Time harmonic wave diffraction problems in materials with sign-shifting coefficients

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Abstract

Some electromagnetic materials present, in a given frequency range, an effective dielectric permittivity and/or magnetic permeability which are negative. We are interested in the reunion of such a "negative" material and a classical one. More precisely, we consider here a scalar model problem for the simulation of a wave transmission between two such materials. This model is governed by a Helmholtz equation with a weight function in the Δ principal part which takes positive and negative real values. Introducing additional unknowns, we have already proposed in [2] some new variational formulations of this problem, which are of Fredholm type provided the absolute value of the contrast of permittivities is large enough, and therefore suitable for a finite element discretization. We prove here that, under similar conditions on the contrast, the natural variational formulation of the problem, although not "coercive plus compact", is nonetheless suitable for a finite element discretization. This leads to a numerical approach which is straightforward, less costly than the previous ones, and very accurate.

Key words: wave diffraction problem, sign shifting dielectric constant, left-handed materials, meta-materials, finite elements, error estimate.

1 Introduction

In electromagnetics, a number of materials is currently modeled at a given frequency ω by considering negative real values for their dielectric permittivity and/or magnetic permeability [13,9,11]. In the London phenomenological model, a super-conductor is represented as a medium with a negative dielectric permittivity, whereas homogenization theory applied to meta-materials leads to negative

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effective dielectric permittivity and/or magnetic permeability (the so-called left-handed materials). These ”negative” nondissipative materials raise many unusual questions. In particular, the simulation of a wave transmission between a classical medium and a ”negative” one must be handled carefully, from both mathematical and numerical points of view [12,17]. Let us consider the reunion – called Ω from now on – of two such materials. In two dimensional configurations, the electromagnetic wave transmission problem can be reduced to a scalar problem of the form

$$\operatorname{div}\left(\epsilon^{-1}\nabla u\right) + \omega^2 \mu u = f \text{ in } \Omega \tag{1}$$

where f is an L^2 source function and ϵ, μ are respectively the electric permittivity and the magnetic permeability. The same model arises when one considers the electrostatic equations in two or three dimensional configurations, with $\omega = 0$.
Without loss of generality, we choose to apply a homogeneous Dirichlet condition on $\partial\Omega$: $u|_{\partial\Omega} = 0$. The scalar problem with a homogeneous Neumann condition can be treated in the same way.
More precisely, assume that the domain Ω is split in two parts Ω_1, Ω_2. For the dielectric constant $\epsilon(x)$, one writes $\epsilon_i = \epsilon|_{\Omega_i}$, for $i = 1, 2$: $\epsilon_1(x)$ is strictly positive over Ω_1, whereas $\epsilon_2(x)$ is strictly negative over Ω_2.
The difficulty due to the ϵ sign-shift is obvious when considering the natural variational formulation equivalent to (1):

$$\begin{aligned} &\text{find } u \in H_0^1(\Omega) \text{ such that} \\ &\forall v \in H_0^1(\Omega), \quad \left(\epsilon^{-1}\nabla u, \nabla v\right)_{L^2(\Omega)} - \omega^2(\mu u, v)_{L^2(\Omega)} = -(f, v)_{L^2(\Omega)}\,; \end{aligned} \tag{2}$$

Since ϵ exhibits a sign-shift, $\left(\epsilon^{-1}\nabla u, \nabla v\right)_{L^2(\Omega)}$ has no specific sign, so its coercivity does not hold. Note that this difficulty disappears if one studies ”negative” and dissipative materials, for which ϵ is a complex number, with $\Re(\epsilon) < 0$ and, for instance, $\Im(\epsilon) > 0$ (cf. [14]).
The scalar problem (1) has already been studied in the case of a piecewise constant ϵ, such that $\epsilon_1 > 0$ and $\epsilon_2 < 0$. In [8] it has been shown, using integral equations, that for a smooth interface $\partial\Omega_1 \cap \partial\Omega_2$, the model problem fits into the Fredholm framework if the contrast $\kappa := \epsilon_2/\epsilon_1$ is not equal to -1. In [15], using Dirichlet to Neumann operators, it has been shown that the model fits into the Fredholm framework if $|\kappa| >> 1$ or $|\kappa| << 1$ (no regularity assumption on the interface). The effect of a geometrical singularity of the interface has been investigated more precisely in [4]. It has been proved there that, for an interface which exhibits a right angle, the problem is ill-posed in $H^1(\Omega)$ if $\kappa \in]-3, -1/3[$ (similar results can be derived for any angle).
We are interested more generally by the Helmholtz equation with sign-shifting and varying constants $\epsilon(x)$ and $\mu(x)$. Moreover, we want to introduce and analyze a finite elements discretization of this model. A first variational approach of the problem is presented in [2]: well-posedness in the Fredholm sense has been obtained, under weak assumptions (Lipschitz interface and L^∞ coefficient ϵ), when the abso-

lute value of the generalized contrast function $\kappa(x)$ is small or large enough. This new formulation is well adapted for a discretization with the finite element method. The extension to the Maxwell three-dimensional case is presented in [3].
The drawback of this approach, especially in three-dimensional configurations, is its cost, since an additional vector unknown is introduced. This led us to consider more carefully the direct approximation of the natural variational formulation (2), which gave surprisingly accurate numerical results (see [18], Chapter 4). The subject of this paper is to explain rigorously this phenomenon. In section 2, we introduce the abstract framework. Then, we are going to fit, under some suitable conditions, the natural variational formulation into a well-posed variational setting (section 3) and prove that a standard finite element discretization converges in a classic manner (section 4). Finally in section 5 we give some concluding remarks.

2 The abstract problem

In the sequel, V is a Hilbert space, with scalar product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$. To a continuous bilinear form a defined on $V \times V$, one associates a unique continuous and linear operator $\mathbb{A}$ ($\mathbb{A} \in \mathcal{L}(V)$): $\forall u, v \in V,\ a(u,v) = (\mathbb{A}u, v)_V$.
Given $l \in V'$, let us focus on the variational problem:

$$\text{find } u \in V \text{ such that} \quad \forall v \in V, \quad a(u,v) = l(v)\,; \tag{3}$$

we assume that the form a can be split as $a = b + c$, where the forms b and c are both continuous and bilinear on $V \times V$. It is well known that (3) is well-posed (if uniqueness holds) as soon as b is coercive and the operator $\mathbb{C}$ (associated to the bilinear form c) is compact. We will extend this result to a class of non-coercive forms b.

Definition 2.1 ($\mathbb{T}$-coercivity) Let $\mathbb{T}$ be a continuous linear operator on V. A bilinear form b is $\mathbb{T}$-coercive on $V \times V$ if

$$\exists \gamma > 0,\ \forall v \in V, \quad |b(v, \mathbb{T}v)| \geq \gamma \|v\|_V^2\,.$$

Hereafter we assume that

(H1) there exists $\mathbb{T} \in \mathcal{L}(V)$, bijective, such that the form b is $\mathbb{T}$-coercive on $V \times V$;
(H2) the operator $\mathbb{C}$ is compact.

Theorem 2.1 If the conditions (H1) and (H2) are fulfilled, the variational problem (3) is well-posed if and only if the uniqueness principle of the solution to (3) holds (i.e. $l = 0 \Longrightarrow u = 0$).

Proof: Since $\mathbb{T}$ is bijective, problem (3) is clearly equivalent to the following:

$$\text{find } u \in V \text{ such that } \quad \forall v \in V, \quad b(u, \mathbb{T}v) + c(u, \mathbb{T}v) = l(\mathbb{T}v)\,. \tag{4}$$

The usual framework is recovered: $b(\cdot, \mathbb{T}\cdot)$ is coercive, $c(\cdot, \mathbb{T}\cdot)$ is a compact perturbation ($\mathbb{T}$ is continuous and $\mathbb{C}$ is compact) and $l(\mathbb{T}\cdot)$ is continuous. ■

The discretized (conforming) version of the problem (3) is

$$\text{find } u^h \in V^h \text{ such that } \quad \forall v^h \in V^h, \quad a(u^h, v^h) = l(v^h)\,, \tag{5}$$

where $(V^h)_h$ is a family of finite dimensional subspaces of V such that, for all $v \in V$, one has $\lim_{h\to 0} \inf_{v^h \in V^h} \|v - v^h\|_V = 0\,.$

The approach we propose is inspired by the finite element theory for the coercive plus compact problems. The idea is to prove the stability of the form a over $(V^h)_h$:

$$\exists \sigma > 0,\ \forall h,\ \forall v^h \in V^h, \quad \sup_{w^h \in V^h} \frac{|a(v^h, w^h)|}{\|w^h\|_V} \geq \sigma \|v^h\|_V\,. \tag{6}$$

Then the standard error estimate is recovered with the help of the Strang Lemma [16].

Theorem 2.2 Assume that hypotheses (H1) and (H2) hold, together with the uniqueness principle so that problem (3) is well-posed.
Assume further that:
$\exists\, \delta > 0, \gamma > 0$, such that $\forall h$, $\exists \mathbb{T}^h \in \mathcal{L}(V^h)$, satisfying

(a) $\|\mathbb{T}^h\| := \sup_{v^h \in V^h} \dfrac{\|\mathbb{T}^h v^h\|_V}{\|v^h\|_V} \leq \delta$,

(b) the form b is $\mathbb{T}^h$-coercive over $V^h \times V^h$ with a coercivity constant equal to γ.

Then, the bilinear form a is stable and the discrete problem (5) is well-posed for h small enough. Moreover the following error estimate holds:

$$\exists\, \mathcal{C} > 0\,, \exists h_0 > 0\,, \forall h \in]0, h_0] \quad \|u - u^h\|_V \leq \mathcal{C} \inf_{v^h \in V^h} \|u - v^h\|_V\,. \tag{7}$$

Proof: The stability of a is proved *by contradiction*: if (6) does not hold, there exists a sequence of subspaces – still called [1] $(V^h)_h$ – together with a sequence of elements $(v^h)_h$, with $v^h \in V^h$, such that

$$(i) \quad \|v^h\|_V = 1 \quad (ii) \quad \sup_{s^h \in V^h} \frac{|a(v^h, s^h)|}{\|s^h\|_V} < \mu_h, \ \text{ with } \lim_{h\to 0} \mu_h = 0\,.$$

[1] The sequence is indexed by the parameter h, whereas it should be indexed by $m \in \mathbb{N}$, with $h = h_m$ such that $\lim_{m\to\infty} h_m = 0$.

Let us now consider $w \in V \setminus \{0\}$. For all $w^h \in V^h$ we have

$$|a(v^h, w)| = |a(v^h, w - w^h) + a(v^h, w^h)| \leq |a(v^h, w - w^h)| + |a(v^h, w^h)|$$
$$\leq \|\mathbb{A}\| \|w - w^h\|_V + \mu_h \|w^h\|_V.$$

Let us choose $\varepsilon > 0$. On the one hand, for h smaller than a given h_0 (which depends on both ε and w), we have $\inf_{w^h \in V^h} \|w - w^h\|_V < \varepsilon$. So, for $h \leq h_0$, there exists $w^h \in V^h$ satisfying $\|w - w^h\|_V < \varepsilon$ (and $\|w^h\|_V < \varepsilon + \|w\|_V$).
On the other hand, according (ii), if h is small enough, one has $\mu_h < \varepsilon$.
As a consequence, for every $w \in V \setminus \{0\}$ and for all $\varepsilon > 0$, there exists h'_0 such that, for all $h \leq h'_0$, $|a(v^h, w)| < \varepsilon^2 + \varepsilon(\|\mathbb{A}\| + \|w\|_V)$. This is true for every element w of V, therefore $\mathbb{A}v^h \rightharpoonup 0$ (weakly) in V. We deduce, since $\mathbb{A}^{-1}$ is continuous by the well-posedness of problem (3), that $v^h \rightharpoonup 0$ (weakly) and, since the operator C is compact, $Cv^h \to 0$ (strongly) in V. In order to conclude, we are going to prove that $v^h \to 0$ (strongly) in V. Indeed we note that $b(v^h, \mathbb{T}^h v^h) = a(v^h, \mathbb{T}^h v^h) - (Cv^h, \mathbb{T}^h v^h)$ and, from (ii) and the $\mathbb{T}^h$-coercivity of b, we obtain

$$\gamma \|v^h\|_V^2 \leq |b(v^h, \mathbb{T}^h v^h)| \leq \mu_h \|v^h\|_V \|\mathbb{T}^h v^h\|_V + \|Cv^h\|_V \|\mathbb{T}^h v^h\|_V \, .$$

This last inequality leads straightforwardly to

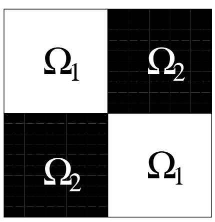
$$\gamma \leq (\mu_h + \|Cv^h\|_V) \|\mathbb{T}^h\| \leq \delta \left(\mu_h + \|Cv^h\|_V\right),$$

which contradicts hypothesis (i). Indeed (6) holds. ■

3 A well-posed variational setting for the natural variational formulation

3.1 Some notations and functional spaces

Let Ω be an open bounded subset of $\mathbb{R}^d$, $d = 2, 3$. It is assumed that this domain can be split in two sub-domains Ω_1 and Ω_2 with pseudo-Lipschitz boundaries [1]: $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, $\Omega_1 \cap \Omega_2 = \emptyset$. In particular Ω_1 and Ω_2 can be disconnected and allow checkerboard-like situations [14], see below.



Moreover, if we let $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$ be the interface, we define $\Gamma_i = \partial\Omega_i \setminus \Sigma$. Throughout this paper we will consider that the constants verify

$$\epsilon, \mu \in L^\infty(\Omega), \quad \epsilon^{-1}, \mu^{-1} \in L^\infty(\Omega).$$

Hereafter we adopt the notation, for all quantities v defined on Ω, $v_i := v_{|\Omega_i}$, for $i = 1, 2$. Furthermore, we use the notations

$$\begin{cases} \text{If } v_i > 0 \text{ a. e. in } \Omega_i\text{: } v_i^{max} = \sup_{x\in\Omega_i} v_i(x),\ v_i^{min} = \inf_{x\in\Omega_i} v_i(x). \\ \text{If } v_i < 0 \text{ a. e. in } \Omega_i\text{: } v_i^{+} = \sup_{x\in\Omega_i} |v_i(x)|,\ v_i^{-} = \inf_{x\in\Omega_i} |v_i(x)|. \end{cases}$$

In what follows, we will use the following Sobolev spaces: $H^{s+1/2}_{0,\Gamma_i}(\Omega_i)$, $H^s_0(\Sigma)$ and $H^s_{00}(\Sigma)$ ($s > 0$), respectively defined by

$$\begin{aligned} &H^{s+1/2}_{0,\Gamma_i}(\Omega_i) && := \left\{v \in H^{s+1/2}(\Omega_i) \text{ s.t. } \exists \tilde{v} \in H^{s+1/2}_0(\Omega),\ \tilde{v}|_{\Omega_i} = v\right\},\ i = 1, 2\,, \\ &\left.\begin{aligned} &H^s_{00}(\Sigma)\ (s - 1/2 \in \mathbb{N}) \\ &H^s_0(\Sigma)\ \ (s - 1/2 \notin \mathbb{N}) \end{aligned}\right\} && := \{v|_\Sigma \text{ s.t. } v \in H^{s+1/2}_{0,\Gamma_1}(\Omega_1)\} = \{v|_\Sigma \text{ s.t. } v \in H^{s+1/2}_{0,\Gamma_2}(\Omega_2)\}\,. \end{aligned}$$

For simplicity, we suppose that $\Gamma_i \neq \emptyset$, $i = 1, 2$, so that the first spaces can be endowed with the $H^{s+1/2}$ semi-norm $\|v\|_{H^{s+1/2}_{0,\Gamma_i}(\Omega_i)} := \|\nabla v\|_{H^{s-1/2}(\Omega_i)}$. Next, we can measure the elements of the second space thanks to either of the norms

$$_i\|p\|_{H^s_{0(0)}(\Sigma)} := \inf_{\substack{v \in H^{s+1/2}_{0,\Gamma_i}(\Omega_i) \\ v|_\Sigma = p}} \|v\|_{H^{s+1/2}_{0,\Gamma_i}(\Omega_i)}\,,\ i = 1, 2\,.$$

The equivalence constants between the two norms $_1\|\cdot\|_{H^s_{00}(\Sigma)}$ and $_2\|\cdot\|_{H^s_{00}(\Sigma)}$ are completely determined by the geometry of $\partial\Omega_1$ and $\partial\Omega_2$ near the interface Σ. When $s = 1/2$, they are denoted by $C_{1\leftarrow 2}$ and $C_{2\leftarrow 1}$, with

$$_i\|v\|_{H^{1/2}_{00}(\Sigma)} \leq C_{i\leftarrow j}\ {}_j\|v\|_{H^{1/2}_{00}(\Sigma)}\,, \quad \forall v \in H^{1/2}_{00}(\Sigma)\,,\ (i, j) \in \{(1, 2), (2, 1)\}\,.$$

3.2 The general result

In this subsection, with the help of the theorem 2.1, we are going to fit the problem (3) into a well-posed variational framework. Let us rewrite the natural variational formulation (2) as

$$\text{find } u \in H^1_0(\Omega) \text{ such that} \quad \forall v \in H^1_0(\Omega)\,, \quad a(u, v) = l(v)\,, \tag{8}$$

where $l(v) := -(f, v)_{L^2(\Omega)}$ and a is split into $a = b + c$, with

$$b(u, v) := (\epsilon^{-1}\nabla u, \nabla v)_{L^2(\Omega)}\,, \quad c(u, v) := -\omega^2(\mu u, v)_{L^2(\Omega)}\,.$$

We are going to build an *ad hoc* operator $\mathbb{T} \in \mathcal{L}(H^1_0(\Omega))$, bijective, such that b is $\mathbb{T}$-coercive over $H^1_0(\Omega)\times H^1_0(\Omega)$. Let us introduce the operator $\mathbb{T}: \ H^1_0(\Omega) \to H^1_0(\Omega)$, defined by

$$\mathbb{T}v = \begin{cases} v_1 \text{ in } \Omega_1 \\ -v_2 + 2\mathcal{R}(v|_\Sigma) \text{ in } \Omega_2 \end{cases},$$

where $\mathcal{R}$ is a continuous and linear operator from $H^{1/2}_{00}(\Sigma)$ to $H^1_{0,\Gamma_2}(\Omega_2)$ such that

$$\mathcal{R}(\varphi)|_\Sigma = \varphi\,, \quad \forall\, \varphi \in H^{1/2}_{00}(\Sigma)\,.$$

By construction, $\mathbb{T}v$ belongs to $H^1_0(\Omega)$. Since the trace mapping is continuous and linear from $H^1_0(\Omega)$ into $H^{1/2}_{00}(\Sigma)$, one has $\mathbb{T} \in \mathcal{L}(H^1_0(\Omega))$. Moreover, one can easily check that $\mathbb{T}^{-1} = \mathbb{T}$, thus $\mathbb{T}$ is a bijective operator.
Let us introduce, for $i = 1, 2$, the notation $b_i(\cdot,\cdot) = (\epsilon_i^{-1}\nabla\cdot, \nabla\cdot)_{L^2(\Omega_i)}$ and the parameter $K_\mathcal{R} > 0$ defined by

$$K_\mathcal{R} := \sup_{v_1 \in H^1_{0,\Gamma_1}(\Omega_1),\, v_1 \neq 0} \frac{|b_2(\mathcal{R}(v_1|_\Sigma), \mathcal{R}(v_1|_\Sigma))|}{b_1(v_1, v_1)}\,. \tag{9}$$

Proposition 3.1 The bilinear form b is $\mathbb{T}$-coercive under the condition $K_\mathcal{R} < 1$.

Proof: To begin with, let us evaluate $b(v, \mathbb{T}v)$, $\forall v \in H^1_0(\Omega)$:

$$b(v, \mathbb{T}v) = b_1(v_1, v_1) - b_2(v_2, v_2) + 2b_2(v_2, \mathcal{R}(v_1|_\Sigma))\,.$$

By the assumption on the signs of ϵ_1 and ϵ_2, it readily follows that

$$|b(v, \mathbb{T}v)| \geq b_1(v_1, v_1) + |b_2(v_2, v_2)| - 2\,|b_2(v_2, \mathcal{R}(v_1|_\Sigma))|\,. \tag{10}$$

Then, the term $|b_2(v_2, \mathcal{R}(v_1|_\Sigma))|$ can be bounded from above by applying Young's inequality and recalling the definition (9). For $\eta > 0$ we have

$$\begin{aligned} |b_2(v_2, \mathcal{R}(v_1|_\Sigma))| &\leq \frac{\eta}{2}|b_2(v_2, v_2)| + \frac{1}{2\eta}|b_2(\mathcal{R}(v_1|_\Sigma), \mathcal{R}(v_1|_\Sigma))| \\ &\leq \frac{\eta}{2}|b_2(v_2, v_2)| + \frac{K_\mathcal{R}}{2\eta} b_1(v_1, v_1)\,. \end{aligned}$$

By combining this last inequality with (10), we obtain

$$|b(v, \mathbb{T}v)| \geq \left(1 - \frac{K_\mathcal{R}}{\eta}\right) b_1(v_1, v_1) + (1-\eta)\,|b_2(v_2, v_2)|\,. \tag{11}$$

Therefore, the form b is $\mathbb{T}$-coercive if both conditions $\eta > K_\mathcal{R}$ and $\eta < 1$ hold simultaneously. It turns out that we can choose a suitable $\eta(K_\mathcal{R})$ satisfying these two conditions, if and only if $K_\mathcal{R} < 1$. ■

Corollary 1 The natural variational formulation (8) fits into the Fredholm well-posed framework if $K_\mathcal{R} < 1$.

Proof: First, we recall that, as $K_{\mathcal{R}} < 1$, the bilinear form b is $\mathbb{T}$-coercive over $H_0^1(\Omega) \times H_0^1(\Omega)$.
According to the Sobolev embedding theorem, the form $c(\cdot, \mathbb{T}\cdot)$ is a compact perturbation of $b(\cdot, \mathbb{T}\cdot)$. Then, we can apply straightforwardly theorem 2.1. ■

This corollary can be related to existing results in the literature [8,2]. Indeed, since the operator $\mathcal{R}$ is continuous from $H_{00}^{1/2}(\Sigma)$ into $H^1_{0,\Gamma_2}(\Omega_2)$, we have

$$|b_2(\mathcal{R}(v_1|_\Sigma), \mathcal{R}(v_1|_\Sigma))| \leq \left(\frac{1}{\epsilon_2^-}\right) {}_2\|\mathcal{R}\|^2{}_2\|v_1|_\Sigma\|^2_{H_{00}^{1/2}(\Sigma)},$$

with ${}_2\|\mathcal{R}\| := \sup_{v \in H_{00}^{1/2}(\Sigma)} \dfrac{\|\mathcal{R}v\|_{H^1_{0,\Gamma_2}(\Omega_2)}}{{}_2\|v\|_{H_{00}^{1/2}(\Sigma)}}$. By the definition of $\mathcal{C}_{2\leftarrow 1}$, we reach

$$|b_2(\mathcal{R}(v_1|_\Sigma), \mathcal{R}(v_1|_\Sigma))| \leq \left(\frac{1}{\epsilon_2^-}\mathcal{C}^2_{2\leftarrow 1}\right) {}_2\|\mathcal{R}\|^2\|v_1\|^2_{H^1_{0,\Gamma_1}(\Omega_1)}.$$

As a consequence, the parameter $K_{\mathcal{R}}$ is bounded from above by $(\epsilon_1^{max}/\epsilon_2^-)_2\|\mathcal{R}\|^2\,\mathcal{C}^2_{2\leftarrow 1}$. Therefore, if $(\epsilon_1^{max}/\epsilon_2^-)_2\|\mathcal{R}\|^2\,\mathcal{C}^2_{2\leftarrow 1} < 1$ (or equivalently, if $\epsilon_2^-/\epsilon_1^{max} > {}_2\|\mathcal{R}\|^2\,\mathcal{C}^2_{2\leftarrow 1}$), the parameter $K_{\mathcal{R}}$ is strictly smaller then 1, and problem (8) is well-posed. This last condition is in accordance with the conditions required in theorems 3.3 and 4.3 of [2]. Moreover, in the case of a piecewise constant dielectric permittivity – with $\epsilon_1 > 0$ and $\epsilon_2 < 0$ – the ratio $\epsilon_2^-/\epsilon_1^{max}$ is equal to the absolute value of the contrast κ (recall $\kappa = \epsilon_2/\epsilon_1$). Then we recover that problem (1) is well-posed for large values of $|\kappa|$, generalizing the results of [8] to the case of (pseudo-)Lipschitz interfaces.
To derive a similar result in the case of a big value of the ratio $\epsilon_1^{min}/\epsilon_2^+$ (i. e. a small value of $|\kappa|$), one proceeds symmetrically, with the roles of Ω_1 and Ω_2 reversed.

3.3 The particular choice of the operator $\mathcal{R}$

The optimal choice of the operator $\mathcal{R}$ which minimizes the value of $K_{\mathcal{R}}$ is $\mathcal{R} = \mathcal{R}_{opt}$, whose action is defined, $\forall \varphi \in H_{00}^{1/2}(\Sigma)$, by $\mathcal{R}_{opt}\varphi = \psi$, where $\psi \in H^1_{0,\Gamma_2}(\Omega_2)$ solves

$$\mathrm{div}(\epsilon_2^{-1}\nabla\psi) = 0 \text{ in } \Omega_2\,, \quad \psi|_\Sigma = \varphi. \tag{12}$$

As a matter of fact, with this choice of ψ, it is well-known that

$$b_2(\psi,\psi) = \min_{v \in H^1_{0,\Gamma_2}(\Omega_2),\ v|_\Sigma = \varphi} b_2(v,v).$$

Since (12) is well-posed, the operator $\mathcal{R}_{opt}$ is bounded and continuous from $H_{00}^{1/2}(\Sigma)$ into $H^1_{0,\Gamma_2}(\Omega_2)$.

In the next section we will need extra regularity of $\psi = \mathcal{R}\varphi$ ($\psi \in H^{s+1/2}(\Omega_2)$, $s > 1/2$). Unfortunately, this does not hold for $\mathcal{R} = \mathcal{R}_{opt}$ if $\epsilon_2 \in L^\infty(\Omega_2)$ has no additional regularity. This leads us to introduce the operator $\mathcal{R}_p$ whose action is defined, $\forall\varphi \in H^{1/2}_{00}(\Sigma)$, by $\mathcal{R}_p\varphi = \psi$, where $\psi \in H^1_{0,\Gamma_2}(\Omega_2)$ solves

$$\Delta\psi = 0 \text{ in } \Omega_2\,, \quad \psi|_\Sigma = \varphi. \tag{13}$$

In this case, provided $\varphi|_\Sigma \in H^s_{00}(\Sigma)$, $\frac{1}{2} \le s \le 1$, then $\psi \in H^{s+1/2}(\Omega_2)$: this property is very important for the finite element error estimate of (8). Indeed in the proof of proposition 4.1, we will apply to the problem (13) the standard finite element error estimate and the inverse inequalities. Again, one has the optimality characterization

$$\|\mathcal{R}_p\phi\|_{H^1_{0,\Gamma_2}(\Omega_2)} = \min_{v\in H^1_{0,\Gamma_2}(\Omega_2),\ v|_\Sigma=\varphi} \|v\|_{H^1_{0,\Gamma_2}(\Omega_2)}\,, \tag{14}$$

and the right-hand side is equal to ${}_2\|\phi\|_{H^{1/2}_{00}(\Sigma)}$ according to the definition of norms. As a consequence, ${}_2\|\mathcal{R}_p\| = 1$ ($\mathcal{R}_p$ is an isometry), and problem (8) is well-posed under the condition

$$\epsilon_2^-/\epsilon_1^{max} > \mathcal{C}^2_{2\leftarrow 1}\,. \tag{15}$$

4 Finite element approximation

As we anticipated in the introduction, even with a sign-shifting permittivity, the standard finite element discretization of (8) gives accurate results, although it does not fit into the usual (coercive, or coercive plus compact) framework. In this section our aim is to explain rigorously why, without any modification, this method is convergent in case of a sign-shifting ϵ.
We are going to approximate the continuous problem (8) with the help of the standard nodal finite element method, both in two- and three-dimensional configurations. Let $(\mathcal{T}_h)_h$ be a regular family of triangulations [5,6] of $\bar\Omega$, made of triangles in 2D and tetrahedra in 3D. Moreover, we suppose that $(\mathcal{T}_h)_h$ fulfills the conditions:

(T1) For all h, for all $T \in \mathcal{T}_h$, there holds either $T \subset \bar\Omega_1$ or $T \subset \bar\Omega_2$.
(T2) The family of triangulations of the interface – $(\mathcal{T}_h|_\Sigma)_h$ – is quasi-uniform [5,6].

For every T, let $P_k(T)$ be the set of polynomials defined on T of degree less than or equal to k. Let us introduce the discrete functional spaces

$$\mathcal{H}^h := \left\{v^h \in \mathcal{C}^0(\bar\Omega) \text{ s.t. } v^h|_T \in P_k(T)\,,\ \forall T \in \mathcal{T}_h\right\}\,; \quad \mathcal{H}^h_0 := \mathcal{H}^h \cap H^1_0(\Omega)\,;$$
$$\mathcal{H}^h_i := \left\{v^h \in \mathcal{C}^0(\bar\Omega_i) \text{ s.t. } v^h|_T \in P_k(T)\,,\ \forall T \in \mathcal{T}_h \text{ and } T \subset \bar\Omega_i\right\}\,,\ i = 1,2\,;$$
$$\mathcal{H}^h_{0,i} := \mathcal{H}^h_i \cap H^1_{0,\Gamma_i}(\Omega_i)\,,\ i = 1,2\,; \quad \mathcal{H}^h_\Sigma := \left\{v^h|_\Sigma \text{ s.t. } v^h \in \mathcal{H}^h\right\}\,.$$

The discretized version of (8) is:

$$\text{find } u^h \in \mathcal{H}_0^h \text{ such that} \quad \forall v^h \in \mathcal{H}_0^h\,, \quad a(u^h, v^h) = l(v^h)\,. \tag{16}$$

In order to apply the theorem 2.2 and recover an error estimate, we must exhibit an operator $\mathbb{T}^h \in \mathcal{L}(\mathcal{H}_0^h)$, whose norm is independent of h and such that b is $\mathbb{T}^h$-coervive over $\mathcal{H}_0^h \times \mathcal{H}_0^h$. Such an operator can be obtained by considering a discretized version of the operator $\mathbb{T}$ introduced in subsection 3.2. Let us now introduce the discrete operator $\mathbb{T}^h : \mathcal{H}_0^h \to \mathcal{H}_0^h$ whose action is defined by

$$\mathbb{T}^h v^h := \begin{cases} v_1^h \text{ in } \Omega_1 \\ -v_2^h + 2\mathcal{R}^h(v^h|_\Sigma) \text{ in } \Omega_2 \end{cases}, \tag{17}$$

where $\mathcal{R}^h$ is a suitably discretized version of the operator $\mathcal{R}$.
Now, the difficulty is to show that $\mathbb{T}^h$ is uniformly bounded from $\mathcal{H}_0^h$ (endowed with the $\|\cdot\|_{H^1(\Omega)}$ norm) into $\mathcal{H}_0^h$. For this one must prove that $\mathcal{R}^h$ is uniformly bounded from from $\mathcal{H}_\Sigma^h$ (endowed with the norm ${}_2\|\cdot\|_{H_{00}^{1/2}(\Sigma)}$) into $\mathcal{H}_{0,2}^h$ (endowed with the norm $\|\cdot\|_{H^1_{0,\Gamma_2}(\Omega_2)}$).
To carry on, let us focus on the operator $\mathcal{R}_p$ and consider its discretized version: let $\mathcal{R}_p^h : \mathcal{H}_\Sigma^h \to \mathcal{H}_{0,2}^h$, whose action is defined, $\forall \varphi^h \in \mathcal{H}_\Sigma^h$, by $\mathcal{R}_p^h \varphi^h = \psi^h$, where ψ^h is the solution to the problem:
Find $\psi^h \in \mathcal{H}_{0,2}^h$ such that

$$(\nabla\psi^h, \nabla v^h)_{L^2(\Omega_2)} = 0,\ \forall v^h \in \mathcal{H}_2^h \cap H_0^1(\Omega_2),\ \psi^h|_\Sigma = \varphi^h\,. \tag{18}$$

Proposition 4.1 The discrete operator $\mathcal{R}_p^h$ is uniformly bounded from $\mathcal{H}_\Sigma^h$ (endowed with the norm ${}_2\|\cdot\|_{H_{00}^{1/2}(\Sigma)}$) into $\mathcal{H}_{0,2}^h$ (endowed with the norm $\|\cdot\|_{H^1_{0,\Gamma_2}(\Omega_2)}$).

Proof: Since the operator $\mathcal{R}_p$ is bounded, we have to prove that $\mathcal{R}_p^h - \mathcal{R}_p$ is uniformly bounded.
Let us recall the definition of ${}_2\|\mathcal{R}_p - \mathcal{R}_p^h\|$:

$${}_2\|\mathcal{R}_p - \mathcal{R}_p^h\| = \sup_{\varphi^h \in \mathcal{H}_\Sigma^h} \frac{\|(\mathcal{R}_p - \mathcal{R}_p^h)\varphi^h\|_{H^1_{0,\Gamma_2}(\Omega_2)}}{{}_2\|\varphi^h\|_{H_{00}^{1/2}(\Sigma)}}\,.$$

Then, let us evaluate $\|(\mathcal{R}_p - \mathcal{R}_p^h)\varphi^h\|_{H^1_{0,\Gamma_2}(\Omega_2)}$: $\mathcal{R}_p\varphi^h$ is the solution ψ to (13) with $\varphi = \varphi^h$ as datum, whereas $\mathcal{R}_p^h\varphi^h$ is the H^1-conforming finite element solution ψ^h to the discrete variational formulation of the same problem, i.e. (18).
As $\partial\Omega_2$ is pseudo-Lipschitz, the solution ψ of (13) exhibits extra-regularity. Indeed, since by construction φ^h belongs to $H_0^1(\Sigma)$, it follows that ψ belongs to $H^{3/2}(\Omega_2)$ (cf. [7], lemma 1).
Moreover, there exists a constant $c_0 > 0$ such that $\|\psi\|_{H^{3/2}(\Omega_2)} \le c_0\, {}_2\|\varphi^h\|_{H_0^1(\Sigma)}$.
Then, according to the standard interpolation theory (cf. [5], Chapter 12), there

exists $C > 0$ such that

$$\|\psi - \psi^h\|_{H^1_{0,\Gamma_2}(\Omega_2)} \le C h^{1/2} \|\psi\|_{H^{3/2}(\Omega_2)} .$$

Next, according to theorem 4.5.11 of [5], the family $(\mathcal{T}_h|_\Sigma)_h$ being quasi-uniform (see (T2)), there exists a strictly positive constant C' such that ${}_2\|\varphi^h\|_{H^1_0(\Sigma)} \le C' h^{-1/2}\, {}_2\|\varphi^h\|_{H^{1/2}_{00}(\Sigma)}$. By combining these inequalities we obtain

$$\|(\mathcal{R}_p - \mathcal{R}_p^h)\varphi^h\|_{H^1_{0,\Gamma_2}(\Omega_2)} = \|\psi - \psi^h\|_{H^1_{0,\Gamma_2}(\Omega_2)} \le c_0\, C\, C'\, {}_2\|\varphi^h\|_{H^{1/2}_{00}(\Sigma)} ,$$

and then the expected result follows, as ${}_2\|\mathcal{R}_p - \mathcal{R}_p^h\| \le c_0\, C\, C'$. ■

Let $\mathbb{T}_p^h$ be the discrete operator defined in (17) with $\mathcal{R}^h$ replaced by $\mathcal{R}_p^h$. We note that, since the trace mapping is linear and continuous from $\mathcal{H}^h_{0,2}$ (endowed with the norm $\|\cdot\|_{H^1_{0,\Gamma_2}(\Omega_2)}$) into $\mathcal{H}^h_\Sigma$ (endowed with the norm ${}_2\|\cdot\|_{H^{1/2}_{00}(\Sigma)}$), the operator $\mathbb{T}_p^h$ is (by construction) bijective and uniformly bounded from $\mathcal{H}^h_0$ into this same space.

Proposition 4.2 Let $\alpha := \sup_h {}_2\|\mathcal{R}_p^h\|$. Under the condition

$$\frac{\epsilon_2^-}{\epsilon_1^{max}} > \alpha^2\, \mathcal{C}^2_{2\leftarrow 1} , \tag{19}$$

the form b is uniformly $\mathbb{T}_p^h$-coercive over $\mathcal{H}_0^h \times \mathcal{H}_0^h$.

Proof: Since the operator $\mathcal{R}_p^h$ is uniformly bounded, one has to follow the proof of the proposition 3.1 (replace $\mathcal{R}$ by $\mathcal{R}_p^h$ there). ■

Theorem 4.1 Suppose condition (19) is fulfilled, then for h small enough, problem (16) is well-posed. Let u^h, u be respectively the solutions to (16) and (8). Then there exists a strictly positive constant C, independent of h such that

$$\|u - u^h\|_{H^1(\Omega)} \le C \inf_{v^h \in \mathcal{H}_0^h} \|u - v^h\|_{H^1(\Omega)} . \tag{20}$$

Remark 2 *Notice that $\alpha \ge 1$, so that condition (19) is more restrictive than condition (15). Indeed, the optimality characterization of $\mathcal{R}_p^h$ is similar to (14), but it is set on $\mathcal{H}^h_{0,2} \subset H^1_{0,\Gamma_2}(\Omega_2)$, so that for all $\phi^h \in \mathcal{H}^h_{0,2}$,*

$$\|\mathcal{R}_p^h \phi^h\|_{H^1_{0,\Gamma_2}(\Omega_2)} \ge \|\mathcal{R}_p \phi^h\|_{H^1_{0,\Gamma_2}(\Omega_2)} = {}_2\|\phi^h\|_{H^{1/2}_{00}(\Sigma)}.$$

It is now possible to recover the usual finite element H^1 error estimate (cf. [6]): one can follow the proof of this last theorem by replacing the Céa's Lemma with theorem 4.1. Then, for a family of triangulations fulfilling conditions (T1) and (T2), and for h small enough, one obtains easily the general result:

$$\lim_{h\to 0} \|u - u^h\|_{H^1(\Omega)} = 0 .$$

On the other hand, we have that, under some extra regularity assumptions, that is if the solution u is such that u_i belongs to $H^{1+\eta}(\Omega_i)$, for $i = 1, 2$, with $\eta > 0$, then the following estimate holds:

$$\inf_{v^h \in \mathcal{H}_0^h} \|u - v^h\|_{H^1(\Omega)} \leq C' \, h^{\min(1,\eta)} \max_i \|u_i\|_{H^{1+\eta}(\Omega_i)} \, .$$

By applying theorem 4.1 again, one recovers the improved error estimate:

$$\|u - u^h\|_{H^1(\Omega)} \leq h^{\min(1,\eta)} C C' \, \max_i \|u_i\|_{H^{1+\eta}(\Omega_i)} \, . \tag{21}$$

5 Concluding Remarks

In this paper, we focused on solving a scalar wave transmission problem between media with opposite sign dielectric and/or magnetic constants. We proved that the natural variational formulation, although not *coercive plus compact* is nevertheless suitable for a finite element discretization, due to the $\mathbb{T}$-coercivity property. What is more, we proved that the continuous Lagrange finite element method yields a converging discretization. Evidently, other finite elements could be used.
We carried out some numerical experiments: we implemented this model on basic geometry and with piecewise constant coefficient ϵ, using P_1 and P_2 Lagrange finite elements. We recovered the expected convergence rates (cf. [18], Chapter 4).

Methods based on the natural variational formulation can be applied to other situations. For instance, when the cavity is a torus as in [10] (periodic boundary conditions). Also, in [18] the approximation of the eigen-frequencies and eigen-modes in resonant cavities (built with meta-materials and dielectrics) is studied.
Last, the natural continuation of the present work is to extend the approach followed here to the magnetostatic and/or time-harmonic Maxwell equations.

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