbf{E}_j) + \alpha\gamma_i^T \mathbf{E}_j - \alpha\gamma_i^T \mathbf{E}_i, \gamma_i^T \mathbf{F}_i\right)_{\Sigma_{ij}} = \sum_i \left(\mathbf{f}_i | \mathbf{F}_i\right)_{\Omega_i} - \left(\mathbf{g}_i, \gamma_i^T \mathbf{F}_i\right)_{\Gamma_i}. \end{aligned} \right. \tag{7.15}$$

Proof. The proof is similar to the one of Theorem 5.3.1 repeated on each subdomain. Note that, by assumption, both traces $\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_i^T \mathbf{E}_i$ and $\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) + \alpha\gamma_i^T \mathbf{E}_j$ belong to $\mathbf{L}^2_t(\Sigma_{ij})$. Therefore, on each subdomain, let us introduce

$$\widetilde{\mathbf{g}}_i \in \mathbf{L}^2_t(\partial\Omega_i) \text{ s.t. } \begin{cases} \widetilde{\mathbf{g}}_i = \mathbf{g}_i & \text{on } \Gamma_i; \\ \widetilde{\mathbf{g}}_i = \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) + \alpha\gamma_i^T \mathbf{E}_j & \text{on } \Sigma_{ij}. \end{cases}$$

Hence, multiplying by a test function $\mathbf{F}_i \in \mathbf{H}^+(\mathbf{curl}, \Omega_i)$ and integrating by parts on each subdomain, one has successively

$$\begin{aligned}
\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i|\,\mathbf{curl}\,\mathbf{F}_i\right)_{\Omega_i} - \omega^2\,(\underline{\boldsymbol{\varepsilon}}\mathbf{E}_i|\mathbf{F}_i)_{\Omega_i} + {}_\pi\langle\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i), \gamma_i^T\mathbf{F}_i\rangle_\gamma &= (\mathbf{f}_i|\mathbf{F}_i)_{\Omega_i}\\
\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i|\,\mathbf{curl}\,\mathbf{F}_i\right)_{\Omega_i} - \omega^2\,(\underline{\boldsymbol{\varepsilon}}\mathbf{E}_i|\mathbf{F}_i)_{\Omega_i} + {}_\pi\langle\widetilde{\mathbf{g}}_i - \alpha\gamma_i^T\mathbf{E}_i, \gamma_i^T\mathbf{F}_i\rangle_\gamma &= (\mathbf{f}_i|\mathbf{F}_i)_{\Omega_i}\\
\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i|\,\mathbf{curl}\,\mathbf{F}_i\right)_{\Omega_i} - \omega^2\,(\underline{\boldsymbol{\varepsilon}}\mathbf{E}_i|\mathbf{F}_i)_{\Omega_i} + \left(\widetilde{\mathbf{g}}_i - \alpha\gamma_i^T\mathbf{E}_i, \gamma_i^T\mathbf{F}_i\right)_{\partial\Omega_i} &= (\mathbf{f}_i|\mathbf{F}_i)_{\Omega_i}
\end{aligned}$$

This is legitimate because $\widetilde{\mathbf{g}}_i$, $\gamma_i^T\mathbf{E}_i \in \mathbf{L}_t^2(\partial\Omega_i)$. Therefore, one can split the inner product on the different parts of $\partial\Omega_i$, and, using the interface conditions of (7.12):

$$\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i|\,\mathbf{curl}\,\mathbf{F}_i\right)_{\Omega_i} - \omega^2\,(\underline{\boldsymbol{\varepsilon}}\mathbf{E}_i|\mathbf{F}_i)_{\Omega_i} + \left(\mathbf{g}_i - \alpha\gamma_i^T\mathbf{E}_i, \gamma_i^T\mathbf{F}_i\right)_{\Gamma_i} + \sum_{j\neq i}\left(\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) + \alpha\gamma_i^T\mathbf{E}_j - \alpha\gamma_i^T\mathbf{E}_i, \gamma_i^T\mathbf{F}_i\right)_{\Sigma_{ij}} = (\mathbf{f}_i|\mathbf{F}_i)_{\Omega_i}.$$

Summing over the subdomains gives formulation (7.15).

Conversely, one can also proceed by subdomain. Taking $\mathbf{F}_i \in \boldsymbol{\mathcal{D}}(\Omega_i)$ in (7.15) and differentiating in $\boldsymbol{\mathcal{D}}'(\Omega_i)$, one recovers the volume equation

$$\mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E}_i = \mathbf{f}_i \quad \text{in } \Omega_i.$$

In the following, letting Γ' an open subset of Γ, we recall the spaces (cf. [7])

$$\widetilde{\mathbf{H}}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma') := \pi_{\Gamma'}^T\left(\mathbf{H}(\mathbf{curl}, \Omega)\right), \tag{7.16}$$

$$\mathbf{H}_{\parallel,0}^{-1/2}(\mathrm{div}_\Gamma, \Gamma') := \gamma_{\Gamma'}^T\left(\mathbf{H}_{0,\Gamma\setminus\Gamma'}(\mathbf{curl}, \Omega)\right), \tag{7.17}$$

where $\pi_{\Gamma'}^T$, $\gamma_{\Gamma'}^T$ denote the restrictions of traces on Γ'. Both spaces are in duality, and the duality product is denoted for short ${}_{\Gamma',\pi}\langle\cdot,\cdot\rangle_{\gamma,0}$.

First, we take $\Gamma' = \Gamma_i$. We choose test functions $\mathbf{F}_i \in \{\mathbf{v} \in \boldsymbol{\mathcal{C}}^\infty(\overline{\Omega}_i)$, that vanish in a neighbourhood of $\partial\Omega_i \setminus \Gamma_i\}$, and get that ${}_{\Gamma_i,\pi}\langle\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_i^T\mathbf{E}_i, \gamma_i^T\mathbf{F}_i\rangle_{\gamma,0} = {}_{\Gamma_i,\pi}\langle\mathbf{g}_i, \gamma_i^T\mathbf{F}_i\rangle_{\gamma,0}$. Because the space of test functions is dense in $\mathbf{H}_{0,\partial\Omega_i\setminus\Gamma_i}(\mathbf{curl}, \Omega)$ (see [7, Def. 2.2.27]), and γ_i^T is surjective from the latter to $\mathbf{H}_{\parallel,0}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i)$, by density arguments, one gets that

$$\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_i^T\mathbf{E}_i = \mathbf{g}_i$$

holds in $\widetilde{\mathbf{H}}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_i) = \left(\mathbf{H}_{\parallel,0}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_i)\right)'$. Because $\mathbf{g}_i$, $\gamma_i^T\mathbf{E}_i$, $\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) \in \mathbf{L}_t^2(\Gamma_i)$, the result also holds in $\mathbf{L}_t^2(\Gamma_i)$.

We proceed similarly on the interfaces. Taking $\Gamma' = \Sigma_{ij}$, and choosing test functions $\mathbf{F}_i \in \boldsymbol{\mathcal{C}}^\infty(\overline{\Omega}_i)$ that vanish in a neighbourhood of $\partial\Omega_i \setminus \Sigma_{ij}$, one has ${}_{\Sigma_{ij},\pi}\langle\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_i^T\mathbf{E}_i, \gamma_i^T\mathbf{F}_i\rangle_{\gamma,0} = {}_{\Sigma_{ij},\pi}\langle\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) + \alpha\gamma_i^T\mathbf{E}_j, \gamma_i^T\mathbf{F}_i\rangle_{\gamma,0}$. By density,

$$\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_i^T\mathbf{E}_i = \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) + \alpha\gamma_i^T\mathbf{E}_j$$

holds in $\widetilde{\mathbf{H}}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Sigma_{ij})$, and in $\mathbf{L}_t^2(\Sigma_{ij})$ under hypothesis H.0. □

Lemma 7.3.8. The solution to the variational formulation (7.15) satisfies the global variational formulation (7.3).

Proof. Let $\mathbf{E} \in \mathbf{PH}^+(\mathbf{curl}, \Omega)$ solution to the the variational formulation (7.15). First of all, let us note that $\mathbf{E}$ satisfies the interface conditions of (7.12), then of (7.10). In particular, the Dirichlet trace is continuous over all the interfaces, so one has in fact $\mathbf{E} \in \mathbf{H}^+(\mathbf{curl}, \Omega)$. Let us take $\mathbf{F} \in \mathbf{H}^+(\mathbf{curl}, \Omega)$ in (7.15), and look at the interface terms. For each interface Σ_{ij}, there are two contributions, coming from the terms associated to the subproblems in Ω_i and Ω_j. Grouping both terms yields

$$\left(\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) + \alpha\gamma_i^T\mathbf{E}_j - \alpha\gamma_i^T\mathbf{E}_i, \gamma_i^T\mathbf{F}_i\right)_{\Sigma_{ij}} + \left(\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_j^T\mathbf{E}_i - \alpha\gamma_j^T\mathbf{E}_j, \gamma_j^T\mathbf{F}_j\right)_{\Sigma_{ij}}.$$

Note that, since $\mathbf{F} \in \mathbf{H}^{+}(\mathbf{curl}, \Omega)$, one has $\gamma_i^T \mathbf{F}_i = -\gamma_j^T \mathbf{F}_j$ (because of the normal convention). Therefore, the previous line becomes

$$\left(\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) + \alpha\gamma_i^T\mathbf{E}_j - \alpha\gamma_i^T\mathbf{E}_i, \gamma_i^T\mathbf{F}_i\right)_{\Sigma_{ij}} - \left(\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) + \alpha\gamma_j^T\mathbf{E}_i - \alpha\gamma_j^T\mathbf{E}_j, \gamma_i^T\mathbf{F}_i\right)_{\Sigma_{ij}}$$

and, because of the normal convention,

$$\left(\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_j) + \alpha\gamma_i^T\mathbf{E}_j - \alpha\gamma_i^T\mathbf{E}_i, \gamma_i^T\mathbf{F}_i\right)_{\Sigma_{ij}} - \left(\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i) - \alpha\gamma_i^T\mathbf{E}_i + \alpha\gamma_i^T\mathbf{E}_j, \gamma_i^T\mathbf{F}_i\right)_{\Sigma_{ij}},$$

which reduces to

$$\left(\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}(\mathbf{E}_j - \mathbf{E}_i)), \gamma_i^T\mathbf{F}_i\right)_{\Sigma_{ij}}.$$

We recall that $\mathbf{E}$ satisfies the interface conditions of (7.10), and now use the fact that the Neumann trace is continuous over the interface. Hence, the above term is null. Therefore, all interface terms in (7.15) vanish, and the variational formulation (7.15) reduces to

$$\sum_i \left((\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E}_i |\, \mathbf{curl}\,\mathbf{F}_i)_{\Omega_i} - \omega^2\,(\underline{\boldsymbol{\varepsilon}}\mathbf{E}_i|\mathbf{F}_i)_{\Omega_i} + \left(\mathbf{g}_i - \alpha\gamma_i^T\mathbf{E}_i, \gamma_i^T\mathbf{F}_i\right)_{\Gamma_i} \right) = \sum_i (\mathbf{f}_i|\mathbf{F}_i)_{\Omega_i}.$$

Grouping altogether the scalar products on all Ω_i, resp. Γ_i, one recovers the variational formulation (7.3) of the global problem. □

Theorem 7.3.9. Under hypothesis H.0, the global problem (7.2), the global variational formulation (7.3), the decomposed problems (7.10) and (7.12) and the associated variational formulation (7.15) are all equivalent.

Proof. We have shown that the global problem (7.2) and its variational formulation (7.3) are equivalent (Theorem 5.3.1); that the solution of the global problem (7.2) satisfies the decomposed problem (7.10) (Lemma 7.3.4); that the decomposed problems (7.10) and (7.12) are equivalent (Lemma 7.3.6); and that the problem (7.12) is equivalent to the variational formulation (7.15) (Lemma 7.3.7); and that the solution to the variational formulation (7.15) also satisfies the global variational formulation (7.3) (Lemma 7.3.8). Therefore, all these problems are equivalent. □

Remark 7.3.10. In principle, the results of this section could be extended even if hypothesis H.0 does not hold, i.e. if one has less than $\mathbf{L}_t^2$-regularity on the interfaces. However, the proofs become more tedious. Indeed, one can no longer split the surface duality products into the various parts of the boundaries, and ${}_{\Gamma',\pi}\langle\cdot,\cdot\rangle_{\gamma,0}$-type duality products have to be considered all along the way.

7.3.3 A comment on a formulation with Lagrange multiplier

In this section, we make some remarks on a domain decomposition method proposed by Back *et al.* [8] for a plasma physics problem. They consider the homogeneous Dirichlet problem

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\;\,\mathbf{E}) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E} = \mathbf{f} & \text{in } \Omega, \\ \gamma^T\mathbf{E} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{7.18}$$

where the material tensors $\underline{\boldsymbol{\mu}}$ and $\underline{\boldsymbol{\varepsilon}}$ are elliptic, the angular frequency $\omega > 0$, and the source term $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Similarly to Section 7.3.1, the starting point is to consider the decomposed problem

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\;\,\mathbf{E}_i) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E}_i = \mathbf{f}_i & \text{in } \Omega_i, \\ \gamma_i^T\mathbf{E}_i = \mathbf{0} & \text{on } \Gamma_i, \\ \gamma_i^T(\mathbf{E}_i - \mathbf{E}_j) = \mathbf{0} & \text{on } \Sigma_{ij}, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}(\mathbf{E}_i - \mathbf{E}_j)) = \mathbf{0} & \text{on } \Sigma_{ij}, \end{cases} \tag{7.19}$$

where the last two equations enforce the continuity of the Dirichlet and Neumann traces on the interface Σ_{ij}. The continuity of the Neumann trace can be considered as a natural condition, whereas the continuity of the Dirichlet trace can be considered as an essential condition (usually enforced in the variational space).

As before, the idea is to set the variational formulation not in $\mathbf{H}_0(\mathbf{curl}, \Omega)$, but in the appropriate piecewise $\mathbf{H}(\mathbf{curl})$ space,

$$\mathbf{V} := \left\{\mathbf{v} \in \mathbf{L}^2(\Omega), \mathbf{v}_i \in \mathbf{H}_{0,\Gamma_i}(\mathbf{curl}, \Omega_i) \text{ on each } \Omega_i\right\} \tag{7.20}$$

(that corresponds to the space $\mathbf{PH}_{\mathrm{Dir}}(\mathbf{curl}, \Omega)$ defined in (7.7)). With this choice of function space, [8] relaxes the "essential" Dirichlet continuity condition. Instead, the continuity of the Dirichlet trace is seen as a constraint, which is taken in account by a saddle-point approach. One dualizes the constraint by introducing a Lagrange multiplier, which leads to a mixed formulation. For each Σ_{ij}, one introduces the jump across Σ_{ij}, denoted $[\mathbf{v} \times \mathbf{n}]_{\Sigma_{ij}} := \mathbf{v}_i \times \mathbf{n}_i + \mathbf{v}_j \times \mathbf{n}_j$. The collection of jumps is denoted $[\mathbf{v} \times \mathbf{n}]_{\Sigma}$. One also introduces the space of jumps of fields of $\mathbf{V}$,

$$\mathbf{S} := \{[\mathbf{v} \times \mathbf{n}]_{\Sigma}, \mathbf{v} \in \mathbf{V}\}. \tag{7.21}$$

The space of Lagrange multipliers is chosen as $\boldsymbol{\Lambda} = \mathbf{S}'$, the dual space of $\mathbf{S}$. The saddle-point formulation reads

$$\left|\begin{array}{l} \text{Find } (\mathbf{E}, \boldsymbol{\lambda}) \in \mathbf{V} \times \mathbf{S}' \text{ s.t., } \forall (\mathbf{F}, \boldsymbol{\mu}) \in \mathbf{V} \times \mathbf{S}', \\ \displaystyle\sum_i \left((\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}_i | \mathbf{curl}\, \mathbf{F}_i)_{\Omega_i} - \omega^2 \left(\underline{\varepsilon} \mathbf{E}_i | \mathbf{F}_i\right)_{\Omega_i} \right) + \langle \boldsymbol{\lambda}, [\mathbf{F} \times \mathbf{n}]_{\Sigma} \rangle_{\mathbf{S}} = \sum_i (\mathbf{f} | \mathbf{F}_i)_{\Omega_i}, \\ \hfill \overline{\langle \boldsymbol{\mu}, [\mathbf{E} \times \mathbf{n}]_{\Sigma} \rangle_{\mathbf{S}}} = 0. \end{array}\right. \tag{7.22}$$

In [8], the authors propose a proof for the well-posedness of (7.22) and for its equivalence with the global problem (7.18). In their proof, they decompose the $\mathbf{S}/\mathbf{S}'$-duality product in the following way:

$$\langle \boldsymbol{\lambda}, [\mathbf{v} \times \mathbf{n}]_{\Sigma} \rangle_{\mathbf{S}} = \sum_{i,j \neq i} \left\langle \boldsymbol{\lambda}^{ij}, [\mathbf{v} \times \mathbf{n}]_{\Sigma_{ij}} \right\rangle = \sum_i \langle \boldsymbol{\lambda}, \mathbf{v}_i \times \mathbf{n}_i \rangle_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma}, \partial\Omega_i)}, \tag{7.23}$$

the duality in the second term being *"between the suitable spaces"* (not defined in [8]). However, we believe that it is unlikely that the relation (7.23) can be meaningful in general. To us, it is unclear whether $\mathbf{S}$ is a Banach, or Hilbert space, and how the $\mathbf{S}/\mathbf{S}'$-duality is defined. Moreover, given an element of $\mathbf{S}$, we do not really understand what is the correct space for each interface term $[\mathbf{v} \times \mathbf{n}]_{\Sigma_{ij}}$, and we wonder how one can justify the splitting of the $\mathbf{S}/\mathbf{S}'$- duality product into duality products on each interface in (7.23).

Remark 7.3.11. Alternatively, a possible framework to address this difficulty would be the multi-trace framework developed by Claeys and Hiptmair, cf. [30, 29].

Clarifications on the function space S

On each Σ_{ij}, there are two well-known traces spaces (cf. [7]):

$$\widetilde{\mathbf{H}}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma}, \Sigma_{ij}) := \gamma^{T}_{\Sigma_{ij}} \left(\mathbf{H}(\mathbf{curl}, \Omega_i)\right), \tag{7.24}$$

$$\mathbf{H}_{\parallel,0}^{-1/2}(\mathrm{div}_{\Gamma}, \Sigma_{ij}) := \gamma^{T}_{\Sigma_{ij}} \left(\mathbf{H}_{0,\partial\Omega_i \setminus \Sigma_{ij}}(\mathbf{curl}, \Omega_i)\right), \tag{7.25}$$

which are both Hilbert spaces. For a better understanding of the space $\mathbf{S}$, we have the next result.

Proposition 7.3.12.

$$\bigoplus \mathbf{H}_{\parallel,0}^{-1/2}(\mathrm{div}_{\Gamma}, \Sigma_{ij}) \subset \mathbf{S} \subset \bigoplus \widetilde{\mathbf{H}}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma}, \Sigma_{ij}). \tag{7.26}$$

Proof. The first embedding is proven in [8, Lemma 5.3]. Let $\boldsymbol{\varphi} \in \mathbf{H}_{\parallel,0}^{-1/2}(\mathrm{div}_{\Gamma}, \Sigma_{ij})$. There exists $\mathbf{v}_i$ in $\mathbf{H}_{0,\partial\Omega_i \setminus \Sigma_{ij}}(\mathbf{curl}, \Omega_i)$ such that $\boldsymbol{\varphi} = \mathbf{v}_i \times \mathbf{n}_{|\Sigma_{ij}}$. Letting $\mathbf{v} = \mathbf{v}_i$ in Ω_i and $\mathbf{v} = \mathbf{0}$ in $\Omega \setminus \Omega_i$, one has $\mathbf{v} \in \mathbf{V}$, with $[\mathbf{v} \times \mathbf{n}]_{\Sigma_{ij}} = \boldsymbol{\varphi}$, and all other jumps being null. The second embedding is straightforward: for all $\mathbf{v} \in \mathbf{V}$, each jump $[\mathbf{v} \times \mathbf{n}]_{\Sigma_{ij}}$ belongs to $\widetilde{\mathbf{H}}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma}, \Sigma_{ij})$. □

We believe that none of the inclusions above is surjective in general. According to [8], *"in addition, they have to satisfy some compatibility conditions"*, which are however not explicited. Nevertheless, one can be more precise in some specific cases:

- If the decomposition has no (interior nor boundary) cross edges/points *(e.g. onion-like decomposition)*, there holds $\mathbf{H}^{-1/2}_{\parallel,0}(\mathrm{div}_\Gamma, \Sigma_{ij}) = \widetilde{\mathbf{H}}^{-1/2}_{\parallel}(\mathrm{div}_\Gamma, \Sigma_{ij})\ \forall i, j$.
- If the decomposition has no interior cross edges/points *(e.g. slice-like decomposition)*, and a homogeneous Dirichlet condition is prescribed (i.e. $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on Γ), there holds $\mathbf{S} = \bigoplus \mathbf{H}^{-1/2}_{\parallel,0}(\mathrm{div}_\Gamma, \Sigma_{ij})$ (in this case, boundary cross points are allowed). Indeed, each jump belongs to $\mathbf{H}^{-1/2}_{\parallel,0}(\mathrm{div}_\Gamma, \Sigma_{ij})$.

In these cases, $\mathbf{S}$ is indeed a Hilbert space, and its duality is the one of $\bigoplus \mathbf{H}^{-1/2}_{\parallel,0}(\mathrm{div}_\Gamma, \Sigma_{ij})$, so that the saddle-point formulation (7.22) has a clear meaning. However, in other instances (if, for example, the decomposition has interior cross edges/points), this remains quite unclear.

In the light of our work, a more precise meaning could be given to the duality brackets in (7.23), depending on the regularity of traces on the interfaces, which is driven by the regularity of the field itself. Indeed, under hypothesis H.0, there holds $[\mathbf{E} \times \mathbf{n}]_{\Sigma_{ij}} \in \mathbf{H}^{s-1/2}_t(\Sigma_{ij}) \subset \mathbf{L}^2_t(\Sigma_{ij})$ for all i, j. In this case, one can choose

$$\mathbf{\Lambda} = \bigoplus \mathbf{L}^2_t(\Sigma_{ij}), \tag{7.27}$$

and the duality is given by the $\mathbf{L}^2_t$-inner product, s.t. the constraint in (7.22) is realized by

$$\forall \boldsymbol{\mu} \in \mathbf{\Lambda}, \quad (\boldsymbol{\mu}, [\mathbf{E} \times \mathbf{n}]_\Sigma)_{\mathbf{L}^2_t(\Sigma)} = 0. \tag{7.28}$$

It is then legitimate to split the duality products (which are integrals) on each interface, and the relation (7.23) holds with $\mathbf{L}^2_t$-inner products. One can further note that, in [8], the authors showed that the solution, as well as its curl, belong to $\mathbf{H}^1(\Omega)$ (this was initially done to justify the use of Lagrange finite elements). Hence, hypothesis H.0 is fulfilled, and the reasoning above applies, which justifies the use of (7.23) in [8].

Remark 7.3.13. Alternatively, it would be possible to consider a formulation with $\mathbf{L}^2_t$ jumps, whatever the solution's regularity. Indeed, even if hypothesis H.0 is not fulfilled, one has that the jumps in (7.19) vanish, so that one can consider them in particular in $\mathbf{L}^2_t(\Sigma_{ij})$. Then, it is possible to set the variational problem in the space

$$\left\{\mathbf{v} \in \mathbf{V} \text{ s.t. } [\mathbf{v} \times \mathbf{n}]_{\Sigma_{ij}} \in \mathbf{L}^2_t(\Sigma_{ij})\right\}, \tag{7.29}$$

s.t., once again, one can choose the space $\mathbf{\Lambda} = \bigoplus \mathbf{L}^2_t(\Sigma_{ij})$ for the saddle-point formulation. This path has been proposed by Ciarlet, Jamelot and Kpadonou for diffusion problems [28].

7.4 Convergence of iterative procedure

We are now interested in an iterative procedure to solve the decomposed problem, where the subproblems are solved in parallel at each iteration. This generates a sequence of local solutions that is expected to converge to the global solution. This was proposed first by Lions [83] as an improvement of Schwarz algorithm (for Laplace equation). For time-harmonic Maxwell problems, the seminal work is the algorithm proposed by Després et al. [42, 43]. For analysis purposes, we focus on this well-known procedure in our work (although more elaborate methods have later been proposed).

One chooses an initial guess $\mathbf{E}^0 \in \mathbf{PH}^+(\mathbf{curl}, \Omega)$ s.t. $\gamma_i^T \mathbf{E}_i,\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}_i^0) \in \mathbf{L}^2_t(\partial\Omega_i)\ \forall i$. Then one solves, $\forall n \geq 0, \forall i, j$,

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}_i^{n+1}) - \omega^2 \underline{\boldsymbol{\varepsilon}} \mathbf{E}_i^{n+1} = \mathbf{f}_i & \text{in } \Omega_i, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}_i^{n+1}) + \alpha \gamma_i^T \mathbf{E}_i^{n+1} = \mathbf{g}_i & \text{on } \Gamma_i, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}_i^{n+1}) + \alpha \gamma_i^T \mathbf{E}_i^{n+1} = \pi_j^T(\underline{\boldsymbol{\mu}}^{-1} \mathbf{curl}\, \mathbf{E}_j^{n}) - \alpha \gamma_j^T \mathbf{E}_j^{n} & \text{on } \Sigma_{ij}. \end{cases} \tag{7.30}$$

This algorithm was first proposed for isotropic and constant tensors $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$, and a scalar and constant impedance coefficient $\alpha = i\omega$ [43]. The authors proved that, in this case, the algorithm converges to the global solution. In this section we propose an extension of their proof to the case of real symmetric definite positive tensors, still with $\alpha = i\omega$. Then, we investigate other cases with elliptic (non-Hermitian) tensors.

To that aim, we need some additional hypotheses:

H.1 The assumptions of Proposition 4.3.1 are satisfied in each sudomain Ω_i, i.e.

- the interfaces are defined s.t. there are no pathological vertices (which is easy to realise in practice);
- the parameter α is extended from Γ to Σ in a manner such that α is elliptic on $\Gamma \cup \Sigma$, and, for all i, the problem

$$\text{Find } \phi^{\text{sing}} \in H^{1/2}_{\text{zmv}}(\partial\Omega_i) \text{ s.t. } \operatorname{curl}_\Gamma(\alpha \, \mathbf{curl}_\Gamma \, \phi) = 0 \text{ on } \partial\Omega_i \tag{7.31}$$

has no singular solution.

This ensures that all Robin traces have $\mathbf{L}^2_t$-regularity on all subdomain boundaries, and in particular on all interfaces. In particular, we are able to split the surface duality products (which are integrals) on the different parts of the boundaries (exterior border and/or interfaces). In the following of the section, we generally focus on cases where the parameter α is constant over the boundary and interfaces. In this case, both assumptions of H.1 are clearly satisfied as soon as there are no pathological vertices on $\Sigma \cup \Gamma$ (cf. Theorem 4.2.1).

Remark 7.4.1. There are other, less obvious cases in which the hypothesis is fulfilled; for example, if α is smooth all over the boundary and interfaces, or if there are no cross edges/points and α is constant by interfaces. If there are cross edges/points *and* coefficient discontinuities, one has to check by hand that hypothesis H.1 is fulfilled, following the proof techniques of e.g. Section 4.2.3.

7.4.1 The real symmetric definite positive case

In this subsection, we extend the original proof of [43] to problems with real-valued symmetric definite positive tensor fields. As in the original proof we specifically assume that $\alpha = i\omega$.

Proposition 7.4.2. Each of the local problems appearing in (7.30) is well-posed.

Proof. The proof is obtained recursively. Assume that at step n, all Dirichlet and Neumann traces belong to $\mathbf{L}^2_t(\Sigma_{ij})$ on any interface. Then the $n+1$-th local problem in Ω_i enters Fredholm alternative (Theorem 5.3.3). Uniqueness is obtained thanks to a unique continuation principle [113]. Therefore, each local problem at step $n+1$ is well-posed. Moreover, because of Theorem 4.1.4 and assumption H.1, the boundary and interface conditions of (7.30) ensure that

$$\pi_i^T(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{E}_i^{n+1}), \; \gamma_i^T \mathbf{E}_i^{n+1} \in \mathbf{L}^2_t(\partial\Omega_i) \quad \forall i. \tag{7.32}$$

Therefore, all boundary data at step $n+1$ also belong to $\mathbf{L}^2_t(\Sigma_{ij})$ on all interfaces. ☐

For the convergence study, let us introduce the error $\mathbf{e}^n = \mathbf{E} - \mathbf{E}^n \in \mathbf{PH}^+(\mathbf{curl}, \Omega)$, where $\mathbf{E}$ is the solution to the global problem (7.2). It satisfies the sequence of problems

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{e}_i^{n+1}) - \omega^2 \underline{\varepsilon} \mathbf{e}_i^{n+1} = \mathbf{0} & \text{in } \Omega_i, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{e}_i^{n+1}) + i\omega\gamma_i^T \mathbf{e}_i^{n+1} = \mathbf{0} & \text{on } \Gamma_i, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{e}_i^{n+1}) + i\omega\gamma_i^T \mathbf{e}_i^{n+1} = \pi_j^T(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{e}_j^{n}) - i\omega\gamma_j^T \mathbf{e}_j^{n} & \text{on } \Sigma_{ij}. \end{cases} \tag{7.33}$$

Let us introduce

$$U^n := \sum_i \sum_{j \neq i} \left\| \pi_i^T(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{e}_i^n) + i\omega\gamma_i^T \mathbf{e}_i^n \right\|^2_{\mathbf{L}^2_t(\Sigma_{ij})}. \tag{7.34}$$

The quantity U^n can be understood as a kind of energy on the interfaces. To study the convergence of $\mathbf{e}^n$, we will investigate the variations of U^n.

First, let us note the following property.

Proposition 7.4.3. Let $\mathcal{O}$ a domain, $\mathbf{v} \in \mathbf{H}^+(\mathbf{curl}, \mathcal{O})$ s.t. $\mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{v}) - \omega^2 \underline{\varepsilon} \mathbf{v} = \mathbf{0}$ in $\mathcal{O}$ and $\pi^T(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{v}) \in \mathbf{L}^2_t(\partial\mathcal{O})$. Then

$$\left\| \pi^T(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{v}) + i\omega\gamma^T \mathbf{v} \right\|^2_{\mathbf{L}^2_t} = \left\| \pi^T(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{v}) - i\omega\gamma^T \mathbf{v} \right\|^2_{\mathbf{L}^2_t} = \left\| \pi^T(\underline{\boldsymbol{\mu}}^{-1} \, \mathbf{curl} \, \mathbf{v}) \right\|^2_{\mathbf{L}^2_t} + \omega^2 \left\| \gamma^T \mathbf{v} \right\|^2_{\mathbf{L}^2_t}. \tag{7.35}$$

Proof. In all generality, there holds

$$\left\|\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}) + i\omega\gamma^T\mathbf{v}\right\|^2_{\mathbf{L}^2_t} = \left\|\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v})\right\|^2_{\mathbf{L}^2_t} + \omega^2\left\|\gamma^T\mathbf{v}\right\|^2_{\mathbf{L}^2_t} + 2\Re\left(\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}), i\omega\gamma^T\mathbf{v}\right)_{\partial\mathcal{O}}. \quad (7.36)$$

Besides, one has

$$\begin{aligned}\left(\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}), i\omega\gamma^T\mathbf{v}\right)_{\partial\mathcal{O}} &= -i\omega\left(\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}), \gamma^T\mathbf{v}\right)_{\partial\mathcal{O}}\\ &= i\omega\left[(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\,\mathbf{curl}\,\mathbf{v}) - (\mathbf{curl}\,\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\mathbf{v})\right]\\ &= i\omega\left[(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\,\mathbf{curl}\,\mathbf{v}) - \omega^2(\underline{\boldsymbol{\varepsilon}}\mathbf{v}|\mathbf{v})\right].\end{aligned}$$

As $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ are Hermitian, the terms $(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\,\mathbf{curl}\,\mathbf{v})$ and $(\underline{\boldsymbol{\varepsilon}}\mathbf{v}|\mathbf{v})$ are real. Thus $\Re\left(\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}), i\omega\gamma^T\mathbf{v}\right)_{\partial\mathcal{O}} = 0$, and hence the result. □

Remark 7.4.4. It is a kind of orthogonality property. The result relies strongly on the Hermitian character of the coefficients; in a more general case one only has (7.36).

The next lemma shows that the energy U^n decreases over the iterations.

Lemma 7.4.5. One has the estimate

$$U^{n+1} - U^n = -4\,\omega^2\|\gamma^T\mathbf{e}^n\|^2_{\mathbf{L}^2_t(\Gamma)}. \quad (7.37)$$

Proof. One has

$$U^{n+1} = \sum_i\sum_{j\neq i}\left\|\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n+1}) + i\omega\gamma_i^T\mathbf{e}_i^{n+1}\right\|^2_{\mathbf{L}^2_t(\Sigma_{ij})}.$$

Using the interface condition of (7.33),

$$\begin{aligned}U^{n+1} &= \sum_j\sum_{i\neq j}\left\|\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) - i\omega\gamma_j^T\mathbf{e}_j^{n}\right\|^2_{\mathbf{L}^2_t(\Sigma_{ij})}\\ &= \sum_j\left\|\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) - i\omega\gamma_j^T\mathbf{e}_j^{n}\right\|^2_{\mathbf{L}^2_t(\partial\Omega_j)} - \sum_j\left\|\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) - i\omega\gamma_j^T\mathbf{e}_j^{n}\right\|^2_{\mathbf{L}^2_t(\Gamma_j)},\end{aligned}$$

and, because of (7.35) for the first part, and of the exterior boundary condition for the second part,

$$\begin{aligned}U^{n+1} &= \sum_j\left\|\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) + i\omega\gamma_j^T\mathbf{e}_j^{n}\right\|^2_{\mathbf{L}^2_t(\partial\Omega_j)} - \sum_j\left\|-2i\omega\gamma_j^T\mathbf{e}_j^{n}\right\|^2_{\mathbf{L}^2_t(\Gamma_j)}\\ &= U^n - 4\,\omega^2\|\gamma^T\mathbf{e}^n\|^2_{\mathbf{L}^2_t(\Gamma)},\end{aligned}$$

which concludes the proof. □

Then, one can state the main result of this subsection.

Theorem 7.4.6. For all i, $\mathbf{e}_i^n$ converges to $\mathbf{0}$ weakly in $\mathbf{H}(\mathbf{curl},\Omega_i)$.

Proof. The sequence U^n is positive and decreasing, so it admits a limit $U^n \longrightarrow U$, and $U^{n+1} - U^n \longrightarrow 0$. Therefore, by Lemma 7.4.5,

$$\|\gamma^T\mathbf{e}^n\|^2_{\mathbf{L}^2_t(\Gamma)} \longrightarrow 0. \quad (7.38)$$

Besides, the sequence U^n is bounded, and $\|\mathbf{e}^n\|_{\mathbf{PH}^+(\mathbf{curl},\Omega)} \lesssim U^n$, because the problems (7.33) are well-posed with continuous dependence w.r.t. boundary data in $\mathbf{L}^2_t(\Gamma)$-norm, cf. Theorem 5.3.3. So, $\|\mathbf{e}^n\|_{\mathbf{PH}^+(\mathbf{curl},\Omega)}$ is also bounded, and it admits a subsequence (still denoted $\mathbf{e}^n$) that converges weakly to a certain $\mathbf{e} \in \mathbf{PH}^+(\mathbf{curl},\Omega)$: $\forall i$, $\mathbf{e}_i^n \rightharpoonup \mathbf{e}_i$ in $\mathbf{H}^+(\mathbf{curl},\Omega_i)$. We shall show that $\mathbf{e} = \mathbf{0}$, and that the whole sequence $\mathbf{e}^n$ converges weakly to $\mathbf{e}$.

The proof is obtained first on the subdomains neighbourhing the exterior boundary, then on the other subdomains by recursion. Let us first consider a subdomain Ω_i s.t. $\Gamma_i \neq \emptyset$. By continuity arguments, one has that

$$\mathbf{curl}\,\mathbf{e}_i^n \rightharpoonup \mathbf{curl}\,\mathbf{e}_i \ \text{ in } \mathbf{L}^2(\Omega_i), \quad (7.39)$$

and that

$$\gamma^T \mathbf{e}_i^n \rightharpoonup \gamma^T \mathbf{e}_i \quad \text{in } \mathbf{L}_t^2(\partial\Omega_i). \tag{7.40}$$

Besides, $\forall n$, $\mathbf{curl}\,\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^n - \omega^2 \underline{\varepsilon}\mathbf{e}_i^n = \mathbf{0}$ in $\boldsymbol{\mathcal{D}}'(\Omega_i)$, so

$$\mathbf{curl}\,\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i - \omega^2 \underline{\varepsilon}\mathbf{e}_i = \mathbf{0} \text{ in } \boldsymbol{\mathcal{D}}'(\Omega_i) \tag{7.41}$$

and $\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i \in \mathbf{H}(\mathbf{curl}, \Omega_i)$. Moreover, subtracting both relations and integrating by parts, one gets

$$_\pi\langle \pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}(\mathbf{e}_i^n - \mathbf{e}_i)), \gamma^T \mathbf{F}\rangle_\gamma = \left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}(\mathbf{e}_i^n - \mathbf{e}_i) | \,\mathbf{curl}\,\mathbf{F}\right)_{\Omega_i} - \omega^2 \left(\underline{\varepsilon}(\mathbf{e}_i^n - \mathbf{e}_i) | \mathbf{F}\right)_{\Omega_i} \longrightarrow 0, \quad \forall \mathbf{F} \in \mathbf{H}(\mathbf{curl}, \Omega_i). \tag{7.42}$$

Taking test functions $\mathbf{F} \in \boldsymbol{\mathcal{C}}^\infty(\overline{\Omega}_i)$ that vanish in a neighbourhood of $\partial\Omega_i \setminus \Gamma_i$, one then has (see the proof of Lemma 7.3.7) that $\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^n) \rightharpoonup \pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i)$ in $\widetilde{\mathbf{H}}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_i)$. Furthermore, because of the boundary condition of (7.33) and the uniqueness of the weak limit,

$$\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i) + i\omega\gamma_i^T \mathbf{e}_i = \mathbf{0} \quad \text{ on } \Gamma_i \tag{7.43}$$

holds in $\widetilde{\mathbf{H}}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma_i)$. Moreover, $\gamma^T \mathbf{e}_i = \mathbf{0}$ on Γ_i, because of (7.38). Hence, both traces vanish on Γ_i, and we conclude by the unique continuation principle [113] that $\mathbf{e}_i = \mathbf{0}$.

Let us show that $\mathbf{e}_i$ is the weak limit of the whole sequence $(\mathbf{e}_i^n)$. Indeed, for all $\mathbf{v} \in \mathbf{H}^+(\mathbf{curl}, \Omega_i)$, the sequence $(\mathbf{e}_i^n, \mathbf{v})_{\mathbf{H}^+(\mathbf{curl},\Omega_i)}$ is bounded, and admits a unique accumulation point, which is 0. So, the whole sequence $(\mathbf{e}_i^n, \mathbf{v})_{\mathbf{H}^+(\mathbf{curl},\Omega_i)}$ converges to 0, $\forall \mathbf{v} \in \mathbf{H}^+(\mathbf{curl}, \Omega_i)$. We conclude that the whole sequence $\mathbf{e}_i^n$ converges weakly to $\mathbf{0}$: $\mathbf{e}_i^n \rightharpoonup \mathbf{0}$ in $\mathbf{H}^+(\mathbf{curl}, \Omega_i)$. Moreover, because of (7.41), one has also $\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^n \rightharpoonup \mathbf{0}$ in $\mathbf{H}^+(\mathbf{curl}, \Omega_i)$, and both traces $\gamma^T \mathbf{e}_i^n$, $\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^n) \rightharpoonup \mathbf{0}$ in $\mathbf{L}_t^2(\partial\Omega_i)$.

Finally, we proceed by recursion on the other subdomains. Let us consider a subdomain Ω_j that neighbours a subdomain Ω_i in which $\mathbf{e}_i = \mathbf{0}$ has been proven. Because of what preceeds, $\gamma^T \mathbf{e}_i^n$, $\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^n) \rightharpoonup \mathbf{0}$ in $\mathbf{L}_t^2(\partial\Omega_i)$. Besides, one has the two interface conditions on Σ_{ij}

$$\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n+1}) + i\omega\gamma_j^T \mathbf{e}_j^{n+1} = \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^n) - i\omega\gamma_i^T \mathbf{e}_i^n,$$
$$\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n+1}) - i\omega\gamma_j^T \mathbf{e}_j^{n+1} = \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n+2}) + i\omega\gamma_i^T \mathbf{e}_i^{n+2}.$$

Taking the sum or the difference, we find that

$$\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j),\ \gamma_j^T \mathbf{e}_j \longrightarrow \mathbf{0} \quad \text{ in } \mathbf{L}_t^2(\Sigma_{ij}). \tag{7.44}$$

Along with (7.41) in Ω_j and the unique continuation principle, we conclude that $\mathbf{e}_j = \mathbf{0}$ in $\mathbf{H}^+(\mathbf{curl}, \Omega_j)$. □

Remark 7.4.7. The proof would extend to complex-valued Hermitian tensors, provided that the unique continuation principle holds. However, to the best of our knowledge, this result has been established only for real symmetric tensors [113, 90].

A complement for problems with more regular solutions

Corollary 7.4.8. Assume moreover that there exists $s > 0$ and a constant $C > 0$ independent of $\mathbf{e}_i^n$ s.t. $\mathbf{e}_i^n$, $\mathbf{curl}\,\mathbf{e}_i^n \in \mathbf{H}^s(\Omega_i)$, with

$$\|\mathbf{e}_i^n\|_{\mathbf{H}^s(\Omega_i)} + \|\,\mathbf{curl}\,\mathbf{e}_i^n\|_{\mathbf{H}^s(\Omega_i)} \leq C\ \|\mathbf{e}_i^n\|_{\mathbf{H}^+(\mathbf{curl},\Omega_i)};\ ^a \tag{7.45}$$

Then $\mathbf{e}_i^n$, $\mathbf{curl}\,\mathbf{e}_i^n$ converge to $\mathbf{0}$ strongly in $\mathbf{H}^\sigma(\Omega_i)$, for all $\sigma \in [0, s[$.

[a] the constant C may depend on the geometry of Ω, of the partition, the frequency ω, and the coefficients $\underline{\boldsymbol{\mu}}$, $\underline{\varepsilon}$ and α.

Proof. By continuity, one has that

$$\mathbf{e}_i^n \rightharpoonup \mathbf{0} \quad \text{ in } \mathbf{H}^s(\Omega_i); \tag{7.46}$$
$$\mathbf{curl}\,\mathbf{e}_i^n \rightharpoonup \mathbf{0} \quad \text{ in } \mathbf{H}^s(\Omega_i). \tag{7.47}$$

Finally, by compact Sobolev embeddings, we conclude that

$$\mathbf{e}_i^n,\ \mathbf{curl}\,\mathbf{e}_i^n \longrightarrow \mathbf{0} \quad \text{in } \mathbf{H}^\sigma(\Omega_i), \tag{7.48}$$

for all $\sigma \in [0, s[$. □

Remark 7.4.9. The validity of the assumption in Corollary 7.4.8 depends on regularity results *with continuous dependence* w.r.t. the $\mathbf{H}^+(\mathbf{curl}, \Omega_i)$-norm, in the spirit of, e.g., Corollary 6.3.3 or Proposition 6.3.5 which, however, are *a priori* valid only in domains of class $\mathcal{C}^2$. Nevertheless, counterparts to these results might exist in other types of domains, e.g. in convex domains. When the domain is not convex, one has additionally to take into account corners, cf. [36].

7.4.2 Investigations in the elliptic case

In this subsection, we discuss the generalisation of the results of previous section in the case of elliptic, possibly non-Hermitian tensors. The impedance coefficient α is assumed to be scalar and constant, but its value can be other than $i\omega$. We begin by revisiting energy estimates.

Energy estimates

For general non-Hermitian tensors $\underline{\varepsilon}$, $\underline{\boldsymbol{\mu}}$, Proposition 7.4.3 does not hold. Instead, one has the following more general results.

Proposition 7.4.10. Let $\mathbf{v} \in \mathbf{H}^+(\mathbf{curl}, \mathcal{O})$ s.t. $\mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}) - \omega^2\underline{\varepsilon}\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$ and $\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}) \in \mathbf{L}_t^2(\partial\mathcal{O})$. Then

$$\left\|\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}) + \alpha\gamma^T\mathbf{v}\right\|_{\mathbf{L}_t^2}^2 = \left\|\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v})\right\|_{\mathbf{L}_t^2}^2 + \left\|\alpha\gamma^T\mathbf{v}\right\|_{\mathbf{L}_t^2}^2 + 2\Re\left(\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}), \alpha\gamma^T\mathbf{v}\right)_{\partial\mathcal{O}}; \tag{7.49}$$

$$\left\|\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}) - \alpha\gamma^T\mathbf{v}\right\|_{\mathbf{L}_t^2}^2 = \left\|\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v})\right\|_{\mathbf{L}_t^2}^2 + \left\|\alpha\gamma^T\mathbf{v}\right\|_{\mathbf{L}_t^2}^2 - 2\Re\left(\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}), \alpha\gamma^T\mathbf{v}\right)_{\partial\mathcal{O}}, \tag{7.50}$$

with, moreover, if α is constant, (α^* denoting the compex conjugate of α),

$$\left(\pi^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}), \alpha\gamma^T\mathbf{v}\right)_{\partial\mathcal{O}} = \alpha^*\left[(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\,\mathbf{curl}\,\mathbf{v}) - (\underline{\varepsilon}\mathbf{v}|\mathbf{v})\right]. \tag{7.51}$$

Proof. We refer to the proof of Proposition 7.4.3. □

Remark 7.4.11. If one does not require α to be constant, the computations become more complex. In particular, relation (7.51) does not hold, because one cannot *a priori* integrate by parts. Provided one can introduce a lifting of α in $\mathcal{O}$, the result could possibly be adapted to a scalar heterogeneous α.

As a consequence, instead of Lemma 7.4.5, one has in general the following result.

Lemma 7.4.12. One has the estimate

$$U^{n+1} - U^n = -4\Re\left[\alpha^*\left(\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}^n|\,\mathbf{curl}\,\mathbf{e}^n\right)_\Omega - \omega^2\left(\underline{\varepsilon}\mathbf{e}^n|\mathbf{e}^n\right)_\Omega\right)\right] - 4\|\alpha\gamma^T\mathbf{e}^n\|_{\mathbf{L}_t^2(\Gamma)}^2. \tag{7.52}$$

Proof. One has

$$
\begin{aligned}
U^{n+1} &= \sum_i \sum_{j\neq i} \left\| \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n+1}) + \alpha\gamma_i^T \mathbf{e}_i^{n+1} \right\|^2_{\mathbf{L}_t^2(\Sigma_{ij})} \\
&= \sum_j \sum_{i\neq j} \left\| \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) - \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\Sigma_{ij})} \\
&= \sum_j \left(\left\| \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) - \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\partial\Omega_j)} - \left\| \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) - \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\Gamma_j)} \right) \\
&= \sum_j \left(\left\| \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) + \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\partial\Omega_j)} - 4\Re\left(\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}), \alpha\gamma_j^T \mathbf{e}_j^{n}\right)_{\partial\Omega_j} - \left\| -2\alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\Gamma_j)} \right).
\end{aligned}
$$

So, using (7.51),

$$
\begin{aligned}
U^{n+1} - U^n &= \sum_j \left(-4\Re\left[\alpha^* \left(\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}^{n}), \gamma_j^T \mathbf{e}^{n}\right)_{\partial\Omega_j}\right] - 4\left\| \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\Gamma_j)} \right) \\
&= \sum_j \left(-4\Re\left[\alpha^* \left(\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n} \,|\, \mathbf{curl}\,\mathbf{e}_j^{n}\right)_{\Omega_j} - \left(\mathbf{curl}\,\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n} | \mathbf{e}_j^{n}\right)_{\Omega_j} \right)\right] - 4\left\| \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\Gamma_j)} \right) \\
&= \sum_j \left(-4\Re\left[\alpha^* \left(\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n} \,|\, \mathbf{curl}\,\mathbf{e}_j^{n}\right)_{\Omega_j} - \omega^2 \left(\underline{\boldsymbol{\varepsilon}}\mathbf{e}_j^{n} | \mathbf{e}_j^{n}\right)_{\Omega_j} \right)\right] - 4\left\| \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\Gamma_j)} \right),
\end{aligned}
$$

hence the result. □

Moreover, one can extend the study to other types of exterior boundary conditions, namely Dirichlet or Neumann. Note that, in any case, the error satisfies an homogeneous exterior boundary condition. Hence, for the Dirichlet problem, the error is governed by

$$
\begin{cases}
\mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n+1}) - \omega^2 \underline{\boldsymbol{\varepsilon}}\mathbf{e}_i^{n+1} = \mathbf{0} & \text{in } \Omega_i, \\
\gamma_i^T \mathbf{e}_i^{n+1} = \mathbf{0} & \text{on } \Gamma_i, \\
\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n+1}) + i\omega\gamma_i^T \mathbf{e}_i^{n+1} = \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) - i\omega\gamma_j^T \mathbf{e}_j^{n} & \text{on } \Sigma_{ij}.
\end{cases}
\tag{7.53}
$$

The energy U^n, still defined as in (7.34), becomes

$$
\begin{aligned}
U^n &= \sum_i \sum_{j\neq i} \left\| \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n}) + i\omega\gamma_i^T \mathbf{e}_i^{n} \right\|^2_{\mathbf{L}_t^2(\Sigma_{ij})} \\
&= \sum_i \left\| \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n}) + i\omega\gamma_i^T \mathbf{e}_i^{n} \right\|^2_{\mathbf{L}_t^2(\partial\Omega_i)} - \sum_i \left\| \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n}) \right\|^2_{\mathbf{L}_t^2(\Gamma_i)}.
\end{aligned}
\tag{7.54}
$$

Then, one gets an estimate comparable to the one of Lemma 7.4.12.

Lemma 7.4.13. For the Dirichlet problem, one has the estimate

$$
U^{n+1} - U^n = -4\Re\left[\alpha^* \left(\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}^{n} \,|\, \mathbf{curl}\,\mathbf{e}^{n}\right)_{\Omega} - \omega^2 \left(\underline{\boldsymbol{\varepsilon}}\mathbf{e}^{n} | \mathbf{e}^{n}\right)_{\Omega} \right)\right]. \tag{7.55}
$$

Proof. One has

$$
\begin{aligned}
U^{n+1} &= \sum_i \sum_{j\neq i} \left\| \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n+1}) + \alpha\gamma_i^T \mathbf{e}_i^{n+1} \right\|^2_{\mathbf{L}_t^2(\Sigma_{ij})} \\
&= \sum_j \sum_{i\neq j} \left\| \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) - \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\Sigma_{ij})} \\
&= \sum_j \left(\left\| \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) - \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\partial\Omega_j)} - \left\| \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) - \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\Gamma_j)} \right) \\
&= \sum_j \left(\left\| \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) + \alpha\gamma_j^T \mathbf{e}_j^{n} \right\|^2_{\mathbf{L}_t^2(\partial\Omega_j)} - 4\Re\left(\pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}), \alpha\gamma_j^T \mathbf{e}_j^{n}\right)_{\partial\Omega_j} - \left\| \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^{n}) \right\|^2_{\mathbf{L}_t^2(\Gamma_j)} \right).
\end{aligned}
$$

Then,

$$U^{n+1} - U^n = -4\Re\left[\alpha^*\left(\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}^n|\,\mathbf{curl}\,\mathbf{e}^n\right)_\Omega - \omega^2\left(\underline{\boldsymbol{\varepsilon}}\mathbf{e}^n|\mathbf{e}^n\right)_\Omega\right)\right].$$

□

One can proceed similarly for the Neumann problem. The error is governed by

$$\begin{cases} \mathbf{curl}(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n+1}) - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{e}_i^{n+1} = \mathbf{0} & \text{in } \Omega_i, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n+1}) = \mathbf{0} & \text{on } \Gamma_i, \\ \pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^{n+1}) + i\omega\gamma_i^T\mathbf{e}_i^{n+1} = \pi_j^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_j^n) - i\omega\gamma_j^T\mathbf{e}_j^n & \text{on } \Sigma_{ij}, \end{cases} \tag{7.56}$$

and the energy becomes

$$\begin{aligned} U^n &= \sum_i\sum_{j\neq i}\left\|\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^n) + i\omega\gamma_i^T\mathbf{e}_i^n\right\|^2_{\mathbf{L}_t^2(\Sigma_{ij})} \\ &= \sum_i\left\|\pi_i^T(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}_i^n) + i\omega\gamma_i^T\mathbf{e}_i^n\right\|^2_{\mathbf{L}_t^2(\partial\Omega_i)} - \sum_i\left\|\alpha\gamma_i^T\mathbf{e}_i^n\right\|^2_{\mathbf{L}_t^2(\Gamma_i)}. \end{aligned} \tag{7.57}$$

Lemma 7.4.14. For the Neumann problem, one has the estimate

$$U^{n+1} - U^n = -4\Re\left[\alpha^*\left(\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}^n|\,\mathbf{curl}\,\mathbf{e}^n\right)_\Omega - \omega^2\left(\underline{\boldsymbol{\varepsilon}}\mathbf{e}^n|\mathbf{e}^n\right)_\Omega\right)\right]. \tag{7.58}$$

Proof. The proof is as for the Dirichlet case. □

Remark 7.4.15. For both Dirichet and Neumann cases, the result is similar to the one with Robin condition, but without the exterior boundary term $-4\|\alpha\gamma^T\mathbf{e}^n\|^2_{\mathbf{L}_t^2(\Gamma)}$.

Discussion

Because of the non-Hermitianity of the tensors, one now has an additional term in the increase rate of U^n, making its sign hard to determine. However, when the bilinear form is coercive, one can go further. For the Robin problem, one has the next result.

Theorem 7.4.16. Assume that $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\varepsilon}}$ are simultaneously elliptic, i.e. $\Theta_{\mu^{-1}} \cap \Theta_{-\varepsilon} \neq \emptyset$, and that $\arg(\alpha^*) \in \Theta_{\mu^{-1}} \cap \Theta_{-\varepsilon}$. Then $\mathbf{e}_i^n \longrightarrow \mathbf{0}$ strongly in $\mathbf{H}(\mathbf{curl}, \Omega_i)$.

Proof. Let $\theta = \arg(\alpha^*)$, the condition means that $\exists C > 0$, $\forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$,

$$\Re\left[e^{i\theta}\left(\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{v}|\,\mathbf{curl}\,\mathbf{v}\right)_\Omega - \omega^2\left(\underline{\boldsymbol{\varepsilon}}\mathbf{v}|\mathbf{v}\right)_\Omega\right)\right] \geq C\|\mathbf{v}\|^2_{\mathbf{H}(\mathbf{curl})}.$$

Combining it with the result of Lemma 7.4.12, one gets

$$\begin{aligned} U^{n+1} - U^n &= -4\Re\left[|\alpha|e^{i\theta}\left(\left(\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{e}^n|\,\mathbf{curl}\,\mathbf{e}^n\right)_\Omega - \omega^2\left(\underline{\boldsymbol{\varepsilon}}\mathbf{e}^n|\mathbf{e}^n\right)_\Omega\right)\right] - 4\|\alpha\gamma^T\mathbf{e}^n\|^2_{\mathbf{L}_t^2(\Gamma)} \\ &\leq -4|\alpha|C\ \|\mathbf{e}^n\|^2_{\mathbf{PH}(\mathbf{curl})} - 4\|\alpha\gamma^T\mathbf{e}^n\|^2_{\mathbf{L}_t^2(\Gamma)} < 0. \end{aligned}$$

Therefore, (U^n) is decreasing. It is also positive, so it admits a limit. Then, $U^n - U^{n+1} \longrightarrow 0$, and, for all i,

$$\begin{aligned} \|\mathbf{e}_i^n\|^2_{\mathbf{L}^2(\Omega_i)} &\longrightarrow 0; \\ \|\,\mathbf{curl}\,\mathbf{e}_i^n\|^2_{\mathbf{L}^2(\Omega_i)} &\longrightarrow 0; \\ \|\gamma_i^T\mathbf{e}_i^n\|^2_{\mathbf{L}_t^2(\Gamma_i)} &\longrightarrow 0. \end{aligned}$$

Hence, for all i, $\mathbf{e}_i^n \longrightarrow \mathbf{0}$ strongly in $\mathbf{H}(\mathbf{curl}, \Omega_i)$. □

Remark 7.4.17. This result is stronger than the one obtained for Hermitian tensors, because one gets *strong* convergence in $\mathbf{H}(\mathbf{curl}, \Omega_i)$. This is thanks to the coercivity of the bilinear form, that makes the convergence easier – a common phenomenon in domain decomposition.

We point out that this result requires an interplay condition between all three parameters $\underline{\boldsymbol{\mu}}$, $\underline{\boldsymbol{\varepsilon}}$ and α, *which is different from the condition ensuring the coercivity of the Robin problem* (see Theorem 5.3.6). There, the condition is simultaneous ellipticity between $\underline{\boldsymbol{\mu}}^{-1}$, $-\underline{\boldsymbol{\varepsilon}}$ and $-\alpha$, not α^*. However, one would like both conditions to be fulfilled, in order to get the well-posedness of the local problems *and* the decrease of the error over iterations. At first sight, it is unclear if both conditions can be fulfilled together.

We believe that one can do so in a simple manner. Indeed, one can multiply the variational formulation by some $e^{i\theta}$. This will shift arguments of all three coefficients $\underline{\boldsymbol{\mu}}^{-1}$, $-\underline{\boldsymbol{\varepsilon}}$ and $-\alpha$ from an angle θ, and will not affect their simultaneous ellipticity. On the other hand, it will shift $\arg(\alpha^*)$ from an angle $-\theta$ (α^* turns "the other way around" in the complex plane). So, one could choose θ such that α^* encounters the region $\Theta_{\mu^{-1}} \cap \Theta_{-\varepsilon}$, and both conditions are satisfied. This should tend to bring closer from one another α^* and $-\alpha$, that is, to bring α near the imaginary axis.

For the Dirichlet or Neumann problem, one has the next result.

Theorem 7.4.18. Assume that $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\varepsilon}}$ are simultaneously elliptic, i.e. $\Theta_{\mu^{-1}} \cap \Theta_{-\varepsilon} \neq \emptyset$, and choose α s.t. $\arg(\alpha^*) \in \Theta_{\mu^{-1}} \cap \Theta_{-\varepsilon}$. Then Després algorithm for Dirichlet or Neumann exterior boundary conditions converges, i.e. $\mathbf{e}_i^n \longrightarrow \mathbf{0}$ strongly in $\mathbf{H}(\mathbf{curl}, \Omega_i)$.

Proof. It is the same proof as in Theorem 7.4.16. ☐

Remark 7.4.19. This result for the Dirichlet and Neumann problems calls a number of comments.

1. As with the Robin problem, the coercivity of the problem makes convergence easier, as one gets *strong* convergence in $\mathbf{H}(\mathbf{curl}, \Omega_i)$, compared to the original result of Després *et al.*

2. Because one does not have a Robin exterior condition, the parameter α now has to be constant only on the interfaces. So, it is independent of the global problem and, in principle, it could be tuned to fit the coercivity of the problem accordingly.

3. The counterpart of the previous point is that one now has to deal with mixed problems in subdomains that have an exterior border. The proofs of well-posedness have to be adapted. Moreover, to ensure that the interface terms hold in $\mathbf{L}^2_t$, the assumption H.1 must be adapted and the results of Chapter 4 must be revisited (if there are boundary cross points). This is partially done in [7, Section 5.1.2.2] for the Dirichlet case.

4. Coming back to the assumptions of subsection 7.4.1, i.e. $\underline{\boldsymbol{\varepsilon}}$, $\underline{\boldsymbol{\mu}}$ Hermitian tensors and $\alpha \in i\mathbb{R}$, one finds surprisingly that the algorithm does not converge with Dirichlet or Neumann exterior boundary condition. Indeed, in this case, there holds
$$U^{n+1} - U^n = 0 \tag{7.59}$$
because of Proposition 7.4.3. Then, U^n remains constant over the iterations. Moreover, U^n vanishes only if all Neumann and Dirichlet traces match on all interfaces, that is, if $\mathbf{E}^n$ is the global solution. So, except if the initial guess is the global solution itself, the algorithm does not converge.

7.5 Numerical experimentations

In this section, we investigate the influence of the impedance condition in DDMs for anisotropic Maxwell problems. The standard Robin condition with $\alpha = i\omega$ is a natural choice for isotropic problems (that is related to the low-order transparent condition). However, when it comes to anisotropic problems, the way to choose this coefficient becomes less clear. In this section, we aim at exploring the behaviour of the ORAS method (that uses impedance transmission conditions) on a simple manufactured benchmark, for different elliptic materials and transmission coefficients. An illustration on a more applicative benchmark is also proposed. Due to code restrictions, ORAS with overlapping DD (with an overlap of one mesh size) have been used. Nevertheless, we believe the results in this section provide an interesting glimpse in the perspective of more advanced studies.

7.5.1 Manufactured benchmark with several elliptic media

Reference solution

In this section, Ω is the unit cube, $\Omega =]0,1[^3$. We consider the time-harmonic Maxwell equation

$$\mathbf{curl}\,\underline{\boldsymbol{\mu}}^{-1}\,\mathbf{curl}\,\mathbf{E} - \omega^2\underline{\boldsymbol{\varepsilon}}\mathbf{E} = \mathbf{f} \text{ in } \Omega, \tag{7.60}$$

with right-hand side $\mathbf{f} = (1,1,1)^T$, and completed with a boundary condition on Γ. Moreover, we assume that the tensors $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\mu}}^{-1}$ are diagonal:

$$\underline{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix}, \qquad \underline{\boldsymbol{\mu}}^{-1} = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix}.$$

For simplicity, we look for a solution with separate variables of the form:

$$\mathbf{E} = \begin{pmatrix} f_1(y)g_1(z) \\ f_2(z)g_2(x) \\ f_3(x)g_3(y) \end{pmatrix}.$$

Plugging this into the Maxwell equation, one gets, for E_x, that

$$-m_3 f_1''(y)g_1(z) - m_2 f_1(y)g_1''(z) - \omega^2\varepsilon_1 f_1(y)g_1(z) = 1.$$

A particular solution is given by

$$E_x = \frac{-1}{\omega^2\varepsilon_1} + \exp\left(\sqrt{\frac{-\omega^2\varepsilon_1}{2}}\left(\frac{y}{\sqrt{m_3}} + \frac{z}{\sqrt{m_2}}\right)\right),$$

where $\sqrt{\cdot}$ has to be understood as the complex square root (with cutoff on $\mathbb{R}^-$). We proceed similarly for the other components. Then, a solution is given by

$$\mathbf{E}_{\text{ref}} = \begin{pmatrix} \frac{-1}{\omega^2\varepsilon_1} + \exp\left(\sqrt{\frac{-\omega^2\varepsilon_1}{2}}\left(\frac{y}{\sqrt{m_3}} + \frac{z}{\sqrt{m_2}}\right)\right) \\ \frac{-1}{\omega^2\varepsilon_2} + \exp\left(\sqrt{\frac{-\omega^2\varepsilon_2}{2}}\left(\frac{z}{\sqrt{m_1}} + \frac{x}{\sqrt{m_3}}\right)\right) \\ \frac{-1}{\omega^2\varepsilon_3} + \exp\left(\sqrt{\frac{-\omega^2\varepsilon_3}{2}}\left(\frac{x}{\sqrt{m_2}} + \frac{y}{\sqrt{m_1}}\right)\right) \end{pmatrix}, \tag{7.61}$$

The curl is given by

$$\mathbf{curl}\,\mathbf{E}_{\text{ref}} = \begin{pmatrix} \sqrt{\frac{-\omega^2\varepsilon_3}{2m_1}}\exp\left(\sqrt{\frac{-\omega^2\varepsilon_3}{2}}\left(\frac{x}{\sqrt{m_2}} + \frac{y}{\sqrt{m_1}}\right)\right) - \sqrt{\frac{-\omega^2\varepsilon_2}{2m_1}}\exp\left(\sqrt{\frac{-\omega^2\varepsilon_2}{2}}\left(\frac{z}{\sqrt{m_1}} + \frac{x}{\sqrt{m_3}}\right)\right) \\ \sqrt{\frac{-\omega^2\varepsilon_1}{2m_2}}\exp\left(\sqrt{\frac{-\omega^2\varepsilon_1}{2}}\left(\frac{y}{\sqrt{m_3}} + \frac{z}{\sqrt{m_2}}\right)\right) - \sqrt{\frac{-\omega^2\varepsilon_3}{2m_2}}\exp\left(\sqrt{\frac{-\omega^2\varepsilon_3}{2}}\left(\frac{x}{\sqrt{m_2}} + \frac{y}{\sqrt{m_1}}\right)\right) \\ \sqrt{\frac{-\omega^2\varepsilon_2}{2m_3}}\exp\left(\sqrt{\frac{-\omega^2\varepsilon_2}{2}}\left(\frac{z}{\sqrt{m_1}} + \frac{x}{\sqrt{m_3}}\right)\right) - \sqrt{\frac{-\omega^2\varepsilon_1}{2m_3}}\exp\left(\sqrt{\frac{-\omega^2\varepsilon_1}{2}}\left(\frac{y}{\sqrt{m_3}} + \frac{z}{\sqrt{m_2}}\right)\right) \end{pmatrix}. \tag{7.62}$$

One checks that this field is indeed a solution of (7.60). In our numerical experiments, we will impose $\mathbf{E}_{\text{ref}}$ as the Dirichlet boundary condition (the same could be done with Neumann or Robin). In the following, we set $\omega = \pi$.

Material tensors

We will consider essentially three different materials in our numerical experiments. The first one is a standard isotropic medium,

$$\underline{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{\mu}}^{-1} = \underline{\mathbf{I}}. \tag{7.63}$$

In this case, the bilinear form associated to the problem is not coercive. However, one has Hermitianity, and, if ω^2 is not an eigenvalue of the problem, the uniqueness of the solution can be established thanks to a unique continuation principle.

For the second material, we choose

$$\underline{\boldsymbol{\varepsilon}} = \begin{pmatrix} 1+\eta i & & \\ & 1+\eta i & \\ & & -2+\eta i \end{pmatrix}, \qquad \underline{\boldsymbol{\mu}}^{-1} = \begin{pmatrix} 1-i & & \\ & 2 & \\ & & 2.5 \end{pmatrix} \tag{7.64}$$

where η is a real parameter for which we consider two values: $\eta = 1$ (Material 2a) and $\eta = 0.1$ (Material 2b). Here, the tensors are not Hermitian. They are elliptic, and diagonal, therefore normal: their ellipticity is entirely driven by the position of their eigenvalues in the complex plane. Moreover, $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\varepsilon}}$ are simultaneously elliptic (the eigenvalues of $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\varepsilon}}$ are contained in the same open half-plane). This is pictured on Figure 7.1 (a) and (b). Therefore, the bilinear form of the problem is coercive. The range of admissible directions, $\Theta_{\mu^{-1}} \cap \Theta_{-\varepsilon}$, is mostly constrained by the eigenvalues 2 and $-1-\eta i$. For short, we denote it Θ. One finds that

$$\Theta := \Theta_{\mu^{-1}} \cap \Theta_{-\varepsilon} = \left]\arctan\frac{-1}{\eta}; \frac{\pi}{2}\right[. \tag{7.65}$$

When η goes to 0, $\underline{\boldsymbol{\varepsilon}}$ becomes "less elliptic": ε_- goes to 0. The whole problem also becomes "less coercive", as $\Theta_{\mu^{-1}} \cap \Theta_{-\varepsilon}$ reduces to $\pi/2$ and the coercivity constant goes to 0. In the limit case $\eta = 0$, $\underline{\boldsymbol{\varepsilon}}$ is no longer elliptic. The material becomes a hyperbolic metamaterial, a configuration in which there are reasons to believe that the problem is in fact ill-posed. To vary the ellipticity directions, we will also use sometimes the material whose tensor coefficients are the opposite of material 2a. This will be simply denoted material -2a.

The third material is a variation of the second one. We choose

$$\underline{\boldsymbol{\varepsilon}} = \begin{pmatrix} 1+\eta i & & \\ & 1+\eta i & \\ & & -2+\eta i \end{pmatrix}, \qquad \underline{\boldsymbol{\mu}}^{-1} = \begin{pmatrix} 1+i & & \\ & 2 & \\ & & 2.5 \end{pmatrix} \tag{7.66}$$

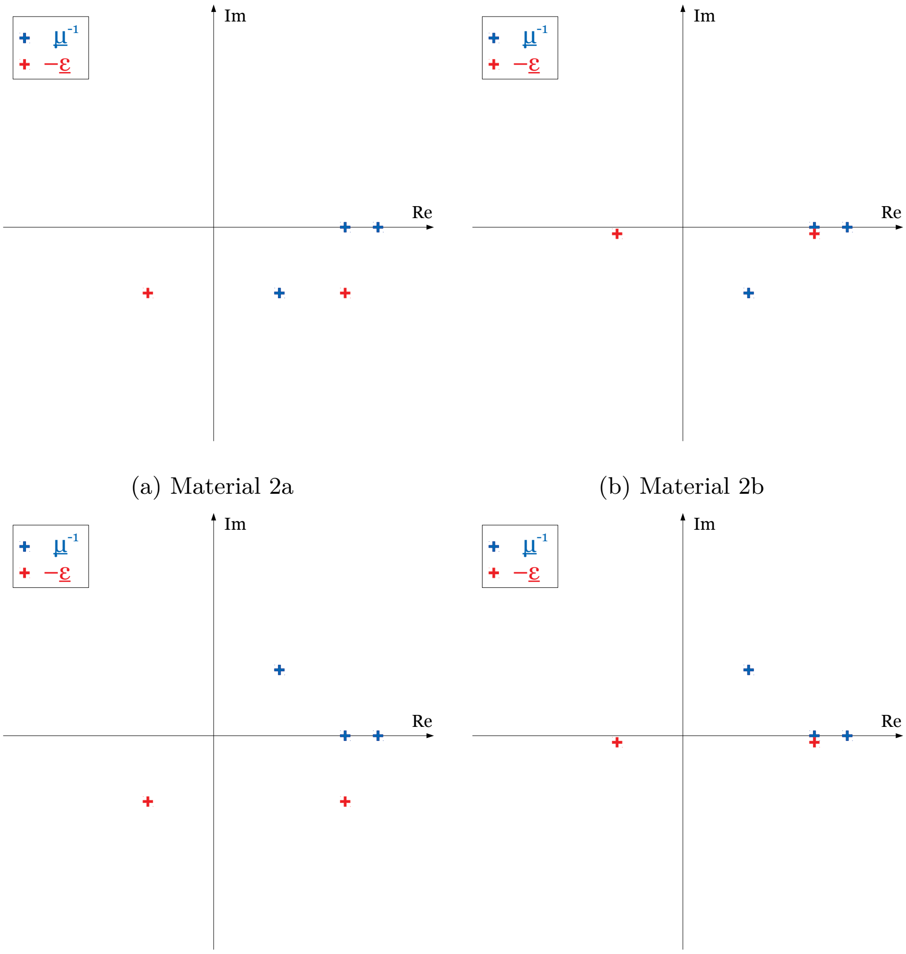
Again, we consider two subcases depending on the value of η: $\eta = 1$ (Material 3a) and $\eta = 0.1$ (Material 3b). Both tensors are still normal and elliptic, but not simultaneously elliptic. Unlike case 2, the relative position of their eigenvalues does not allow coercivity. When $\eta = 1$, $\underline{\boldsymbol{\mu}}^{-1}$ and $\underline{\boldsymbol{\varepsilon}}$ share a common eigenvalue, $1+i$. Then, all eigenvalues are contained in the same closed (not open) half-complex plane, taking $\theta = -\pi/4$ (see Figure 7.1 (c)). This situation is similar to the one of isotropic media. When $\eta = 0.1$, the situation is worse: there is no half-plane in $\mathbb{C}$ that contains all eigenvalues of $\underline{\boldsymbol{\mu}}^{-1}$ and $\underline{\boldsymbol{\varepsilon}}$ (see Figure 7.1 (d)). Therefore, we are only able to state that the problem enters Fredholm Alternative. In the following, we assume that this problem admits a unique solution.

Validation and preliminary discussion

The numerical simulations are performed with FreeFem++ and the library PETSc. This code provides two classical overlapping, preconditioning methods. The method we are mostly interested in is ORAS *(Optimized Restricted Additive Schwarz)*, a preconditioner that takes advantage of transmission conditions; at each step, local problems are solved with homogeneous Robin conditions on the interfaces. However, the impedance coefficient has to be tuned. Before investigating that matter, we simply choose $\alpha = i\omega$ as a first naive approach. We also experiment ASM *(Additive Schwarz Method)*, which is a purely algebraic preconditioner (see e.g. [47]); at each step, local problems are solved with homogeneous Dirichlet conditions, and there is no parameter to be tuned. As a preliminary study, we first compare these two naive approaches, before investigating further the influence of impedance condition.

The problem is discretized by order 2 edge elements (where the solution is approximated by order 2 polynomials, while the curl is approximated by order 1 polynomials). The number of degrees of freedom in the benchmark is 86508 ($h = 0.075$). The overlap between subdomains is set to its minimal value, i.e. one mesh size. At first, ORAS is used with the classical impedance coefficient $\alpha = i\omega$. The global system is solved by an iterative procedure thanks to GMRES (without restart), and the numerical solution is compared to the reference solution $\mathbf{E}_{\text{ref}}$.

The results presented on Figure 7.2 show the convergence of both methods ASM and ORAS. This is confirmed by the fact that we reach the same error (compared to the reference solution) than in the mono-domain case. For a



(a) Material 2a (b) Material 2b

(c) Material 3a (d) Material 3b

Figure 7.1: Eigenvalues of the material tensors

standard isotropic material (material 1), ORAS converges clearly better than ASM. For more complex materials 2a and 2b, the results are not as clear. We observe that with ASM, the residual decreases first quickly then more slowly; for ORAS, the contrary is observed. As a result, both curves intersect: ASM takes fewer iterations if the tolerance is relatively high, and ORAS becomes better for small tolerances. The $\mathbf{H}(\mathbf{curl})$ relative error behaves in a similar way. Note that for material 2b, with ORAS, this error even increases at first before decreasing rapidly. For materials 3a and 3b, we observe results (not presented here) that are similar to those obtained for materials 2a and 2b, respectively. In the following of the results, the relative tolerance is set to 1e-5, for which the error between the DDM solution and the reference solution is comparable to the error obtained in mono-domain.

The number of iterations obtained for the different preconditioners, materials, and number of subdomains are summarized in Table 7.1. They do not allow us to adjucate firmly which method is the most efficient between ASM and ORAS. We observe that ORAS generally provides better results than ASM, at least with 2 or 3 subdomains. However, we note that when going from 3 to 4 subdomains, the number of iterations of ASM increases only slowly, and even decreases in several cases, in particular 2b and 3b. We do not really understand this behaviour. As a result, in these configurations, ASM converges in less iterations than ORAS.

These results mostly show the high sensitivity of both methods to the properties of the material tensors. The convergence is worsened when the problems are not coercive (materials 3a and 3b), and, mostly, when the ellipticity constant of one of the material tensors is small (materials 2b and 3b).

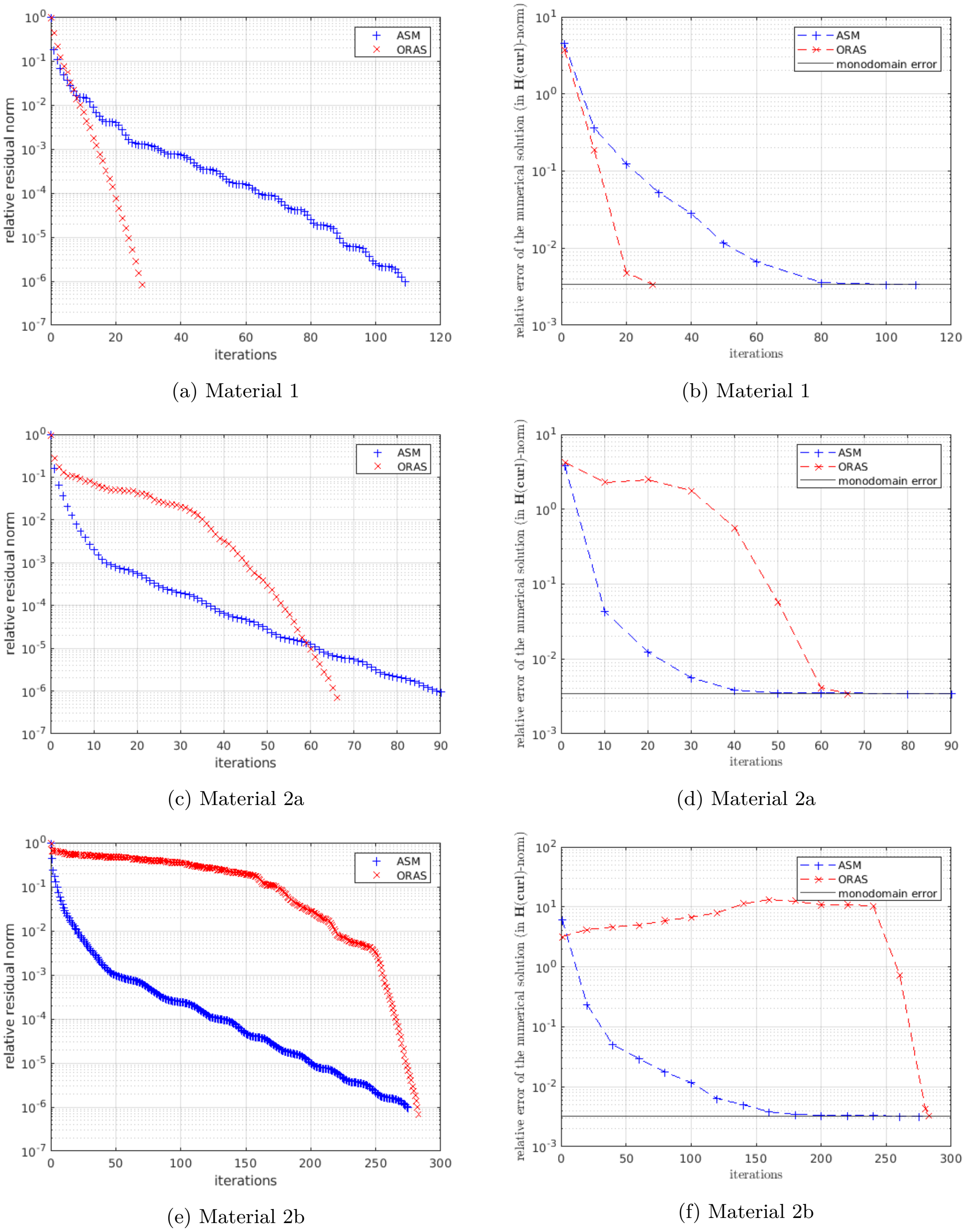


Figure 7.2: Evolution of the residual and numerical solution norms across iterations (for 4 subdomains)

Influence of the impedance coefficient

In what follows, we investigate the influence of different impedance coefficients in the transmission condition for ORAS method. In particular, instead of the standard coefficient $\alpha = i\omega$, we test different arguments while conserving a constant modulus: $\alpha = \omega e^{i\varphi}$, for different values of φ. Results are presented in Figure 7.3 for materials 2a, 2b and -2a (which all correspond to coercive problems). We observe that $i\omega$ (i.e. $\varphi = \pi/2$) does not always provide the smaller number of iterations. In the following, for each material, we denote $\varphi_\star$ the argument providing

	Material	N_d	ASM	ORAS
(1)	$\underline{\boldsymbol{\varepsilon}} = \underline{\mathbf{I}}$, $\underline{\boldsymbol{\mu}}^{-1} = \underline{\mathbf{I}}$	2	58	11
		3	84	19
		4	89	24
(2a)	$\underline{\boldsymbol{\varepsilon}} = \begin{pmatrix} 1+i & & \\ & 1+i & \\ & & -2+i \end{pmatrix}$, $\underline{\boldsymbol{\mu}}^{-1} = \begin{pmatrix} 1-i & & \\ & 2 & \\ & & 2.5 \end{pmatrix}$	2	40	8
		3	65	30
		4	62	60
(2b)	$\underline{\boldsymbol{\varepsilon}} = \begin{pmatrix} 1+0.1i & & \\ & 1+0.1i & \\ & & -2+0.1i \end{pmatrix}$, $\underline{\boldsymbol{\mu}}^{-1} = \begin{pmatrix} 1-i & & \\ & 2 & \\ & & 2.5 \end{pmatrix}$	2	112	14
		3	215	136
		4	201	274
(-2a)	$\underline{\boldsymbol{\varepsilon}} = -\begin{pmatrix} 1+i & & \\ & 1+i & \\ & & -2+i \end{pmatrix}$, $\underline{\boldsymbol{\mu}}^{-1} = -\begin{pmatrix} 1-i & & \\ & 2 & \\ & & 2.5 \end{pmatrix}$	2	40	17
		3	65	54
		4	62	112
(3a)	$\underline{\boldsymbol{\varepsilon}} = \begin{pmatrix} 1+i & & \\ & 1+i & \\ & & -2+i \end{pmatrix}$, $\underline{\boldsymbol{\mu}}^{-1} = \begin{pmatrix} 1+i & & \\ & 2 & \\ & & 2.5 \end{pmatrix}$	2	50	9
		3	76	29
		4	77	62
(3b)	$\underline{\boldsymbol{\varepsilon}} = \begin{pmatrix} 1+0.1i & & \\ & 1+0.1i & \\ & & -2+0.1i \end{pmatrix}$, $\underline{\boldsymbol{\mu}}^{-1} = \begin{pmatrix} 1+i & & \\ & 2 & \\ & & 2.5 \end{pmatrix}$	2	133	17
		3	261	144
		4	246	279

Table 7.1: Number of iterations with ASM or ORAS ($\alpha = i\omega$) for $N_d = 2, 3, 4$ subdomains

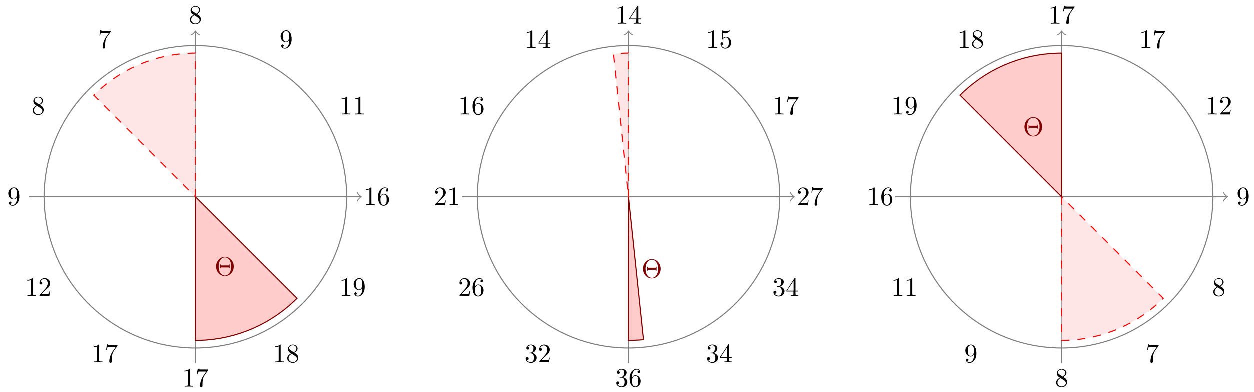
the smallest number of iterations among those tested, and $\alpha_\star = \omega e^{i\varphi_\star}$. For material 2a, we find $\varphi_\star = 2\pi/3$; for material -2a, the opposite is observed, $\varphi_\star = -\pi/3$.

On Figure 7.3, we have plotted in dark red the range Θ of simultaneous ellipticity directions of $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\varepsilon}}$. For more convenience, we have also plotted in light red the opposite directions to Θ. These results suggest a link between the choice of the coefficient argument and the tensors ellipticity directions. The convergence seems to be best if α is chosen in opposition to the coercivity directions Θ (i.e. $\arg(-\alpha) \in \Theta$); more precisely, if all three coefficients $\underline{\boldsymbol{\mu}}^{-1}$, $-\underline{\boldsymbol{\varepsilon}}$ and $-\alpha$ are *simultaneously elliptic*, which is the condition for the local problems to be coercive (see Theorem 5.3.6). On the contrary, when $-\alpha$ is chosen opposed to Θ, the convergence worsens.

Although not presented extensively here, this have also been tested for materials 1, 3a and 3b. Table 7.2 summarizes the best impedance argument $\varphi_\star$ obtained (among those tested) for all materials. For material 1, $\alpha_\star$ is either $i\omega$ or $-i\omega$. The problem is not coercive, but the tensors eigenvalues are all contained in a closed half-space of $\mathbb{C}$ (either $\Im z \leq 0$ or $\Im z \geq 0$). Taking either $\alpha = i\omega$ or $\alpha = -i\omega$ preserves one of these. For materials 3a and 3b, there is no simultaneous ellipticity. We observe that the best directions are respectively $2\pi/3$ and $\pi/2$. However, we do not have a real conjecture for these results.

A summary of the results with impedance coefficient $\alpha_\star$ is provided in Table 7.3. In most cases, choosing $\alpha_\star$ instead of $i\omega$ does not have a huge impact on convergence, because both of them are on the same side of the circle according to the tensors ellipticity directions. However, material -2a is an interesting counterexample: in this case, the choice $\alpha = i\omega$ does not match with the simultaneous ellipticity condition, so the convergence is degraded. On the contrary, the choice $\alpha_\star = \exp(-i\pi/3)\omega$ does, and provides a significantly better convergence.

The last experimentations concern a choice of coefficient that is more motivated by physical concerns. Indeed, in isotropic media, the impedance coefficient is typically given by $\alpha = i\omega\sqrt{\varepsilon\mu^{-1}}$. For an anistropic medium, a naive idea is to use an "equivalent impedance" that would correspond to the value of impedance in the direction normal to the interface: $\alpha_{\mathbf{n}} = i\omega\sqrt{\mathbf{n}(\underline{\boldsymbol{\varepsilon}}\underline{\boldsymbol{\mu}}^{-1})\mathbf{n}}$. Then, it takes a bit of the geometry into account, contrarily to our previous choices where α were constant all over the interfaces. The results are presented in the last column of Table 7.3. This coefficient appears to be generally the one giving the best results among those we have experimented, except for materials 2b and 3b. Moreover, it is easy to compute, independently of considerations on the tensors ellipticity directions.



(a) Material 2a (b) Material 2b (c) Material -2a

Figure 7.3: Number of iterations of ORAS depending on the argument of impedance coefficient (for 2 subdomains). The range of directions Θ of simultaneous ellipticity of $\underline{\boldsymbol{\mu}}^{-1}$ and $-\underline{\boldsymbol{\varepsilon}}$ is plotted in dark red.

Material	1	2a	2b	-2a	3a	3b
$\varphi_\star$	$\pm\pi/2$	$2\pi/3$	$\pi/2$	$-\pi/3$	$2\pi/3$	$\pi/2$

Table 7.2: Best observed argument for the impedance coefficient (for 2 subdomains)

7.5.2 An illustrative benchmark: cloaking of a sphere

In order to test the method on a more illustrative case, we conclude the section by considering the cloaking of a sphere by an ideal optical metamaterial.

Ideal optical metamaterial

We consider the scattering of a plane wave $\mathbf{E}_{\text{inc}} = [\exp(ikz), 0, 0]^T$ by a sphere of radius R_0. In order to make it invisible, we surround the sphere by a layer of an ideal optical metamaterial, in the region $R_0 < r < R_1$ where r denotes the radial coordinate.

The (ideal) metamaterial layer is designed by using a geometric transformation that stretches all the electric field of region $r < R_1$ in the cloak region. It is given by the change of variables

$$r' = R_0 + \frac{R_1 - R_0}{R_1} r, \tag{7.67}$$

where r is the radial coordinate, and r' the "stretched" radial coordinate. Introduce $\mathbf{r}$ the radial unit vector field and $\underline{\mathbf{R}} = \mathbf{r} \otimes \mathbf{r}$, and let $s_1 = \frac{R_1}{R_1 - R_0}$, and $s_2 = \frac{R_1(r-R_0)}{r(R_1-R_0)}$. The Jacobian of this transformation is given by

$$\underline{\mathbf{J}} = s_1 \underline{\mathbf{R}} + s_2(\underline{\mathbf{I}} - \underline{\mathbf{R}}), \tag{7.68}$$

where the part $s_1\underline{\mathbf{R}}$ corresponds to the transformation of the radial coordinate, and the part $s_2(\underline{\mathbf{I}} - \underline{\mathbf{R}})$ corresponds to the transformation of the tangential coordinates (this part is non-zero because of the spherical system of coordinates; the different coordinates are coupled).

With this at hand, it is possible to determine the values of the metamaterial tensors (see [93] for details, and [89, 116] for examples in other contexts). Then $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\boldsymbol{\mu}}$ are given by

$$\underline{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{\mu}} = \frac{s_2^2}{s_1} \underline{\mathbf{R}} + s_1(\underline{\mathbf{I}} - \underline{\mathbf{R}}). \tag{7.69}$$

Remark 7.5.1. It is to be noted that, while s_1 is constant, s_2 is a scalar field whose value goes to 0 when r goes to R_0. Therefore, the tensors are not strictly speaking elliptic (they are elliptic outside a neighbourhood of $r = R_0$). To deal with this problem, one can threshold the value of s_2 in a neighbourhood of R_0 to a small (strictly positive) value. In practice, this lack of ellipticity does not affect the convergence of the method.

	Material	N_d	$i\omega$	$\alpha_\star$	$i\omega\sqrt{\mathbf{n}(\underline{\varepsilon}\underline{\mu}^{-1})\mathbf{n}}$
(1)	$\underline{\varepsilon} = \underline{\mathbf{I}},\ \underline{\mu}^{-1} = \underline{\mathbf{I}}$	2	11	11	11
		3	19	19	19
		4	24	24	24
(2a)	$\underline{\varepsilon} = \begin{pmatrix} 1+i & & \\ & 1+i & \\ & & -2+i \end{pmatrix},\ \underline{\mu}^{-1} = \begin{pmatrix} 1-i & & \\ & 2 & \\ & & 2.5 \end{pmatrix}$	2	8	7	7
		3	30	29	21
		4	60	62	44
(2b)	$\underline{\varepsilon} = \begin{pmatrix} 1+0.1i & & \\ & 1+0.1i & \\ & & -2+0.1i \end{pmatrix},\ \underline{\mu}^{-1} = \begin{pmatrix} 1-i & & \\ & 2 & \\ & & 2.5 \end{pmatrix}$	2	14	14	14
		3	136	136	163
		4	274	274	293
(-2a)	$\underline{\varepsilon} = -\begin{pmatrix} 1+i & & \\ & 1+i & \\ & & -2+i \end{pmatrix},\ \underline{\mu}^{-1} = -\begin{pmatrix} 1-i & & \\ & 2 & \\ & & 2.5 \end{pmatrix}$	2	17	7	17
		3	54	29	50
		4	112	62	84
(3a)	$\underline{\varepsilon} = \begin{pmatrix} 1+i & & \\ & 1+i & \\ & & -2+i \end{pmatrix},\ \underline{\mu}^{-1} = \begin{pmatrix} 1+i & & \\ & 2 & \\ & & 2.5 \end{pmatrix}$	2	9	8	8
		3	29	29	19
		4	62	58	39
(3b)	$\underline{\varepsilon} = \begin{pmatrix} 1+0.1i & & \\ & 1+0.1i & \\ & & -2+0.1i \end{pmatrix},\ \underline{\mu}^{-1} = \begin{pmatrix} 1+i & & \\ & 2 & \\ & & 2.5 \end{pmatrix}$	2	17	17	18
		3	144	144	153
		4	279	279	278

Table 7.3: Number of iterations for ORAS preconditioner with various impedance coefficients

The same principle allows to compute the analytic expression of the solution in the metamaterial region. In the metamaterial, there holds

$$\mathbf{E}_{\text{exa}} = \underline{\mathbf{J}}[\exp(iks_2z), 0, 0]^T. \tag{7.70}$$

On the other hand, in the air, we have $\underline{\varepsilon} = \underline{\mu} = \underline{\mathbf{I}}$, and $\mathbf{E}_{\text{exa}} = \mathbf{E}_{\text{inc}}$. A visual of the reference solution is provided in Fig. 7.4: in the external layer (air), the solution behaves like a plane wave (as if not scattered); in the internal layer (metamaterial) the wave is distorted, such that the sphere appears invisible.

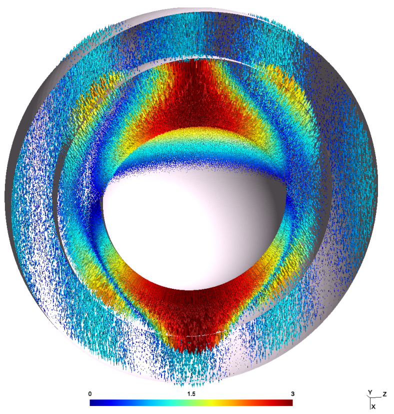
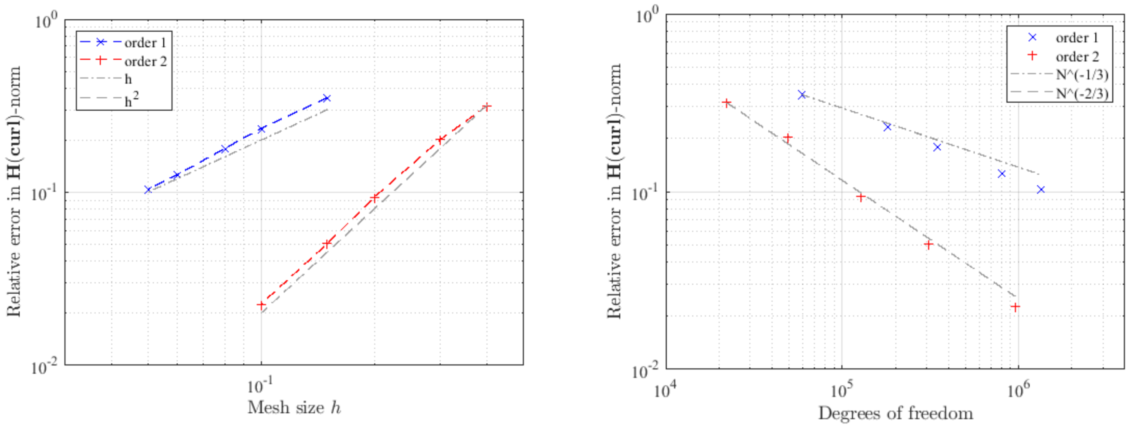


Figure 7.4: Reference solution (courtesy of A. Modave)

The problem to be solved is then the following:

$$\begin{cases} \mathbf{curl}(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\underline{\varepsilon}\mathbf{E} = \mathbf{0} & \text{in } \Omega, \\ \underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_{\text{int}}, \\ \pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}) + ik\gamma^T\mathbf{E} = \mathbf{g} & \text{on } \Gamma_{\text{ext}}. \end{cases} \tag{7.71}$$

The computational domain is $\Omega = \mathcal{B}(R_2)\backslash\mathcal{B}(R_0)$, the sphere of radius $R_0 = 1$ being surrounded by two layers: an ideal optical metamaterial in the layer between R_0 and $R_1 = 1.5$, and air in the layer between R_1 and $R_2 = 2$. No volume source is present, $\mathbf{f} = \mathbf{0}$. On the interior boundary Γ_{int}, we impose a homogeneous Neumann condition. On the other hand, on the exterior boundary Γ_{ext}, we impose an inhomogeneous impedance condition corresponding to the incident plane wave, $\mathbf{g} = \pi^T(\underline{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_{\text{inc}}) + ik\gamma^T\mathbf{E}_{\text{inc}}$, with a wavenumber $k = \pi$.



(a) Error vs mesh size h

(b) Error vs degrees of freedom

Figure 7.5: Convergence in mono-domain for cloaking benchmark

N_d	ASM	ORAS
2	191	16
3	284	51
4	353	99

Table 7.4: Number of iterations for cloaking benchmark (with 49k dof)

N_d	ASM	ORAS
2	312	21
3	475	66
4	533	90

Table 7.5: Number of iterations for cloaking benchmark (with 128k dof)

Validation of monodomain benchmark

The problem is discretized using order 1 and 2 edge finite elements (where the order refers to the polynomial order of approximation of the solution itself), and solved thanks to the finite element library FreeFem++ [68]. We first check the convergence of the mono-domain solver. As the solution is smooth enough, as well as its curl, we expect the error $\|\mathbf{E}_{\text{exa}} - \mathbf{E}_{\text{num}}\|_{\mathbf{H}(\mathbf{curl})}$ to decrease in the order $O(h)$ for low-order edge finite elements, and $O(h^2)$ for high-order elements.

The convergence results presented on Fig. 7.5 (a) show that we get the expected convergence rate. Fig. 7.5 (b) represents the same results in terms of degrees of freedom, showing the huge interest of using high-order elements. Indeed, for comparable sizes of problems, high-order elements provide a much better precision; conversely, to get a fixed precision, the computational cost is reduced by using high-order elements.

DDM experimentations

For the DDM experimentations, we use order 2 edge finite elements, and focus on two meshes: the ones with meshsize $h = 0.3$ and $h = 0.2$, with 49064 and 128146 degrees of freedom, respectively. As before, the overlap is set to 1 mesh size, and the relative tolerance in the GMRES to 1e-5.

With this benchmark, we observe that ORAS converges significantly better than ASM. Here, we used the classical impedance coefficient $\alpha = i\omega$. Using the "smarter" coefficient $\alpha_{\mathbf{n}} = i\omega\sqrt{\mathbf{n}(\underline{\boldsymbol{\varepsilon}}\underline{\boldsymbol{\mu}}^{-1})\mathbf{n}}$ has no impact, because with the ideal optical metamaterial there holds $\underline{\boldsymbol{\varepsilon}}\underline{\boldsymbol{\mu}}^{-1} = \underline{\mathbf{I}}$.

Conclusion

In this chapter, we have investigated several aspects of DDMs for electromagnetic anisotropic problems. Indeed, in the perspective of large-scale simulations, robust and efficient DDMs are required, but only a few contributions are available for what concerns anisotropic problems. In this work, three different directions have been explored:

- We have studied decomposed formulations for anisotropic problems. In particular, we have proven equivalence between a formulation with impedance transmission conditions and the global problem (Theorem 7.3.9). This was done under a regularity hypothesis (whose validity can be discussed as was done in Chapter 6 for domains of class $\mathcal{C}^2$). In Section 7.3.3, we also shed a new light on the saddle-point decomposed formulation proposed in [8], as these regularity matters help giving a more precise meaning to the space of Lagrange multipliers.
- We have also revisited the iterative procedure proposed by Després et al. [43]. We have extended the proof of convergence to the case of real symmetric definite positive tensors (Theorem 7.4.6). First results have also been obtained in other settings: for coercive anisotropic problems (Theorem 7.4.16), as well as for other boundary conditions than Robin (Theorem 7.4.18).
- Finally, we have made various numerical experimentations on DDM, focusing on the influence of the impedance condition on convergence. These results tend to illustrate a link between the coercivity direction of the problem (when it is coercive) and the best choice of direction for the impedance coefficient. They also show the potential interest of using an impedance coefficient that takes into account the anisotropy of the medium. This certainly requires further experiments.

These investigations call to future work in both theoretical and practical directions. For the Neumann or Dirichlet exterior boundary condition, we have obtained partial convergence results, but this analysis is not complete. Indeed, one has to deal with mixed boundary conditions problems. The well-posedness study for these problems must be adapted (see [55]). To ensure that the interface conditions hold in $\mathbf{L}^2_t$, one also has to adapt results of Chapter 4 to the setting of mixed boundary conditions (see section 5.1.2.2 of [7]). For non-coercive problems, further investigations are required, as this was not covered by our work. Moreover, we have mainly restricted our analysis to the case of scalar constant impedance coefficients. This should be pushed further with heterogeneous (and maybe tensor-valued) coefficients.

For more practical aspects, let us note that both tested methods were overlapping. A comparison with the non-overlapping method that truly corresponds to the algorithm described in Section 7.4 would also be interesting. Further explorations should be conducted on the choice of the impedance coefficient. In particular, heterogeneous and/or tensor-valued coefficients could be tested. This also opens the road for more elaborate types of transmission conditions. Indeed, ideally, $\underline{\boldsymbol{\alpha}}$ is a Dirichlet-to-Neumann operator. One could propose methods with second-order or higher-order transmission conditions, such as in [99] and [52], among others. Non-local operators could even be investigated, as was done by Claeys, Collino and Thierry [31], and more recently by Parolin [91].

Chapter 8

Conclusions and perspectives

8.1 Conclusion

In this work, we have conducted an analysis of anisotropic Maxwell problems for a general class of tensors: elliptic tensors defined in (3.1). In Chapter 3, we have developed the mathematical framework allowing to do so. To that aim, we have extended the results already known for the study of isotropic Maxwell problems: the main ones are Helmholtz decompositions and compact embedding results.

In Chapter 4, we have obtained surface regularity results for Robin traces. We have proven that the regularity of traces related by an impedance condition is governed only by the geometry of the boundary and the impedance coefficient. This result was obtained for smooth or piecewise constant scalar coefficients. In principle, the techniques presented would allow to cover the case of a coefficient with jumps, and to get more precise regularity exponent estimates in given configurations. A first result for the case of a tensor-valued coefficient was also provided.

With these results, we have analysed the variational formulations of anisotropic Maxwell problems, and proven their well-posedness in Fredholm sense. We have done so for the three main types of boundary conditions: Dirichlet, Neumann and Robin. Among these, the Robin problem is apart for two reasons. The first one is that one has to ensure that the traces belong to $\mathbf{L}^2_t(\Gamma)$, in order to set the variational formulation in $\mathbf{H}^+(\mathbf{curl}, \Omega)$. This was the point of the studies of Chapter 4. The second one is that, contrarily to Dirichlet and Neumann cases, one needs additionally to assume interplay between both coefficients $\underline{\boldsymbol{\mu}}$ and $\underline{\boldsymbol{\alpha}}$ in order to prove the Fredholm character.

Moreover, we have obtained regularity estimates for both the solution and its curl for all three types of problems. This was done for smooth boundaries and coefficients and for low-regularity data. For Dirichlet and Neumann problems, we recover the classical estimates of isotropic case. Following standard numerical analysis results then gives the convergence rate of the edge finite elements method (for coercive problems). The Robin problem is again apart in this matter. Contrarily to Dirichlet and Neumann cases, we do not obtain in general regularity estimates with continuous dependence on the data only. Moreover, when we are able to do so, there is a gap between the regularity estimates with or without continuous dependence on data.

With the help of the work above, we have explored domain decomposition, which was one of the main original motivations for this work, in different directions. In particular, we have proven equivalence between a decomposed formulation with impedance transmission conditions and the global problem. This was done assuming known regularity of the global problem. Our work helps giving a new light to the choice of the Lagrange multiplier function space for saddle-point formulations in the spirit of [8]. Moreover, we have extended the proof of convergence of the iterative scheme proposed by Després et al. [43] to the case of real symmetric definite positive tensors. Investigations of this convergence in other anisotropic contexts – with scalar, constant impedance coefficient – were also conducted.

Although not the core of our work, some numerical investigations were conducted with FreeFem++ and PETSc on classical domain decomposition methods. They mostly illustrate the difficulty for standard methods to deal with complex anisotropic problems. Having studied several choices of impedance coefficient, the results we have obtained tend to indicate a link between the best possible direction for the impedance coefficient and the coercivity direction of the problem. They also show the potential interest of using an "equivalent impedance" coefficient that takes into account anistropy. This needs to be confirmed by further studies.

8.2 Perspectives

Among the numerous ways of future prospects are the treatment of mixed boundary condition problems. Indeed, this would allow to analyse theoretically the impedance-based DD for exterior boundary conditions other than Robin. This requires a new extension of our work. As the variational formulation will be posed in a different function space, one needs to develop the corresponding Helmholtz decompositions and compact embeddings, following the work of Fernandes and Gilardi [55] for real symmetric tensors. This would allow to study the well-posedness of the problem. For regularity matters, an additional difficulty is that one has to come up with appropriate shift regularity results. For this, we refer to the work of Jochmann [76, 77].

Let us also point out that our regularity results have been obtained for smooth boundaries and material tensors. These results can be extended to less regular problems. In the case of piecewise smooth boundaries, one has to take into account reentrant edges and corners, see e.g. [35]. In the case of coefficients with jumps, one can proceed as in [27]. Both aspects involve the theory of singularities, somehow as the techniques used in Chapter 4 (but in a volume setting). Concerning numerical analysis results, we have limited our results to the case of coercive problems. For non-coercive problems, one has to prove a discrete uniform inf-sup condition. This can be achieved in the spirit of [64].

An important prospect direction concerns impedance coefficients, especially in view of domain decomposition. This encompasses both theoretical questions and applicative ones. Among the theoretical aspects, the results of Chapter 4 could be pushed forward. For scalar coefficients, we have treated the smooth and piecewise constant cases. These results can be extended to deal with piecewise smooth impedance coefficients. The techniques used can be adapted to give more precise results for given configurations. For tensor-valued coefficients, we have proposed a first result that can certainly be enhanced. Another aspect is the proof of convergence of the algorithm in Section 7.4. In our work, the proofs are limited to the case of transmission conditions with constant coefficient, and to particular classes of material tensors. The study still has to be done for more general cases. In practice, heterogeneous coefficients seem more attractive than constant ones, so it would be useful to extend the proofs to scalar heterogeneous coefficients, and maybe even to tensor-valued heterogeneous coefficients.

Our work also raises more practical questions. While our numerical results show that the choice of the coefficient in the transmission condition clearly has an impact on the efficiency of the method, this is not clearly understood yet. Many questions remain open: is there a physical meaning that could be given to the impedance coefficient in anisotropic media? Is there an optimal way to choose the coefficient in the transmission condition to ensure a better convergence of the method (and which one)? Could better results be obtained using a tensor-valued coefficient, and, again, how to tune it? Finally, because this work is, we believe, among the first to address domain decomposition for anisotropic Maxwell problems, it opens a broad range of perspectives in the wide wild world of domain decomposition methods. Thus, more elaborate transmission conditions could be investigated, such as higher-order ones (see, e.g., [52]), or even non-local transmission conditions as in [91]. The multi-trace theory could provide a suitable setting for further developments.

Index of notations

Main operators

$\mathbf{curl}$	curl operator
curl_Γ	surface scalar curl operator
$\mathbf{curl}_\Gamma$	surface vectorial curl operator
div	divergence operator
div_Γ	surface divergence operator
Δ	(scalar) Laplace operator
Δ_Γ	Laplace-Beltrami operator
γ	$v \mapsto v_{\vert\Gamma}$ (trace of scalar fields)
γ^n	$\mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{n}_{\vert\Gamma}$ (normal trace)
γ^T	$\mathbf{v} \mapsto \mathbf{v} \times \mathbf{n}_{\vert\Gamma}$ (tangential trace)
π^T	$\mathbf{v} \mapsto \mathbf{n} \times (\mathbf{v} \times \mathbf{n})_{\vert\Gamma}$ (tangential components trace)
∂_α	partial derivative
∇	gradient
∇_Γ	surface gradient
$\times$	cross product
$\cdot^*$	complex conjugate or adjoint
$\cdot\,'$	dual space of given space
$\cdot_{\vert\mathcal{O}}$	restriction of given field to $\mathcal{O}$
$\widetilde{\cdot}$	continuation by zero
$[\,\cdot\,]_\Sigma$	jump of given quantity across surface Σ
$(\cdot \mid \cdot)$	inner product of $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$
$(\cdot,\cdot)_\Gamma$	inner product of $\mathbf{L}^2_t(\Gamma)$
$\langle\cdot,\cdot\rangle$	duality product in the sense of distributions
$\langle\cdot,\cdot\rangle_V$	duality product between V' and V
${}_\gamma\langle\cdot,\cdot\rangle_\pi$	duality product between $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ and $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$
${}_{\Gamma',\pi}\langle\cdot,\cdot\rangle_{\gamma,0}$	duality product between $\widetilde{\mathbf{H}}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma')$ and $\mathbf{H}_{\parallel,0}^{-1/2}(\mathrm{div}_\Gamma,\Gamma')$
$\Vert\cdot\Vert_\gamma$	norm of $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$
$\Vert\cdot\Vert_\pi$	norm of $\mathbf{H}_\perp^{-1/2}(\mathbf{curl},\Gamma)$

Main function spaces

$\mathcal{C}(\Omega)$	(set of) continuous fields of Ω
$\mathcal{C}^k(\Omega)$	k-differentiable fields of Ω ($k \in \mathbb{N}$)
$\mathcal{C}^\infty(\Omega)$	infinitely differentiable fields of Ω
$\mathcal{D}(\Omega)$	fields of $\mathcal{C}^\infty(\Omega)$ with compact support in Ω

$\mathcal{D}'(\Omega)$	dual space of $\mathcal{D}(\Omega)$ (distributions)
$\mathbf{H}(\mathbf{curl},\Omega)$	$=\{\mathbf{v}\in\mathbf{L}^2(\Omega),\ \mathbf{curl}\,\mathbf{v}\in\mathbf{L}^2(\Omega)\}$
$\mathbf{H}(\mathrm{div},\Omega)$	$=\{\mathbf{v}\in\mathbf{L}^2(\Omega),\ \mathrm{div}\,\mathbf{v}\in L^2(\Omega)\}$
$\mathbf{H}(\mathrm{div}\,0,\Omega)$	$=\{\mathbf{v}\in\mathbf{H}(\mathrm{div},\Omega),\ \mathrm{div}\,\mathbf{v}=0\}$
$\mathbf{H}(\mathrm{div}\,\underline{\boldsymbol{\xi}},\Omega)$	$=\{\mathbf{v}\in\mathbf{L}^2(\Omega),\ \mathrm{div}\,\underline{\boldsymbol{\xi}}\mathbf{v}\in L^2(\Omega)\}$
$\mathbf{H}(\mathrm{div}\,\underline{\boldsymbol{\xi}}0,\Omega)$	$=\{\mathbf{v}\in\mathbf{H}(\mathrm{div}\,\underline{\boldsymbol{\xi}},\Omega),\ \mathrm{div}\,\underline{\boldsymbol{\xi}}\mathbf{v}=0\}$
$\mathcal{H}(\Gamma)$	$=\{v\in H^1_{\mathrm{zmv}}(\Gamma),\ \Delta_\Gamma\, v\in H^{-1/2}(\Gamma)\}$
$\mathbf{H}_0(\mathbf{curl},\Omega)$	$=\{\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega),\ \gamma^T\mathbf{v}=\mathbf{0}\}$
$\mathbf{H}_{0,\Gamma_0}(\mathbf{curl},\Omega)$	$=\{\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega),\ \gamma^T\mathbf{v}_{\lvert\Gamma_0}=\mathbf{0}\}$
$\mathbf{H}_0(\mathrm{div},\Omega)$	$=\{\mathbf{v}\in\mathbf{H}(\mathrm{div},\Omega),\ \gamma^n\mathbf{v}=0\}$
$\mathbf{H}_0(\mathrm{div}\,0,\Omega)$	$=\mathbf{H}_0(\mathrm{div},\Omega)\cap\mathbf{H}(\mathrm{div}\,0,\Omega)$
$\mathbf{H}_0(\mathrm{div}\,\underline{\boldsymbol{\xi}},\Omega)$	$=\{\mathbf{v}\in\mathbf{L}^2(\Omega),\ \underline{\boldsymbol{\xi}}\mathbf{v}\in\mathbf{H}_0(\mathrm{div},\Omega)\}$
$\mathbf{H}_0(\mathrm{div}\,\underline{\boldsymbol{\xi}}0,\Omega)$	$=\mathbf{H}_0(\mathrm{div}\,\underline{\boldsymbol{\xi}},\Omega)\cap\mathbf{H}(\mathrm{div}\,\underline{\boldsymbol{\xi}}0,\Omega)$
$H^s(\Omega)$	Sobolev spaces ($s\in\mathbb{R}$)
$\mathbf{H}^s_t(\Gamma)$	$=\mathbf{H}^s(\Gamma)\cap\mathbf{L}^2_t(\Gamma)$ ($s>0$)
$H^s_{\mathrm{zmv}}(\Omega)$	$=H^s(\Omega)\cap L^2_{\mathrm{zmv}}(\Omega)$ ($s>0$)
$H^1_0(\Omega)$	$=\{v\in H^1(\Omega),\ \gamma v=0\}$
$\mathbf{H}^+(\mathbf{curl},\Omega)$	$=\{\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega),\ \gamma^T\mathbf{v}\in\mathbf{L}^2_t(\Gamma)\}$
$\mathbf{H}^{-1/2}_{\parallel}(\mathrm{div}_\Gamma,\Gamma)$	$=\gamma^T\left(\mathbf{H}(\mathbf{curl},\Omega)\right)$
$\mathbf{H}^{-1/2}_{\perp}(\mathrm{curl}_\Gamma,\Gamma)$	$=\pi^T\left(\mathbf{H}(\mathbf{curl},\Omega)\right)$
$\widetilde{\mathbf{H}}^{-1/2}_{\parallel}(\mathrm{div}_\Gamma,\Gamma')$	$=\gamma^T_{\Gamma'}\left(\mathbf{H}(\mathbf{curl},\Omega)\right)$
$\widetilde{\mathbf{H}}^{-1/2}_{\perp}(\mathrm{curl}_\Gamma,\Gamma')$	$=\pi^T_{\Gamma'}\left(\mathbf{H}(\mathbf{curl},\Omega)\right)$
$\mathbf{H}^{-1/2}_{\parallel,0}(\mathrm{div}_\Gamma,\Gamma')$	$=\gamma^T_{\Gamma'}\left(\mathbf{H}_{0,\Gamma\setminus\Gamma'}(\mathbf{curl},\Omega)\right)$
$\mathbf{H}^{-1/2}_{\perp,0}(\mathrm{curl}_\Gamma,\Gamma')$	$=\pi^T_{\Gamma'}\left(\mathbf{H}_{0,\Gamma\setminus\Gamma'}(\mathbf{curl},\Omega)\right)$
$\mathbf{K}_N(\Omega)$	$=\mathbf{H}_0(\mathbf{curl},\Omega)\cap\mathbf{H}(\mathrm{div}\,0,\Omega)$
$\mathbf{K}_N(\underline{\boldsymbol{\xi}};\Omega)$	$=\mathbf{H}_0(\mathbf{curl},\Omega)\cap\mathbf{H}(\mathrm{div}\,\underline{\boldsymbol{\xi}}0,\Omega)$
$\mathbf{K}_T(\Omega)$	$=\mathbf{H}(\mathbf{curl},\Omega)\cap\mathbf{H}_0(\mathrm{div}\,0,\Omega)$
$\mathbf{K}_T(\underline{\boldsymbol{\xi}};\Omega)$	$=\mathbf{H}(\mathbf{curl},\Omega)\cap\mathbf{H}_0(\mathrm{div}\,\underline{\boldsymbol{\xi}}0,\Omega)$
$L^2(\Omega)$	$=\{v \text{ s.t. } \int_\Omega \lvert v\rvert^2<\infty\}$
$\mathbf{L}^2_t(\Gamma)$	$=\{\mathbf{v}\in\mathbf{L}^2(\Gamma),\ \mathbf{v}\cdot\mathbf{n}=0\}$
$L^2_{\mathrm{zmv}}(\Omega)$	$=\{v\in L^2(\Omega),\ (v\lvert 1)=0\}$
$L^\infty(\Omega)$	essentially bounded fields of Ω
$W^{1,\infty}(\Omega)$	$=\{v\in L^\infty(\Omega),\ \forall k, \partial_k v\in L^\infty(\Omega)\}$
$\mathbf{W}_N(\Omega)$	$=\mathbf{H}^+(\mathbf{curl},\Omega)\cap\mathbf{H}(\mathrm{div}\,0,\Omega)$
$\mathbf{W}_N(\underline{\boldsymbol{\xi}};\Omega)$	$=\mathbf{H}^+(\mathbf{curl},\Omega)\cap\mathbf{H}(\mathrm{div}\,\underline{\boldsymbol{\xi}}0,\Omega)$
$\mathbf{X}_N(\Omega)$	$=\mathbf{H}_0(\mathbf{curl},\Omega)\cap\mathbf{H}(\mathrm{div},\Omega)$
$\mathbf{X}_N(\underline{\boldsymbol{\xi}};\Omega)$	$=\mathbf{H}_0(\mathbf{curl},\Omega)\cap\mathbf{H}(\mathrm{div}\,\underline{\boldsymbol{\xi}},\Omega)$
$\mathbf{X}_T(\Omega)$	$=\mathbf{H}(\mathbf{curl},\Omega)\cap\mathbf{H}_0(\mathrm{div},\Omega)$
$\mathbf{X}_T(\underline{\boldsymbol{\xi}};\Omega)$	$=\mathbf{H}(\mathbf{curl},\Omega)\cap\mathbf{H}_0(\mathrm{div}\,\underline{\boldsymbol{\xi}},\Omega)$

Various symbols

$\underline{\mathbf{I}}$	identity tensor (of $\mathbb{R}^{3\times 3}$)
$\mathbf{n}$	unit outward normal
Γ	boundary of Ω
Ω	domain of $\mathbb{R}^3$

$\overline{\mathcal{O}}$	closure of domain $\mathcal{O}$
$\partial\mathcal{O}$	boundary of domain $\mathcal{O}$
$\lesssim$	lesser up to a constant

Abbreviations

a.e.	almost everywhere
cf.	*confer*
e.g.	*exempli gratia*
i.e.	*id est*
iff	if and only if
s.t.	such that
w.r.t.	with respect to

Acronyms

ASM	Additive Schwarz Method
DDM	Domain Decomposition Method
dofs	Degrees of freedom
FEM	Finite Element Method
ORAS	Optimized Restrictive Additive Schwarz
PDE	Partial Differential Equation

Bibliography

[1] G. S. Alberti. Hölder regularity for Maxwell's equations under minimal assumptions on the coefficients. *Calculus of Variations and Partial Differential Equations*, 57(3):71, 2018.

[2] G. S. Alberti and Y. Capdeboscq. Elliptic regularity theory applied to time-harmonic anisotropic Maxwell's equations with less than Lipschitz complex coefficients. *SIAM Journal on Mathematical Analysis*, 46(1):998–1016, 2014.

[3] A. Alonso and A. Valli. A domain decomposition approach for heterogeneous time-harmonic Maxwell equations. *Computer methods in applied mechanics and engineering*, 143(1-2):97–112, 1997.

[4] A. Alonso and A. Valli. Unique solvability for high-frequency heterogeneous time-harmonic Maxwell equations via the Fredholm alternative theory. *Mathematical Methods in the Applied Sciences*, 21(6):463–477, 1998.

[5] A. Alonso Rodriguez and L. Gerardo-Giorda. New nonoverlapping domain decomposition methods for the harmonic Maxwell system. *SIAM Journal on Scientific Computing*, 28(1):102–122, 2006.

[6] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Mathematical Methods in the Applied Sciences*, 21(9):823–864, 1998.

[7] F. Assous, P. Ciarlet, and S. Labrunie. *Mathematical foundations of computational electromagnetism.* Springer, 2018.

[8] A. Back, T. Hattori, S. Labrunie, J.-R. Roche, and P. Bertrand. Electromagnetic wave propagation and absorption in magnetised plasmas: variational formulations and domain decomposition. *ESAIM: Mathematical Modelling and Numerical Analysis*, 49(5):1239–1260, 2015.

[9] J. M. Ball, Y. Capdeboscq, and B. Tsering-Xiao. On uniqueness for time harmonic anisotropic Maxwell's equations with piecewise regular coefficients. *Mathematical Models and Methods in Applied Sciences*, 22(11):1250036, 2012.

[10] H. Barucq and B. Hanouzet. Asymptotic behavior of solutions to Maxwell's system in bounded domains with absorbing Silver–Müller's condition on the exterior boundary. *Asymptotic Analysis*, 15(1):25–40, 1997.

[11] A. Bermúdez, R. Rodríguez, and P. Salgado. Numerical solution of eddy current problems in bounded domains using realistic boundary conditions. *Computer methods in applied mechanics and engineering*, 194(2-5):411–426, 2005.

[12] M. S. Birman and M. Z. Solomyak. L^2-theory of the Maxwell operator in arbitrary domains. *Russian Mathematical Surveys*, 42(6):R03, 1987.

[13] M. Bonazzoli, V. Dolean, I. Graham, E. Spence, and P.-H. Tournier. Domain decomposition preconditioning for the high-frequency time-harmonic Maxwell equations with absorption. *Mathematics of Computation*, 88(320):2559–2604, 2019.

[14] A. Bonito, J.-L. Guermond, and F. Luddens. Regularity of the Maxwell equations in heterogeneous media and Lipschitz domains. *Journal of Mathematical Analysis and applications*, 408(2):498–512, 2013.

[15] A.-S. Bonnet-Ben Dhia, C. Hazard, and F. Monteghetti. Complex-scaling method for the plasmonic resonances of planar subwavelength particles with corners. *Journal of Computational Physics*, 440:110433, 2021.

[16] A. Buffa. Hodge decompositions on the boundary of nonsmooth domains: the multi-connected case. *Mathematical Models and Methods in Applied Sciences*, 11(09):1491–1503, 2001.

[17] A. Buffa and S. H. Christiansen. The electric field integral equation on Lipschitz screens: definitions and numerical approximation. *Numerische Mathematik*, 94(2):229–267, 2003.

[18] A. Buffa and P. Ciarlet Jr. On traces for functional spaces related to Maxwell's equations. Part I: An integration by parts formula in Lipschitz polyhedra. *Mathematical Methods in the Applied Sciences*, 24(1):9–30, 2001.

[19] A. Buffa and P. Ciarlet Jr. On traces for functional spaces related to Maxwell's equations. Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications. *Mathematical Methods in the Applied Sciences*, 24(1):31–48, 2001.

[20] A. Buffa, M. Costabel, and C. Schwab. Boundary element methods for Maxwell's equations on non-smooth domains. *Numerische Mathematik*, 92(4):679–710, 2002.

[21] A. Buffa, M. Costabel, and D. Sheen. On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. *Journal of Mathematical Analysis and Applications*, 276(2):845–867, 2002.

[22] D. Chicaud, P. Ciarlet Jr, and A. Modave. Perturbed edge finite element method for the simulation of electromagnetic waves in magnetised plasmas. In *Proceedings of the 14th International Conference on Mathematical and Numerical Aspects of wave Propagation Phenomena (Vienna, Austria)*, pages 434–435, 2019.

[23] D. Chicaud, P. Ciarlet Jr, and A. Modave. Analysis of variational formulations and low-regularity solutions for time-harmonic electromagnetic problems in complex anisotropic media. *SIAM Journal on Mathematical Analysis*, 53(3):2691–2717, 2021.

[24] P. Ciarlet Jr. Augmented formulations for solving Maxwell equations. *Computer Methods in Applied Mechanics and Engineering*, 194(2-5):559–586, 2005.

[25] P. Ciarlet Jr. On the approximation of electromagnetic fields by edge finite elements. Part 1: Sharp interpolation results for low-regularity fields. *Computers & Mathematics with Applications*, 71(1):85–104, 2016.

[26] P. Ciarlet, Jr. Mathematical and numerical analyses for the div-curl and div-curlcurl problems with a sign-changing coefficient. https://hal.archives-ouvertes.fr/hal-02567484/, May 2020.

[27] P. Ciarlet Jr. On the approximation of electromagnetic fields by edge finite elements. Part 3: sensitivity to coefficients. *SIAM Journal on Mathematical Analysis*, 52(3):3004–3038, 2020.

[28] P. Ciarlet Jr, E. Jamelot, and F. D. Kpadonou. Domain decomposition methods for the diffusion equation with low-regularity solution. *Computers & Mathematics with Applications*, 74(10):2369–2384, 2017.

[29] X. Claeys. *Boundary integral equations of time harmonic wave scattering at complex structures*. Université Pierre et Marie Curie (UPMC Paris 6), 2016.

[30] X. Claeys and R. Hiptmair. Electromagnetic scattering at composite objects: a novel multi-trace boundary integral formulation. *ESAIM: Mathematical Modelling and Numerical Analysis*, 46(6):1421–1445, 2012.

[31] X. Claeys, B. Thierry, and F. Collino. Integral equation based optimized Schwarz method for electromagnetics. In *International Conference on Domain Decomposition Methods*, pages 187–194. Springer, 2017.

[32] F. Collino, G. Delbue, P. Joly, and A. Piacentini. A new interface condition in the non-overlapping domain decomposition method for the Maxwell equations. *Computer methods in applied mechanics and engineering*, 148(1-2):195–207, 1997.

[33] F. Collino, S. Ghanemi, and P. Joly. Domain decomposition method for harmonic wave propagation: a general presentation. *Computer methods in applied mechanics and engineering*, 184(2-4):171–211, 2000.

[34] M. Costabel. A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. *Mathematical Methods in the Applied Sciences*, 12(4):365–368, 1990.

[35] M. Costabel and M. Dauge. Singularities of Electromagnetic Fields in Polyhedral Domains. *Archive for Rational Mechanics and Analysis*, 151(3):221–276, 2000.

[36] M. Costabel and M. Dauge. Weighted regularization of Maxwell equations in polyhedral domains. *Numerische Mathematik*, 93(2):239–277, 2002.

[37] M. Costabel, M. Dauge, and S. Nicaise. Singularities of Maxwell interface problems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 33(3):627–649, 1999.

[38] M. Costabel, M. Dauge, and S. Nicaise. Corner singularities and analytic regularity for linear elliptic systems. Part I: Smooth domains. https://hal.archives-ouvertes.fr/hal-00453934/, Feb. 2010.

[39] M. Dauge. *Elliptic boundary value problems on corner domains*. Lecture Notes in Mathematics, **1341**. Springer-Verlag, 1988.

[40] M. Dauge. Singularities of corner problems and problems of corner singularities. In *ESAIM: Proceedings*, volume 6, pages 19–40. EDP Sciences, 1999.

[41] A. Dello Russo and A. Alonso. Finite element approximation of Maxwell eigenproblems on curved Lipschitz polyhedral domains. *Applied numerical mathematics*, 59(8):1796–1822, 2009.

[42] B. Després. *Méthodes de décomposition de domaine pour la propagation d'ondes en régime harmonique. Le théorème de Borg pour l'équation de Hill vectorielle.* PhD thesis, Université Paris IX Dauphine, 1991.

[43] B. Després, P. Joly, and J. E. Roberts. A domain decomposition method for the harmonic Maxwell equations. Iterative methods in linear algebra. In *Iterative Methods in Linear Algebra*, pages 245–252, 1992.

[44] C. R. Dohrmann and O. B. Widlund. An iterative substructuring algorithm for two-dimensional problems in $\mathbf{H}(\mathbf{curl})$. *SIAM Journal on Numerical Analysis*, 50(3):1004–1028, 2012.

[45] V. Dolean, M. J. Gander, and L. Gerardo-Giorda. Optimized Schwarz methods for Maxwell's equations. *SIAM Journal on Scientific Computing*, 31(3):2193–2213, 2009.

[46] V. Dolean, M. J. Gander, S. Lanteri, J.-F. Lee, and Z. Peng. Effective transmission conditions for domain decomposition methods applied to the time-harmonic curl–curl Maxwell's equations. *Journal of Computational Physics*, 280:232–247, 2015.

[47] V. Dolean, P. Jolivet, and F. Nataf. *An introduction to domain decomposition methods: algorithms, theory, and parallel implementation*, volume 144. SIAM, 2015.

[48] V. Dolean, S. Lanteri, and R. Perrussel. A domain decomposition method for solving the three-dimensional time-harmonic Maxwell equations discretized by discontinuous Galerkin methods. *Journal of Computational Physics*, 227(3):2044–2072, 2008.

[49] H. Duan, Z. Du, W. Liu, and S. Zhang. New mixed elements for Maxwell equations. *SIAM Journal on Numerical Analysis*, 57(1):320–354, 2019.

[50] M. El Bouajaji, X. Antoine, and C. Geuzaine. Approximate local magnetic-to-electric surface operators for time-harmonic maxwell's equations. *Journal of Computational Physics*, 279:241–260, 2014.

[51] M. El Bouajaji, V. Dolean, M. Gander, and S. Lanteri. Optimized Schwarz methods for the time-harmonic Maxwell equations with damping. *SIAM Journal on Scientific Computing*, 34(4):A2048–A2071, 2012.

[52] M. El Bouajaji, B. Thierry, X. Antoine, and C. Geuzaine. A quasi-optimal domain decomposition algorithm for the time-harmonic Maxwell's equations. *Journal of Computational Physics*, 294:38–57, 2015.

[53] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159. Springer, 2004.

[54] O. G. Ernst and M. J. Gander. Why it is difficult to solve Helmholtz problems with classical iterative methods. *Numerical analysis of multiscale problems*, pages 325–363, 2012.

[55] P. Fernandes and G. Gilardi. Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions. *Mathematical Models and Methods in Applied Sciences*, 7(07):957–991, 1997.

[56] G. N. Gatica and S. Meddahi. Finite element analysis of a time-harmonic Maxwell problem with an impedance boundary condition. *IMA Journal of Numerical Analysis*, 32(2):534–552, 2012.

[57] Y.-L. Geng and C.-W. Qiu. Extended Mie theory for a gyrotropic-coated conducting sphere: An analytical approach. *IEEE transactions on antennas and propagation*, 59(11):4364–4368, 2011.

[58] Y.-L. Geng, X.-B. Wu, L.-W. Li, and B.-R. Guan. Electromagnetic scattering by an inhomogeneous plasma anisotropic sphere of multilayers. *IEEE transactions on antennas and propagation*, 53(12):3982–3989, 2005.

[59] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms*, volume 5. Springer, 1986.

[60] J. Gopalakrishnan and J. Pasciak. Overlapping Schwarz preconditioners for indefinite time-harmonic Maxwell equations. *Mathematics of computation*, 72(241):1–15, 2003.

[61] P. Grisvard. *Elliptic problems in nonsmooth domains.* SIAM, 1985.

[62] P. Grisvard. *Singularities in boundary value problems*, volume 22. Springer, 1992.

[63] P. W. Gross and P. R. Kotiuga. *Electromagnetic theory and computation: a topological approach.* Number 48. Cambridge University Press, 2004.

[64] M. Halla. Electromagnetic Steklov eigenvalues: approximation analysis. *ESAIM: Mathematical Modelling and Numerical Analysis*, 2021. In press.

[65] R. Haller-Dintelmann, H.-C. Kaiser, and J. Rehberg. Direct computation of elliptic singularities across anisotropic, multi-material edges. *Journal of Mathematical Sciences*, 172(4):589–622, 2011.

[66] T. Hattori. *Décomposition de domaine pour la simulation Full-Wave dans un plasma froid.* PhD thesis, Université de Lorraine, 2014.

[67] Y.-X. He, L. Li, S. Lanteri, and T.-Z. Huang. Optimized Schwarz algorithms for solving time-harmonic Maxwell's equations discretized by a hybridizable discontinuous Galerkin method. *Computer Physics Communications*, 200:176–181, 2016.

[68] F. Hecht. New development in FreeFem++. *Journal of Numerical Mathematics*, 20(3-4):251–265, 2012.

[69] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, 11:237, 2002.

[70] R. Hiptmair, A. Moiola, and I. Perugia. Stability results for the time-harmonic Maxwell equations with impedance boundary conditions. *Mathematical Models and Methods in Applied Sciences*, 21(11):2263–2287, 2011.

[71] R. Hiptmair and A. Toselli. Overlapping Schwarz methods for vector valued elliptic problems in three dimensions. In *IMA Volumes in Mathematics and its Applications*, 1997.

[72] R. Hiptmair and J. Xu. Nodal auxiliary space preconditioning in H(curl) and H(div) spaces. *SIAM Journal on Numerical Analysis*, 45(6):2483–2509, 2007.

[73] Q. Hu and J. Zou. A nonoverlapping domain decomposition method for Maxwell's equations in three dimensions. *SIAM Journal on Numerical Analysis*, 41(5):1682–1708, 2003.

[74] Q. Hu and J. Zou. Substructuring preconditioners for saddle-point problems arising from Maxwell's equations in three dimensions. *Mathematics of Computation*, 73(245):35–61, 2004.

[75] L.-M. Imbert-Gérard. *Mathematical and numerical problems of some wave phenomena appearing in magnetic plasmas.* PhD thesis, Université Pierre et Marie Curie-Paris VI, 2013.

[76] F. Jochmann. Uniqueness and regularity for the two-dimensional drift-diffusion model for semiconductors coupled with Maxwell's equations. *Journal of Differential Equations*, 147(2):242–270, 1998.

[77] F. Jochmann. Regularity of weak solutions of Maxwell's equations with mixed boundary-conditions. *Mathematical Methods in the Applied Sciences*, 22(14):1255–1274, 1999.

[78] A. Kirsch and F. Hettlich. *Mathematical Theory of Time-harmonic Maxwell's Equations*. Springer, 2016.

[79] V. Kondrat'ev and O. Oleinik. Boundary value problems for partial differential equations in non-smooth domains. *Russian Math. Surveys*, 38:1–86, 1983.

[80] V. Kozlov, V. Maz'ya, and J. Rossmann. *Spectral problems associated with corner singularities of solutions to elliptic equations*. Number 85. American Mathematical Society, 2001.

[81] L. Li, S. Lanteri, and R. Perrussel. A hybridizable discontinuous Galerkin method combined to a Schwarz algorithm for the solution of 3D time-harmonic Maxwell's equation. *Journal of Computational Physics*, 256:563–581, 2014.

[82] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications: Vol. 1*, volume 181. Springer, 1968.

[83] P.-L. Lions. On the Schwarz alternating method. III: a variant for nonoverlapping subdomains. In *Third international symposium on domain decomposition methods for partial differential equations*, volume 6, pages 202–223. SIAM Philadelphia, PA, 1990.

[84] N. Marsic, C. Waltz, J.-F. Lee, and C. Geuzaine. Domain decomposition methods for time-harmonic electromagnetic waves with high-order Whitney forms. *IEEE Transactions on Magnetics*, 52(3):1–4, 2015.

[85] T. P. Mathew. *Domain decomposition methods for the numerical solution of partial differential equations*, volume 61. Springer, 2008.

[86] P. Monk. *Finite element methods for Maxwell's equations*. Oxford University Press, 2003.

[87] C. Müller. *Foundations of the mathematical theory of electromagnetic waves*, volume 155. Springer, 1969.

[88] J.-C. Nédélec. *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*. Springer, 2001.

[89] A. Nicolet, F. Zolla, and S. Guenneau. Electromagnetic analysis of cylindrical cloaks of an arbitrary cross section. *Optics letters*, 33(14):1584–1586, 2008.

[90] T. Okaji. Strong unique continuation property for time-harmonic Maxwell equations. *Journal of the Mathematical Society of Japan*, 54(1):89–122, 2002.

[91] É. Parolin. *Non-overlapping domain decomposition methods with non-local transmission operators for harmonic wave propagation problems*. PhD thesis, Institut Polytechnique de Paris, 2020.

[92] J. E. Pasciak and J. Zhao. Overlapping Schwarz methods in $\mathbf{H}(\mathbf{curl})$ on polyhedral domains. *Journal of Numerical Mathematics*, 10(3):221–234, 2002.

[93] J. B. Pendry, D. Schurig, and D. R. Smith. Controlling electromagnetic fields. *Science*, 312(5781):1780–1782, 2006.

[94] Z. Peng and J.-F. Lee. Non-conformal domain decomposition method with second-order transmission conditions for time-harmonic electromagnetics. *Journal of Computational Physics*, 229(16):5615–5629, 2010.

[95] Z. Peng and J.-F. Lee. A Scalable Nonoverlapping and Nonconformal Domain Decomposition Method for Solving Time-Harmonic Maxwell Equations in $\mathbb{R}^3$. *SIAM Journal on Scientific Computing*, 34(3):A1266–A1295, 2012.

[96] Z. Peng, V. Rawat, and J.-F. Lee. One way domain decomposition method with second order transmission conditions for solving electromagnetic wave problems. *Journal of Computational Physics*, 229(4):1181–1197, 2010.

[97] A. Quarteroni and A. Valli. *Domain decomposition methods for partial differential equations*. Oxford University Press, 1999.

[98] F. Rapetti and A. Toselli. A FETI preconditioner for two dimensional edge element approximations of Maxwell's equations on nonmatching grids. *SIAM Journal on Scientific Computing*, 23(1):92–108, 2001.

[99] V. Rawat and J.-F. Lee. Nonoverlapping domain decomposition with second order transmission condition for the time-harmonic Maxwell's equations. *SIAM Journal on Scientific Computing*, 32(6):3584–3603, 2010.

[100] J. E. Santos and D. Sheen. On the existence and uniqueness of solutions to Maxwell's equations in bounded domains with application to magnetotellurics. *Mathematical Models and Methods in Applied Sciences*, 10(04):615–628, 2000.

[101] A. Schädle, L. Zschiedrich, S. Burger, R. Klose, and F. Schmidt. Domain decomposition method for Maxwell's equations: scattering off periodic structures. *Journal of Computational Physics*, 226(1):477–493, 2007.

[102] E. Sébelin, Y. Peysson, X. Litaudon, D. Moreau, J. Miellou, and O. Lafitte. Uniqueness and existence result around Lax-Milgram lemma: application to electromagnetic waves propagation in tokamak plasmas. Technical report, Association Euratom-CEA, 1997. online: http://www.iaea.org/inis/collection/NCLCollectionStore/_Public/30/017/30017036.pdf.

[103] S. Silver. *Microwave antenna theory and design*. McGraw-Hill, 1949.

[104] T. H. Stix. *Waves in plasmas*. Springer, 1992.

[105] I. Terrasse. *Résolution mathématique et numérique des équations de Maxwell instationnaires par une méthode de potentiels retardés*. PhD thesis, Palaiseau, Ecole polytechnique, 1993.

[106] A. Toselli. *Domain decomposition methods for vector field problems*. PhD thesis, New York University, Graduate School of Arts and Science, 1999.

[107] A. Toselli. Overlapping Schwarz methods for Maxwell's equations in three dimensions. *Numerische Mathematik*, 86(4):733–752, 2000.

[108] A. Toselli. Dual-primal FETI algorithms for edge finite-element approximations in 3D. *IMA Journal of Numerical Analysis*, 26(1):96–130, 2006.

[109] A. Toselli and A. Klawonn. A FETI domain decomposition method for Maxwell's equations with discontinuous coefficients in two dimensions. In *Courant Institute, New York University*, 1999.

[110] A. Toselli and O. Widlund. *Domain decomposition methods-algorithms and theory*, volume 34. Springer, 2006.

[111] A. Toselli, O. Widlund, and B. Wohlmuth. An iterative substructuring method for Maxwell's equations in two dimensions. *Mathematics of Computation*, 70(235):935–949, 2001.

[112] B. Tsering-xiao and W. Xiang. Regularity of solutions to time-harmonic Maxwell's system with various lower than Lipschitz coefficients. *Nonlinear Analysis*, 192:111693, 2020.

[113] V. Vogelsang. On the strong unique continuation principle for inequalities of Maxwell type. *Mathematische Annalen*, 289(1):285–295, 1991.

[114] C. Weber. A local compactness theorem for Maxwell's equations. *Mathematical Methods in the Applied Sciences*, 2(1):12–25, 1980.

[115] X. Xiang. On L^r Estimates for Maxwell's Equations with Complex Coefficients in Lipschitz Domains. *SIAM Journal on Mathematical Analysis*, 52(6):6140–6154, 2020.

[116] F. Zolla, S. Guenneau, A. Nicolet, and J. Pendry. Electromagnetic analysis of cylindrical invisibility cloaks and the mirage effect. *Optics Letters*, 32(9):1069–1071, 2007.

ECOLE DOCTORALE DE MATHEMATIQUES HADAMARD

Titre: Analyse de problèmes électromagnétiques harmoniques en temps dans des milieux anisotropes elliptiques

Mots clés: équations de Maxwell, milieux anisotropes, ondes électromagnétiques, éléments finis, étude de régularité, décomposition de domaine

Résumé: La simulation numérique de problèmes électromagnétiques dans des configurations physiques complexes est largement utilisée pour de nombreuses applications scientifiques et industrielles, telles que la conception de métamatériaux optiques ou l'étude des plasmas froids. L'analyse mathématique et numérique des problèmes de Maxwell est bien connue dans des contextes physiques simples, où les paramètres du milieu sont isotropes. Des résultats en milieux anistropes existent, mais se limitent généralement au cas des tenseurs réels symétriques (ou complexes hermitiens) définis positifs. Cependant, pour certains milieux plus complexes, les problèmes ne sont pas couverts par la théorie standard. De nouveaux outils mathématiques doivent donc être développés pour analyser ces problèmes.
Dans cette thèse, nous analysons des problèmes électromagnétiques harmoniques en temps pour une classe générale de tenseurs matériels anistropes, appelés elliptiques. Nous développons un cadre fonctionnel étendu adapté à ces problèmes anisotropes, en généralisant les résultats connus. Nous étudions le caractère bien posé de problèmes avec conditions limites de Dirichlet, Neumann ou Robin. Dans le cas Robin, un intérêt particulier est porté à la caractérisation des espaces fonctionnels pour les traces de Robin. Nous étudions la régularité de la solution et de son rotationnel, et donnons des éléments d'analyse numérique. Dans la perspective de l'utilisation de méthodes de décomposition de domaine (DDM) pour une résolution accélérée, nous proposons et étudions différentes formulations décomposées, en nous focalisant sur leurs espaces fonctionnels et leur équivalence avec le problème global. Quelques expérimentations numériques sur la DDM complètent ce travail.

Title: Analysis of time-harmonic electromagnetic problems in elliptic anisotropic media

Keywords: Maxwell equations, anisotropic media, electromagnetic waves, finite elements, regularity analysis, domain decomposition

Abstract: The numerical simulation of electromagnetic problems in complex physical settings is a trending topic which conveys many scientific and industrial applications, such as the design of optical metamaterials, or the study of cold plasmas. The mathematical and numerical analysis of Maxwell problems is well-known in simple physical contexts, when the material parameters are isotropic. Some results in anisotropic media exist, but they generally tend to focus on the case where the material tensors are real symmetric (or complex Hermitian) definite positive. However, problems in more complex media are not covered by the standard theory. Therefore, new mathematical tools need to be developed to analyse these problems. This thesis aims at analysing time-harmonic electromagnetic problems for a general class of complex anisotropic material tensors. These are called elliptic materials. We derive an extended functional framework well-suited for these anisotropic problems, generalizing well-known results. We study the well-posedness of Maxwell boundary value problems for Dirichlet, Neumann, and Robin boundary conditions. For the Robin case, the characterization of appropriate function spaces for Robin traces is addressed. The regularity of the solution and its curl is studied, and elements of numerical analysis for edge finite elements are provided. In the perspective of the use of Domain Decomposition Methods (DDM) for accelerated numerical computing, various decomposed formulations are proposed and studied, focusing on their right meaning in terms of function spaces and equivalence with the global problem. These results are complemented with some numerical DDM experimentations in anisotropic media.

Institut Polytechnique de Paris
91120 Palaiseau, France