Introduction to interval analysis

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Introduction

Rump polynomial

$f(a,b) = 333.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + a/(2b)$
If we compute $f(77617.0, 33096.0) = ?$

With floating number

$-1.18 \cdot 10^{21}$ (matlab) or $1.18 \cdot 10^{21}$ or 1.172 (depends on implementation)

With intervals

Roundoff is guaranteed ($1/3 \in [0.33...3, 0.33...4]$) and thus
$f(a,b) \in [-0.8273960599469, -0.8273960599467]$

Notations

- An interval $[x] \in \mathbb{IR} : [x] = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} \mid \underline{x} \leq x \leq \overline{x}\}$
- $\underline{x}$ the lower bound and $\overline{x}$ the upper bound
- $\mathrm{m}([x])$ the midpoint such that $\mathrm{m}([x]) = \underline{x} + (\overline{x} - \underline{x})/2$
- $\mathrm{w}([x])$ the diameter of $[x]$ such that $\mathrm{w}([x]) = \overline{x} - \underline{x}$
- A box (Cartesian product) $[\mathbf{x}] \in \mathbb{IR}^n$: $[\mathbf{x}] = ([x_1], ..., [x_n])^T$

Interval Arithmetic

- $[\underline{x},\overline{x}] + [\underline{y},\overline{y}] = [\underline{x}+\underline{y},\overline{x}+\overline{y}]$
- $[\underline{x},\overline{x}] - [\underline{y},\overline{y}] = [\underline{x}-\overline{y},\overline{x}-\underline{y}]$
- $[\underline{x},\overline{x}] * [\underline{y},\overline{y}] = [min(\underline{x}*\underline{y},\underline{x}*\overline{y},\overline{x}*\underline{y},\overline{x}*\overline{y}), max(\underline{x}*\underline{y},\underline{x}*\overline{y},\overline{x}*\underline{y},\overline{x}*\overline{y})]$
- $[\underline{x},\overline{x}]/[\underline{y},\overline{y}] = [\underline{x},\overline{x}] * (1/[\underline{y},\overline{y}])$
- $1/[\underline{y},\overline{y}] = [min(1/\underline{y},1/\overline{y}), max(1/\underline{y},1/\overline{y})]$ if $0 \notin [\underline{y},\overline{y}]$

Problem

$[x] - [x] = \{x - y \mid x \in [x], y \in [x]\}$

Example

$[x] = [2,3]$, $[x] - [x] = ?$

Interval Arithmetic

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- $[\underline{x}, \overline{x}] - [\underline{y}, \overline{y}] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$
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$[x] = [2, 3]$, $[x] - [x] = [2, 3] - [2, 3] = [-1, 1] \neq 0$!

Interval Arithmetic

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Problem

$$[x] - [x] = \{x - y \mid x \in [x], y \in [x]\}$$

Example

$[x] = [2, 3]$, $[x] - [x] = [2, 3] - [2, 3] = [-1, 1] \neq 0$!

$\Rightarrow$ But $0 \in [-1, 1]$

Interval Arithmetic

Other issues

- Addition $\neq$ Soustraction^{-1}
- Multiplication $\neq$ Division^{-1}
- Multiplication is sub-distributive wrt addition: $x \times (y + z) \subset x \times y + x \times z$

But strong advantage

Always correct by inclusion ! (even wrt roundoff errors)

General form

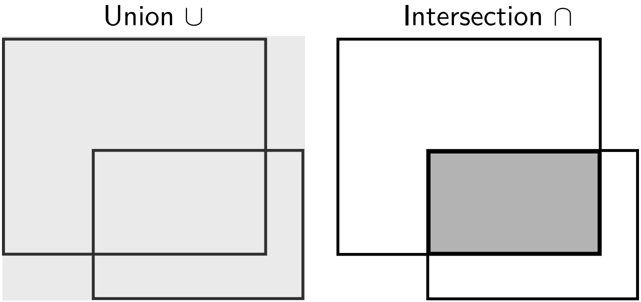
$[x] \diamond [y] = [\{x \diamond y \mid x \in [x], y \in [y]\}] =$
$[min(\underline{x} \diamond \underline{y}, \underline{x} \diamond \overline{y}, \overline{x} \diamond \underline{y}, \overline{x} \diamond \overline{y}), max(\underline{x} \diamond \underline{y}, \underline{x} \diamond \overline{y}, \overline{x} \diamond \underline{y}, \overline{x} \diamond \overline{y})]$

Interval extensions

of the elementary functions

By the help of monotonicity: cos, sin, exp, acoth, log, etc...

of the set operators



Interval-valued extension of Real functions

- Considering $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, an extension or *inclusion function is the interval function* $[f] : \mathbb{IR}^n \mapsto \mathbb{IR}^m$ *if*

$$\forall [x] \in \mathbb{IR}^n, f([x]) \subset [f]([x])$$

- $[f]$ *is inclusion monotonic if*

$$[x] \subset [y] \Rightarrow [f]([x]) \subset [f]([y])$$

Natural extension

Easy: replace each operators (*,-,+,/) and elementary functions by its interval counterpart !

Example

Evaluate with interval natural extention $f(x) = x(x+1)$, evaluate natural inclusion for $[x] = [-1, 1]$

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$$[f]_N([x]) = [x]([x]+1) = [-2, 2]$$

But in function of the syntax of the expression we can have different results!

$$f(x) = xx + x$$
$$f(x) = x^2 + x$$

Example

Evaluate with interval natural extention $f(x) = x(x+1)$, evaluate natural inclusion for $[x] = [-1, 1]$

$$[f]_N([x]) = [x]([x]+1) = [-2, 2]$$

But in function of the syntax of the expression we can have different results!

$$\begin{aligned} f(x) = xx + x &= \text{[-2,2]} \\ f(x) = x^2 + x &= \text{[-1,2]} \end{aligned}$$

Interval-valued extension of Real functions

Centered inclusion function

$$[f_C]([x]) \triangleq f(\mathrm{m}([x])) + [J_f^T]([x])([x] - \mathrm{m}([x]))$$

where $[J_f^T]([x])$ is an interval inclusion of the Jacobian matrix of f

Taylor inclusion function

Same as previous one, but at order 2

$$[f_T]([x]) \triangleq f\left(\mathrm{m}([x])\right) + [J_f^T]\big(\mathrm{m}([x])\big)\left([x] - \mathrm{m}([x])\right) + \frac{1}{2}(([x] - \mathrm{m}([x]))^2)^T [H_f]([x])([x] - \mathrm{m}([x]))^2$$

where $[H_f^T]([x])$ is an interval inclusion of the Hessian matrix of f

Comparison of interval extensions

Comparison (in general)

- Natural inclusion is competitive for the larger interval
- Taylor and centered are more efficient for smaller ones
- Taylor is always more efficient than centered, but more expensive computationally

Evaluation of $f(x) = 4x^2 - 2x + cos(x)$

$[x] = [-1, 2.5]$: $f_N([x]) = [-15.8, 28]$; $f_C([x]) = [-31.4, 34.5]$
$[x] = [1.1, 1.4]$: $f_N([x]) = [2.2, 6.1]$; $f_C([x]) = [2.8, 5.3]$

Constraint propagation

Forward / Backward propagation

Isolation of each variables, contraction of its domain, and propagate

Example

Consider the constraint $x_3 = x_1 x_2$, and the box $[\mathbf{x}] = [1,4] \times [1,4] \times [8,40]$ Constraint can be rewritten in three ways: $x_1 = x_3/x_2$; $x_2 = x_3/x_1$; $x_3 = x_1 x_2$ which can be seen as contractors:

- $(x_3/x_2) \cap x_1 = [8,40]/[1,4] \cap [1,4] = [2,4]$
- $(x_3/x_1) \cap x_2 = [2,4]$
- $(x_1 x_2) \cap x_3 = ([1,4] \times [1,4]) \cap [8,40] = [8,16]$

The new domain of $[\mathbf{x}]$ is then $[2,4] \times [2,4] \times [8,16]$

Constraint propagation

Constraint Satisfaction Problem (CSP)

A (continuous or numerical) CSP $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ is defined as follows:

- $\mathcal{X} = \{x_1, \dots, x_n\}$ is a set of variables
- $\mathcal{D} = \{\mathbf{x_1}, \dots, \mathbf{x_n}\}$ is a set of domains
- $\mathcal{C} = \{c_1, \dots, c_m\}$ is a set of constraints

Example of CSP

$$\mathcal{H} : \begin{pmatrix} \mathcal{X} : & \{x_1, x_2, x_3, x_4\} \\ \mathcal{D} : & [\mathbf{x}] \in [-10, 10] \times [-10, 10] \times [-1, 1] \times [-1, 1] \\ \mathcal{C} : & \{x_1 + 2x_2 - x_3 = 0, \quad x_1 - x_2 - x_4 = 0\} \end{pmatrix}$$

Solving procedure

Forward / Backward on each constraint, propagate the result to the others, till a fixed point

Constraint propagation

Iterations of fwd/bwd

k	$[x_1](k)$	$[x_2](k)$	$[x_3](k)$	$[x_4](k)$
0	$[-10, 10]$	$[-10, 10]$	$[-1, 1]$	$[-1, 1]$
1	$[-6.5, 6.5]$	$[-5.5, 5.5]$	$[-1, 1]$	$[-1, 1]$
2	$[-4.75, 4.75]$	$[-3.75, 3.75]$	$[-1, 1]$	$[-1, 1]$
∞	$[-3, 3]$	$[-2, 2]$	$[-1, 1]$	$[-1, 1]$

$\Rightarrow$ Fixed point reached, but too large !

Solution

Bisection: Divide and rule ! (without solution loss)

$[\mathbf{x}](1) \in ([-10, 0]; [-10, 10]; [-1, 1]; [-1, 1])$

$[\mathbf{x}](2) \in ([0, 10]; [-10, 10]; [-1, 1]; [-1, 1])$

And start again :

$[\mathbf{x}](1) \in ([-1.5, 0]; [-0.5, 1]; [-1, 1]; [-1, 1])$

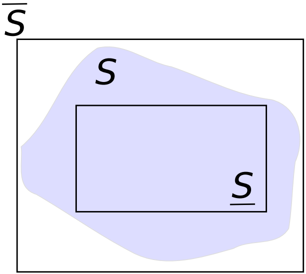
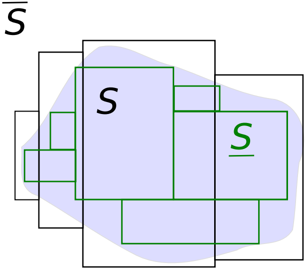
$[\mathbf{x}](2) \in ([0, 1.5]; [-1, 0.5]; [-1, 1]; [-1, 1])$

$\Rightarrow [\mathbf{x}] = [\mathbf{x}](1) \cup [\mathbf{x}](2) = ([-1.5, 1.5]; [-1, 1]; [-1, 1]; [-1, 1])$

Bisection-subpaving

How represent a set $\mathcal{S}$?

With two subpavings (without overlaping): $\underline{\mathcal{S}} \subset \mathcal{S} \subset \overline{\mathcal{S}}$

with two boxes	with two subpavings of N boxes
	

Volume: $Vol(\underline{\mathcal{S}}) < Vol(\mathcal{S}) < Vol(\overline{\mathcal{S}})$

$N \nearrow$ implies $Vol(\underline{\mathcal{S}}) \nearrow$ and $Vol(\overline{\mathcal{S}}) \searrow$, then better surrounding !

Bisection-subpaving

Paving of set

Start by initial domain and bisect till the required accuracy

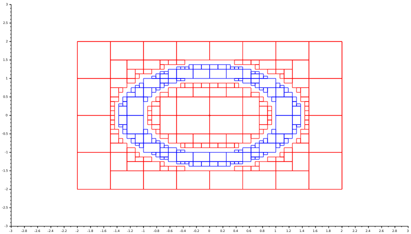
Different way to bisect a box

- Always on the same dimension
- Round Robin
- Largest First
- Smear (wrt to impact on function evaluation)
- In two or more sub boxes
- At the middle or not (90%-10%)
- etc...

Branch & Prune

The algorithm for solving $f(\mathcal{X}) = \mathcal{Y}$ from $[X]_0$ (SIVIA)

Require: $Stack = \emptyset, Stack_{acc} = \emptyset, Stack_{rej} = \emptyset, Stack_{unc} = \emptyset, [X]_0 \subset \mathbb{R}^n$
Push $[X]_0$ in *Stack*
while $Stack \neq \emptyset$ **do**
Pop a $[X]$ from *Stack*
if $[f]([X]) \subset [Y]$ **then**
Push $[X]$ in $Stack_{acc}$
else if $[f]([X]) \cap [Y] = \emptyset$ **then**
Push $[X]$ in $Stack_{rej}$
else if $\text{width}([X]) > \tau$ **then**
$([X_{left}], [X_{right}]) = \text{Bisect}([X]$
Push $[X_{left}]$ in *Stack*
Push $[X_{right}]$ in *Stack*
else
Push $[X]$ in $Stack_{unc}$
end if
end while



Optimization with Branch & Prune

With a basic cost on X

Optimization by ordering the stack:

- $([X_{left}], [X_{right}]) = \text{Bisect}([X])$
- If $Cost([X_{left}]) < Cost([X_{right}])$
-     PushFront $[X_{left}]$ and PushBack $[X_{right}]$
- else PushBack $[X_{left}]$ and PushFront $[X_{right}]$

Then the first solution minimizes the cost and valids the constraints

Contractors

Formalism of contractors

Contracting a CSP $\mathcal{H}$: replacing $[x]$ by a smaller domain $[x']$, s.t. solution set stays unchanged: $\mathcal{S} \subset [x'] \subset [x]$

Properties of contractors:

- Contractance: $\forall [x], C([x]) \subset [x]$
- Correctness: $\forall [x], [x] \cap \mathcal{S} \subset C([x])$

Contractors can be combined: $C_1(C_2([x]))$ or embeded $C_1(C_2, [x])$

Most used

- $Ctc_{fwd/bwd}([x])$, based on constraint propagation
- $Ctc_N([x])$, based on Newton method (for square problem)
- $Ctc_{FixedPoint}(Ctc, [x])$, which calls Ctc till a fixed point
- $Ctc_{cid}(Ctc, [x])$, which slices $[x]$, calls Ctc on each sub-boxes, and return the union of results

Branch & Contract

The algorithm for solving $f(x) = 0$ from $[X]_0$

Require: $Stack = \emptyset, Stack_{acc} = \emptyset, [X]_0 \subset \mathbb{R}^n$

Push $[X]_0$ in $Stack$
while $Stack \neq \emptyset$ **do**
  Pop a $[X]$ from $Stack$
  $[X] := Ctc([X])$
  if $[X] \neq \emptyset$ **then**
    if width$([X]) < \tau$ **then**
      Push $[X]$ in $Stack_{acc}$
    else
      $([X_{left}], [X_{right}]) = \text{Bisect}([X])$
      Push $[X_{left}]$ in $Stack$
      Push $[X_{right}]$ in $Stack$
    end if
  end if
end while

Heurisitic of bisection and choice of contractor (w.r.t. problem) !

For further

- Branch and Bound algorithm
- Inner boxes
- Quantifiers $\forall, \exists$
- Existence and uniqueness proof (last course)
- Overconstrained systems
- Q-intersection
- Many problems in interval analysis are NP-hard
- Distance between boxes (Hausdorff)
- Validated integration (next courses)
- etc...

Bibliography

- Applied Interval Analysis - Jaulin et al. - 2001 - Springer
- Introduction to Interval Analysis - Moore et al. - 2003 - SIAM
- Interval Analysis - Moore - 1966 - Prentice-hall
- Interval methods for systems of equations - Neumaier - 1990 - Cambridge press
- Global optimization using interval analysis - Hansen - 2003 - Marcel Dekker
- Introduction à l'arithmétique par intervalles - Revol - 2004 - `http://perso.ens-lyon.fr/nathalie.revol/polys/ArithIntervalles.pdf`
- Documentation of Ibex `http://www.ibex-lib.org/doc/`

Configuration to start quickly

Download exo_empty.cpp and makefile, then in a terminal:

```
>> export PKG_CONFIG_PATH=/home/uei/chapoutot/Public/share/pkgconfig
>> make exo_empty
>> ./exo_empty
```

Do it yourself

Solve the CSP:

$$\mathcal{H} : \begin{pmatrix} x_1 + x_1 + x_2 + x_3 + x_4 + x_5 = 6; \\ x_2 + x_1 + x_2 + x_3 + x_4 + x_5 = 6; \\ x_3 + x_1 + x_2 + x_3 + x_4 + x_5 = 6; \\ x_4 + x_1 + x_2 + x_3 + x_4 + x_5 = 6; \\ x_1 * x_2 * x_3 * x_4 * x_5 = 1; \\ x_i \in [-1e8, 1e8] \end{pmatrix}$$

As fast as possible !

Do it yourself

- Write a function $f(x) = 0$ in Ibex
- Build a contractor CtcFwdBwd
- Write a Branch&Contract
- Choose the Bisector
- Print solutions
- Optional: clean the solution set (3 solutions)

Do it yourself

Proove that the following CSP has no solution:

$$\mathcal{H} : \begin{pmatrix} x_1 - x_2 = 0 \\ x_1^2 + x_2^2 - 1 = 0 \\ x_2 - \sin(\pi x_1) = 0 \\ x_1 - \sin(\pi x_2) = 0 \\ x_2 - x_1^2 = 0 \\ x_1 \in [0, 1], x_2 \in [0, 1] \end{pmatrix}$$

In the simplest manner !