Research Report

The Fourier Singular Complement Method for the Poisson Problem in Prismatic Domains

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CUHK-2003-04 (278)

March 2003

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The Fourier Singular Complement Method for the Poisson problem in prismatic domains

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Abstract

This paper proposes a Fourier Singular Complement Method for numerically solving the Poisson problem in a three-dimensional prismatic domain. The method is based on a Fourier expansion in the direction parallel to the singular edge of the domain, and an improved variant of the Singular Complement Method in the 2d section perpendicular to the singular edge. Neither refinements near the singular edges of the domain nor cut-off functions are required in the computations to achieve an optimal convergence order in terms of the meshsize and the number of Fourier modes used.

1 Introduction

The *Singular Complement Method* (SCM) was originally introduced by Assous *et al.* [6, 7], for the 2D static or instationary Maxwell equations without charges. The cases with charges have been recently solved by Garcia [13], including the numerical solution to the 2D Vlasov-Maxwell system of equations. The SCM has been extended in [10] to the 2D Poisson problem. Further extensions to the 2D heat or (instationary) wave equations, or to similar problems with piecewise constant coefficients, can be obtained naturally.

The primary basis of the SCM is the decomposition of the solution into a regular and singular part. Methodologically speaking, the SCM consists in adding some singular test functions to the usual P_1 Lagrange FEM so that it recovers the optimal H^1-convergence rate, even in non-convex domains. In 2D, one may simply add one singular test function per reentrant corner.

There exist a couple of numerical methods in the literature for accurately solving 2D Poisson problems in non-convex domains. It was shown in [10] that the SCM can be reformulated so that it coincides with the approach of Moussaoui [20] when L-shaped domains are considered. The SCM differs from the *Dual Singular Function Method* (DSFM) of Blum and Dobrowolski [8] in that it requires no cut-off functions.

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Actually, when the numerical implementation of the SCM is carried out, the cut-off function is traded for a non homogeneous boundary condition. Note that Cai and Kim recently proposed a new SFM [9], which involves the evaluations of singular and cut-off functions and the solution of a nonsymmetric elliptic problem. The SCM is clearly different from (anisotropic) mesh refinement techniques [21, 2, 18, 3, 1], and in principle can be applied efficiently to instationary problems, since it does not need the refinements of the mesh and thus large timesteps may be allowed. However the anisotropic mesh refinement methods have one advantage: they require only a partial knowledge of the most singular part of the solution.

The numerical solution of 3D singular Poisson problems is quite different from the 2D case, and much more difficult. This is a relatively new field of research: most existing approaches rely on anisotropic mesh refinements, see for instance [2, 4, 18, 3, 19], as well as [1] and the references therein.

This paper is the first attempt to generalize the SCM for three-dimensional singular Poisson problems. Specifically, we shall consider the numerical solution of the Poisson problem: Find $u \in H_0^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in} \quad \Omega, \tag{1}$$

where $f \in L^2(\Omega)$, and Ω is a *prismatic domain* described by

$$\Omega = \omega \times Z\,, \tag{2}$$

and ω is a two-dimensional general polygonal domain, Z is an interval varying from 0 to a positive constant z_0 on the x_3-axis. Non-homogeneous Dirichlet boundary conditions, or (non-)homogeneous Neumann boundary conditions can be handled in exactly the same manner.

The rest of the paper is organized as follows. In Section 2, some theoretical results concerning the regularity of the solution to the Poisson problem in prismatic domains are recalled. In particular, *a priori* regularity results of the solution u to (1), and the relevance of the Fourier expansion along x_3, are emphasized. Then a *Fourier Singular Complement Method* (FSCM) is proposed for accurately solving problem (1). The FSCM is based on a Fourier expansion in x_3, and an improved variant of the Singular Complement Method in the 2d section ω. In Section 3, we study a theoretical splitting (into regular and singular parts) of the solution u_μ to 2D problems of the form $-\Delta u_\mu + \mu\, u_\mu = f_\mu$ in ω (with a parameter $\mu \geq 0$ related to the Fourier modes). The regular-singular splitting is chosen independent of μ. Estimates on Sobolev norms of u_μ and its splitting are established. In Section 4, the SCM is introduced to approximate u_μ accurately, *via* the discretization of the splitting. Numerical aspects are then considered, and the optimal H^1-norm convergence of the order $O(h)$ is recovered. In the last Section, we show that the FSCM has the optimal convergence of order $O(h+N^{-1})$, where h is the 2D meshsize and N is the number of Fourier modes used.

We remark that C will be frequently used in the sequel to denote the generic constant which depends only on the geometry of the domain.

2 Singular Poisson problem

We assume, for ease of exposition, that the polygon ω has only one reentrant corner C with an interior angle larger than π, denoted as π/α, with $1/2 < \alpha < 1$.

It is known that the Poisson problem (1) has a unique weak solution $u \in H_0^1(\Omega)$, but the solution has some singular behaviour near the edge $E = \{C\} \times Z$ of the domain Ω. More accurately, one can decompose, see e.g. [1, 15], the solution u into a singular part u_s and a regular part u_r as follows:

$$u = u_s + u_r, \text{ with } u_r \in H^2(\Omega) \text{ and } u_s = \gamma(\vec{x})\, r^\alpha \sin(\alpha\theta)\,,$$

where r and θ are the polar coordinates in a plane perpendicular to the edge, that is $r = \mathrm{dist}(\vec{x}, E)$ and $\theta \in (0, \pi/\alpha)$. The regularity of u, and therefore of the singular part u_s, can be characterized accurately as follows (cf. [1]):

$$\begin{aligned} & u \in H^{1+\alpha-\varepsilon}(\Omega), \ \forall \varepsilon > 0, \text{ and} \\ & r^\beta \frac{\partial u}{\partial x_1}, \ r^\beta \frac{\partial u}{\partial x_2} \in H^1(\Omega), \ \forall \beta > 1 - \alpha; \quad \frac{\partial u}{\partial x_3} \in H^1(\Omega). \end{aligned}$$

The function $\gamma(\vec{x})$ in the expression of u_s is often called *the stress intensity distribution*, and it depends *a priori* on all three variables $\vec{x} = (x_1, x_2, x_3)$. Further, γ can be explicitly characterized by a convolution integral:

$$\gamma(r, x_3) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{r}{r^2 + s^2} q(x_3 - s)\, ds. \tag{3}$$

It is shown in [15] that q has the regularity $H^{1-\alpha}(\mathbb{R})$. This characterization is important as it gives a possible way to compute the stress intensity distribution if we can find some method to approximate the function q used in (3).

Next, we shall discuss two possible approximations of the singular part u_s: one uses a more general *non-tensor-like* form, the other uses a specialized *tensor-like* form. To do so, we first approximate q in (3) by the Fourier sine series:

$$q_N(x_3) = \sqrt{2\pi} \sum_{k=0}^{N} c_k \sin\left(\frac{k\pi}{z_0} x_3\right). \tag{4}$$

Substituting this expression into (3), we have

$$\gamma_N(r, x_3) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{r}{r^2 + s^2}\, q_N(x_3 - s)\, ds\,,$$

then taking the Fourier transform on both sides gives

$$\begin{aligned} \hat{\gamma}_N(r, \xi) &= \exp(-r|\xi|) \hat{q}_N(\xi) \\ &= \frac{\sqrt{2\pi}}{2\,i} \exp(-r|\xi|) \sum_{k=0}^{N} c_k \Big\{ \delta\Big(\xi + \frac{k\pi}{z_0}\Big) - \delta\Big(\xi - \frac{k\pi}{z_0}\Big) \Big\}. \end{aligned}$$

Now by taking the inverse Fourier transform on both sides, we derive

$$\gamma_N(r,x_3) = \sum_{k=0}^{N} c_k \exp(-\frac{rk\pi}{z_0}) \sin\left(\frac{k\pi}{z_0}x_3\right).$$

Plugging this into the expression of the singular part u_s leads to

$$u_s^N(r,\theta,x_3) = \sum_{k=0}^{N} c_k r^\alpha \exp(-\frac{rk\pi}{z_0}) \sin\left(\frac{k\pi}{z_0}x_3\right) \sin(\alpha\theta).$$

This is called a *non-tensor-like* approximation [17], in the sense that u_s^N can not be expressed as the product of an (r,θ)-function and an x_3-function. According to [17, (6.23-6.24) p. 853] (in the axisymmetric case *without conical points*), one has

$$\lim_{N\to+\infty} u_s^N = u_s \quad \text{in} \quad \{v \in H^1(\Omega) \ : \ \Delta v \in L^2(\Omega)\}.$$

Remark 2.1 *By plugging (4) into the expression of $\gamma_N(r,x_3)$, one can see directly that the limit $\gamma(r,x_3)$ of γ_N vanishes at $x_3 = 0$ and z_0. As a matter of fact, for any (relevant) r, the integrand of $\gamma_N(r,0)$ is an odd function of s, so $\gamma_N(r,0) = 0$ since the integral is over $\mathbb{R}$. The same is true for $\gamma_N(r,z_0)$, which vanishes for all r. Passing to the limit in $H^1(\Omega)$ yields the result.*

If the right-hand side f of (1) is slightly more regular, that is, if we have

$$\frac{\partial f}{\partial x_3} \in L^2(\Omega) \text{ and } \frac{\partial^2 f}{\partial {x_3}^2} \in L^2(\Omega), \tag{5}$$

then it is possible to derive a *tensor-like* approximation (see again [17] for the axisymmetric case *without conical points* [3], and the subsequent Section 5 here). To see this, we expand the solution u and the right-hand side f in (1) in the Fourier sine series:

$$u(x_1,x_2,x_3) = \sum_{k=0}^{\infty} u_k(x_1,x_2) \sin\frac{k\pi}{z_0}x_3\,, \tag{6}$$

$$f(x_1,x_2,x_3) = \sum_{k=0}^{\infty} f_k(x_1,x_2) \sin\frac{k\pi}{z_0}x_3\,. \tag{7}$$

By substituting the expressions (6) and (7) into the equation (1) we see that the coefficients $u_k(x_1,x_2)$ satisfy the two-dimensional elliptic problems:

$$-\Delta u_k + \left(\frac{k\pi}{z_0}\right)^2 u_k = f_k \quad \text{in} \quad \omega; \quad u_k = 0 \quad \text{on} \quad \partial\omega. \tag{8}$$

[3] Heinrich proved in [17] that the tensor-like form can be obtained, under the conditions

$$r^{1/2} f \in L^2(\Omega_a),\ r^{1/2}\frac{\partial f}{\partial \phi} \in L^2(\Omega_a) \text{ and } r^{1/2}\frac{\partial^2 f}{\partial \phi^2} \in L^2(\Omega_a),$$

where (r,ϕ,z) are the cylindrical coordinates in an axisymmetric domain Ω_a *without conical points*. This corresponds precisely to the conditions that $f \in L^2(\Omega)$ and satisfies (5) in our current prismatic case.

It is interesting to notice that the elliptic problems (8) become more coercive when k increases. This is certainly an advantage for the numerical solutions. We will apply the Singular Complement Method to solve the two-dimensional elliptic problems (8). To that aim we rewrite the solutions as follows:

$$u_k = u_{r,k} + c_k u_s \tag{9}$$

where the *same singular function* is used for all Fourier coefficients u_k. Using the decomposition (9), we obtain the following approximation

$$u_N = u_r^N + u_s^N$$

with

$$u_r^N = \sum_{k=0}^{N} u_{r,k}(x_1, x_2) \sin \frac{k\pi}{z_0} x_3, \quad u_s^N = \Big(\sum_{k=0}^{N} c_k \sin \frac{k\pi}{z_0} x_3 \Big) u_s(x_1, x_2).$$

One may use two different representations for the singular function $u_s(x_1, x_2)$ (see [10]). The first one is to use a global singular function which belongs to $H_0^1(\omega)$:

$$u_s(x_1, x_2) = \phi_s(x_1, x_2); \tag{10}$$

the second one is to keep only the principal part of the singularity:

$$u_s(x_1, x_2) = r^\alpha \sin(\alpha\theta). \tag{11}$$

For both representations, the resulting approximations u_s^N are *tensor-like*. As stated before, under the assumption that f is slightly more regular than $f \in L^2(\Omega)$, see (5), one can prove that

$$\lim_{N \to +\infty} \sum_{k=0}^{N} u_r^N \to u_r \text{ in } H^2(\Omega).$$

In the case (11), one can pass to the limit in u_s^N and write the limit as

$$u_s = \gamma(x_3) r^\alpha \sin(\alpha\theta). \tag{12}$$

The regularity of u implies that $\gamma \in H^2(]0, z_0[)$. Thanks to the Sobolev imbedding theorem [14, Chapter I], we find that $\gamma \in \mathcal{C}^1([0, z_0])$. In addition, one easily derives that there holds $\gamma(0) = \gamma(z_0) = 0$, since

$$\gamma_N = \sum_{k=0}^{N} c_k \sin \frac{k\pi}{z_0} x_3$$

converges to γ in $H^2(]0, z_0[)$.

For the case (10), similar results hold, which yields in turn $u_s' = \gamma(x_3)\phi_s(x_1, x_2)$, with the same stress intensity distribution γ as in (12). In this case, as u_s' belongs to $H_0^1(\Omega)$ by construction, one gets the splitting $u = u_r' + u_s'$, with $u_r' \in H^2(\Omega) \cap H_0^1(\Omega)$.

If the boundary conditions for u is the homogeneous Neumann boundary condition $\partial_n u = 0$ on $\partial\Omega$, then one can take the following Fourier cosine series:

$$\begin{aligned} u(x_1, x_2, x_3) &= \sum_{k=0}^{\infty} u_k(x_1, x_2) \cos \frac{k\pi}{z_0} x_3, \\ f(x_1, x_2, x_3) &= \sum_{k=0}^{\infty} f_k(x_1, x_2) \cos \frac{k\pi}{z_0} x_3. \end{aligned}$$

Under the same regularity assumptions on f, i.e. (5), one finds

$$u = u_r + \gamma_n(x_3) r^{\alpha} \cos(\alpha\theta) = u'_r + \gamma_n(x_3)\psi_s(x_1, x_2).$$

Here ψ_s is a global singular function in the Neumann case. Again, u_r and u'_r both belong to $H^2(\Omega)$ (with $\partial_n u'_r = 0$ on $\partial\Omega$), and $\gamma_n \in H^2(]0, z_0[)$ satisfies that $\gamma'_n(0) = \gamma'_n(z_0) = 0$.

Remark 2.2 *If the boundary conditions for u on the top and bottom faces of the physical domain Ω are the non-homogeneous Dirichlet boundary condition:*

$$u = g \quad at \quad x_3 = 0 \quad and \quad x_3 = z_0,$$

one can set $w = u - \tilde{g}$ with $\tilde{g}$ being an extension of g into Ω. Then the problem reduces to the case with the solution w satisfying the homogeneous Dirichlet boundary condition.

If we have the non-homogeneous Neumann boundary condition:

$$\frac{\partial u}{\partial n} = g \quad at \quad x_3 = 0 \quad or \quad x_3 = z_0,$$

one may then study the solution

$$w(\vec{x}) = u(\vec{x}) - \int_0^{x_3} \tilde{g}(x_1, x_2, x_3)\, dx_3$$

first, which satisfies the homogeneous Neumann boundary conditions at the top and bottom faces of the domain Ω. Here $\tilde{g}$ is an extension of g into Ω.

The regularity assumptions on g are such that the problem in w is posed with a right-hand side of sufficient regularity (in $L^2(\Omega)$, and possibly with the additional assumptions (5)).

2.1 Fourier expansion

We devote this section to some justifications about the Fourier series expansion of the Poisson solution to (1). First, one can show following the proof of Lemma 3.2 in Heinrich [17]:

Lemma 2.1 *For any $u \in L^2(\Omega)$, there exist Fourier coefficients defined by*

$$u_k(x_1,x_2) = \frac{2}{z_0}\int_0^{z_0} u(x_1,x_2,x_3)\sin\frac{k\pi}{z_0}x_3\,dx_3, \quad k=1,2,3,\cdots,$$

such that $u_k \in L^2(\omega)$ and

$$u(x_1,x_2,x_3) = \sum_{k=0}^{\infty} u_k(x_1,x_2)\sin\frac{k\pi}{z_0}x_3 \quad \textit{a.e. in } \Omega, \tag{13}$$

and

$$\|u\|^2_{L^2(\Omega)} = \frac{z_0}{2}\sum_{k=1}^{\infty}\|u_k\|^2_{L^2(\omega)} < \infty.$$

If $u \in H^1(\Omega)$, then $u_k \in H^1(\omega)$ for all k and

$$\|\nabla u\|^2_{L^2(\Omega)} = \frac{z_0}{2}\sum_{k=1}^{\infty}\left\{\|\nabla u_k\|^2_{L^2(\omega)} + \left(\frac{k\pi}{z_0}\right)^2\|u_k\|^2_{L^2(\omega)}\right\} < \infty. \tag{14}$$

Consider the weak form of the Poisson problem (1): *Find $u \in H_0^1(\Omega)$ such that*

$$a(u,v) = f(v) \quad \forall\, v \in H_0^1(\Omega) \tag{15}$$

where $a(\cdot,\cdot)$ and $f(\cdot)$ are given by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx, \quad f(v) = \int_\Omega f\,v\,dx.$$

Lemma 2.2 *For any $u, v \in H_0^1(\Omega)$, we have*

$$a(u,v) = \frac{z_0}{2}\sum_{k=1}^{\infty} a_k(u_k,v_k), \quad f(v) = \frac{z_0}{2}\sum_{k=1}^{\infty} f_k(v_k),$$

where a_k and f_k are given by

$$a_k(u_k,v_k) = \int_\omega \left\{\nabla u_k \cdot \nabla v_k + \left(\frac{k\pi}{z_0}\right)^2 u_k v_k\right\}dx_1dx_2, \quad f_k(v_k) = \int_\omega f_k\,v_k\,dx_1dx_2,$$

and u_k, v_k and f_k are Fourier coefficients of u, $v \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$ respectively.

Lemma 2.3 *For any $f \in L^2(\Omega)$, let $u \in H_0^1(\Omega)$ be the unique weak solution of (15) and u_k and f_k be the Fourier coefficients of u and f. Then $u_k \in H_0^1(\omega)$ is the unique solution of the following 2D weak problem:* Find $u_k \in H_0^1(\omega)$ such that

$$a_k(u_k,v) = f_k(v) \quad \forall\, v \in H_0^1(\omega). \tag{16}$$

Moreover, u_k satisfies the following a priori *estimates:*

$$\begin{aligned}\int_\omega \left\{|\nabla u_k|^2 + \left(\frac{k\pi}{z_0}\right)^2 u_k^2\right\}dx_1dx_2 &\le \left(\frac{z_0}{k\pi}\right)^2\|f_k\|^2_{L^2(\omega)}, \quad k=1,2,\cdots,\\ \sum_{k=1}^{\infty}k^2\left\{\|\nabla u_k\|^2_{L^2(\omega)} + \left(\frac{k\pi}{z_0}\right)^2\|u_k\|^2_{L^2(\omega)}\right\} &\le \frac{2z_0}{\pi^2}\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

3 Regular-singular decomposition in the 2D domain ω: theoretical study

The main interest of this paper is to propose some efficient numerical method for solving the three-dimensional singular Poisson problem (1) in a prismatic domain. Basically, the method reduces the 3D problem into a series of 2D Poisson like problems, see (8), by the Fourier expansion of the 3D solution along the x_3-direction.

This section will first study the 2D singular Poisson problem: Find $u_\mu \in H_0^1(\omega)$ such that

$$-\Delta u_\mu \ + \mu(k)\, u_\mu \ = f(k) \quad \text{in } \ \omega \tag{17}$$

where $\mu(k) = k^2\pi^2/z_0^2$ and $f(k)$ depends on k. Due to the presence of the Fourier mode index k, the coefficient $\mu(k)$ in the equation (17) varies in a large range, from $O(1)$ to $O(N^2)$, where N will be the number of Fourier modes required subsequently in the numerical approximation (cf. Section 5). This brings in one of the main difficulties in the subsequent error estimates, which should hold for all k's in a large range.

Let $\gamma_1, \gamma_2, \cdots, \gamma_K$ be the line segments of $\partial\omega$, where γ_1 and γ_2 are two line segments which form the single re-entrant corner of ω. Our numerical method is based on the following important decomposition of the space $L^2(\omega)$ [16]:

$$L^2(\omega) = \Delta[H^2(\omega) \cap H_0^1(\omega)] \overset{\perp}{\oplus} N\,, \tag{18}$$

where N is a space of singular harmonic functions defined by

$$N = \Big\{ p \in L^2(\omega) \ : \ \Delta p = 0, \ p|_{\gamma_k} = 0 \text{ in } (H_{00}^{1/2}(\gamma_k))', \ 1 \le k \le K \Big\}.$$

As the domain ω has only one re-entrant corner, we know $\dim(N) = 1$, and N=span$\{p_s\}$ for some $p_s \in N \setminus \{0\}$, see Grisvard [16].

Let ϕ_s be an element in $H_0^1(\omega)$, which solves the Poisson problem

$$-\Delta\phi_s = p_s \quad \text{in} \quad \omega\,. \tag{19}$$

Then by the decomposition (18), we can split the solution u_μ to equation (17) as

$$u_\mu = \tilde{u}_\mu + c_\mu \phi_s, \tag{20}$$

where $\tilde{u}_\mu \in H^2(\omega) \cap H_0^1(\omega)$, and is called the regular part of u_μ.

We will devote the rest of this section to derive some *a priori* estimates for the solution u_μ, its regular part $\tilde{u}_\mu$ and the singular coefficient c_μ, as well as the solvability of $\tilde{u}_\mu$ and c_μ. Let us first introduce some notation.

Throughout the rest of the paper, α_0 will be a frequently used fixed positive constant lying in the interval $]\frac{1}{2}, \alpha[$, where $\alpha \in]\frac{1}{2}, 1[$ is the singularity index. $|\cdot|_s$ is used to denote the semi-norm of the Sobolev space $H^s(\omega)$ for any $s > 0$, $(\cdot, \cdot)$ and $\|\cdot\|_0$ are used to denote the inner product and the norm in the space $L^2(\omega)$. Also, $(\cdot, \cdot)$ will be used for the dual pairing between the space $H_0^1(\omega)$ and $H^{-1}(\omega)$ when necessary.

The following lemma summarizes some a priori estimates on u_μ and c_μ.

Lemma 3.1 *Let u_μ be the the solution u_μ to the Poisson problem (17), then we have the following* a priori *estimates:*

$$\mu \|u_\mu\|_0 \leq \|f\|_0 , \quad \sqrt{\mu}\, |u_\mu|_1 \leq \frac{1}{\sqrt{2}} \|f\|_0 , \quad \|\Delta u_\mu\|_0 \leq 2 \|f\|_0 , \tag{21}$$

$$|c_\mu| \leq C\mu^{-\frac{1-\alpha}{2}} \|f\|_0 \tag{22}$$

$$|u_\mu|_{1+\alpha_0} \leq C \mu^{-\frac{1-\alpha_0}{2}} \|f\|_0 . \tag{23}$$

Proof. Multiplying equation (17) by u_μ and integrating over ω yield

$$|u_\mu|_1^2 + \mu \|u_\mu\|_0^2 \leq \|f\|_0 \|u_\mu\|_0 ,$$

this proves the first estimate in (21). Then applying the Cauchy-Schwarz inequality, we further obtain

$$|u_\mu|_1^2 + \mu \|u_\mu\|_0^2 \leq \frac{1}{2} \mu \|u_\mu\|_0^2 + \frac{1}{2\mu} \|f\|_0^2,$$

which leads to the H^1 semi-norm estimate in (21).

The last estimate in (21) follows immediately from $\Delta u_\mu = \mu u_\mu - f$ and the first inequality in (21).

As far as (22) is concerned, it is a simple matter to check that the singular coefficient c_μ, multiplied by some constant $\beta^\star$, equals the singular coefficient $c(\mu)$ of [16, pp. 62-69]. Indeed, in the works of Grisvard, u_μ is decomposed into:

$$u_\mu = u_\mu^G + c(\mu) \mathrm{e}^{-\sqrt{\mu} r} \xi(r) r^\alpha \sin(\alpha\theta), \quad u_\mu^G \in H^2(\omega) \cap H_0^1(\omega) \tag{24}$$

where ξ is a smooth cut-off function, equal to one in a neighborhood of 0.

On the othe hand one can decompose the singular part in (20) as (cf. [10] or (43) below)

$$c_\mu \phi_s = c_\mu \left(\tilde{\phi} + \beta^\star r^\alpha \sin(\alpha\theta) \right) , \quad \tilde{\phi} \in H^2(\omega), \quad \beta^\star = \frac{1}{\pi} \|p_s\|_0^2 .$$

Using this, (20) and (24), we can write

$$\begin{aligned} & (c_\mu \beta^\star - c(\mu) \xi(r)) r^\alpha \sin(\alpha\theta) \\ = \;& u_\mu - (\tilde{u}_\mu + c_\mu \tilde{\phi}) - c(\mu) \xi(r) r^\alpha \sin(\alpha\theta) \\ = \;& u_\mu^G + c(\mu) \left(\mathrm{e}^{-\sqrt{\mu} r} - 1 \right) \xi(r) r^\alpha \sin(\alpha\theta) - (\tilde{u}_\mu + c_\mu \tilde{\phi}). \end{aligned} \tag{25}$$

Noting that each term on the right-hand side of (25) belongs to in $H^2(\omega)$, we must have $c_\mu = c(\mu)/\beta^\star$. But it is shown in [16, ineq. (2.5.5)] that

$$|c(\mu)| \leq C\mu^{-\frac{1-\alpha}{2}} \|f\|_0, \tag{26}$$

this implies (22).

In order to derive the estimate (23), we shall use (24-27), with the additional norm estimate [16, ineq. (2.5.4)] on the regular part u_μ^G, namely

$$|u_\mu^G|_2 + \sqrt{\mu}|u_\mu^G|_1 + \mu\|u_\mu^G\|_0 \leq C\,\|f\|_0. \tag{27}$$

Indeed, from the estimates

$$|u_\mu^G|_1 \leq C\,\mu^{-1/2}\|f\|_0, \qquad |u_\mu^G|_2 \leq C\,\|f\|_0\,,$$

we have then by standard interpolation theory that

$$|u_\mu^G|_{1+\alpha_0} \leq C\,\mu^{-\frac{1-\alpha_0}{2}}\|f\|_0.$$

Next, we use (26) and a direct estimate of the $H^{1+\alpha_0}$ semi-norm to bound the singular part in (24). Actually, there holds

$$|v|_{1+\alpha_0}^2 = \int_{\vec{x}\in\omega}\int_{\vec{x}'\in\omega} \frac{\|\nabla v(\vec{x}) - \nabla v(\vec{x}')\|^2}{\|\vec{x}-\vec{x}'\|^{2+2\alpha_0}}d\omega(\vec{x})\,d\omega(\vec{x}'), \quad \forall\, v \in H^{1+\alpha_0}(\omega).$$

Due to the uniform smoothness (in μ) of $\mathrm{e}^{-\sqrt{\mu}r}\xi(r)r^\alpha\sin(\alpha\theta)$ for $r \geq r_0 > 0$, it is possible to evaluate the integrals only on $\omega_\infty = \{(r,\theta) \in]0, r_0[\times]0,\pi/\alpha[\}$. Then, one performs the changes of variables $s = \sqrt{\mu}r$, $s' = \sqrt{\mu}r'$, to find

$$|\mathrm{e}^{-\sqrt{\mu}r}\xi(r)r^\alpha\sin(\alpha\theta)|_{H^{1+\alpha_0}(\omega_\infty)} \leq C(\alpha_0)\mu^{-\frac{\alpha-\alpha_0}{2}}.$$

This with (22) leads to (23)). $\diamond$

Remark 3.1 *Both ϕ_s and p_s in (19) are chosen independent of $f(k)$, u_μ and the Fourier mode index k, so their norms will be regarded as some generic constants independent of $f(k)$, u_μ and the index k in the subsequent analysis.*

Remark 3.2 *Instead of the decomposition (20), it seems more natural [15, 16] to take the decomposition $u_\mu = \tilde{u}_\mu' + c_\mu\phi_\mu$, where $\phi_\mu \in H_0^1(\omega)$ depends on the Fourier mode index $\mu(k)$, and it is the solution to the problem: $-\Delta\phi_\mu + \mu\,\phi_\mu = p_\mu$ in ω, with $p_\mu \in N_\mu \setminus \{0\}$, where N_μ is given by*

$$N_\mu = \left\{p \in L^2(\omega) \;:\; (-\Delta + \mu\,\mathrm{I})\,p = 0,\; p|_{\gamma_k} = 0 \text{ in } (H_{00}^{1/2}(\gamma_k))',\; 1 \leq k \leq K\right\}.$$

But the decomposition (20) has an important advantage: the singular part ϕ_s is independent of the Fourier mode index $\mu(k)$. As we shall see, this will be much less expensive than using the above more natural decomposition.

Next, we study the solvability of $\tilde{u}_\mu$ and c_μ in decomposition (20). For convenience, we introduce the notation $a_\mu(\cdot,\cdot)$ and the norm $\|\cdot\|_a$:

$$a_\mu(w,v) = (\nabla w, \nabla v) + \mu\,(w,v)\,, \quad \|v\|_a^2 = a_\mu(v,v)\,,$$

and the operator $A_\mu = -\Delta + \mu\,\mathrm{I}$. Clearly, A_μ is an operator from $H_0^1(\omega)$ into $H^{-1}(\omega)$, and

$$(A_\mu w, v) = a_\mu(w, v) \quad \forall\, w, v \in H_0^1(\omega).$$

It is not difficult to verify that A_μ is a one-to-one and onto operator, so it is invertible.

Now we claim that $\tilde{u}_\mu$ and c_μ solve the following coupled system:

$$a_\mu(\tilde{u}_\mu, v) + c_\mu\, a_\mu(\phi_s, v) = (f, v) \quad \forall v \in H_0^1(\omega)\,, \tag{28}$$

$$\left(\|p_s\|_0^2 + \mu|\phi_s|_1^2\right) c_\mu + \mu\,(\tilde{u}_\mu, p_s) = (f, p_s)\,. \tag{29}$$

In fact, by multiplying the equation (17) by p_s and integrating over ω we obtain

$$-(\Delta u_\mu, p_s) + \mu\,(u_\mu, p_s) = (f, p_s)\,,$$

then (29) follows readily from the decomposition (20), the orthogonality between p_s and $\Delta\tilde{u}_\mu$, along with the relation (19) and its following direct consequence

$$|\phi_s|_1^2 = (\phi_s, p_s). \tag{30}$$

On the other hand, the solution u_μ of (17) also satisfies the weak form:

$$(\nabla u_\mu, \nabla v) + \mu\,(u_\mu, v) = (f, v) \quad \forall v \in H_0^1(\omega),$$

this and the decomposition (20) lead to the equation (28).

Next, we show the well-posedness of the system (28)-(29).

Lemma 3.2 *There exists a unique solution $(\tilde{u}_\mu, c_\mu)$ to the coupled system (28)-(29) and the following stability estimates hold:*

$$\|\tilde{u}_\mu\|_a \;\le\; \sqrt{2}\left(2\sqrt{\mu}C_P^2 + \frac{1}{\sqrt{\mu}}\right)\|f\|_0\,,$$

$$|c_\mu| \;\le\; 2\,\frac{\|f\|_0}{\|p_s\|_0}\,, \qquad |\tilde{u}_\mu|_2 \le 4\,\|f\|_0\,,$$

where C_P is the constant in the Poincaré inequality.

Proof. To see the unique existence, we rewrite (28) as the following operator form:

$$A_\mu \tilde{u}_\mu + c_\mu\, A_\mu \phi_s = f \quad \text{in} \quad H^{-1}(\omega). \tag{31}$$

As the inverse of A_μ exists, we know from (31) that $\tilde{u}_\mu$ can be determined if c_μ is available:

$$\tilde{u}_\mu = A_\mu^{-1} f - c_\mu\,\phi_s\,. \tag{32}$$

This is exactly our original decomposition (20). Substituting this into (29),

$$\left(\|p_s\|_0^2 + \mu|\phi_s|_1^2\right)c_\mu + \mu\,(A_\mu^{-1} f - c_\mu\,\phi_s, p_s) = (f, p_s)\,.$$

With (30), we obtain that

$$c_\mu = \frac{(f - \mu\, A_\mu^{-1} f,\, p_s)}{\|p_s\|_0^2}\,. \tag{33}$$

With c_μ uniquely determined, $\tilde{u}_\mu$ is clearly uniquely determined by (28) or (32).

Next, we derive the stability estimates in Lemma 3.2. We show that these estimates are the consequences of (32-33) and the following inequality

$$\|A_\mu^{-1}\, g\|_0 \le \frac{1}{\mu}\, \|g\|_0 \quad \forall\, g \in L^2(\omega). \tag{34}$$

In fact, if (34) is true, then the desired estimate on c_μ follows from (33):

$$|c_\mu| \le \frac{\|f\| + \mu\, \|A_\mu^{-1} f\|_0}{\|p_s\|_0} \le 2\frac{\|f\|_0}{\|p_s\|_0}\,.$$

On the other hand, we have from (30) and the Poincaré inequality that

$$\|\phi_s\|_0 \le C_P\, \|\nabla \phi_s\|_0 \le C_P^2 \|p_s\|_0\,.$$

Using this and the bound of c_μ, we derive from (28) by taking $v = \tilde{u}_\mu$ that

$$\begin{aligned} \|\nabla \tilde{u}_\mu\|_0^2 + \mu \|\tilde{u}_\mu\|_0^2 &\le \|f\|_0 \|\tilde{u}_\mu\|_0 + |c_\mu|\, (\|\nabla \phi_s\|_0 \|\nabla \tilde{u}_\mu\|_0 + \mu\, \|\phi_s\|_0 \|\tilde{u}_\mu\|_0) \\ &\le \|f\|_0 \|\tilde{u}_\mu\|_0 + 2C_P \|f\|_0 \|\nabla \tilde{u}_\mu\|_0 + 2\, \mu\, C_P^2 \|f\|_0 \|\tilde{u}_\mu\|_0\,. \end{aligned}$$

Then the application of the Young inequality yields

$$\|\nabla \tilde{u}_\mu\|_0^2 + \mu \|\tilde{u}_\mu\|_0^2 \le \frac{1}{2}\mu\, \|\tilde{u}_\mu\|_0^2 + \frac{1}{\mu}\|f\|_0^2 + \frac{1}{2}\|\nabla \tilde{u}_\mu\|_0^2 + 2C_P^2 \|f\|_0^2 + 4\mu\, C_P^4 \|f\|_0^2\,.$$

This implies

$$\frac{1}{2}\|\tilde{u}_\mu\|_a^2 \le \left(\frac{1}{\mu} + 2C_P^2 + 4\mu C_P^4\right) \|f\|_0^2 \le \left(\frac{1}{\sqrt{\mu}} + 2\sqrt{\mu} C_P^2\right)^2 \|f\|_0^2\,,$$

so the desired estimate on $\|\tilde{u}_\mu\|_a$ follows.

We now show the H^2-norm estimate. By the decomposition (32), we have $u_\mu = A_\mu^{-1} f = \tilde{u}_\mu + c_\mu \phi_s$, and

$$-\Delta \tilde{u}_\mu = -\Delta u_\mu + c_\mu \Delta \phi_s = f - \mu\, u_\mu - c_\mu p_s,$$

this gives

$$\|\Delta \tilde{u}_\mu\|_0 \le \|f\|_0 + \mu \|u_\mu\|_0 + |c_\mu|\, \|p_s\|_0.$$

But we know from Lemma 3.1 that $\mu \|u_\mu\|_0 \le \|f\|_0$. This, along with the previous bound for c_μ, leads to

$$\|\Delta \tilde{u}_\mu\|_0 \le 4 \|f\|_0.$$

Now, for any $\vec{v} \in H^1(\omega)^2$ such that $\vec{v} \cdot \vec{\tau} = 0$ on $\partial\omega$, with $\vec{\tau}$ the vector tangential to $\partial\omega$, it is well-known (cf. [12]) that (since ω is a polygon)

$$\sum_{1 \le k,l \le 2} \|\partial_k v_l\|_0^2 = \|\mathrm{curl}\vec{v}\|_0^2 + \|\mathrm{div}\vec{v}\|_0^2.$$

So, by taking $\vec{v} = \nabla \tilde{u}_\mu$, one actually finds

$$|\tilde{u}_\mu|_2 = \|\Delta \tilde{u}_\mu\|_0 \le 4\|f\|_0.$$

Finally, it remains to prove (34). By the definition of $a_\mu(\cdot,\cdot)$, we easily see the following lower bound:

$$(A_\mu v, v) = a_\mu(v, v) \ge \mu \|v\|_0^2 \quad \forall\, v \in H_0^1(\omega)\,. \tag{35}$$

Then for any $g \in L^2(\omega) \subset H^{-1}(\omega)$, as A_μ is invertible, let $v = A_\mu^{-1} g$, then $v \in H_0^1(\omega)$ and $A_\mu v = g$. It follows from (35) that

$$\|A_\mu^{-1} g\|_0^2 = \|v\|_0^2 \le \frac{1}{\mu}(A_\mu v, v) = \frac{1}{\mu}(g, A_\mu^{-1} g) = \frac{1}{\mu}\|g\|_0 \, \|A_\mu^{-1} g\|_0.$$

This proves (34). ◇

4 Discrete formulation in the 2D domain ω: the SCM

In this section we shall formulate the generalized SCM method for solving the coupled system (28)-(29) and derive the error estimates of the approximate solutions. The SCM method was first introduced by Assous *et al* [6] for solving the 2D static or unsteady Maxwell equations without charges. As we will see, the formulation of the SCM for the 2D Poisson like problem (8) is quite different here due to the involvement of the Fourier mode number k.

Let $\mathcal{T}_h$ be a regular triangulation of the domain ω, with vertices $\{M_j\}_{j=1}^{N_i+N_b}$ and the last N_b vertices lying on the boundary $\partial\omega$. We define V^h to be the continuous piecewise linear finite element space on $\mathcal{T}_h$ with the standard basis functions $\{\psi_j\}_{j=1}^{N_i+N_b}$ (cf. [11]). We further define V_0^h to be the subspace of V^h with all functions vanishing on the boundary of ω. The interpolation associated with the space V_h will be denoted by Π_h.

4.1 Approximation of the singular function p_s

We start with the finite element approximation of the singular function $p_s \in N$ in (19). Recall the splitting (see [10])

$$p_s = \tilde{p} + p_P, \quad \tilde{p} \in H^1(\omega), \quad p_P = r^{-\alpha}\sin(\alpha\theta)\,.$$

As p_s is harmonic in ω, one can directly verify that the regular part $\tilde{p}$ in the splitting solves the problem: Find $\tilde{p} \in H^1(\omega)$ such that $\tilde{p} = s$ on $\partial\omega$ and

$$(\nabla\tilde{p}, \nabla v) = 0 \quad \forall\, v \in H_0^1(\omega) \tag{36}$$

where the boundary function s is given by

$$s = 0 \quad \text{on} \quad \gamma_1 \cup \gamma_2; \quad s = -p_P \quad \text{on} \quad \gamma_k \ (3 \le k \le K).$$

For the finite element approximation of the problem (36), we shall use the simple treatment of the boundary condition:

$$\pi_h\, s = \sum_{j=N_i+1}^{N_i+N_b} s(M_j)\psi_j\,. \tag{37}$$

Then we approximate p_s by $p_s^h = \tilde{p}_h + p_P$, where $\tilde{p}_h$ is the piecewise linear finite element solution to the problem (36). Namely, $\tilde{p}_h = \pi_h s + p_h^0$ where $p_h^0 \in V_h^0$ solves

$$(\nabla\tilde{p}_h, \nabla v_h) = 0 \quad \forall\, v_h \in V_0^h\,. \tag{38}$$

The error estimates for the singular function p_s and its finite element approximation p_s^h are summarized in the following lemma.

Lemma 4.1 *We have* [4]

$$|p_s - p_s^h|_1 \le C\, h^{\alpha_0}\,, \quad \|p_s - p_s^h\|_0 \le C\, h^{2\alpha_0}\,.$$

Proof. We introduce a smooth extension of s into ω:

$$\tilde{s} = -p_P(1 - \xi(r))\,.$$

Clearly, $\tilde{s} = s$ on $\partial\omega$ and $\tilde{s} \in H^2(\omega)$. Let $p^0 = \tilde{p} - \tilde{s}$. It is known that $\tilde{p} \in H^{1+\alpha_0}(\omega)$, so we have $p^0 \in H^{1+\alpha_0}(\omega) \cap H_0^1(\omega)$. It follows from (36) that

$$(\nabla p^0, \nabla v) = -(\nabla\tilde{s}, \nabla v) \quad \forall\, v \in H_0^1(\omega)\,. \tag{39}$$

Recall Π_h is the interpolant associated with V^h, thus we can rewrite the finite element solution $\tilde{p}_h$ to the system (38) as $\tilde{p}_h = \Pi_h\tilde{s} + p_h^0$ with $p_h^0 \in V_0^h$ now solving

$$(\nabla p_h^0, \nabla v_h) = -(\nabla\Pi_h\tilde{s}, \nabla v_h) \quad \forall\, v_h \in V_0^h\,, \tag{40}$$

by noting $\Pi_h\tilde{s} = \pi_h\, s$ on $\partial\omega$.

[4] By construction, neither p_s nor p_s^h belong to $H^1(\omega)$, due to the presence of p_P, but the following holds:

$$p_s - p_s^h = \tilde{p} - \tilde{p}_h \in H^1(\omega).$$

Now we are ready to derive the error estimates. It is clear from (39) and (40) that

$$(\nabla(p^0 - p_h^0), \nabla v_h) = (\nabla(\Pi_h \tilde{s} - \tilde{s}), \nabla v_h) \quad \forall v_h \in V_0^h \,. \tag{41}$$

Using this, we obtain for any $q_h \in V_0^h$ that

$$\|\nabla(p^0 - q_h)\|^2 \ge \|\nabla(p^0 - p_h^0)\|^2 + 2(\nabla(\Pi_h \tilde{s} - \tilde{s}), \nabla(p_h^0 - q_h)),$$

taking $q_h = \Pi_h p^0$ above and using the Young inequality leads to

$$\begin{aligned} |p^0 - p_h^0|_1^2 &\le |p^0 - \Pi_h p^0|_1^2 + 2\,|\Pi_h \tilde{s} - \tilde{s}|_1\,(|p_h^0 - p^0|_1 + |p^0 - \Pi_h p^0|_1) \\ &\le 2\,|p^0 - \Pi_h p^0|_1^2 + \frac{1}{2}|p_h^0 - p^0|_1^2 + 3\,|\Pi_h \tilde{s} - \tilde{s}|_1^2 \,. \end{aligned}$$

Then by the standard interpolation results we obtain

$$\begin{aligned} |p^0 - p_h^0|_1^2 &\le 4\,|p^0 - \Pi_h p^0|_1^2 + 6\,|\Pi_h \tilde{s} - \tilde{s}|_1^2 \,, \\ &\le C\,(h^{2\alpha_0}|p^0|_{1+\alpha_0}^2 + h^2\,|\tilde{s}|_2^2), \end{aligned}$$

this leads to the desired H^1-norm error estimate:

$$\begin{aligned} |p_s - p_s^h|_1 &= |\tilde{p} - \tilde{p}_h|_1 = |p^0 + \tilde{s} - p_h^0 - \Pi_h \tilde{s}|_1 \\ &\le |p^0 - p_h^0|_1 + |\tilde{s} - \Pi_h \tilde{s}|_1 \le C\,h^{\alpha_0} + C\,h|\tilde{s}|_2 \le C\,h^{\alpha_0}. \end{aligned}$$

Finally, we apply the Nitsche trick to derive the L^2-norm error estimate. Let $w \in H_0^1(\omega)$ be the solution to the variational problem

$$(\nabla w, \nabla v) = (p^0 - p_h^0, v) \quad \forall v \in H_0^1(\omega). \tag{42}$$

By the elliptic theory, we know $w \in H^{1+\alpha_0}(\omega)$ and

$$|w|_{1+\alpha_0} \le C\|p^0 - p_h^0\|_0.$$

Let w_h be the finite element approximation of w: $w_h \in V_0^h$ solves

$$(\nabla w_h, \nabla v_h) = (p^0 - p_h^0, v_h) \quad \forall v_h \in V_0^h.$$

Taking $v_h = w_h$ above and using the Poincaré inequality, we know

$$|w_h|_1 \le C\,\|p^0 - p_h^0\|_0.$$

Also, by the standard error estimate, we have

$$|w - w_h|_1 \le C\,h^{\alpha_0}|w|_{1+\alpha_0} \le C\,h^{\alpha_0}\|p^0 - p_h^0\|_0.$$

Now, taking $v = p^0 - p_h^0$ in (42) and using (41) and the duality argument, we obtain

$$\begin{aligned} \|p^0 - p_h^0\|_0^2 &= (\nabla w, \nabla(p^0 - p_h^0)) \\ &= (\nabla(w - w_h), \nabla(p^0 - p_h^0)) + (\nabla w_h, \nabla(p^0 - p_h^0)) \\ &= (\nabla(w - w_h), \nabla(p^0 - p_h^0)) + (\nabla(\Pi_h \tilde{s} - \tilde{s}), \nabla(w_h - w)) \\ &\quad +(\nabla(\Pi_h \tilde{s} - \tilde{s}), \nabla w) \\ &\le |w - w_h|_1\,|p^0 - p_h^0|_1 + |\Pi_h \tilde{s} - \tilde{s}|_1\,|w_h - w|_1 + |\Pi_h \tilde{s} - \tilde{s}|_{1-\alpha_0}|w|_{1+\alpha_0} \\ &\le C\,h^{2\alpha_0}\|p^0 - p_h^0\|_0 + C\,h^{1+\alpha_0}|\tilde{s}|_2\|p^0 - p_h^0\|_0 \,, \end{aligned}$$

this leads to the desired L^2-norm error estimate:

$$\|p_s - p_s^h\|_0 \le \|p^0 - p_h^0\|_0 + \|\tilde{s} - \Pi_h \tilde{s}\|_0 \le C\,h^{2\alpha_0} + C\,h^2|\tilde{s}|_2 \le C\,h^{2\alpha_0}.$$

◇

4.2 Approximation of the singular part ϕ_s

To approximate the singular part ϕ_s in the decomposition $u_\mu = \tilde{u}_\mu + c_\mu \phi_s$, we recall (cf. [10]) that $\phi_s \in H_0^1(\omega)$ solves the elliptic problem (19) and has the following decomposition:

$$\phi_s = \tilde{\phi} + \beta^\star \phi_P, \quad \tilde{\phi} \in H^2(\omega), \quad \beta^\star = \frac{1}{\pi}\|p_s\|_0^2, \quad \phi_P = r^\alpha \sin(\alpha\theta). \tag{43}$$

Using (19), we see that $\tilde{\phi}$, satisfying $\tilde{\phi} = -\beta^\star \phi_P$ on $\partial\omega$, solves the variational problem:

$$(\nabla\tilde{\phi}, \nabla v) = (p_s, v) \quad \forall\, v \in H_0^1(\omega). \tag{44}$$

Next, we consider the finite element approximation of $\tilde{\phi}$ in V^h:

$$\tilde{\phi}_h = -\beta_h^\star \pi_h \phi_P + \phi_h^0,$$

where π_h is defined as in (37), $\beta_h^\star$ is computed using $\beta_h^\star = \frac{1}{\pi}\int_\omega (p_s^h)^2 d\omega$, and $\phi_h^0 \in V_0^h$ is the solution to the problem:

$$(\nabla\tilde{\phi}_h, \nabla v_h) = (p_s^h, v_h) \quad \forall\, v_h \in V_0^h. \tag{45}$$

Then we propose to compute the finite element approximation of ϕ_s by

$$\phi_s^h = \tilde{\phi}_h + \beta_h^\star \phi_P\,.$$

Next, we derive the error estimates for this approximation.

Lemma 4.2 *The following error estimates hold*

$$|\phi_s - \phi_s^h|_1 \le C\,h, \quad \|\phi_s - \phi_s^h\|_a \le C\sqrt{\mu}\,h\,.$$

Proof. We first estimate the error $\tilde{\phi} - \tilde{\phi}_h$. Subtracting (45) from (44) yields

$$(\nabla(\tilde{\phi} - \tilde{\phi}_h), \nabla v_h) = (p_s - p_s^h, v_h) \quad \forall\, v_h \in V_0^h\,,$$

thus we obtain for any $w_h \in V^h$ satisfying $w_h - \tilde{\phi}_h \in V_0^h$,

$$|\tilde{\phi} - w_h|_1^2 = |\tilde{\phi} - \tilde{\phi}_h|_1^2 + |\tilde{\phi}_h - w_h|_1^2 + 2(p_s - p_s^h, \tilde{\phi}_h - w_h),$$

which with the Young inequality and the Poincaré inequality gives

$$\begin{aligned} |\tilde{\phi} - \tilde{\phi}_h|_1^2 &\le |\tilde{\phi} - w_h|_1^2 + 2C_P\|p_s - p_s^h\|_0(|\tilde{\phi} - \tilde{\phi}_h|_1 + |\tilde{\phi} - w_h|_1) \\ &\le 2\,|\tilde{\phi} - w_h|_1^2 + \frac{1}{2}|\tilde{\phi} - \tilde{\phi}_h|_1^2 + C\,\|p_s - p_s^h\|_0^2. \end{aligned} \tag{46}$$

Noting that $\tilde{\phi} = -\beta^\star\phi_P$ on $\partial\omega$, so $\beta_h^\star \Pi_h \tilde{\phi} = \beta^\star \tilde{\phi}_h$ on $\partial\omega$. Let $w_h = \beta_h^\star \Pi_h \tilde{\phi}/\beta^\star$, then $w_h - \tilde{\phi}_h \in V_0^h$. With this w_h, we derive from (46) and Lemma 4.1 that

$$\begin{aligned} |\tilde{\phi} - \tilde{\phi}_h|_1^2 &\le C\,h^2 + C(\beta^\star)^{-2}|\beta^\star\tilde{\phi} - \beta_h^\star \Pi_h \tilde{\phi}|_1^2 \\ &\le C\,h^2 + C\Big\{|\beta^\star - \beta_h^\star|^2\,|\tilde{\phi}|_1^2 + |\beta_h^\star|^2\,|\tilde{\phi} - \Pi_h\tilde{\phi}|_1^2\Big\}. \end{aligned} \tag{47}$$

But using the definitions of $\beta^\star$ and $\beta^\star_h$, we have

$$|\beta^\star - \beta^\star_h| = \frac{1}{\pi}\Big| \|p_s\|_0^2 - \|p_s^h\|_0^2 \Big| \le C\|p_s - p_s^h\|_0 \le C\, h^{2\alpha_0}\,, \tag{48}$$

then it follows from (47) and the fact that $\tilde\phi \in H^2(\omega)$ that

$$|\tilde\phi - \tilde\phi_h|_1 \le C\, h\,.$$

This with (48) and the decompositions of ϕ_s and ϕ_s^h gives the desired H^1-norm estimate:

$$|\phi_s - \phi_s^h|_1 \quad \le \quad |\tilde\phi - \tilde\phi_h|_1 + |\beta^\star - \beta^\star_h|\, |\phi_P|_1 \le C\, h\,.$$

Finally, by noting that both ϕ_s and ϕ_s^h vanish on γ_1 and γ_2, we can apply the Poincaré inequality to the function $\phi_s - \phi_s^h$ to get

$$\|\phi_s - \phi_s^h\|_0 \le C'_P |\phi_s - \phi_s^h|_1\,.$$

Then the desired estimate on $\|\phi_s - \phi_s^h\|_a$ follows from

$$\|\phi_s - \phi_s^h\|_a^2 = |\phi_s - \phi_s^h|_1^2 + \mu\, \|\phi_s - \phi_s^h\|_0^2 \le C\,(h^2 + \mu\, h^2).$$

◇

4.3 Approximation of $\tilde u_\mu$ and c_μ in decomposition (20)

Noting that $\tilde u_\mu$ and c_μ solve the coupled system (28) and (29), it is natural to formulate their finite element approximations as follows:

Find $\tilde u_\mu^h \in V_0^h$ and $c_\mu^h \in \mathbb{R}^1$ such that

$$a_\mu(\tilde u_\mu^h, v) + c_\mu^h\, a_\mu(\phi_s^h, v) = (f, v) \quad \forall v \in V_0^h, \tag{49}$$

$$\Big(\|p_s^h\|_0^2 + \mu|\phi_s^h|_1^2\Big) c_\mu^h + \mu\,(\tilde u_\mu^h, p_s^h) = (f, p_s^h)\,, \tag{50}$$

where ϕ_s^h and p_s^h are the finite element approximations of ϕ_s and p_s, see Subsect. 4.1-4.2.

However, this formulation requires one to solve a coupled system, and it poses some difficulty in getting the error estimates as it does not fall into any existing saddle-point like framework. Instead, we are going to propose a more efficient approximation which enables us to find $\tilde u_\mu^h \in V_0^h$ and c_μ^h separately. In fact, we can use the formula (33) to first find c_μ^h, and then use (49) to find $\tilde u_\mu^h \in V_0^h$. This leads to the following algorithm to find $\tilde u_\mu^h \in V_0^h$ and c_μ^h.

SCM Algorithm for finding $\tilde u_\mu^h \in V_0^h$ and $c_\mu^h \in \mathbb{R}^1$.

Step 1. Find $z_\mu^h \in V_0^h$ such that

$$a_\mu(z_\mu^h, v) = (f, v) \quad \forall\, v \in V_0^h\,. \tag{51}$$

Compute c_μ^h as follows:

$$c_\mu^h = \frac{(f - \mu\, z_\mu^h,\, p_s^h)}{\|p_s^h\|_0^2} \quad \text{if} \quad k < C\, h^{-\frac{1}{2-\alpha_0}}\,; \tag{52}$$

and

$$c_\mu^h = 0 \quad \text{if} \quad k \ge C\, h^{-\frac{1}{2-\alpha_0}}\,. \tag{53}$$

<u>**Step 2**</u>. Find $\tilde{u}_\mu^h \in V_0^h$ such that

$$a_\mu(\tilde{u}_\mu^h, v) + c_\mu^h\, a_\mu(\phi_s^h, v) = (f, v) \quad \forall v \in V_0^h\,. \tag{54}$$

Below, we shall derive the error estimates on $(c_\mu - c_\mu^h)$ and $(\tilde{u}_\mu^h - \tilde{u}_\mu^h)$. Recall the formula (33) for c_μ:

$$c_\mu = \frac{(f - \mu\, z_\mu,\, p_s)}{\|p_s\|_0^2}\,, \tag{55}$$

where $z_\mu = A_\mu^{-1} f \in H_0^1(\omega)$ solves

$$a_\mu(z_\mu, v) = (f, v) \quad \forall\, v \in H_0^1(\omega)\,. \tag{56}$$

Clearly $z_\mu = u_\mu$, the solution to the equation (17). But a different notation z_μ is used here for convenience, since the numerical approximation z_μ^h is derived with the *standard* piecewise linear FEM.

Lemma 4.3 *For the solution z_μ to the problem (56) and its piecewise linear finite element approximation z_μ^h in (51), we have the following error estimates*

$$\|z_\mu - z_\mu^h\|_0 \ \le\ \mu^{-1}\|f\|_0\,, \tag{57}$$
$$\|z_\mu - z_\mu^h\|_0 \ \le\ C\, h^{2\alpha_0}\mu^{\alpha_0 - 1}\|f\|_0\,, \tag{58}$$

while for the coefficient c_μ in (55) and its approximation c_μ^h in (52), we have

$$|c_\mu - c_\mu^h| \le C\,(h^{2\alpha_0}\mu^{\alpha_0} + h)\,\|f\|_0\,. \tag{59}$$

Proof. It follows from (56) and (51) that

$$a_\mu(z_\mu - z_\mu^h, z_\mu - z_\mu^h) = a_\mu(z_\mu, z_\mu - z_\mu^h) = (f, z_\mu - z_\mu^h).$$

This implies

$$|z_\mu - z_\mu^h|_1^2 + \mu\,\|z_\mu - z_\mu^h\|_0^2 \le \|f\|_0\,\|z_\mu - z_\mu^h\|_0,$$

thus (57) follows by the Young inequality.

We next show (58). Again it follows from (51) and (56) that

$$\|z_\mu - z_\mu^h\|_a \le \|z_\mu - v_h\|_a \quad \forall\, v_h \in V_0^h.$$

But, by standard interpolation theory, we know that

$$|z_\mu - \Pi_h z_\mu|_1 \le C\, h^{\alpha_0} |z_\mu|_{1+\alpha_0}, \text{ and } \|z_\mu - \Pi_h z_\mu\|_0 \le C\, h^{1+\alpha_0} |z_\mu|_{1+\alpha_0}.$$

Therefore,

$$\|z_\mu - z_\mu^h\|_a \le C\,(1+\sqrt{\mu}\,h) h^{\alpha_0} |z_\mu|_{1+\alpha_0} \le C\, h^{\alpha_0} |z_\mu|_{1+\alpha_0}\,. \tag{60}$$

For any $g \in L^2(\omega)$, define $w \in H_0^1(\omega)$ such that

$$a_\mu(w, v) = (g, v) \quad \forall\, v \in H_0^1(\omega). \tag{61}$$

Using the duality and (61), we have

$$\begin{aligned} \|z_\mu - z_\mu^h\|_0 &= \sup_{g\in L^2(\omega)} \frac{(z_\mu - z_\mu^h, g)}{\|g\|_0} = \sup_{g\in L^2(\omega)} \frac{a_\mu(w, z_\mu - z_\mu^h)}{\|g\|_0} \\ &= \sup_{g\in L^2(\omega)} \frac{a_\mu(w - \Pi_h w, z_\mu - z_\mu^h)}{\|g\|_0} \le \sup_{g\in L^2(\omega)} \frac{\|w - \Pi_h w\|_a\, \|z_\mu - z_\mu^h\|_a}{\|g\|_0}. \end{aligned}$$

Using the interpolation result and the same derivation as in (60) and the *a priori* estimate (23) (with u and f replaced by w and g), we obtain

$$\begin{aligned} \|z_\mu - z_\mu^h\|_0 &\le \sup_{g\in L^2(\omega)} \frac{C\, h^{2\alpha_0} |w|_{1+\alpha_0} |z_\mu|_{1+\alpha_0}}{\|g\|_0} \\ &\le C\, h^{2\alpha_0} \mu^{\alpha_0 - 1} \|f\|_0\,. \end{aligned}$$

This proves (58).

It remains to prove (59). We have from (52) and (55) that

$$\begin{aligned} c_\mu - c_\mu^h &= \frac{(f - \mu\, z_\mu,\, p_s)}{\|p_s\|_0^2} - \frac{(f - \mu\, z_\mu^h,\, p_s^h)}{\|p_s^h\|_0^2} \\ &= \left\{ \frac{(f,\, p_s)}{\|p_s\|_0^2} - \frac{(f,\, p_s^h)}{\|p_s^h\|_0^2} \right\} + \mu \left\{ \frac{(z_\mu^h,\, p_s^h)}{\|p_s^h\|_0^2} - \frac{(z_\mu,\, p_s)}{\|p_s\|_0^2} \right\} := I_1 + I_2\,. \end{aligned}$$

For I_1, we have from Lemma 4.1 that

$$|I_1| \le C\, h\, \|f\|_0\,.$$

For I_2, we further write it as follows

$$I_2 = \mu\, \frac{(z_\mu^h - z_\mu,\, p_s^h)}{\|p_s^h\|_0^2} + \mu\, \frac{(z_\mu,\, p_s^h - p_s)}{\|p_s^h\|_0^2} + \mu\,(z_\mu,\, p_s) \left\{ \frac{1}{\|p_s^h\|_0^2} - \frac{1}{\|p_s\|_0^2} \right\}.$$

Then using estimate (58) and Lemma 4.1, we can derive

$$|I_2| \le \mu\, \|z_\mu - z_\mu^h\|_0 + C\, h\, \|f\|_0 \le C\,(h^{2\alpha_0}\mu^{\alpha_0} + h)\, \|f\|_0\,.$$

This with the estimate of I_1 gives (59). ◇

In the rest of this section, we shall estimate the error between the solution u_μ to the elliptic problem (17) and its SCM approximation u_μ^h. Noting the decomposition of u_μ:

$$u_\mu = \tilde{u}_\mu + c_\mu\, \phi_s = \tilde{u}_\mu + c_\mu\, (\tilde{\phi} + \beta^\star \phi_P)\,, \tag{62}$$

we propose its SCM approximation u_μ^h of the form:

$$u_\mu^h = \tilde{u}_\mu^h + c_\mu^h\, \phi_s^h = \tilde{u}_\mu^h + c_\mu^h\, (\tilde{\phi}_h + \beta_h^\star \phi_P). \tag{63}$$

We shall derive the error estimate on $u_\mu - u_\mu^h$. Let us start with the estimate of $(\tilde{u}_\mu - \tilde{u}_\mu^h)$. We have

Lemma 4.4 *The following error estimate holds*

$$\|\tilde{u}_\mu - \tilde{u}_\mu^h\|_a^2 \le C\,\sqrt{\mu}\,(h^2\,\|f\|_0^2 + |c_\mu - c_\mu^h|^2)\,.$$

Proof. Subtracting (49) from (28) we have

$$a_\mu(\tilde{u}_\mu - \tilde{u}_\mu^h, v_h) + c_\mu a_\mu(\phi_s, v_h) - c_\mu^h a_\mu(\phi_s^h, v_h) = 0 \quad \forall\, v_h \in V_0^h.$$

Using this we obtain for any $w_h \in V_0^h$,

$$\|\tilde{u}_\mu - w_h\|_a^2 = \|\tilde{u}_\mu - \tilde{u}_\mu^h\|_a^2 + \|\tilde{u}_\mu^h - w_h\|_a^2 + 2c_\mu^h a_\mu(\phi_s^h, \tilde{u}_\mu^h - w_h) - 2c_\mu a_\mu(\phi_s, \tilde{u}_\mu^h - w_h),$$

this implies

$$\begin{aligned} & \|\tilde{u}_\mu - \tilde{u}_\mu^h\|_a^2 \\ \le\; & \|\tilde{u}_\mu - w_h\|_a^2 + 2c_\mu\, a_\mu(\phi_s - \phi_s^h, \tilde{u}_\mu^h - w_h) + 2(c_\mu - c_\mu^h) a_\mu(\phi_s^h, \tilde{u}_\mu^h - w_h) \\ \le\; & \|\tilde{u}_\mu - w_h\|_a^2 + 2\,|c_\mu|\,\|\phi_s - \phi_s^h\|_a\,\|\tilde{u}_\mu^h - w_h\|_a + 2\,|c_\mu - c_\mu^h|\,\|\phi_s^h\|_a\,\|\tilde{u}_\mu^h - w_h\|_a\,. \end{aligned} \tag{64}$$

Now, there holds $\|\phi_s\|_a - \|\phi_s - \phi_s^h\|_a \le \|\phi_s^h\|_a \le \|\phi_s\|_a + \|\phi_s - \phi_s^h\|_a$. Using Lemma 4.2 and $\|\phi_s\|_a^2 = |\phi_s|_1^2 + \mu\,\|\phi_s\|_0^2$, we find $\|\phi_s^h\|_a \sim \sqrt{\mu}\,\|\phi_s\|_0$. Using the interpolation results, we obtain

$$\|\tilde{u}_\mu - \Pi_h \tilde{u}_\mu\|_a^2 \le |\tilde{u}_\mu - \Pi_h \tilde{u}_\mu|_1^2 + \mu\,\|\tilde{u}_\mu - \Pi_h \tilde{u}_\mu\|_0^2 \le C\,h^2\,|\tilde{u}_\mu|_2^2,$$

thus letting $w_h = \Pi_h \tilde{u}_\mu$ in (64) and using Lemma 3.2, we derive

$$\begin{aligned} \|\tilde{u}_\mu - \tilde{u}_\mu^h\|_a^2 &\le C\,h^2\,|\tilde{u}_\mu|_2^2 + C\,(\sqrt{\mu} h^2)\|f\|_0\,|\tilde{u}_\mu|_2 + C\,\sqrt{\mu}\,h\,|c_\mu - c_\mu^h|\,|\tilde{u}_\mu|_2 \\ &\le C\,\sqrt{\mu}\,(h^2\,\|f\|_0^2 + |c_\mu - c_\mu^h|^2)\,. \end{aligned}$$

◇

Theorem 4.1 *Let u_μ be the solution to the equation (17) and u_μ^h be its finite element approximation given in (63). Then the following error error estimate holds:*

$$\|u_\mu - u_\mu^h\|_a \le C\,\mu\,h\,\|f\|_0$$

for all $\sqrt{\mu} = k \in \{1, 2, \cdots, N\}$.

Proof. It follows from (62) and (63) that

$$u_\mu - u_\mu^h = (\tilde{u}_\mu - \tilde{u}_\mu^h) + c_\mu(\phi_s - \phi_s^h) + \phi_s^h(c_\mu - c_\mu^h).$$

Then we have using Lemmas 4.4, 4.2 and 3.2 that

$$\begin{aligned} \|u_\mu - u_\mu^h\|_a^2 &\leq 3\Big\{\|\tilde{u}_\mu - \tilde{u}_\mu^h\|_a^2 + |c_\mu|^2\,\|\phi_s - \phi_s^h\|_a^2 + \|\phi_s^h\|_a^2|c_\mu - c_\mu^h|^2\Big\} \\ &\leq C\,(\mu\,h^2\,\|f\|_0^2 + \mu\,|c_\mu - c_\mu^h|^2)\,. \end{aligned}$$

To prove the desired estimate, we need simply

$$|c_\mu - c_\mu^h|^2 \leq C\,\mu\,h^2\,\|f\|_0^2\,. \tag{65}$$

First consider the case (53), i.e. $\sqrt{\mu} = k \geq C\,h^{-\frac{1}{2-\alpha_0}}$. This condition is equivalent to

$$h^{-2}\mu^{\alpha_0-2} \leq C\,.$$

Then (65) comes directly from this condition, $c_\mu^h = 0$ and (22) as follows:

$$|c_\mu - c_\mu^h|^2 = c_\mu^2 \leq C\,\mu^{\alpha_0-1}\|f\|_0^2 \leq C\,\mu\,h^2\,(h^{-2}\mu^{\alpha_0-2})\|f\|_0^2 \leq C\,\mu\,h^2\,\|f\|_0^2\,.$$

For the remaining case (52), we have $\sqrt{\mu} = k < C\,h^{-\frac{1}{2-\alpha_0}}$, or $h^2 < C\,\mu^{-(2-\alpha_0)}$. Therefore, since $2\alpha_0 - 1 > 0$,

$$h^{4\alpha_0-2} \leq C\,\mu^{-(2\alpha_0-1)(2-\alpha_0)}\,.$$

But it follows from (59) and this condition that

$$\begin{aligned} |c_\mu - c_\mu^h|^2 &\leq C\,(h^{4\alpha_0}\mu^{2\alpha_0} + h^2)\,\|f\|_0^2 = C\,(h^2\,h^{4\alpha_0-2}\mu^{2\alpha_0} + h^2)\,\|f\|_0^2 \\ &\leq C\,h^2\,(\mu^{2\alpha_0-(2\alpha_0-1)(2-\alpha_0)} + 1)\,\|f\|_0^2\,. \end{aligned}$$

Now (65) follows from this and the fact that, as $\alpha_0 \in]\frac{1}{2}, 1[$, the exponent of μ is bounded by

$$2\alpha_0 - (2\alpha_0 - 1)(2 - \alpha_0) = 2\alpha_0^2 - 3\alpha_0 + 2 = 1 + (2\alpha_0 - 1)(\alpha_0 - 1) < 1\,.$$

◇

5 Fourier Singular Complement Methods

Let u be the solution to the elliptic problem (15) and u_k be its Fourier coefficients in terms of (6). We define

$$u^N(x_1, x_2, x_3) = \sum_{k=1}^{N} u_k(x_1, x_2)\sin\frac{k\pi}{z_0}x_3\,.$$

By Lemma 2.3, we know that $u_k(x_1, x_2)$ solves the 2D problem (16), the weak formulation of the elliptic problem (8). And using (20) we can decompose u_k as follows:

$$u_k = \tilde{u}_k + c_k\, \phi_s$$

where $\tilde{u}_k \in H^2(\omega) \cap H_0^1(\omega)$ and $\phi_s \in H_0^1(\omega)$ solves (19). Let $u_k^h(x_1, x_2)$ be the SCM solution of $u_k(x_1, x_2)$, defined to be the same as u^h in (63) but with μ replaced by $k^2\pi^2/z_0^2$ now, that is,

$$u_k^h = \tilde{u}_k^h + c_k^h\, \phi_s^h.$$

We define the Fourier SCM (FSCM) solution to (15) as follows:

$$u_h^N(x_1, x_2, x_3) = \sum_{k=1}^{N} u_k^h(x_1, x_2) \sin \frac{k\pi}{z_0} x_3 .$$

Then we have

Theorem 5.1 *The following error estimate holds:*

$$\|\nabla(u - u_h^N)\|_{L^2(\Omega)} \leq C\,(h + N^{-1})\Big\{\Big\|f\Big\|_{L^2(\Omega)} + \Big\|\frac{\partial f}{\partial x_3}\Big\|_{L^2(\Omega)} + \Big\|\frac{\partial^2 f}{\partial x_3^2}\Big\|_{L^2(\Omega)}\Big\}.$$

Proof. Using the Fourier expansion of u in (13) and the definition of u_h^N, we have, cf. (14),

$$\begin{aligned}
\|\nabla(u - u_h^N)\|_{L^2(\Omega)}^2 &= \frac{z_0}{2}\sum_{k=1}^{N}\Big(\|\nabla(u_k - u_k^h)\|_0^2 + (\frac{k\pi}{z_0})^2\|u_k - u_k^h\|_0^2\Big) \\
&\quad + \frac{z_0}{2}\sum_{k>N}\Big(\|\nabla u_k\|_0^2 + (\frac{k\pi}{z_0})^2\|u_k\|_0^2\Big) \\
&=: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

By Lemma 2.3, we derive

$$\begin{aligned}
\mathrm{I}_2 &= \frac{z_0}{2}\sum_{k>N}\Big(\|\nabla u_k\|_0^2 + (\frac{k\pi}{z_0})^2\|u_k\|_0^2\Big) \\
&\leq \frac{z_0}{2}N^{-2}\sum_{k>N} k^2\Big(\|\nabla u_k\|_0^2 + (\frac{k\pi}{z_0})^2\|u_k\|_0^2\Big) \\
&\leq \Big(\frac{z_0}{\pi}\Big)^2 N^{-2}\,\|f\|_{L^2(\Omega)}^2 .
\end{aligned}$$

For I_1, we have

$$\mathrm{I}_1 = \frac{z_0}{2}\sum_{k=1}^{N}\|u_k - u_k^h\|_a^2 .$$

By Theorem 4.1 we have

$$\|u_k - u_k^h\|_a^2 \le C\, k^4\, h^2\, \|f_k\|_0^2 .$$

Using this and the completeness relations of the Fourier series, we obtain the estimate of I_1:

$$\mathrm{I}_1 \le C\, h^2 \sum_{k=1}^{N} k^4\, \|f_k\|_0^2 \le C\, h^2 \left\| \frac{\partial^2 f}{\partial x_3^2} \right\|_{L^2(\Omega)}^2 .$$

This with the previous estimate of I_2 leads to the desired error estimate. ◇

6 Conclusion

The optimal convergence rate of the FSCM in prismatic domains, has been proven for the Poisson problem with homogeneous Dirichlet boundary conditions. Assuming that the right-hand side f is slightly more regular than $f \in L^2(\Omega)$, i.e. that $\partial f/\partial x_3$ and $\partial^2 f/\partial x_3^2$ both belong to $L^2(\Omega)$, the convergence rate of the FSCM in H^1-norm is like

$$\|u - u_h^N\|_1 \le C_f (h + N^{-1}),$$

where h is the 2d meshsize, and N is the number of Fourier modes used.

The same result also holds for the discretization of the Poisson problem with a homogeneous Neumann boundary condition. For the Poisson problem with non-homogeneous boundary conditions, one can apply Remark 2.2 to reach similar results.

Further, it is no difficulty to consider the case of a prismatic domain Ω with several reentrant edges, i.e. ω with several reentrant corners.

Finally, the results, which have been proven in this paper, can also be viewed as the first effort towards the discretization of electromagnetic fields in prismatic domains, with *continuous* discrete fields, the importance of which is well-known, cf. [5]. As a matter of fact, the SCM developed in [6, 7, 13] for 2D electromagnetic computations can be generalized, based from the results obtained here.

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