Degenerate elliptic equations for resonant wave problems[*]

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Abstract

The modelling of resonant waves in 2D plasma leads to the coupling of two degenerate elliptic equations with a smooth coefficient α and compact terms. The coefficient α changes sign. The region where $\{\alpha > 0\}$ is propagative, and the region where $\{\alpha < 0\}$ is non propagative and elliptic. The two models are coupled through the line $\Sigma = \{\alpha = 0\}$. Generically, it is an ill-posed problem, and additional information must be introduced to get a satisfactory treatment at Σ. In this work we define the solution by relying on the limiting absorption principle (α is replaced by $\alpha + i0^+$) in an adapted functional setting. This setting lies on the decomposition of the solution in a regular and a singular part, which originates at Σ, and on quasi-solutions. It leads to a new well-posed mixed variational formulation with coupling. As we design explicit quasi-solutions, numerical experiments can be carried out, which illustrate the good properties of this new tool for numerical computation.

Keywords: degenerate elliptic equations, weighted Sobolev, singular solutions, manufactured solutions, mixed variational formulations

1 Introduction

Resonant waves appears in various electromagnetic phenomena, such as: fusion plasma heating, where a wave is sent inside a plasma and transfers energy to the particles in a localized region [9, 7, 19]; cloaking devices where the transition between metamaterials of negative index and non-dissipative dielectrics is exploited [1, 6, 17]; or photoacoustic imaging of biological tissues where metallic nanoparticles' heating is used [22]. The model problem for resonant waves throughout this work is the degenerate equation, elliptic where $\alpha < 0$,

$$\begin{cases} -\operatorname{div}(\alpha \nabla u) - u &= 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \\ \alpha \partial_n u + i\lambda u &= f \quad \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a positive scalar and $f \in L^2(\Gamma)$ is complex valued on the boundary $\Gamma = \partial\Omega$. The resonance occurs at the transition between the propagative and non propagative regions of the domain. It is characterized by a coefficient $\alpha \in \mathcal{C}^2(\overline{\Omega})$ that changes sign inside the domain Ω, typically over a closed curve denoted as Σ. In this study, $\Sigma \neq \emptyset$ is totally enclosed in the domain Ω, and α behaves as a signed distance to $\Sigma = \{\alpha = 0\}$. The equation (1.1) can be seen as two separate degenerate elliptic equations, one based in $\{\alpha < 0\}$, the other in $\{\alpha > 0\}$ with remaining compact terms, both coupled at Σ.

This model is easily obtained from the time harmonic Maxwell's equations in fusion magnetized plasmas, which are often used to model plasma heating in a tokamak [21, 16]. In this context, the system of PDEs is

$$\begin{cases} \mathbf{curl}\, B - \underline{\underline{\epsilon}} \mathbf{E} &= 0, \\ B - \operatorname{curl} \mathbf{E} &= 0. \end{cases} \tag{1.2}$$

where $\underline{\underline{\epsilon}}$ is the so-called cold plasma permittivity tensor, which encodes the specific propagation properties of the magnetized plasma. It is of the form

$$\underline{\underline{\epsilon}} = \begin{pmatrix} \alpha & i\delta \\ -i\delta & \alpha \end{pmatrix}$$

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with real coefficients that depend on the constant frequency $\omega > 0$ of the wave sent into the plasma, and on the space variable $\mathbf{x} \in \Omega$ through the plasma parameters. Precisely,

$$\alpha(\mathbf{x}) = \left(\frac{\omega}{c}\right)^2 \left(1 - \frac{\omega_p^2(\mathbf{x})}{\omega^2 - \omega_c^2}\right) \in \mathbb{R}, \quad \delta(\mathbf{x}) = \left(\frac{\omega}{c}\right)^2 \left(\frac{\omega_c \omega_p^2(\mathbf{x})}{\omega(\omega^2 - \omega_c^2)}\right) \in \mathbb{R}.$$

Here ω_p is the plasma pulsation. It varies in space and corresponds to the frequency of oscillations of slightly perturbed electrons as they return to equilibrium. And ω_c is the cyclotron pulsation, the frequency to which electrons gyrate around the magnetic field. We are interested in the singular behavior of the upper hybrid resonance, localized at $\Sigma = \{\mathbf{x} \in \mathbb{R}^2, \omega^2 = \omega_p^2(\mathbf{x}) + \omega_c^2\} = \{\mathbf{x} \in \mathbb{R}^2, \alpha(\mathbf{x}) = 0\}$. We assume that Σ is a closed curve that separates the domain Ω in two. On Σ, the off-diagonal coefficient is proportional to $\delta = \frac{\omega \omega_c}{c^2} > 0$. We thus consider δ to be positive and bounded below by a non-zero constant. For simplicity, we will consider from now on that δ is constant. In problem (1.2), the normal component of the electric field is expected [7, 19] to have a singularity at Σ of the order of $1/\alpha$. This singularity does not belong to $L^2(\Omega)$, nor to $L^1(\Omega)$. Conclusions drawn from the 1D case led us to consider the auxiliary fields $\tilde{\mathbf{E}} := \mathbf{E} - \nabla \frac{B}{i\delta}$ and $u := \frac{B}{i\delta}$. The field $\tilde{\mathbf{E}}$ is the so-called regular part of the electric field [2, Chap. 6]. Developing the algebra, it yields $i\delta \, \mathbf{curl} \, u + i\delta \underline{\underline{\epsilon}} \nabla u = \alpha \nabla u$. One gets

$$\begin{cases} \alpha \nabla u - \underline{\underline{\epsilon}} \tilde{\mathbf{E}} &= 0, \\ i\delta u - \mathrm{curl} \, \tilde{\mathbf{E}} &= 0. \end{cases}$$

Reformulated on the unknown u, it gives $\mathrm{curl}\left(\alpha \underline{\underline{\epsilon}}^{-1} \nabla u\right) - i\delta u = 0$, where

$$\underline{\underline{\epsilon}}^{-1} = \frac{1}{\alpha^2 - \delta^2} \begin{pmatrix} \alpha & -i\delta \\ i\delta & \alpha \end{pmatrix} = -\sum_{n \geq 0} \frac{\alpha^{2n+1}}{\delta^{2n+2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n \geq 0} \frac{\alpha^{2n}}{\delta^{2n+1}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

which yields the expansion

$$-\frac{1}{\delta^2} \, \mathrm{div} \, (\alpha \nabla u) - \sum_{n \geq 0} \mathrm{div} \left(\frac{\alpha^{2n+2}}{\delta^{2n+3}} \begin{pmatrix} \alpha/\delta & i \\ -i & \alpha/\delta \end{pmatrix} \nabla u\right) - u = 0.$$

Given that α vanishes on Σ, the terms under the summation sign can be neglected as they are factor of α^{2n+2} with $n \geq 0$. Adding mixed boundary conditions, it comes down to

$$\begin{cases} -\frac{1}{\delta^2} \, \mathrm{div} \, (\alpha \nabla u) - u &= 0 \quad \text{in } \Omega, \\ \alpha \partial_n u + i\lambda u &= f \quad \text{on } \Gamma, \end{cases}$$

which is our model problem (1.1) for $\delta = 1$. We chose α to behave as a signed distance to Σ.

The main mathematical consequence of the degeneracy of α at Σ is that singular solutions localized at Σ may occur.

Similar equations and problems arise from the study of metamaterials as stated previously. It is shown in [6, 4, 5] that the Fredholm well-posedness of the problem depends on the contrast between the negative permittivity of the metamaterial and the positive permittivity of the dielectric, as well as the geometry of the interface. The metamaterial problem was studied under a limiting absorption principle point of view in combination with Agmon-Douglis-Niremberg elliptic a priori estimates in [17]. However the permittivity is constant on each side and does not vanish at the interface, so that the situation of different for hybrid resonances in fusion plasma.

In strict terms of partial differential equations, (1.1) is a coupling of two degenerate equations of different sign. There is ellipticity in the non propagative region, and ellipticity up to the addition of compact terms in the propagative region. In each separate case, the degeneracy concerns a part of the border. Both are then coupled through the degenerate border. The class $\mathcal{M}$ of Muckenhoupt weights, made of coefficients which are locally integrables as well as their inverse [23], is commonly used to study degenerate elliptic equations and we refer to [13, 12] for results from the elliptic community. Unfortunately, $\mathrm{dist}_\Sigma \notin \mathcal{M}$. To the best of our knowledge, the coupling of two degenerate elliptic equations with a coefficient proportional to the distance to the subset of the border, as $|\alpha| = \mathrm{dist}_\Sigma$, is not treated in the literature.

A classical mean to recover the correct solution of such time harmonic wave equations is to resort to a limiting absorption principle. In this work, as in previous 1D works [7, 19], a small parameter $\nu > 0$

is introduced, which regularizes the equation and corresponds to a negligible physical quantity. The regularized problem is then

$$\begin{cases} -\operatorname{div}\left((\alpha + i\nu)\nabla u^\nu\right) - u^\nu &=& 0 & \text{in } \Omega, \\ (\alpha + i\nu)\partial_n u^\nu + i\lambda u^\nu &=& f & \text{on } \Gamma. \end{cases} \tag{1.3}$$

Since (1.3) is well-posed for $\nu > 0$ according to the Lax-Milgram theorem, the whole point is to find a way to pass to the limit as $\nu \to 0^+$ to construct a solution to (1.1).

Our contribution in this work is to propose an original variational formulation for the limit problem $\nu = 0^+$. The mathematical idea is to use a domain decomposition approach. Physically, we decouple the equations in the propagative region $\{\alpha > 0\}$ from the equations in the non propagative region $\{\alpha < 0\}$, and focus on the transmission area. Equations (1.1) and (1.3) thus separate into two similar equations written respectively in subdomains $\Omega_1 = \{\mathbf{x} \in \Omega, \alpha(\mathbf{x}) > 0\}$ and $\Omega_2 = \{\mathbf{x} \in \Omega, \alpha(\mathbf{x}) < 0\}$. The main difficulty consists in finding transmission conditions on $\Sigma = \partial\Omega_1 \cap \partial\Omega_2 = \{\mathbf{x} \in \Omega, \alpha(\mathbf{x}) = 0\}$.

To adress these questions, the method that we propose is based on a novel characterization of the singular behaviour on Σ of the solution, with the design of complex logarithmic quasi-solutions in combination with an adapted functional apparatus. Before stating the main results of this work, we develop the type of singular solutions we have in mind on a simple explicit solution in dimension one. The main ideas behind the construction will appear through this example: we will see that although a generic singular solution has no Dirichlet trace at the singular locus Σ, it remains possible to define a Neumann type trace for the flux at Σ.

1.1 An explicit singular solution in 1D

In dimension one with a coefficient $\alpha(x) = x$, the model problem (1.1) writes in a simpler form

$$-(xu'(x))' - u(x) = 0, \quad \forall x \in \Omega = (-1, 1). \tag{1.4}$$

The propagative region is $\Omega_1 = \{0 < x < 1\}$. The non propagative region is $\Omega_1 = \{-1 < x < 0\}$. We first describe the analytical solutions to (1.4), and the singularity occuring at $\Sigma = \{0\}$. Then we motivate a convenient weighted functional setting that is adapted to our representation of the solutions.

One can check that $v : z \mapsto u(z^2/2)$ verifies the Bessel equation of order 0. So v is a linear combination of the Bessel functions of order 0,

$$J_0 : z \in \mathbb{C} \mapsto \sum_{k \geq 0} \frac{(-1)^k}{(k!)^2}\left(\frac{z^2}{4}\right)^k \quad \text{and} \quad Y_0 : z \in \mathbb{C}^* \mapsto \frac{2}{\pi}\left((\log\frac{z}{2} + \gamma)J_0(z) + \sum_{k \geq 1}\frac{(-1)^{k+1}H_k}{(k!)^2}\left(\frac{z^2}{4}\right)^k\right)$$

where γ is the Euler constant and H_k the harmonic sum of order k, see [20] for more details on those special functions. A solution u of (1.4) is a linear combination of $x \mapsto J_0(\sqrt{2x})$ and $x \mapsto Y_0(\sqrt{2x})$. But since equation (1.4) degenerates at Σ, no natural or immediate continuity relations can be deduced at Σ. In consequence, u is a linear combination of $J_0(\sqrt{2\cdot})$ and $Y_0(\sqrt{2\cdot})$ separately on Ω_1 and Ω_2. A specific study is needed at Σ. Also, since Y_0 has a logarithmic singularity, so does a generic solution u, separately on Ω_1 and Ω_2. Once again, this singularity inhibits continuity of u or u' at $x = 0$. Nevertheless, a solution to (1.4) with the minimal requirement that $u \in L^2_{\text{loc}}(\Omega)$ satisfies $xu' \in \mathcal{C}^0(\Omega)$. In particular, the continuous function xu' has a trace at 0, which may be zero or non zero. This information is of tremendous importance in the rest of this work.

In order to exploit this idea, we introduce the spaces $H^1_{1/2}(\Omega_j) := \{v \in L^2(\Omega_j), \int_{\Omega_j}|x||v'(x)|^2\mathrm{d}x < \infty\}$ for $j = 1, 2$. These spaces arise when integrating (1.4) by parts, and are natural in the variational formulation of this problem. We next observe that a function $u_j \in H^1_{1/2}(\Omega_j)$ that verifies weakly equation (1.4) on Ω_j is such that $xu'_j \in H^1(\Omega_j)$. Therefore one can write

$$xu'_j(x)|_{x=0} = \lim_{\epsilon \to 0}\epsilon^{-1}\int_{\Omega_j \cap \{|x| < \epsilon\}} xu'_j(x)\mathrm{d}x,$$

so that the trace of $\sqrt{x}u'_j$ on Σ has a meaning. Indeed, since $\sqrt{x}u'_j \in L^2(\Omega_j)$, a Cauchy-Schwarz inequality and the dominated convergence theorem lead to

$$\frac{1}{\epsilon}\int_{\Omega_j \cap \{|x| < \epsilon\}}\left|xu'_j(x)\right|\mathrm{d}x \leq \frac{1}{\sqrt{2}}\left(\int_{\Omega_j \cap \{|x| < \epsilon\}}|x||u'_j(x)|^2\mathrm{d}x\right)^{1/2} \underset{\epsilon \to 0}{\to} 0.$$

Hence such a function u_j verifies $xu'_j(x)|_{x=0} = 0$, that is, its trace xu'_j vanishes at the origin. One can check that $J_0(\sqrt{2}\cdot) \in H^1_{1/2}(\Omega_j)$. But on the contrary $Y_0(\sqrt{2}\cdot) \notin H^1_{1/2}(\Omega_j)$, since the trace of $x(Y_0(\sqrt{2}\cdot))'$ at the origin is equal to $1/\pi$. To be more precise, on the scale of weighted Sobolev spaces $\left(H^1_s(\Omega_j)\right)_{s>0}$, one can check that the logarithmic function $\log|\cdot|$ belongs to $\bigcap_{\epsilon>0} H^1_{1/2+\epsilon}(\Omega_j)$ but not to $H^1_{1/2}(\Omega_j)$, which is the critical space on this scale. It follows that the generic solution u does not belong to $H^1_{1/2}(\Omega_j)$, even if $u \in \bigcap_{\epsilon>0} H^1_{1/2+\epsilon}(\Omega_j)$.

We thus propose to represent u on each subdomain Ω_j (i.e. on the propagative and non propagative regions separately) as the sum of a function $u_j \in H^1_{1/2}(\Omega_j)$ and of a complement term that is more singular. To characterize this singular term, we introduce the scalar g at $x = 0$ defined by $g = xu'(x)|_{x=0}$. We lift g as a function w_g with support in Ω, in such a way that w_g has a logarithmic derivative and verifies $xw'_g(x)|_{x=0} = g$. The solution u to (1.4) on Ω then easily decomposes in regular and singular parts: one writes

$$u(x) = u_j(x) + w_g(x) \quad x \in \Omega_j, \quad \text{separately for } j = 1, 2. \tag{1.5}$$

In this ansatz, the regular part is $u_j \in H^1_{1/2}(\Omega_j)$, and the singular part is $w_g \notin H^1_{1/2}(\Omega_j)$. This construction in 1D contains all the ideas behind the decomposition used in this work.

In 2D, $\alpha(\mathbf{x})$ behaves like a signed distance to Σ. From the above 1D study, we expect a logarithmic growth in the normal direction to Σ. For a given bounded open set with Lipschitz boundary $\omega \subset \mathbb{R}^2$, we consider the weighted Sobolev space $H^1_{1/2}(\omega)$ containing the functions $v \in L^2(\omega)$ such that $\sqrt{\text{dist}_{\partial\omega}}\nabla v \in L^2(\omega)^2$. With these notations, it will be possible to use ansatz (1.5) in 2D in combination with the limiting absorption principle $\nu = 0^+$.

1.2 Outline and main results

To treat the singularity at Σ which is the main objective of this work, we introduce some preliminary material. The first one is a specific decomposition of the unknown u into a regular part and a singular part, as in (1.5). The second one is a characterization of the singular part following a limiting absorption principle $\nu = 0^+$. This is obtained in Section 4 and it is the most technical and original part of this work. The singular part w_g^+ is characterized by the singular coefficient g, which is defined along Σ. The regular part is denoted as $\mathbf{u} = (u_1, u_2)$ and is defined by local problems on the subdomains Ω_1 and Ω_2 that involve the singular coefficient g. We introduce an auxiliary variable h in the same space as g and a Lagrange multiplier $\boldsymbol{\lambda}$ in the same space as $\mathbf{u}$. These functions are such that $(\mathbf{u}, g, h) \in V = Q \times H^2(\Sigma) \times H^2(\Sigma)$ and $\boldsymbol{\lambda} \in Q = H^1_{1/2}(\Omega_1) \times H^1_{1/2}(\Omega_2)$. The validity of the decomposition of the solution relies on a technical lemma.

Lemma 1. *Let $\omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. The weighted Sobolev space $H^1_{1/2}(\omega)$ is compactly embedded into $L^2(\omega)$.*

This compact embedding is particularly useful to show that the local problem used to define u_1 in the propagative subdomain Ω_1 is well posed. The local problem used to define u_2 in the non propagative subdomain Ω_2 is also well posed, but the analysis can be simplified since the local problem is naturally coercive in Ω_2. Using a family of explicit quasi-solutions for the regularized problem (1.3), we implement the limiting absorption principle in sections 3 and 4. This allows us to formulate and prove the following result.

Theorem 2. *The formal limit $\nu = 0^+$ of problem (1.3) admits a regularized mixed variational formulation*

$$\text{Find } (\mathbf{u}, g, h) \in V \text{ and } \boldsymbol{\lambda} \in Q \text{ such that}$$
$$\begin{cases} a_r^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) & - & \overline{b^+\left((\mathbf{v}, k, l), \boldsymbol{\lambda}\right)} & = & 0, & \forall (\mathbf{v}, k, l) \in V, \\ b^+\left((\mathbf{u}, g, h), \boldsymbol{\mu}\right) & & & = & \ell(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in Q, \end{cases} \tag{1.6}$$

for b^+ and a_r^+ defined in (4.7) and (4.10) respectively. Moreover, this formulation is well-posed, in the sense that for all $f \in L^2(\Gamma)$, there exists a unique solution $(\mathbf{u}, g, h) \in V$ and $\boldsymbol{\lambda} \in Q$, which depends continuously on $\|f\|_{L^2(\Gamma)}$.

The organization of this work is as follows. Newt Section 2 is devoted to 2D geometrical notations and the functional setting. The variational characterization of the singularity with quasi-solutions and the 2D justification of the ansatz (1.5) is provided in Section 3. The miwed variational fomulation (1.6) which combines all these ideas is defined and studied in Section 4 Finally, in Section 5, a discrete formulation

of (1.6) is constructed in the context of the classical finite element method. It leads to a new method for the numerical approximation of (1.1). The numerical results illustrate on the one hand the robustness of the discretized formulation, and on the other hand the accuracy of the discrete solution.
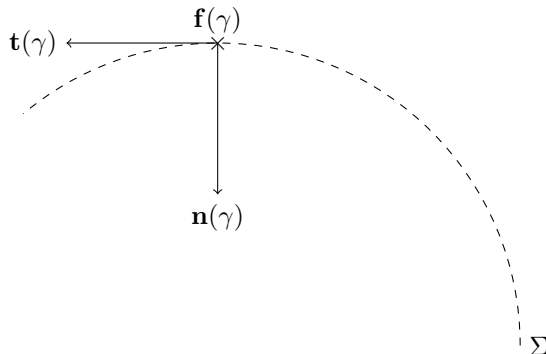
2 Geometry in dimension two

Figure 1: Σ parametrization

We use a standard parametrization of the geometry, see [10].

2.1 Notations

The coefficient α is assumed to be smooth with enough derivatives, typically $\mathcal{C}^2$, and non degenerate in the sense that $\nabla\alpha(\mathbf{x}) \neq 0$ in the domain of interest. Under these conditions, we define the curve $\Sigma := \{\mathbf{x} \in \mathbb{R}^2, \alpha(\mathbf{x}) = 0\}$, and to further simplify, we assume that Σ is a closed simple line. We consider the parametrization $\mathbf{f} : [0, 1] \to \mathbb{R}^2$ of Σ illustrated in Fig. 2, with $\mathbf{f}(0) = \mathbf{f}(1)$ and $\mathbf{f}$ bijective between $[0, 1)$ and Σ. We assume γ is a curvilinear abscissa, that is $|\mathbf{f}'(\gamma)| = 1$. The curvature radius of Σ at $\mathbf{f}(\gamma)$ is denoted $R(\gamma)$, and we note the minimal value of the curvature radius $R_* := \min_\gamma R(\gamma) > 0$. This quantity is well-defined for a continuous R. For a given $\gamma \in [0, 1]$, the ingoing normal and tangent vectors to Σ at $\mathbf{f}(\gamma)$ are denoted respectively $\mathbf{n}(\gamma)$ and $\mathbf{t}(\gamma)$. For all $\gamma \in [0, 1]$ and $\sigma \in \mathbb{R}$, we set $\psi(\gamma, \sigma) := \mathbf{f}(\gamma) + \sigma\mathbf{n}(\gamma)$ which belongs to a neighbourhood of Σ for small values of σ. It is known that ψ is injective on $[0, 1) \times (-R_*, R_*)$. We next define the tubular extension of Σ

$$\Sigma_{\text{tub}} := \psi\left([0, 1) \times (-\tfrac{1}{2}R_*, \tfrac{1}{2}R_*)\right) \cap \Omega.$$

It is convenient to consider the change of variable $\mathbf{x} = \psi(\gamma, \sigma)$ for $\mathbf{x} \in \Sigma_{\text{tub}}$. The Jacobian of the transformation

$$p_\Omega(\gamma, \sigma) := \det \nabla \psi = |\mathbf{f}'(\gamma)| - \frac{\sigma}{R(\gamma)} = 1 - \frac{\sigma}{R(\gamma)} \geq \frac{1}{2} \tag{2.1}$$

is such that for $O \subset \Sigma_{\text{tub}}$

$$\int_O v(\mathbf{x})\mathrm{d}\mathbf{x} = \int_{\psi^{-1}(O)} v \circ \psi(\gamma, \sigma) p_\Omega(\gamma, \sigma) \mathrm{d}\sigma \mathrm{d}\gamma.$$

We will always use the notation $\nabla = (\partial_x, \partial_y)^t$, and one has

$$\nabla\gamma(\mathbf{x}) = \frac{\mathbf{t}(\gamma(\mathbf{x}))}{1 - \sigma(\mathbf{x})/R(\gamma(\mathbf{x}))} \quad \text{and} \quad \nabla\sigma(\mathbf{x}) = \mathbf{n}(\gamma(\mathbf{x})). \tag{2.2}$$

For any function v we make the abuse of notation $v \circ \psi(\gamma, \sigma) = v(\gamma, \sigma)$. So we define $r(\gamma) = \partial_\sigma \alpha(\gamma, 0) \neq 0$, and we have the local expansion for small σ

$$\alpha(\gamma, \sigma) = r(\gamma)\sigma + O(\sigma^2), \tag{2.3}$$

with r of constant sign. We suppose without loss of generality that $r < 0$, therefore on a given $\Sigma_{\text{tub}}^* \subset \Sigma_{\text{tub}}$, one has $0 < c_* \leq -\partial_\sigma\alpha \leq c^*$. In the sequel, we will consider the case $\Omega = \Sigma_{\text{tub}}^*$. In particular,

$$0 < c_* \leq -\frac{\alpha(\gamma, \sigma)}{\sigma} \leq c^*, \quad \forall \psi(\gamma, \sigma) \in \Omega. \tag{2.4}$$

We also define

$$\Omega_1 := \psi\left([0,1) \times (-\tfrac{1}{2}R_*, 0)\right) \cap \Omega \quad \text{and} \quad \Omega_2 := \psi\left([0,1) \times (0, \tfrac{1}{2}R_*)\right) \cap \Omega,$$

such that $\Omega = \Omega_1 \cup \Sigma \cup \Omega_2$, and the exterior boundaries

$$\Gamma_1 := \partial\Omega_1 \cap \Gamma \quad \text{and} \quad \Gamma_2 := \partial\Omega_2 \cap \Gamma,$$

where $\Gamma = \partial\Omega = \Gamma_1 \cup \Gamma_2$. One has $\Omega_1 = \{\alpha > 0\}$ and $\Omega_2 = \{\alpha < 0\}$. Finally, we define the L_w^2 weighted norm on Σ such that

$$\|v\|_{L_w^2(\Sigma)}^2 := \int_0^1 |v(\gamma, 0)|^2 \frac{\mathbf{1}}{|r(\gamma)|} \mathrm{d}\gamma.$$

2.2 Functional setting

The unknowns are complex valued, and so are the considered functional spaces. We use the α-weighted Sobolev spaces defined on Ω_j, $j = 1, 2$ as

$$H_{1/2}^1(\Omega_j) := \{u \in L^2(\Omega_j), \int_{\Omega_j} |\alpha(\mathbf{x})||\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} < \infty\}$$

endowed with the α-weighted norm

$$\|u\|_{H_{1/2}^1(\Omega_j)} := \left(\|u\|_{L^2(\Omega_j)}^2 + \|\sqrt{|\alpha|}\nabla u\|_{L^2(\Omega_j)}^2\right)^{1/2}.$$

This norm is equivalent to the standard weighted $H_{1/2}^1$ norm involving the distance to a boundary [15] according to (2.4). The dual spaces, in the sense of the spaces of anti-linear maps into $\mathbb{C}$, are noted with a prime.

We recall the definition of T-coercivity as introduced in [6] which is an explicit realization of the inf-sup condition.

Definition 3. *Let H be a Hilbert space. Let $\mathsf{T} : H \to H$ be a continuous linear operator. A bilinear form b defined on $H \times H$ is T-coercive if there exists $C > 0$ such that $|b(u, \mathsf{T}u)| \geq C\|u\|_H^2$.*

If T is a bijection, for all continuous linear forms ℓ defined on H, there exists a unique $u \in H$ such that $b(u, v) = \ell(v)$ for all $v \in H$. And as it is the case for coercive forms, up to a compact perturbation, b is associated to a Fredholm operator of index 0. This last property implies that the associated variational formulation admits a unique solution if and only if uniqueness holds. This result will be used further.

For any $x, y \in \mathbb{R}$, we note the complex logarithm $\log(x + iy) := \tfrac{1}{2}\log(x^2 + y^2) - i\arctan(x/y)$.

3 Limit viscosity $\nu \to 0^+$ solution

3.1 Variational formulations

Because the sign of α changes on Σ, it is natural to separate the problem on each side of Σ, where it has a fixed sign. For these subproblems, we show a well-posedness result in the Hilbert space

$$\begin{aligned}
&Q := H_{1/2}^1(\Omega_1) \times H_{1/2}^1(\Omega_2), \text{ equipped with the norm} \\
&\|\mathbf{u}\|_Q := \|u_1\|_{H_{1/2}^1(\Omega_1)} + \|u_2\|_{H_{1/2}^1(\Omega_2)} \text{ for } \mathbf{u} = (u_1, u_2).
\end{aligned} \tag{3.1}$$

Define the following problem

$$\begin{aligned}
&\text{Find } \mathbf{u} \in Q \text{ such that for all } \mathbf{v} \in Q, \\
&b(\mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}),
\end{aligned} \tag{3.2}$$

where for $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2) \in Q$,

$$\begin{aligned}
b((u_1, u_2), (v_1, v_2)) &:= \sum_{j=1,2} \left(\int_{\Omega_j} \left(\alpha \nabla u_j \cdot \overline{\nabla v_j} - u_j \overline{v_j} \right) \mathrm{d}\mathbf{x} + \int_{\Gamma_j} i\lambda u_j \overline{v_j} \mathrm{d}s \right) \\
\ell(v_1, v_2) &:= \int_{\Gamma_1} f\overline{v_1} \mathrm{d}s + \int_{\Gamma_2} f\overline{v_2} \mathrm{d}s.
\end{aligned} \tag{3.3}$$

Proposition 4. *Let $\lambda > 0$ and $f \in L^2(\Gamma)$. Problem (3.2) has a unique solution in Q.*

Before proving Proposition 4, we prove Lemma 1 for $\omega = \Omega_1$.

Lemma 5. *The weighted Sobolev space $H^1_{1/2}(\Omega_1)$ is compactly embedded into $L^2(\Omega_1)$.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset H^1_{1/2}(\Omega_1)$ be a bounded sequence. Up to a subsequence, u_n weakly converges towards a limit in $H^1_{1/2}(\Omega_1)$, and substracting this limit to the sequence, one can consider $u_n \rightharpoonup 0$.
For all $\epsilon > 0$, define $\Omega_1^\epsilon := \{(\gamma, \sigma) \in \Omega_1, |\sigma| < \epsilon\}$. The H^1 and $H^1_{1/2}$ norms are equivalent on any set $\Omega_1 \backslash \Omega_1^\epsilon$ since the weight α is positively bounded below on this domain: so $\|u_n\|_{L^2(\Omega_1\backslash\Omega_1^\epsilon)} \to 0$.
To prove our claim, we show that as ϵ goes to 0,

$$\int_{\Omega_1^\epsilon} |u_n|^2 \mathrm{d}\mathbf{x} \to 0 \quad \text{uniformly in } n.$$

Introducing the value of u_n on Γ_1, it yields

$$
\begin{aligned}
\int_{\Omega_1^\epsilon} |u_n|^2 \mathrm{d}\mathbf{x} &= \int_\gamma \int_{-\epsilon}^0 |u_n(\gamma,\sigma)|^2 p_\Omega(\gamma,\sigma) \mathrm{d}\sigma \mathrm{d}\gamma \\
&\leq 2\int_\gamma \int_{-\epsilon}^0 |u_n(\gamma,\sigma) - u_n(\gamma, -\frac{R^*}{2})|^2 p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\mathrm{d}\gamma + 2\int_\gamma\int_{-\epsilon}^0 |u_n(\gamma,-\frac{R^*}{2})|^2 p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\mathrm{d}\gamma.
\end{aligned}
$$

The second term is uniformly controlled by ϵ as

$$\int_\gamma\int_{-\epsilon}^0 |u_n(\gamma,-\frac{R^*}{2})|^2 p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\mathrm{d}\gamma \leq C\epsilon\|u_n\|^2_{L^2(\Gamma_1)} \leq \tilde{C}\epsilon.$$

For the first term, a Cauchy-Schwarz argument applied twice gives

$$
\begin{aligned}
\int_\gamma\int_{-\epsilon}^0 |u_n(\gamma,\sigma)-u_n(\gamma,-\frac{R^*}{2})|^2 p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\mathrm{d}\gamma &= \int_\gamma\int_{-\epsilon}^0 \left|\int_{-\frac{R^*}{2}}^\sigma \partial_\sigma u_n(\gamma,s)\mathrm{d}s\right|^2 p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\mathrm{d}\gamma \\
&\leq \int_\gamma\int_{-\epsilon}^0 \left(\int_{-\frac{R^*}{2}}^\sigma |s|\,|\partial_\sigma u_n(\gamma,s)|^2\,\mathrm{d}s\right)\log\left|\frac{R^*}{2\sigma}\right| p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\mathrm{d}\gamma \\
&\leq \int_\gamma \left(\int_{-\frac{R^*}{2}}^0 |s|\,|\partial_\sigma u_n(\gamma,s)|^2\,\mathrm{d}s\right)\left(\int_{-\epsilon}^0 \log\left|\frac{R^*}{2\sigma}\right| p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\right)\mathrm{d}\gamma \\
&\leq \sup_\gamma\left(\int_{-\epsilon}^0 \log\left|\frac{R^*}{2\sigma}\right| p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\right)\left(\int_\gamma\int_{-\frac{R^*}{2}}^0 |s|\,|\partial_\sigma u_n(\gamma,s)|^2\,\mathrm{d}s\mathrm{d}\gamma\right).
\end{aligned}
$$

The first term is such that

$$\sup_\gamma \int_{-\epsilon}^0 \log\left|\frac{R^*}{2\sigma}\right| p_\Omega(\gamma,\sigma)\mathrm{d}\sigma \leq C\epsilon(1 + |\log\epsilon|) \underset{\epsilon\to 0}{\to} 0,$$

and the second one verifies

$$\int_\gamma\int_{-\frac{R^*}{2}}^0 |s|\,|\partial_\sigma u_n(\gamma,s)|^2\,\mathrm{d}s\mathrm{d}\gamma \leq C\|\alpha^{1/2}\nabla u_n\|^2_{L^2(\Omega_1)} \leq M,$$

with $M > 0$ a constant independent of n. The result is established. $\qquad\square$

Proof. [Proposition 4] Let $\lambda > 0$ and $f \in L^2(\Gamma)$.
First, ℓ is a continuous antilinear form on Q. In fact, for a given constant $C > 0$,

$$
\begin{aligned}
\|\ell(\mathbf{v})\| &\leq \|f\|_{L^2(\Gamma)}\left(\|v_1\|_{L^2(\Gamma_1)} + \|v_2\|_{L^2(\Gamma_2)}\right) \\
&\leq C\|f\|_{L^2(\Gamma)}\|\mathbf{v}\|_Q,
\end{aligned}
$$

since the L^2 norm on Γ is controlled by the H^1 norm in a neighbourhood of Γ in Ω, and that H^1 and $H^1_{1/2}$ norms are equivalent away from Σ.
Second, the sesquilinear form b is continuous on $Q \times Q$ since there exists a constant $C > 0$ such that for all $\mathbf{u}, \mathbf{v} \in Q$,

$$|b(\mathbf{u},\mathbf{v})| \leq (2 + \lambda C)\|\mathbf{u}\|_Q\|\mathbf{v}\|_Q.$$

Third, b is the sum of coercive and compact forms. Denoting b_0 and b_1 the forms such that for $\mathbf{u}, \mathbf{v} \in Q$,

$$b_0(\mathbf{u}, \mathbf{v}) := -2 \int_{\Omega_1} u_1 \overline{v_1} \mathrm{d}\mathbf{x} \quad \text{and} \quad b_1(\mathbf{u}, \mathbf{v}) := (u_1, v_1)_{H_{1/2}^1(\Omega_1)} - (u_2, v_2)_{H_{1/2}^1(\Omega_2)} + \sum_{j=1,2} \int_{\Gamma_j} i\lambda u_j \overline{v_j} \mathrm{d}s,$$

one has the decomposition $b(\mathbf{u}, \mathbf{v}) = b_0(\mathbf{u}, \mathbf{v}) + b_1(\mathbf{u}, \mathbf{v})$. For all $\mathbf{v} \in Q$,

$$\operatorname{Re} b_1(\mathbf{v}, \mathsf{T}\mathbf{v}) = \|\mathbf{v}\|_Q^2 \quad \text{for} \quad \mathsf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

hence b_1 is T-coercive for the bijective operator T.

The form b_0, equivalent to the L^2 scalar product on Ω_1, is a compact perturbation of b_1 on Q according to Lemma 5: for a bounded sequence $(\mathbf{v}^n)_{n \in \mathbb{N}} \subset Q$, and up to a subsequence, its first component v_1^n converges in $L^2(\Omega_1)$.

Thus b is associated to a Fredholm operator of zero index. The Fredholm alternative indicates it suffices to prove injectivity, in the sense that if $b(\mathbf{v}, .) = 0$ for a given $\mathbf{v} \in Q$, then $\mathbf{v} = 0$, to have bijectivity. Testing against $(0, v_2)$ and taking the real part, we obtain $\|v_2\|_{H_{1/2}^1(\Omega_1)} = 0$. Testing against $(v_1, 0)$ we obtain first $\|v_1\|_{L^2(\Gamma_1)} = 0$. For all $\epsilon > 0$, the function v_1 also verifies the Helmholtz equation on $\Omega_1 \backslash \Omega_1^\epsilon = \{\mathbf{x} \in \Omega_1, \operatorname{dist}_\Sigma(\mathbf{x}) \geq \epsilon\}$

$$-\operatorname{div}(\alpha \nabla v) - v = 0.$$

Going back to the variational formulation one finds that $\partial_n v_1|_{\Gamma_1} = 0$. The uniqueness continuation principle from partial Cauchy data implies that in $\Omega_1 \backslash \Omega_1^\epsilon$ one has $v_1 = 0$. Letting ϵ go to 0, the claim follows. $\qquad \square$

Since the weak formulation (3.2) of (1.1) is restricted to $H_{1/2}^1$ solutions, it excludes $\log |\sigma|$ singularities as seen in Section 1.1. Thus, it will only allow us to describe the regular part. For the singular part, we will follow a limiting absorption principle which relies on the regularized problem. The classical way is to introduce a complex shift $\alpha + i\nu$ [19, 5], and then pass to the limit $\nu \to 0^+$. We will prove that for $\nu = 0^+$ the limit solution decomposes into a regular part in the weighted space Q plus a complementary singular part.

The remaining part of this section is devoted to show that the problem for $\nu > 0$ is well-posed in $H^1(\Omega)$, which poses no real difficulties. For any $\nu > 0$, problem (1.3) can be formulated in a variational way as

$$\begin{aligned} &\text{Find } u \in H^1(\Omega) \text{ such that for all } v \in H^1(\Omega), \\ &b^\nu(u, v) = \ell(v). \end{aligned} \tag{3.4}$$

The sesquilinear form is

$$b^\nu(u, v) = \int_\Omega \left((\alpha + i\nu)\nabla u \cdot \overline{\nabla v} - u\overline{v} \right) \mathrm{d}\mathbf{x} + \int_\Gamma i\lambda u\overline{v} \mathrm{d}s, \tag{3.5}$$

and for purpose of simplicity we redefine another function ℓ that coincides with (3.3) on $H^1(\Omega)$

$$\ell(v) := \int_\Gamma f\overline{v} \mathrm{d}s. \tag{3.6}$$

Proposition 6. *Let $\nu > 0$, $\lambda > 0$ and $f \in L^2(\Gamma)$. The weak formulation (3.4) of problem (1.3) has a unique solution u^ν in $H^1(\Omega)$.*

Proof. The continuity of forms b^ν and ℓ are straightforward, α being bounded and the L^2 norm on Γ being controlled by the H^1 norm on Ω. Let us show b^ν is coercive. For all $u \in H^1(\Omega)$,

$$\operatorname{Im} b^\nu(u, u) \geq \nu \|\nabla u\|_{L^2(\Omega)^2}^2 \quad \text{and} \quad \operatorname{Re} b^\nu(u, u) \leq \|\alpha\|_{L^\infty(\Omega)} \int_\Omega |\nabla u|^2 \mathrm{d}\mathbf{x} - \|u\|_{L^2(\Omega)}^2,$$

so that for all $C > \|\alpha\|_{L^\infty(\Omega)}/\nu$, $\operatorname{Re}\left(-(1 + iC)b^\nu(u, u) \right) \geq \min(1, C\nu - \|\alpha\|_{L^\infty(\Omega)})\|u\|_{H^1(\Omega)}^2$ and b^ν is coercive. The Lax-Milgram Theorem can be applied to (3.4), which thus has a unique solution u^ν in $H^1(\Omega)$. $\qquad \square$

3.2 A family of quasisolutions

For any $\nu > 0$, we define a family of quasisolutions to problem (1.3). It is composed of functions w_g^ν approximating the expected logarithmic singular behaviour as $\nu \to 0$ with data $g \in H^2(\Sigma)$:

$$w_g^\nu(\gamma, \sigma) := \frac{g(\gamma)}{r(\gamma)} \left(\frac{\log\left(r(\gamma)^2 \sigma^2 + \nu^2\right)}{2} - i \arctan\left(\frac{r(\gamma)\sigma}{\nu}\right) \right). \tag{3.7}$$

When applying the differential operators $(-\nabla \cdot ((\alpha + i\nu)\nabla) - \mathrm{id})$ in Ω and $((\alpha + i\nu)\partial_n + i\lambda\mathrm{id})$ on Γ to the family, we define the resulting quantities

$$\begin{cases} q_g^\nu & := & -\nabla \cdot ((\alpha + i\nu)\nabla w_g^\nu) - w_g^\nu & \text{in } \Omega, \\ z_g^\nu & := & (\alpha + i\nu)\partial_n w_g^\nu + i\lambda w_g^\nu & \text{on } \Gamma. \end{cases} \tag{3.8}$$

Developing the equation in Ω from (3.7) and using (2.2), one has

$$\begin{aligned} q_g^\nu(\gamma, \sigma) & = & -\partial_\gamma \left((\alpha + i\nu)\partial_\gamma w_g^\nu \right) |\nabla\gamma|^2 + (\alpha + i\nu)\partial_\gamma w_g^\nu (\partial_\gamma \nabla\gamma) \cdot \nabla\gamma \\ && + (\alpha + i\nu)\partial_\sigma w_g^\nu (\partial_\gamma \nabla\sigma) \cdot \nabla\gamma - \partial_\sigma \left((\alpha + i\nu)\partial_\sigma w_g^\nu \right) |\nabla\sigma|^2 - w_g^\nu \\ \\ & = & \left[\frac{-\partial_\gamma}{(1 - \sigma/R(\gamma))^2} + \frac{\sigma R'(\gamma)R(\gamma)}{(R(\gamma) - \sigma)^3} \right] \left((\alpha + i\nu) \left(\left(g'(\gamma) - \frac{g(\gamma)r'(\gamma)}{r(\gamma)} \right) w_1^\nu + \frac{g(\gamma)}{r(\gamma)} \frac{r'(\gamma)\sigma}{r(\gamma)\sigma + i\nu} \right) \right) \\ && + \left[-\partial_\sigma + \frac{1}{R(\gamma) - \sigma} \right] \left(g(\gamma) \frac{\alpha + i\nu}{r(\gamma)\sigma + i\nu} \right) - w_g^\nu(\gamma, \sigma), \end{aligned} \tag{3.9}$$

where $w_1^\nu(\gamma, \sigma) = r(\gamma)^{-1} \left(\frac{1}{2}\log\left(r(\gamma)^2\sigma^2 + \nu^2\right) - i \arctan(\frac{r(\gamma)\sigma}{\nu}) \right)$.

Proposition 7. *Let $\nu \in (0,1)$ and $g \in H^2(\Sigma)$. The manufactured solution w_g^ν, the right hand side q_g^ν belong to $L^2(\Omega)$ and the boundary term z_g^ν belongs to $L^2(\Gamma)$. Moreover, the bounds are uniform with respect to ν.*

Proof. Let $\nu \in (0,1)$ and $g \in H^2(\Sigma)$. The manufactured solution is defined in (3.7) as a product of g which is an H^2 function with respect to γ, of $1/r$ which is a bounded coefficient, and of the sum of a logarithm and of a bounded term. As a consequence, w_g^ν belongs to $L^2(\Omega)$, and the bound is uniform with respect to ν.

Assumption (2.3) ensures that the fractions

$$\frac{\alpha + i\nu}{r(\gamma)\sigma + i\nu}, \quad \frac{\sigma}{r(\gamma)\sigma + i\nu}, \quad \frac{\sigma(\alpha + i\nu)}{(r(\gamma)\sigma + i\nu)^2},$$

are bounded, i.e. $O(1)$, for small σ with a constant independent of ν. For the term

$$\partial_\sigma \left[\frac{\alpha + i\nu}{r(\gamma)\sigma + i\nu} \right] = \frac{\partial_\sigma \alpha}{r(\gamma)\sigma + i\nu} - \frac{r(\gamma)(\alpha + i\nu)}{(r(\gamma)\sigma + i\nu)^2},$$

one finds once again that

$$\left| \frac{\partial_\sigma \alpha}{r(\gamma)\sigma + i\nu} - \frac{r(\gamma)(\alpha + i\nu)}{(r(\gamma)\sigma + i\nu)^2} \right| \leq \frac{|r(\gamma)\sigma\partial_\sigma \alpha - r(\gamma)\alpha|}{(r(\gamma)\sigma)^2} + \frac{\nu|\partial_\sigma \alpha - r(\gamma)|}{2\nu|r(\gamma)\sigma|} = O(1) \quad \text{for small } \sigma.$$

Referring to (3.9), q_g^ν thus amounts to a sum of square integrable terms independent of the value of ν. Finally, for z_g^ν, since it involves the function w_g^ν and its derivatives away from the curve Σ, it is indeed in $L^2(\Gamma)$ with a bound that is uniform with respect to ν. $\qquad\square$

Lemma 8. *Let $g \in H^2(\Sigma)$. As $\nu \to 0^+$, the L^2 limit of the manufactured functions defined above are*

$$\begin{cases} w_g^+(\gamma, \sigma) & = & \frac{g(\gamma)}{r(\gamma)} \left(\log|r(\gamma)\sigma| - \frac{i\pi}{2}\mathrm{sign}(r(\gamma)\sigma) \right) & \text{in } L^2(\Omega), \\ q_g^+(\gamma, \sigma) & = & \left[\frac{-\partial_\gamma}{(1 - \sigma/R(\gamma))^2} + \frac{\sigma R'(\gamma)R(\gamma)}{(R(\gamma) - \sigma)^3} \right] \\ && \left(\left(\frac{g'(\gamma)}{r(\gamma)} - \frac{g(\gamma)r'(\gamma)}{r^2(\gamma)} \right) \alpha(\log|r(\gamma)\sigma| - \frac{i\pi}{2}\mathrm{sign}(r(\gamma)\sigma)) + \frac{g(\gamma)r'(\gamma)}{r(\gamma)}\sigma\frac{\alpha}{r(\gamma)\sigma} \right) \\ && + \left[-\partial_\sigma + \frac{1}{R(\gamma) - \sigma} \right] \left(g(\gamma)\frac{\alpha}{r(\gamma)\sigma} \right) - w_g^+(\gamma, \sigma) & \text{in } L^2(\Omega), \\ z_g^+(\gamma, \sigma) & = & \alpha\partial_n w_g^+ + i\lambda w_g^+ & \text{in } L^2(\Gamma). \end{cases} \tag{3.10}$$

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Proposition 9. *For $g \in H^2(\Sigma)$, the associated manufactured solutions depend in the following way on g*

$$\|w_g^+\|_{L^2(\Omega)} \le C\|g\|_{L^2(\Sigma)}, \ \|\alpha\nabla w_g^+\|_{L^2(\Omega)^2} \le C\|g\|_{H^1(\Sigma)}, \ \|q_g^+\|_{L^2(\Omega)} \le C\|g\|_{H^2(\Sigma)}, \ and \ \|z_g^+\|_{L^2(\Gamma)} \le C\|g\|_{H^1(\Sigma)},$$

for four positive constants C independent of g.

Remark 10. *For any non trivial $g \in H^2(\Sigma)$, function w_g^+ does not belong to $H^1_{1/2}(\Omega)$. On the other hand, for any smooth function ϕ that vanishes on Σ, the term $\left(|\alpha|^{1/2}\nabla w_g^+\right)\phi$ is square integrable. In fact, the most singular term in $\int_\Omega |\alpha||\nabla w_g^+|^2\phi^2$ is*

$$\int_0^1 \int_{-R}^R |\alpha|\frac{|g(\gamma)|^2}{r^2(\gamma)\sigma^2}\phi^2(\gamma,\sigma)p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\mathrm{d}\gamma,$$

which is indeed integrable since $\alpha(\gamma,\sigma) = r(\gamma)\sigma + O(\sigma^2)$. Therefore $w_g^+ \in \bigcap_{\epsilon>0} H^1_{1/2+\epsilon}(\Omega)$.

Proposition 11. *Let $g \in H^2(\Sigma)$. The α-weighted flux of w_g^+ on Σ is equal to g, in the sense that for all $h \in L^2(\Sigma)$,*

$$\int_\gamma \alpha(\gamma,0)\nabla w_g^+(\gamma,0) \cdot \mathbf{n}(\gamma)\overline{h(\gamma)}\mathrm{d}\gamma = \int_\gamma g(\gamma)\overline{h(\gamma)}\mathrm{d}\gamma.$$

Proof. Let $\nu > 0$. For all $g \in H^2(\Sigma)$, and $\gamma \in [0,1]$,

$$(\alpha(\gamma,0) + i\nu)\nabla w_g^\nu(\gamma,0) \cdot \mathbf{n}(\gamma) = \frac{\alpha(\gamma,0) + i\nu}{r(\gamma) \times 0 + i\nu}g(\gamma)|\mathbf{n}(\gamma)|^2 = g(\gamma).$$

According to Proposition 9, $(\alpha + i\nu)\nabla w_g^\nu$ converges weakly in $L^2(\Sigma)$ towards $\alpha\nabla w_g^+$, so the result is proven. $\square$

Remark 12. *This last proposition is essential for the decomposition in regular and singular parts. We will introduce a singular coefficient g supported on Σ representing the α-weighted flux over the curve Σ of the whole solution of (1.1). Going back to the 1D case, it corresponds to the fact xY_0' has a trace at $x = 0$ and in this case the unknown g is reduced to the coefficient b, see subSection 1.1.*

3.3 Decomposition of the solution in regular and singular parts

According to the ansatz introduced in Section 1.1, we decompose u as

$$u = \left|\begin{array}{ll} u_1 - w_g^+, & \text{in } \Omega_1, \\ u_2 - w_g^+, & \text{in } \Omega_2, \end{array}\right. \tag{3.11}$$

with the singular coefficient $g \in H^2(\Sigma)$ yet to be characterized, and for $j = 1, 2$, the regular part $u_j \in H^1_{1/2}(\Omega_j)$ such that

$$\left\{\begin{array}{rll} -\nabla \cdot (\alpha\nabla u_j) - u_j &= q_g^+, & \text{in } \Omega_j \\ \alpha\partial_n u_j + i\lambda u_j &= f + z_g^+, & \text{on } \Gamma_j \\ \alpha\partial_n u_j &= 0, & \text{on } \Sigma. \end{array}\right. \tag{3.12}$$

For a fixed g, these equations rewrite in a classical variational way

$$\begin{array}{l} \text{Find } \mathbf{u} \in Q \text{ such that for all } \mathbf{v} \in Q, \\ b(\mathbf{u}, \mathbf{v}) = \ell_g(\mathbf{v}), \end{array} \tag{3.13}$$

where this time the antilinear form ℓ_g is defined by

$$\ell_g(v_1, v_2) := \sum_{j=1,2} \int_{\Omega_j} q_g^+ \overline{v_j}\mathrm{d}\mathbf{x} + \int_{\Gamma_j} (f + z_g^+)\overline{v_j}\mathrm{d}s. \tag{3.14}$$

For $f = 0$ and $g \in H^2(\Sigma)$, we denote the solution $\mathbf{u}(g) = (u_1(g), u_2(g))$.

Proposition 13. *Let $\lambda > 0$, $f \in L^2(\Gamma)$ and $g \in H^2(\Sigma)$. There exists a unique solution $(u_1, u_2) \in Q$ solution of the weak formulation (3.13) of (3.12). The solution is such that*

$$\|u_1\|_{H^1_{1/2}(\Omega_1)} \le C(\|g\|_{H^2(\Sigma)} + \|f\|_{L^2(\Gamma)}) \quad and \quad \|u_2\|_{H^1_{1/2}(\Omega_2)} \le C(\|g\|_{H^2(\Sigma)} + \|f\|_{L^2(\Gamma)}), \tag{3.15}$$

for constants $C > 0$ that are independent of f and g.

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Proof. Under the assumption that ℓ_g is continuous, the proof of Prop. 4 shows that the problem is well-posed for any $\lambda > 0$, $f \in L^2(\Gamma)$ and $g \in H^2(\Sigma)$. And the continuity of ℓ_g is immediate considering Prop. 9.

Testing against $(u_1, 0)$ and $(0, u_2)$ respectively, we obtain the bounds

$$\|u_1\|_{H^1_{1/2}(\Omega_1)} - \|u_1\|_{L^2(\Omega_1)} \leq C(\|g\|_{H^2(\Sigma)} + \|f\|_{L^2(\Gamma)}), \tag{3.16}$$

and

$$\|u_2\|_{H^1_{1/2}(\Omega_2)} \leq C(\|g\|_{H^2(\Sigma)} + \|f\|_{L^2(\Gamma)}). \tag{3.17}$$

Let us now precise the bound (3.16). We show that there exists a constant $C > 0$ that is independent of f and g such that

$$\|u_1\|_{L^2(\Omega_1)} \leq C(\|f\|_{L^2(\Gamma_1)} + \|g\|_{H^2(\Sigma)}).$$

We proceed by contradiction as in *e.g.* [11]. Consider there exists sequences $(f_k)_{k\in\mathbb{N}} \subset L^2(\Gamma)$, $(g_k)_{k\in\mathbb{N}} \subset H^2(\Sigma)$ and $(u_{1,k})_{k\in\mathbb{N}} \subset H^1_{1/2}(\Omega_1)$ that verify for all $k \in \mathbb{N}$,

$$\begin{cases} -\nabla \cdot (\alpha \nabla u_{1,k}) - u_{1,k} &= q^+_{g_k}, & \text{in } \Omega_1 \\ \alpha \partial_n u_{1,k} + i\lambda u_{1,k} &= f_k + z^+_{g_k}, & \text{on } \Gamma_1 \\ \alpha \partial_n u_{1,k} &= 0, & \text{on } \Sigma \end{cases} \tag{3.18}$$

and such that $\|u_{1,k}\|_{L^2(\Omega_1)} = 1$ for all k and $\|f_k\|_{L^2(\Gamma)} + \|g_k\|_{H^2(\Sigma)} \to 0$. Using relation (3.16), we get that $(u_{1,k})_{k\in\mathbb{N}}$ is bounded in $H^1_{1/2}(\Omega_1)$ norm. Therefore there exists $u_1^* \in H^1_{1/2}(\Omega_1)$ towards which, up to a subsequence, $u_{1,k}$ converges weakly in $H^1_{1/2}(\Omega_1)$ and strongly in $L^2(\Omega_1)$ according to Lemma 5. Thus u_1^* is the weak solution of

$$\begin{cases} -\nabla \cdot (\alpha \nabla u_1) - u_1 &= 0, & \text{in } \Omega_1 \\ \alpha \partial_n u_1 + i\lambda u_1 &= 0, & \text{on } \Gamma_1 \\ \alpha \partial_n u_1 &= 0, & \text{on } \Sigma \end{cases} \tag{3.19}$$

which means $u_1^* = 0$. But this contradicts the fact that $\|u_1^*\|_{L^2(\Omega_1)} = 1$.

So the bound (3.16) can be expressed exclusively in terms of the $H^1_{1/2}(\Omega_1)$ norm of u_1 as

$$\|u_1\|_{H^1_{1/2}(\Omega_1)} \leq C(\|g\|_{H^2(\Sigma)} + \|f\|_{L^2(\Gamma)})$$

and the proof is ended. $\qquad\square$

It is now necessary to find how to characterize the transmission condition on Σ, quantified by the unknown g, to close the system.

4 A mixed variational formulation of the limit problem

4.1 Energy estimates

Let u^ν denote the solution of (1.3). We introduce the set of compactly supported, smooth and positive cutoff around Σ functions depending only on σ, and not on γ

$$\mathcal{C}^1_{0,+}(\Omega) = \left\{ \varphi \in \mathcal{C}^1_0(\Omega), \partial_\gamma \varphi = 0, \ \varphi \geq 0, \ \varphi|_\Sigma = 1 \right\}. \tag{4.1}$$

Let $\varphi \in \mathcal{C}^1_{0,+}(\Omega)$. We introduce a new unknown $h \in H^2(\Sigma)$, which corresponds to a dual variable associated to the unknown g. For a given h, testing the equation verified by $u^\nu + w^\nu_h$ against $\overline{(u^\nu + w^\nu_h)}\varphi$ gives

$$\int_\Omega \left(-\nabla \cdot ((\alpha + i\nu)\nabla(u^\nu + w^\nu_h)) \overline{(u^\nu + w^\nu_h)}\varphi - |u^\nu + w^\nu_h|^2 \varphi \right) d\mathbf{x} = \int_\Omega q^\nu_h \overline{(u^\nu + w^\nu_h)}\varphi d\mathbf{x}.$$

Integrating by parts, it yields

$$\int_\Omega \left((\alpha + i\nu)|\nabla(u^\nu + w^\nu_h)|^2 \varphi + (\alpha + i\nu)\overline{(u^\nu + w^\nu_h)}\nabla(u^\nu + w^\nu_h) \cdot \nabla\varphi - |u^\nu + w^\nu_h|^2 \varphi \right) d\mathbf{x} = \int_\Omega q^\nu_h \overline{(u^\nu + w^\nu_h)}\varphi d\mathbf{x},$$

which implies

$$\operatorname{Im} \int_\Omega \left((\alpha + i\nu)(u^\nu + w^\nu_h)\nabla\overline{(u^\nu + w^\nu_h)} \cdot \nabla\varphi + q^\nu_h \overline{(u^\nu + w^\nu_h)}\varphi \right) d\mathbf{x} = \int_\Omega \nu |\nabla(u^\nu + w^\nu_h)|^2 d\mathbf{x} \geq 0. \tag{4.2}$$

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Definition 14. *Let $\nu > 0$, $\varphi \in \mathcal{C}^1_{0,+}(\Omega)$. For all $u \in H^1(\Omega)$ and $h \in H^2(\Sigma)$, define the quadratic form*

$$\mathcal{J}^\nu(u,h) = \operatorname{Im} \int_\Omega \left((\alpha + i\nu)(u + w_h^\nu)\nabla\overline{(u + w_h^\nu)} \cdot \nabla\varphi + q_h^\nu \overline{(u + w_h^\nu)}\varphi \right) \mathrm{d}\mathbf{x}. \tag{4.3}$$

We also define the Hilbert space

$$V := Q \times H^2(\Sigma) \times H^2(\Sigma), \text{ equipped with the norm} \\ \|(\mathbf{u}, g, h)\|_V := \|\mathbf{u}\|_Q + \|g\|_{H^2(\Sigma)} + \|h\|_{H^2(\Sigma)}. \tag{4.4}$$

and the limit quadratic form such that for all $(\mathbf{u}, g, h) \in V$ with $\mathbf{u} = (u_1, u_2)$

$$\mathcal{J}^+(\mathbf{u}, g, h) = \sum_{j=1,2} \operatorname{Im} \int_{\Omega_j} \left(\alpha(u_j - w^+_{g-h})\nabla\overline{(u_j - w^+_{g-h})} \cdot \nabla\varphi + q_h^+ \overline{(u_j - w^+_{g-h})}\varphi \right) \mathrm{d}\mathbf{x}. \tag{4.5}$$

For a given $(\mathbf{u}, g, h)$, the quantity $\mathcal{J}^+(\mathbf{u}, g, h)$ is the formal pointwise limit of $\mathcal{J}^\nu(u, h)$ where $u|_{\Omega_j}$ is a regularization of $u_j - w_g^+$ for $j = 1, 2$. Since relation (4.2) holds, the idea is now to minimize $\mathcal{J}^+$ under the constraint that $(\mathbf{u}, g)$ verifies equation (3.13). We define for this purpose the following Lagrangian on $V \times Q$, such that for all $(\mathbf{u}, g, h) \in V$ and $\mathbf{v} \in Q$

$$\mathcal{L}^+(\mathbf{u}, g, h, \mathbf{v}) = \mathcal{J}^+(\mathbf{u}, g, h) + \operatorname{Im}\left(b(\mathbf{u}, \mathbf{v}) - \ell_g(\mathbf{v}) \right). \tag{4.6}$$

4.2 Mixed variational formulation

To begin with, define b^+ the sesquilinear form on $V \times Q$ such that for all $(\mathbf{u}, g, h) \in V$ and $\mathbf{v} \in Q$,

$$b^+\left((\mathbf{u}, g, h), \mathbf{v}\right) = \sum_{j=1,2} \left(\int_{\Omega_j} \left(\alpha\nabla u_j \cdot \nabla\overline{v_j} - (u_j + q_g^+)\overline{v_j} \right) \mathrm{d}\mathbf{x} + \int_{\Gamma_j} (i\lambda u_j - z_g^+)\overline{v_j}\mathrm{d}s \right). \tag{4.7}$$

Remark 15. *On $V \times Q$, it holds that $b^+((\mathbf{u}, g, h), \mathbf{v}) = b^+((\mathbf{u}, g, 0), \mathbf{v}) = b(\mathbf{u}, \mathbf{v}) - \ell_g(\mathbf{v}) + \ell(\mathbf{v})$.*

Define a^+ a sesquilinear form on $V \times V$ that verifies $\operatorname{Im} a^+ = d\mathcal{J}^+$. We choose the form defined for all $(\mathbf{u}, g, h), (\mathbf{v}, k, l) \in V$ by

$$\begin{aligned} a^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) = & \sum_{j=1,2} \int_{\Omega_j} \left(\alpha(u_j - w^+_{g-h})\nabla\overline{(v_j - w^+_{k-l})} \cdot \nabla\varphi + q_h^+ \overline{(v_j - w^+_{k-l})}\varphi \right) \mathrm{d}\mathbf{x} \\ & - \int_{\Omega_j} \left(\alpha\overline{(v_j - w^+_{k-l})}\nabla(u_j - w^+_{g-h}) \cdot \nabla\varphi + \overline{q_l^+}(u_j - w^+_{g-h})\varphi \right) \mathrm{d}\mathbf{x}. \end{aligned} \tag{4.8}$$

Note that a^+ is anti-hermitian.
The Euler-Lagrange equations associated to the minimization of (4.6) have the following structure

$$\begin{aligned} & \text{Find } (\mathbf{u}, g, h) \in V \text{ and } \boldsymbol{\lambda} \in Q \text{ such that} \\ & \begin{cases} a^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) & - & \overline{b^+\left((\mathbf{v}, k, l), \boldsymbol{\lambda}\right)} & = & 0, & \forall(\mathbf{v}, k, l) \in V, \\ b^+\left((\mathbf{u}, g, h), \boldsymbol{\mu}\right) & & & = & \ell(\boldsymbol{\mu}), & \forall\boldsymbol{\mu} \in Q. \end{cases} \end{aligned} \tag{4.9}$$

We will see that with an arbitrary small regularization in g and h for the form a^+, this form is T-coercive on V, which allows us to apply the classical results of [3] and lead to the conclusion that the regularized problem is well-posed.
For $\rho, \mu \in \mathbb{R}^+$, we introduce the regularized form on $V \times V$

$$a_r^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) = a^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) + i\left(-\rho\, (g, k)_{H^2(\Sigma)} + \mu\, (h'', l'')_{L^2(\Sigma)} \right). \tag{4.10}$$

Theorem 16. *Let $\lambda > 0$, $f \in L^2(\Gamma)$, and $\rho, \mu > 0$. The regularized formulation of (4.9)*

$$\begin{aligned} & \textit{Find } (\mathbf{u}, g, h) \in V \textit{ and } \boldsymbol{\lambda} \in Q \textit{ such that} \\ & \begin{cases} a_r^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) & - & \overline{b^+\left((\mathbf{v}, k, l), \boldsymbol{\lambda}\right)} & = & 0, & \forall(\mathbf{v}, k, l) \in V, \\ b^+\left((\mathbf{u}, g, h), \boldsymbol{\mu}\right) & & & = & \ell(\boldsymbol{\mu}), & \forall\boldsymbol{\mu} \in Q, \end{cases} \end{aligned} \tag{4.11}$$

admits a unique solution.

Remark 17. *Theorem 16 means that the regularization in ν across the curve Σ has been replaced by a regularization in ρ and μ along Σ.*

4.3 Proof of the well-posedness

Denote $B^+ : V \to Q'$ the linear continuous operator such that for all $(\mathbf{v}, k, l) \in V$ and $\boldsymbol{\mu} \in Q$,

$$\left(B^+(\mathbf{v}, k, l), \boldsymbol{\mu}\right)_{Q',Q} = b^+\left((\mathbf{v}, k, l), \boldsymbol{\mu}\right),$$

and let $K := \ker B^+$. Denote $A^+_{KK'} : K \to K'$ the linear continuous operator such that for all $(\mathbf{u}, g, h), (\mathbf{v}, k, l) \in K$,

$$\left(A^+_{KK'}(\mathbf{u}, g, h), (\mathbf{v}, k, l)\right)_{K',K} = a^+_r\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right), \tag{4.12}$$

The proof of Theorem 16 will consist in applying the following classical result.

Theorem 18 (Theorem 4.2.2 of Boffi-Brezzi-Fortin [3] in $\mathbb{C}$). *For any $\kappa \in V'$ and $\varkappa \in Q'$, the mixed system*

$$
\begin{aligned}
&\textit{Find } (\mathbf{u}, g, h) \in V \textit{ and } \boldsymbol{\lambda} \in Q \textit{ such that}\\
&\left\{
\begin{aligned}
a^+_r\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) &- \overline{b^+\left((\mathbf{v}, k, l), \boldsymbol{\lambda}\right)} &= (\kappa, (\mathbf{v}, k, l))_{V',V}, &\quad \forall (\mathbf{v}, k, l) \in V,\\
b^+\left((\mathbf{u}, g, h), \boldsymbol{\mu}\right) & &= (\varkappa, \boldsymbol{\mu})_{Q',Q}, &\quad \forall \boldsymbol{\mu} \in Q.
\end{aligned}
\right.
\end{aligned}
\tag{4.13}
$$

has a unique solution if and only if $A^+_{KK'}$ is an isomorphism from K to K' and if $\operatorname{Im} B^+ = Q'$.

Proposition 19. *Operator B^+ is onto Q'.*

Proof. This is a consequence of Prop. 13. Indeed, according to (4.7), B^+ is such that

$$\left(B^+(\mathbf{u}, g, h), \boldsymbol{\mu}\right)_{Q',Q} = b\left(\mathbf{u}, \boldsymbol{\mu}\right) - \sum_{j=1,2} \left(\int_{\Omega_j} q^+_g \overline{\mu_j} \, \mathrm{d}\mathbf{x} + \int_{\Gamma_j} z^+_g \overline{\mu_j} \, \mathrm{d}s \right).$$

For any $\varkappa \in Q'$, it has been proven that for all $g \in H^2(\Sigma)$, there exists $\mathbf{u} \in Q$ such that

$$b\left(\mathbf{u}, \boldsymbol{\mu}\right) = \sum_{j=1,2} \left(\int_{\Omega_j} q^+_g \overline{\mu_j} \, \mathrm{d}\mathbf{x} + \int_{\Gamma_j} z^+_g \overline{\mu_j} \, \mathrm{d}s \right) + (\varkappa, \boldsymbol{\mu})_{Q',Q} \quad \forall \boldsymbol{\mu} \in Q.$$

As a result, operator B^+ is onto Q'. $\qquad\square$

Proposition 20. *The kernel K of operator B^+ can be described as*

$$K = \{(\mathbf{u}, g, h) \in V, \quad \mathbf{u}(g) = (u_1(g), u_2(g))\}. \tag{4.14}$$

Proof. As the last component in $H^2(\Sigma)$ is a silent variable for B^+, it is not constrained. Necessarily $(\mathbf{u}, g)$ is such that $b(\mathbf{u}, \boldsymbol{\mu}) = \ell_g(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in Q$, where the boundary term $f \in L^2(\Gamma)$ is taken equal to 0. Referring to Prop. 13, this is verified for all $g \in H^2(\Sigma)$ by $\mathbf{u}(g) = (u_1(g), u_2(g))$. $\qquad\square$

Let us now address the properties verified by a^+.

Proposition 21. *For all $(\mathbf{u}, g, h), (\mathbf{v}, k, l) \in K$,*

$$
\begin{aligned}
a^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) &= \sum_{j=1,2} \int_{\Omega_j} q^+_h \overline{(v_j - w^+_{k-l})} - \overline{q^+_l} (u_j - w^+_{g-h}) \, \mathrm{d}\mathbf{x}\\
&\quad + \int_{\Gamma_j} z^+_h \overline{(v_j - w^+_{k-l})} - \overline{z^+_l}(u_j - w^+_{g-h}) - 2i\lambda (u_j - w^+_{g-h}) \overline{(v_j - w^+_{k-l})} \, \mathrm{d}s
\end{aligned}
\tag{4.15}
$$

and a^+ is independent of the cutoff φ, as long as $\varphi \in \mathcal{C}^1_{0,+}(\Omega)$, see (4.1).

Proof. Let $(\mathbf{u}, g, h)$, $(\mathbf{v}, k, l) \in K$, with $\mathbf{u} = (u_1, u_2) = (u_1(g), u_2(g))$ and $\mathbf{v} = (v_1, v_2) = (u_1(k), u_2(k))$ according to Prop. 20. Remark that

$$
\begin{aligned}
a^+\left((\mathbf{u}, g, h), (\mathbf{v}, k, l)\right) &= \sum_{j=1,2} \int_{\Omega_j} \left(\alpha(u_j - w^+_{g-h}) \nabla \overline{(v_j - w^+_{k-l})} \cdot \nabla(\varphi - 1) + q^+_h \overline{(v_j - w^+_{k-l})}(\varphi - 1) \right) \mathrm{d}\mathbf{x}\\
&\quad - \int_{\Omega_j} \left(\alpha \overline{(v_j - w^+_{k-l})} \nabla(u_j - w^+_{g-h}) \cdot \nabla(\varphi - 1) + \overline{q^+_l}(u_j - w^+_{g-h})(\varphi - 1) \right) \mathrm{d}\mathbf{x}\\
&\quad + \int_{\Omega_j} \left(q^+_h \overline{(v_j - w^+_{k-l})} - \overline{q^+_l}(u_j - w^+_{g-h}) \right) \mathrm{d}\mathbf{x}.
\end{aligned}
$$

Since $\varphi|_\Sigma = 1$ and $\alpha|\nabla w_\cdot^+|^2 = \frac{c_0(\gamma,\sigma)}{\sigma} + c_1(\gamma,\sigma)$ with $c_0, c_1 \in L^2(\Omega)$, the terms in $\alpha|\nabla w_\cdot^+|^2(\varphi-1)$ are integrable thanks to Hardy's inequality, which ensures that $\frac{\varphi-1}{\sigma}$ is square integrable on Ω. Therefore the quantity

$$\sum_{j=1,2}\int_{\Omega_j} \alpha\nabla\overline{(v_j - w_{k-l}^+)}\cdot\nabla(u_j - w_{g-h}^+)(\varphi-1)\mathrm{d}\mathbf{x}$$

is well defined. The first two lines above thus rewrite

$$
\begin{aligned}
A \;:=\; & \sum_{j=1,2}\int_{\Omega_j}\left(\alpha(u_j - w_{g-h}^+)\nabla\overline{(v_j - w_{k-l}^+)}\cdot\nabla(\varphi-1) + q_h^+\overline{(v_j - w_{k-l}^+)}(\varphi-1)\right)\mathrm{d}\mathbf{x}\\
& -\int_{\Omega_j}\left(\alpha\overline{(v_j - w_{k-l}^+)}\nabla(u_j - w_{g-h}^+)\cdot\nabla(\varphi-1) + \overline{q_l^+}(u_j - w_{g-h}^+)(\varphi-1)\right)\mathrm{d}\mathbf{x}\\
=\; & \sum_{j=1,2}\int_{\Omega_j}\left(\alpha\nabla\overline{(v_j - w_{k-l}^+)}\cdot\nabla\left((u_j - w_{g-h}^+)(\varphi-1)\right) + q_h^+\overline{(v_j - w_{k-l}^+)}(\varphi-1)\right)\mathrm{d}\mathbf{x}\\
& -\int_{\Omega_j}\left(\alpha\nabla(u_j - w_{g-h}^+)\cdot\nabla\left(\overline{(v_j - w_{k-l}^+)}(\varphi-1)\right) + \overline{q_l^+}(u_j - w_{g-h}^+)(\varphi-1)\right)\mathrm{d}\mathbf{x}\\
=\; & \sum_{j=1,2}\int_{\Omega_j}\left(\alpha\nabla\overline{(v_j - w_{k-l}^+)}\cdot\nabla\left((u_j - w_{g-h}^+)(\varphi-1)\right) + (u_j - w_{g-h}^+ + q_h^+)\overline{(v_j - w_{k-l}^+)}(\varphi-1)\right)\mathrm{d}\mathbf{x}\\
& -\int_{\Omega_j}\left(\alpha\nabla(u_j - w_{g-h}^+)\cdot\nabla\left(\overline{(v_j - w_{k-l}^+)}(\varphi-1)\right) + \overline{(v_j - w_{k-l}^+ + q_l^+)}(u_j - w_{g-h}^+)(\varphi-1)\right)\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Since on each Ω_j $u_j - w_{g-h}^+$ verifies weakly $-\nabla\cdot(\alpha\nabla(u_j - w_{g-h}^+)) - (u_j - w_{g-h}^+) = q_h^+$ with boundary conditions on Γ_j $\alpha\partial_n(u_j - w_{g-h}^+) + i\lambda(u_j - w_{g-h}^+) = f + z_h^+$, and since $v_j - w_{k-l}^+$ verifies the corresponding relations for $g = k$, $h = l$ and $u_j = v_j = u_j(k)$, integrating by parts and using that $(\varphi-1)|_\Sigma = 0$ it yields

$$
\begin{aligned}
A \;=\; & \sum_{j=1,2}\int_{\Gamma_j}\left(\overline{(v_j - w_{k-l}^+)}\alpha\partial_n(u_j - w_{g-h}^+) - (u_j - w_{g-h}^+)\alpha\partial_n\overline{(v_j - w_{k-l}^+)}\right)\mathrm{d}s\\
=\; & \sum_{j=1,2}\int_{\Gamma_j}\left(\overline{(v_j - w_{k-l}^+)}(-i\lambda(u_j - w_{g-h}^+) + z_h^+) - (u_j - w_{g-h}^+)\overline{(-i\lambda(v_j - w_{k-l}^+) + z_l^+)}\right)\mathrm{d}s
\end{aligned}
$$

and the result is proven. $\qquad\square$

Lemma 22. *For all $g \in H^2(\Sigma)$,*

$$\operatorname{Im}\int_\Omega q_g^+\overline{w_g^+}\,\mathrm{d}\mathbf{x} = \pi\|g\|_{L_w^2(\Sigma)}^2 - \operatorname{Im}\int_\Gamma(z_g^+ - i\lambda w_g^+)\overline{w_g^+}\,\mathrm{d}s. \tag{4.16}$$

Proof. For all $\epsilon > 0$, define ψ_ϵ a cutoff of Σ, a real valued piecewise affine function of σ such that $\psi_\epsilon(\gamma,\sigma) = 0$ if $|\sigma| < \epsilon/2$ and $\psi_\epsilon(\gamma,\sigma) = 1$ if $|\sigma| > \epsilon$. For all $g \in H^2(\Sigma)$, using ψ_ϵ and integrating by parts,

$$
\begin{aligned}
\operatorname{Im}\int_\Omega q_g^+\overline{w_g^+}\,\mathrm{d}\mathbf{x} \;=\; & \lim_{\epsilon\to 0}\operatorname{Im}\int_\Omega q_g^+\overline{w_g^+}\psi_\epsilon\,\mathrm{d}\mathbf{x}\\
=\; & \lim_{\epsilon\to 0}\operatorname{Im}\left(\int_\Omega\left(\alpha|\nabla w_g^+|^2\psi_\epsilon + \alpha\overline{w_g^+}\nabla w_g^+\cdot\nabla\psi_\epsilon - |w_g^+|^2\psi_\epsilon\right)\mathrm{d}\mathbf{x} - \int_\Gamma\alpha\partial_n w_g^+\overline{w_g^+}\,\mathrm{d}s\right)\\
=\; & \lim_{\epsilon\to 0}\operatorname{Im}\left(\int_\Omega\alpha\overline{w_g^+}\nabla w_g^+\cdot\nabla\psi_\epsilon\,\mathrm{d}\mathbf{x} - \int_\Gamma(z_g^+ - i\lambda w_g^+)\overline{w_g^+}\,\mathrm{d}s\right).
\end{aligned}
$$

Let us now compute this first integral on Ω for a given $\epsilon > 0$. Expressing the functions in terms of (γ,σ),

$$
\begin{aligned}
\operatorname{Im}\int_\Omega\alpha\overline{w_g^+}\nabla w_g^+\cdot\nabla\psi_\epsilon\,\mathrm{d}\mathbf{x} \;=\; & \operatorname{Im}\int_\gamma\int_{-\epsilon}^{-\epsilon/2}\frac{-2|g(\gamma)|^2}{\epsilon r(\gamma)}\frac{\alpha}{r(\gamma)\sigma}\left(\log|r(\gamma)\sigma| - i\frac{\pi}{2}\operatorname{sign}(r(\gamma))\right)p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\mathrm{d}\gamma\\
& +\operatorname{Im}\int_\gamma\int_{\epsilon/2}^{\epsilon}\frac{2|g(\gamma)|^2}{\epsilon r(\gamma)}\frac{\alpha}{r(\gamma)\sigma}\left(\log|r(\gamma)\sigma| + i\frac{\pi}{2}\operatorname{sign}(r(\gamma))\right)p_\Omega(\gamma,\sigma)\mathrm{d}\sigma\mathrm{d}\gamma\\
=\; & \operatorname{Im}\int_\gamma\int_{\epsilon/2}^{\epsilon}\Big[\frac{-2|g(\gamma)|^2}{\epsilon r(\gamma)}\frac{\alpha}{r(\gamma)\sigma}\left(\log|r(\gamma)\sigma| - i\frac{\pi}{2}\operatorname{sign}(r(\gamma))\right)p_\Omega(\gamma,-\sigma)\\
& +\frac{2|g(\gamma)|^2}{\epsilon r(\gamma)}\frac{\alpha}{r(\gamma)\sigma}\left(\log|r(\gamma)\sigma| + i\frac{\pi}{2}\operatorname{sign}(r(\gamma))\right)p_\Omega(\gamma,\sigma)\Big]\mathrm{d}\sigma\mathrm{d}\gamma
\end{aligned}
$$

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for the weight $p_\Omega(\gamma, \sigma) = 1 - \sigma/R(\gamma)$ defined in (2.1). Identifying the imaginary part, it follows

$$
\operatorname{Im} \int_\Omega \alpha \overline{w_g^+} \nabla w_g^+ \cdot \nabla \psi_\epsilon \mathrm{d}\mathbf{x} \quad = \quad \int_\gamma \int_{\epsilon/2}^\epsilon \frac{2|g(\gamma)|^2}{\epsilon |r(\gamma)|} \frac{\alpha}{r(\gamma)\sigma} \pi \mathrm{d}\sigma \mathrm{d}\gamma
$$
$$
\underset{\epsilon \to 0}{\rightarrow} \quad \int_\gamma \frac{|g(\gamma)|^2}{|r(\gamma)|} \pi \mathrm{d}\gamma,
$$

which is equal to $\pi \|g\|_{L_w^2(\Sigma)}$ The proof is ended. $\qquad\square$

With this technical lemma we can now state the following.

Proposition 23. *For all $(\mathbf{u}(g), g, h) \in K$,*

$$
a^+ \left((0,0,h),(0,0,h)\right) = 2i\pi \|h\|_{L_w^2(\Sigma)}^2 \quad and \quad a^+ \left((\mathbf{u}(g),g,0),(\mathbf{u}(g),g,0)\right) = -2i\lambda \sum_{j=1,2} \int_{\Gamma_j} |u_j(g) - w_g^+|^2 \mathrm{d}s.
$$

Proof. Let $h \in H^2(\Sigma)$. According to (4.15),

$$
a^+ \left((0,0,h),(0,0,h)\right) \quad = \quad \int_\Omega (q_h^+ \overline{w_h^+} - \overline{q_h^+} w_h^+) \mathrm{d}\mathbf{x} + \int_\Gamma (z_h^+ \overline{w_h^+} - \overline{z_h^+} w_h^+ - 2i\lambda |w_h^+|^2) \mathrm{d}s
$$
$$
= \quad 2i \operatorname{Im} \left(\int_\Omega q_h^+ \overline{w_h^+} \mathrm{d}\mathbf{x} + \int_\Gamma (z_h^+ \overline{w_h^+} - i\lambda |w_h^+|^2) \mathrm{d}s \right).
$$

By Lemma 22, it is thus equal to $2i\pi \|h\|_{L_w^2(\Sigma)}^2$. Now let $g \in H^2(\Sigma)$ and $\mathbf{u} = \mathbf{u}(g) \in Q$. Relying again on (4.15), it follows

$$
a^+ \left((\mathbf{u}(g),g,0),(\mathbf{u}(g),g,0)\right) \quad = \quad \sum_{j=1,2} \int_{\Gamma_j} -2i\lambda |u_j - w_g^+|^2 \mathrm{d}s.
$$

$\qquad\square$

Proposition 24. *Let ρ, $\mu > 0$. The sesquilinear form a_r^+ defined in (4.10) is T-coercive on K in the sense that there exists a positive constant C such that for all $(\mathbf{u}(g), g, h) \in K$,*

$$
\operatorname{Im} a_r^+ \left((\mathbf{u}(g),g,h), \mathsf{T}(\mathbf{u}(g),g,h)\right) \geq C \|(\mathbf{u}(g),g,h)\|_V,
$$

for $\mathsf{T} : (\mathbf{u},g,h) \in V \mapsto (-\mathbf{u},-g,h) \in V$. In particular, the operator $A_{KK'}^+$ defined in (4.12) is an isomorphism.

Proof. Let ρ, $\mu > 0$, and $(\mathbf{u}(g), g, h) \in K$. Using the definition of form a_r^+ and the fact that a^+ is anti-hermitian, it follows from Proposition 23 that

$$
\operatorname{Im} a_r^+ \left((\mathbf{u}(g),g,h),(-\mathbf{u}(g),-g,h)\right) \quad = \quad 2\pi \|h\|_{L_w^2(\Sigma)}^2 + 2\lambda \sum_{j=1,2} \|u_j(g) - w_g^+\|_{L^2(\Gamma_j)}^2
$$
$$
+ \rho \|g\|_{H^2(\Sigma)}^2 + \mu |h|_{H^2(\Sigma)}^2
$$
$$
\geq \quad C(\|g\|_{H^2(\Sigma)}^2 + \|h\|_{H^2(\Sigma)}^2)
$$

for a $C > 0$. Using Proposition 13 on the control of $\|\mathbf{u}(g)\|_Q$ by $\|g\|_{H^2(\Sigma)}$, we are able to conclude. $\qquad\square$

Proof. [Theorem 16] The hypotheses of Theorem 18 have been verified in Propositions 19 and 24. $\qquad\square$

Remark 25. *An equivalence between $g \mapsto \lambda \|u_j(g) - w_g^+\|_{L^2(\Gamma)}$ and the $L^2(\Sigma)$ norm can be obtained on the kernel of a^+, see Proposition 4.3.2. from [18].*

5   Numerical illustration

The general objectives of this section are to confirm the theoretical analysis by showing numerical results for the approximation of singular solutions of system (4.11) and to show that the mixed variational method developed in this work is compatible with standard finite element solvers, such as Freefem++ [14] in our case. To do so, we construct simple reference analytical solutions with and without a logarithmic singularity and use them for numerical error measurements.

5.1 Construction of analytical solutions

We construct an analytical solution on a simplified model. Dropping out the 0 order term, which is a compact perturbation, in (1.1), one has

$$\begin{cases} -\operatorname{div}(\alpha\nabla u) &= 0 \quad \text{in } \Omega, \\ \alpha\partial_n u + i\lambda u &= f \quad \text{on } \Gamma. \end{cases} \tag{5.1}$$

Without the 0 order term, the line of arguments that led to the definition of ansatz (3.11) still holds.
Let Ω be $(-1,1)\times(-1,1)$ with periodic boundary conditions at $y=\pm 1$, with $\Sigma=\{x=0\}$ and $\alpha=x$. A Fourier decomposition in the y-direction $u(\mathbf{x})=\sum_{k\in\mathbb{Z}}u_k(x)\exp(ik\pi y)$ yields for all modes

$$xu_k'' + u_k' - x(k\pi)^2 u_k = 0 \quad \text{in } (-1,1).$$

The general solution is

$$u_k(x) = \left| \begin{array}{ll} a_k I_0(k\pi x) + b_k K_0(k\pi x) & \forall k \neq 0, \\ a_0 + b_0\left(\log|x| - \dfrac{i\pi}{2}\operatorname{sign}(x)\right) & k = 0, \end{array}\right.$$

where the modified Bessel functions are given by

$$\begin{cases} I_0(x) &= J_0(ix), \\ K_0(x) &= -\dfrac{\pi}{2}\left(iJ_0(ix) + Y_0(-ix)\right), \end{cases}$$

see Section 1.1 for the Bessel functions. We consider four test cases in which respective solutions are

$$u^1(\mathbf{x}) = 1, \quad u^2(\mathbf{x}) = \log|x| - \frac{i\pi}{2}\operatorname{sign}(x), \quad u^3(\mathbf{x}) = e^{i\pi y}I_0(\pi x) \quad \text{and} \quad u^4(\mathbf{x}) = e^{i\pi y}K_0(\pi x). \tag{5.2}$$

The functions u^1 and u^2 are solutions for the $k=0$ mode. For these two functions, one computes easily the $(\mathbf{u}, g, h) \in V$ and $\boldsymbol{\lambda} \in Q$ that is a solution of the variational formulation (4.11). According to the decomposition (3.11), one has

$$\begin{cases} u_j^1 = 1 & \text{and} \quad g^1 = 0, \\ u_j^2 = 0 & \text{and} \quad g^2 = -1, \end{cases} \tag{5.3}$$

and in both cases, $h=g$ and $\lambda_j = u_j\varphi$, where φ is the cut-off function that localizes near Σ. In the illustration below, the cut-off function is $\varphi(x) = \exp\left(4x^2/(4x^2-1)\right)\mathbf{1}_{(-1/2,1/2)}(x)$.
On the other hand, the functions u^3 and u^4 are solutions for the $k=1$ mode. One can compute the corresponding $(\mathbf{u}, g, h) \in V$ and $\boldsymbol{\lambda} \in Q$ as well.
The functions u^1 and u^3 are regular, while the functions u^2 and u^4 exhibit the logarithmic singularity which is the object of this study.
With $\lambda = 1$, the boundary condition source term f in (5.1) is determined accordingly and

$$f^1(\mathbf{x}) = i, \ f^2(\mathbf{x}) = (1+\frac{\pi}{2})\operatorname{sign}(x), \ f^3(\mathbf{x}) = e^{i\pi y}\left(\pi I_0'(\pi x) + iI_0(\pi x)\right), \ f^4(\mathbf{x}) = e^{i\pi y}\left(\pi K_0'(\pi x) + iK_0(\pi x)\right).$$

5.2 Principle of the discretization

The tests were implemented using the Freefem++ [14] code. Freefem++ offers a large choice of bidimensional finite elements, but does not allow so far to discretize a generic bilinear form like $\mathsf{a_h(u_h, v_h)}$ where $\mathsf{u_h}$ belongs to a 1D FE space and $\mathsf{v_h}$ to a 2D FE space. Since we need this feature, we have decided to focus on simple geometry and to use a penalization method to constrain a 2D FE space to unidimensionality. For this reason, the $H^2(\Sigma)$ space is discretized using 2D P3 Hsieh-Clough-Tocher (HCT) elements [8] penalized in the x-direction on a 2D triangular mesh denoted Σ^M. The upper script M stands for the number of triangles that lie on Σ. The more standard $H^1(\Omega_j)$ 2D spaces are discretized using P1 elements on uniform triangular meshes of Ω_j denoted Ω_j^N, where the upper script N stands for the number of edges on each Γ_j. In the presented test cases, the parameters are $M=4$ for the P3 elements and $N=40$ for the P1 elements.
The discretization of (4.11) leads to the linear system

$$\mathsf{AU = L} \tag{5.4}$$

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with $\mathsf{U} = (\underline{u}_1, \underline{u}_2, \underline{g}, \underline{h}, \underline{\lambda}_1, \underline{\lambda}_2)$ the coefficients of the solution in the appropriate FE bases, and where A and L have block matricial structures

$$
\mathsf{A} = \begin{pmatrix}
A_1 & 0 & A_{g1} & A_{h1} & B_1 & 0 \\
* & A_2 & A_{g2} & A_{h2} & 0 & B_2 \\
* & * & A_g & A_{hk} & B_{1k} & B_{2k} \\
* & * & * & A_h & 0 & 0 \\
* & * & * & * & 0 & 0 \\
* & * & * & * & * & 0
\end{pmatrix}, \quad
\mathsf{L} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\underline{\ell}_1 \\
\underline{\ell}_2
\end{pmatrix}. \tag{5.5}
$$

As a consequence of the structure (4.11) and because the sesquilinear form a^+ (4.8) is anti-hermitian, the matrix A is anti-hermitian $\mathsf{A} = -\overline{\mathsf{A}}^t$. Note that the penalization used to achieve unidimensionality is performed in a similar way to (4.10), so that the anti-hermitian nature of the matrix is preserved.

5.3 Numerical results

The numerical solution is obtained by solving the linear system (5.4)-(5.5). With the numerical implementation described above, we observe that the matrices are non-singular and the computations run smoothly.

5.3.1 Numerical errors

In Table 1 we present the relative errors in $L^2(\Omega)$ norms for the four test cases on the total solution u of (5.1). We observe a relative error of order 10^{-2} for all problems even for this coarse mesh. We also observe that the error magnitude is slightly smaller for the case 1 in mode 0 and case 3 in mode 1. Our interpretation is that it is due to the regularity of u^1 and u^3, whereas u^2 and u^4 have logarithmic singularities.

	case 1	case 2	case 3	case 4
$\dfrac{\|u_{ex} - u_{num}\|_{L^2}}{\|u_{ex}\|_{L^2}}$	0.017	0.047	0.008	0.024

Table 1: For the four test cases of (5.2), relative $L^2(\Omega)$ error between the exact solution u_{ex} and its approximation $u_{num} = u_j - w_g^+$.

In Table 2, we present the block residual errors of $\mathsf{AU} - \mathsf{L}$. The block residuals are defined from (5.5) as the residuals for each of the 6 unknowns. It allows a more accurate description of the residual error. We perform the test for the test cases 1 and 2, which means that U takes the two exact values

$$
\mathsf{U}^1 = \begin{pmatrix}
1_N^{\Omega_1} \\
1_N^{\Omega_2} \\
0 \\
0 \\
\varphi_N^{\Omega_1} \\
\varphi_N^{\Omega_2}
\end{pmatrix}
\quad \text{and} \quad
\mathsf{U}^2 = \begin{pmatrix}
0 \\
0 \\
-1_M^{\Sigma} \\
-1_M^{\Sigma} \\
0 \\
0
\end{pmatrix}. \tag{5.6}
$$

In this expression cf. (5.3), $1_N^{\Omega_j}$ and $\varphi_N^{\Omega_j}$ are the coefficients of the P1 interpolations of the functions 1 and φ in $H^1(\Omega_j)$, and 1_M^{Σ} are the coefficients of 1 in the HCT space.
A priori, a residual error is the result of three main contributions which are an interpolation error, a penalization error, and errors due to the approximation of the bilinear forms.
We observe in Table 2 that all block residual errors are close to machine precision, except for the first four blocks in test case 1. After inspection of the structure of A and the nature of the exact solutions (5.6), our interpretation is that when machine precision is reached, the only significant error comes from interpolation errors.

5.3.2 Plot of the numerical solutions in cases 3 and 4

The imaginary part of the numerical approximation of the solutions u^3 and u^4 is shown on the right part of figures 2 and 3. The exact solutions, which are Bessel functions modulated in the direction y, are shown on the left part of the figures. The trace of the 2D FE mesh is also visible, together with

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	norm	case 1	case 2
u_1 block	$L^2(\Omega)$	0.0419775	7.76013e-16
u_2 block	$L^2(\Omega)$	0.0491617	7.14962e-16
g block	$L^2(\Sigma)$	0.0360422	3.86971e-12
h block	$L^2(\Sigma)$	0.0361913	4.37914e-12
λ_1 block	$L^2(\Omega)$	7.04875e-15	7.68808e-16
λ_2 block	$L^2(\Omega)$	7.29193e-15	2.70813e-15

Table 2: Residual errors in L^2 norms for the two first test cases.

the vertical line Σ. In Fig. 2, we observe that the numerical solution on the right is qualitatively and quantitatively very similar to the exact one on the left. The Fig. 3 is of greater interest since the exact solution presents the logarithmic singular behaviour. Qualitatively, the results are very similar and the logarithmic singularity seems to be correctly captured by the numerical solution. Quantitatively, the L^2 norm of the relative error is small, as reported in Table 1, even if a small discrepancy is visible, partly due to a different scaling between both representations.
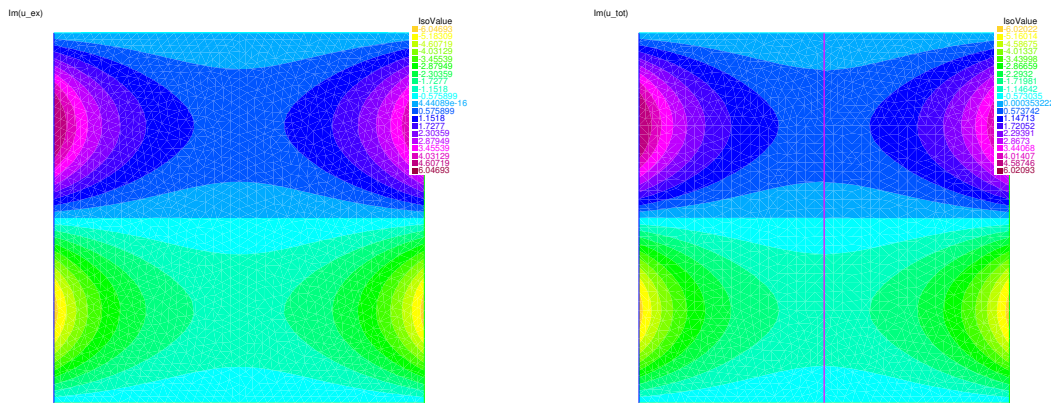

Figure 2: Imaginary parts of the exact solution $u^3(\mathbf{x}) = e^{i\pi y}I_0(\pi x)$ (left) and its approximation (right).
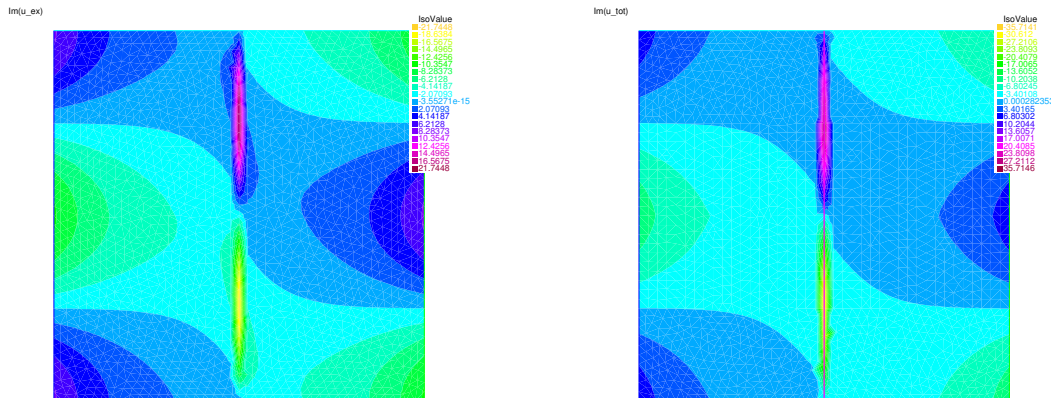

Figure 3: Imaginary parts of the exact solution $u^4(\mathbf{x}) = e^{i\pi y}K_0(\pi x)$ (left) and its approximation (right).

5.3.3 Plot of the numerical solution of the full problem

We now consider the initial equation $-\nabla \cdot (\alpha \nabla u) - u = 0$ from (1.1). For this problem, we do not know any analytical solution, and we stress that our goal is not to observe wave propagation nor absorption but to focus on the interface between these regions. In the matrix A (5.5), only the blocks A_1 and A_2 are modified by adding block diagonal terms corresponding to the discretization of $\int_{\Omega_j} u_j^N \overline{v_j^N}$. The results are displayed in Fig. 4 and can be compared to the results of figures 2 and 3. We observe that the logarithmic singularity behaviour is present in both illustrations in Fig. 4. Little burr can be seen around the logarithmic singularity, however it seems to be related to the size of the first layer around Σ cells and can be expected to resorb as the mesh is refined.
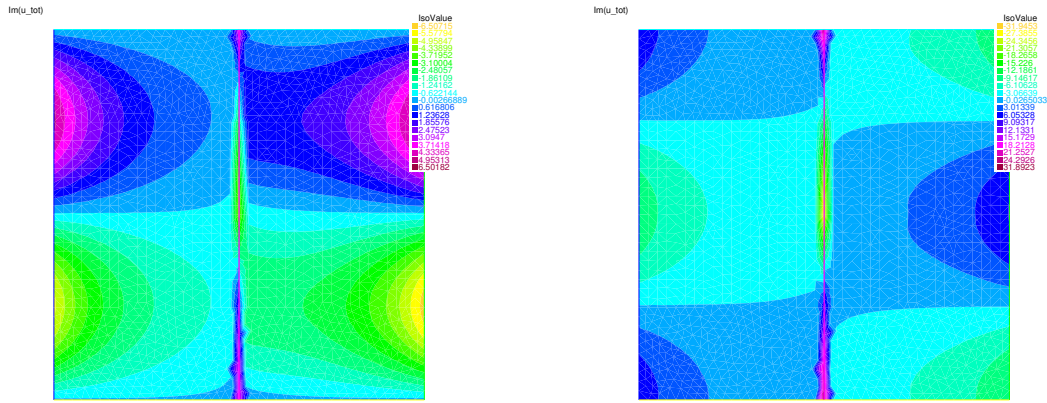
Figure 4: Imaginary parts of the numerical solutions to the complete variational formulation (4.11) with BC f^3 (left) and f^4 (right).

5.3.4   Comments on the value of the penalization parameters

The numerical illutrations presented above have been obtained with small non-zero values for the penalization parameters (4.10). Non-zero values of the penalization parameters are compatible with the theory presented in this work. Arbitrarily, we used the values $\rho = 10^{-2}$ and $\mu = 10^{-4}$. However, other simulations taking these parameters equal to 0 lead to results with similar accuracy.

References

[1] H. Ammari, G. Ciraolo, H. Kang, H. Lee and G. W. Milton, Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance. Arch. Rational Mech. Anal., 218 (2013), 667–692.

[2] F. Assous, P. Ciarlet Jr. and S. Labrunie, Mathematical foundations of computational electromagnetism, Springer-Verlag, 2018.

[3] D. Boffi, F. Brezzi and M. Fortin Mixed finite element methods and applications. Springer-Verlag, Berlin, 2013.

[4] A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., T-coercivity for scalar interface problems between dielectrics and metamaterials. ESAIM: Mathematical Modeling and Numerical Analysis, EDP Sciences, 46 (2012), no. 6, 1363-1387.

[5] A.-S. Bonnet-Ben Dhia, L. Chesnel, X. Claeys, Radiation condition for a non-smooth interface between a dielectric and a metamaterial. M3AS, 23 (2013), no. 9, 1629-1662.

[6] A.-S. Bonnet-Ben Dhia, P. Ciarlet Jr., C.M. Zwölf, Time harmonic wave diffraction problems in materials with sign-shifting coefficients. Journal of Computational and Applied Mathematics, 234 (2010), 1912-1919. Corrigendum 234 (2010), 2616.

[7] M. Campos-Pinto and B. Després, Constructive formulations of resonant Maxwell's equations. SIAM J. Math. Anal. 49 (2017), no. 5, 3637-3670.

[8] P. G. Ciarlet, Interpolation error estimates for the reduced Hsieh-Clough-Tocher triangle. Math. of Comp. 32 (1978), no. 142, 335–344.

[9] B. Després, L.M. Imbert-Gérard and R. Weder, Hybrid resonance of Maxwell's equations in slab geometry. J. Math. Pures Appl., Elsevier, 101 (2014), no. 5, 623-659.

[10] M. P. do Carmo, Differential Geometry of Curves and Surfaces. Prentice-Hall, 1976.

[11] L. C. Evans, Partial Differential Equations. American Mathematical Society, 2010.

[12] G. Di Fazio, M. S. Fanciullo and P. Zamboni, Harnack inequality and smoothness for quasilinear degenerate elliptic equations, Journal of Differential Equations 245 (2008), no. 10, 2939–2957.

[13] E. B. Fabes, C. E. Kenig and R. P. Serapioni, The local regularity of degenerate elliptic equations, Communications in Partial Differential Equations 7 (1982), no. 1, 77–116.

[14] F. Hecht, New development in FreeFem++, J. Numer. Math. 20 (2012), no. 3-4, 251–265.

[15] S. A. Iskhokov, M. G. Gadoev and T. P. Konstantinova, Variational dirichlet problem for degenerate elliptic operators generated by noncoervice forms, Doklady Mathematics 91 (2015), no. 3, 255–258.

[16] L. Lu, K. Crombé, D. Van Eester, L. Colas, J. Jacquot and S. Heuraux, Ion cyclotron wave coupling in the magnetized plasma edge of tokamaks: impact of a finite, inhomogeneous density inside the antenna box. Plasma Physics and Controlled Fusion, 58 (2016), no. 5.

[17] H.-M. Nguyen, Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients. J. Math. Pures Appl., Elsevier, 106 (2016), no. 2, 342-374.

[18] A. Nicolopoulos, Formulations variationnelles d'équations de Maxwell résonantes. PhD Thesis, Sorbonne Université, 2019.

[19] A. Nicolopoulos, M. Campos Pinto and B. Després, A stable formulation of resonant Maxwell's equations in cold plasma. Journal of Computational and Applied Mathematics (2019), https://doi.org/10.1016/j.cam.2019.05.007.

[20] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, NIST Handbook of Mathematical Functions. Cambridge University Press, New York, 2010.

[21] T. H. Stix, Fast-wave heating of a two-component plasma, Technical Report, Plasma Physics Laboratory, Princeton University,1975.

[22] F. Triki and M. Vauthrin, Mathematical modeling of the photoacoustic effect generated by the heating of metallic nanoparticles. Quarterly of Applied Mathematics, 76 (2018), no. 4, 673–698.

[23] B. O. Turesson, Nonlinear Potential Theory and Weighted Sobolev Spaces. Springer-Verlag, Berlin, 2000.