A MATHEMATICAL STUDY OF A HYPERBOLIC METAMATERIAL IN FREE SPACE

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Abstract. Wave propagation in hyperbolic metamaterials is described by the Maxwell equations with a frequency-dependent tensor of dielectric permittivity, whose eigenvalues are of different signs. In this case the problem becomes hyperbolic (Klein-Gordon equation) for a certain range of frequencies. The principal theoretical and numerical difficulty comes from the fact that this hyperbolic equation is posed in a free space, without initial conditions provided. The subject of the work is the theoretical justification of this problem. In particular, this includes the construction of a radiation condition, a well-posedness result, a limiting absorption principle and regularity estimates on the solution.

Key words. Hyperbolic metamaterial, Maxwell equations, Klein-Gordon equation, radiation condition, limiting absorption principle

AMS subject classifications. 35Q60, 35B30, 35L10

1 Introduction and problem setting.

Metamaterials are novel artificial materials [30] which exhibit properties that are important for applications, such as negative refraction and artificial magnetisation. The possibility of their physical realization was predicted in the seminal article by V. Veselago [32]. Typically they are fabricated as periodic structures of metals immersed into dielectrics, and thus electromagnetic wave propagation is modelled with the help of the heterogeneous Maxwell equations. Because the properties of the metamaterials are often revealed in the low-frequency regime, when the wavelength is much larger than the characteristic size of the inclusions, the respective heterogeneous Maxwell equations are further transformed using the homogenization process into homogeneous Maxwell equations with frequency-dependent tensors of dielectric permittivity and magnetic permeability. Numerous works have been devoted to different aspects of the mathematical and numerical analysis of isotropic models, when the dielectric permittivity and magnetic permeability are frequency-dependent scalars [11, 27, 9, 10, 13, 14, 22, 8]. However, up to our knowledge, there exist very few recent articles dedicated to the mathematical analysis of the anisotropic models, especially in the case when the tensors of the dielectric permittivity and/or magnetic permeability are no longer sign definite (so-called hyperbolic metamaterials [29]), with the only exception being the work by E. Bonnetier and H.-M. Nguyen [12]. Let us remark that real materials are always dissipative (which mathematically leads to elliptic models). But, first of all, the dissipation can be small (and much effort is dedicated to its minimization [33, 26, 18]), and, second, the qualitative behaviour of the solutions to the dissipative models approaches the behaviour in models without dissipation. This is especially important for the numerical simulations.

The goal of this work is to perform mathematical analysis of frequency domain wave propagation in the simplest 2D hyperbolic metamaterial, where the frequency-dependent tensor of the dielectric permittivity is diagonal, with eigenvalues of different signs for a range of frequencies, and the magnetic permeability is a positive constant. In this case the respective problem reduces to the Klein-Gordon equation (compare this to the classical case, when the wave propagation is modelled by the Helmholtz

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equation). In this work we are interested in the well-posedness of the respective model in the free space (in particular, existence, uniqueness, limiting absorption principle, regularity of the solution, especially in view of the further numerical analysis applications). The underlying operator is a so-called principal type operator. Some regularity results have been shown by S. Agmon in the classical work [2]. We refine these results to take into account the propagation of singularities along the characteristics. In the context of the limiting absorption principle and the radiation condition, the principal type operators were considered by S. Agmon and L. Hörmander in [4], but, first of all, in our case, the absorption is in the principal symbol of the operator, and, moreover, their proposed radiation condition is provided in the implicit form and does not seem to be suited for the problem we consider.

We present the model under scrutiny in the next section, and provide an outline of the work in Section 1.2.

1.1 The model. One of the simplest models that incorporates distinctive features of the wave propagation phenomena in hyperbolic metamaterials comes from plasma physics and describes wave propagation in a strongly magnetized cold plasma [29]. Mathematically, the corresponding model reduces to the Maxwell's equations supplemented with ODEs. In the case when the electromagnetic field does not depend on the z-coordinate, the model further decouples into the 2D transverse-electric and the transverse-magnetic systems. In this work we will concentrate on the latter system. Its derivation can be found e.g. in [6]; for convenience of the reader, we present it in Appendix A. In the time domain, it reads

$$
\begin{aligned}
&\varepsilon_0\partial_t E_x - \partial_y H_z = 0,\\
&\varepsilon_0\partial_t E_y + \partial_x H_z + j = 0, \qquad \partial_t j - \varepsilon_0\omega_p^2 E_y = 0,\\
&\mu_0\partial_t H_z + \partial_x E_y - \partial_y E_x = 0, \qquad (\mathbf{x},t)\equiv(x,y,t)\in\mathbb{R}^2\times\mathbb{R}.
\end{aligned}
\tag{1.1}
$$

The vector unknown $\mathbf{E} = (E_x, E_y)^T$ is the electric field, the scalar unknown H_z is the magnetic induction, while j plays the role of a current. The coefficients ε_0, μ_0 are the dielectric permittivity and the magnetic permeability of vacuum, and ω_p is the plasma frequency. In what follows we will perform a change of coordinates and rescaling of unknowns in (1.1), chosen so that the coefficients ε_0 and μ_0 disappear from the formulation. This, in particular, implies that the speed of light $c = (\varepsilon_0\mu_0)^{-\frac{1}{2}}$ is rescaled to 1. In these new coordinates (1.1) becomes (where we keep the old notation for simplicity)

$$
\begin{aligned}
&\partial_t E_x - \partial_y H_z = 0,\\
&\partial_t E_y + \partial_x H_z + j = 0, \qquad \partial_t j - \omega_p^2 E_y = 0,\\
&\partial_t H_z + \partial_x E_y - \partial_y E_x = 0, \qquad (\mathbf{x},t)\equiv(x,y,t)\in\mathbb{R}^2\times\mathbb{R}.
\end{aligned}
\tag{1.2}
$$

We denote by $(.,.)$ the L^2-scalar hermitian product, and by $\|.\|$ the respective norm:

$$
(u,v) = \int_{\mathbb{R}^2} u\bar{v}\,d\mathbf{x}, \qquad \|u\| = \left(\int_{\mathbb{R}^2} |u|^2 d\mathbf{x}\right)^{\frac{1}{2}}.
$$

Testing the equations of (1.2) by correspondingly E_x, E_y, $\omega_p^{-2}j$ and H_z, and then summing up the result shows that the energy of (1.2) is conserved:

$$
\frac{d}{dt}\mathcal{E}(t) = 0, \qquad \mathcal{E}(t) := \frac{1}{2}\left(\|E_x(t)\|^2 + \|E_y(t)\|^2 + \|H_z(t)\|^2 + \omega_p^{-2}\|j(t)\|^2\right).
$$

It is thus classical to conclude about the well-posedness and stability of the initial-value problem for (1.2). However the well-posedness of the problem (1.2) in the frequency domain is not as trivial. To see this, let us apply the Fourier-Laplace transform, defined for causal functions of polynomial growth by

$$\hat{u}(\omega) = \int_0^\infty \mathrm{e}^{i\omega t} u(t) dt, \qquad \omega \in \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \tag{1.3}$$

to (1.2). Re-expressing the current $\hat{j}$ via $\hat{E}_x$, we obtain the following system:

$$-i\omega \underline{\underline{\varepsilon}}(\omega)\hat{\mathbf{E}} - \mathbf{curl}\hat{H}_z = 0, \tag{1.4}$$

$$-i\omega \hat{H}_z + \operatorname{curl}\hat{\mathbf{E}} = 0, \tag{1.5}$$

where we denote $\mathbf{curl} = (\partial_y, -\partial_x)^T$, $\operatorname{curl}\boldsymbol{v} = \partial_x v_y - \partial_y v_x$. The 2-by-2 tensor $\underline{\underline{\varepsilon}}(\omega) = \operatorname{diag}(1, \varepsilon(\omega))$ is the relative electric permittivity, with $\varepsilon(\omega)$ defined by

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}. \tag{1.6}$$

As we see, the above model defines a hyperbolic metamaterial [29], since $\varepsilon(\omega) < 0$ for $0 < \omega < \omega_p$. We will simplify it further, by expressing $\hat{\mathbf{E}}$ via $\hat{H}_z$, which results in the following problem for $\hat{H}_z$:

$$\omega^2 \hat{H}_z + \varepsilon(\omega)^{-1}\partial_x^2 \hat{H}_z + \partial_y^2 \hat{H}_z = 0, \qquad (x, y) \in \mathbb{R}^2. \tag{1.7}$$

More generally, we consider the following problem: given f, find u_ω, s.t.

$$\mathcal{L}_\omega u_\omega = f, \qquad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{1.8}$$

where

$$\mathcal{L}_\omega u := \omega^2 u + \varepsilon(\omega)^{-1}\partial_x^2 u + \partial_y^2 u. \tag{1.9}$$

The spaces to which u_ω, f belong will be specified later.

For $0 < \omega < \omega_p$, the above problems reduce to the (hyperbolic) Klein-Gordon equation. Because the theory of hyperbolic problems posed in the free space is much less developed than for elliptic problems, the phenomena of wave propagation governed by (1.2) is not fully understood from the qualitative and quantitative points of view. Our goal is thus to fill some gaps in the mathematical justification of (1.2).

Let us first of all introduce some notations. We define, for $u \in L^1(\mathbb{R}^2)$, s.t. $\hat{u} \in L^1(\mathbb{R}^2)$, its partial and full Fourier transforms:

$$\mathcal{F}_x u(\xi_x, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{i\xi_x x'} u(x', y) dx', \quad \mathcal{F}_y u(x, \xi_y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{i\xi_y y'} u(x, y') dy',$$

$$\mathcal{F}u(\xi_x, \xi_y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{e}^{i\boldsymbol{\xi}\cdot\mathbf{x}} u(x, y) dx dy, \quad \mathcal{F}^{-1}\hat{u}(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{e}^{-i\boldsymbol{\xi}\cdot\mathbf{x}} \hat{u}(\xi_x, \xi_y) d\xi_x \, d\xi_y.$$

At various points of this work, it will be of more convenience to work with weighted Sobolev spaces. In particular, let us define

$$L^2_{s,\perp}(\mathbb{R}^2) \equiv L^2_{s,\perp} := \{v \in L^2_{loc}(\mathbb{R}^2) : \int_{\mathbb{R}^2} (1 + y^2)^s |v(x, y)|^2 dx \, dy < \infty\},$$

with the norm

$$\|v\|^2_{L^2_{s,\perp}} \equiv \|v\|^2_{s,\perp} := \int_{\mathbb{R}^2} (1+y^2)^s |v(x,y)|^2 dx\, dy.$$

The corresponding Sobolev spaces $H^{\mu}_{s,\perp}$ are then defined with the help of the Bessel-like potential

$$\mathcal{J}_\mu v = \mathcal{F}^{-1}\left((1+|\xi_x|^\mu + |\xi_y|^\mu)\, \mathcal{F}v(\xi_x,\xi_y) \right), \quad \mu \in \mathbb{R}^+,$$

namely

$$H^{\mu}_{s,\perp}(\mathbb{R}^2) \equiv H^{\mu}_{s,\perp} := \{v \in L^2_{s,\perp}(\mathbb{R}^2) : \mathcal{J}_\mu v \in L^2_{s,\perp}(\mathbb{R}^2)\}, \quad \|v\|^2_{s,\perp} = \|\mathcal{J}_\mu v\|^2_{s,\perp}.$$

It will be useful to work with the partial $x-$directed Fourier transforms of functions on the above spaces. Remark that for any $v \in L^2_{s,\perp}(\mathbb{R}^2)$, $v(.,y) \in L^2(\mathbb{R})$, a.e. in $y \in \mathbb{R}$. Therefore, equivalent norms on $L^2_{s,\perp}(\mathbb{R}^2), H^1_{s,\perp}(\mathbb{R}^2)$ can be rewritten using the Plancherel theorem in the following form:

$$\|v\|^2_{s,\perp} = \|\mathcal{F}_x v\|^2_{s,\perp} = \int_{\mathbb{R}^2} (1+y^2)^s |\mathcal{F}_x v(\xi_x,y)|^2 d\xi_x dy, \tag{1.10}$$

$$\begin{aligned} \|v\|^2_{H^1_{s,\perp}} &= \int_{\mathbb{R}^2} (1+y^2)^s (1+\xi_x^2) |\mathcal{F}_x v(\xi_x,y)|^2 d\xi_x dy \\ &+ \int_{\mathbb{R}^2} (1+y^2)^s |\partial_y \mathcal{F}_x v(\xi_x,y)|^2 d\xi_x dy. \end{aligned} \tag{1.11}$$

We will use the notation $a \lesssim b$ (resp. $a \gtrsim b$) to indicate that there exists $C>0$ that may depend on ω_p and ω, s.t. $a \leq Cb$ (resp. $a \geq Cb$).

1.2 Outline. The rest of the article is organized as follows. Section 2 is dedicated to the well-posedness and regularity results related to the problem (1.7) in the *hyperbolic regime*, that is for $0 < \omega < \omega_p$. Section 3 is dedicated to the in-depth analysis of the regularity of the solution to (1.7). We demonstrate the optimality of the regularity estimates of Section 2 in the framework of Sobolev spaces, and show how the respective results can be improved when considering spaces adjusted to the way singularities propagate in (1.7). Section 4 is dedicated to the proof of the limiting absorption principle for $0 < \omega < \omega_p$.

2 Well-posedness of (1.8) in the hyperbolic regime.

This section is organized as follows:

- in Section 2.1 we show that (1.8) is well-posed in $L^2(\mathbb{R}^2)$ when $\omega \in \mathbb{C} \setminus \mathbb{R}$;
- in Section 2.2 we prove the existence of the solution to (1.8) by a limiting absorption principle;
- in Section 2.4 we derive the radiation condition;
- Section 2.5 is dedicated to the statement of the main result of this section.

REMARK 1. *Evidently, when $\omega \in \mathbb{R}$, it suffices to consider the well-posedness of the problem for $\omega \geq 0$. We are interested in the case when $\omega \in [0, \omega_p]$, since for $\omega \in \mathbb{R} \setminus [0, \omega_p]$, the model reduces to the Helmholtz equation. In the limiting case $\omega = \omega_p$, it can be shown that the limiting absorption principle holds for the Maxwell's equations (1.4), and the resulting solution vanishes for a sufficiently regular right-hand side. On the other hand, for $\omega = 0$, the application of the limiting absorption to (1.4) yields a non-vanishing solution. More details can be found in [21].*

2.1 Well-posedness for complex frequencies. Let us define the sesquilinear form associated to (1.8):

$$a_\omega(.,.): H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\to\mathbb{C},$$
$$a_\omega(u,v)=\omega^2(u,v)-\varepsilon(\omega)^{-1}(\partial_x u,\partial_x v)-(\partial_y u,\partial_y v).$$

It is possible to show that, whenever $\omega\in\mathbb{C}\setminus\mathbb{R}$, the above form is coercive on $H^1(\mathbb{R}^2)$, thanks to non-vanishing $\operatorname{Im}(\omega\varepsilon(\omega))\neq 0$. This result is summarized in the following lemma, which follows from the proof of Proposition 3.12 and Theorem 5.4 of [7].

LEMMA 2.1. *For all $\omega\in\mathbb{C}\setminus\mathbb{R}$, $\omega=\omega_r+i\omega_i$, $\omega_r,\,\omega_i\in\mathbb{R}$, it holds*

$$|a_\omega(u,v)|\lesssim|\omega|^2\max(1,\omega_i^{-2})\|u\|_{H^1}\|v\|_{H^1},$$
$$|\operatorname{Im}a_\omega(u,\omega u)|\gtrsim|\omega_i|\min(\omega_i^2,1)\|u\|^2_{H^1}.$$

Thus, for all $f\in H^{-1}(\mathbb{R}^2)$, there exists a unique $u_\omega\in H^1(\mathbb{R}^2)$ that satisfies (1.8). Moreover, $\|u_\omega\|_{H^1}\lesssim|\omega_i|^{-1}\max(\omega_i^{-2},1)|\omega|\|f\|_{H^{-1}}$.

We leave the proof of the above result to the reader. The unique solution to (1.8) is given by the convolution of the source f with the fundamental solution $\mathcal{G}_\omega$:

$$u_\omega=\mathcal{N}_\omega f:=\mathcal{G}_\omega * f=\int_{\mathbb{R}^2}\mathcal{G}_\omega(\cdot-\mathbf{x}')f(\mathbf{x}')d\mathbf{x}'. \tag{2.1}$$

A derivation of an explicit form of $\mathcal{G}_\omega$, $\omega\in\mathbb{C}\setminus\mathbb{R}$, is given in Appendix B. Before presenting it, let us make the following remark.

REMARK 2. *All over the article, we use the following convention: for a complex number $z\in\mathbb{C}$, $\sqrt{z}$ denotes the principal branch of the square root, i.e.* $\operatorname{Re}\sqrt{z}>0$ *for all $z\in\mathbb{C}\setminus(-\infty,0]$; respectively,* $\log z=\log|z|+i\operatorname{Arg}z$, $\operatorname{Arg}z\in(-\pi,\pi)$.

Then the fundamental solution for (1.8) is given by

$$\mathcal{G}_\omega(\mathbf{x})=\frac{-i\sqrt{\varepsilon(\omega)}}{4}\begin{cases}H_0^{(1)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2}), & \operatorname{Re}\omega>0,\,\operatorname{Im}\omega>0,\\[2ex] H_0^{(2)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2}), & \operatorname{Re}\omega>0,\,\operatorname{Im}\omega<0,\end{cases} \tag{2.2}$$

where $H_0^{(1)},H_0^{(2)}$ are Hankel functions of the first and second kind.

2.2 Existence of solutions Because the solution to (1.8) is well-defined when $\omega\in\mathbb{C}\setminus\mathbb{R}$, to prove the existence, for now we will make use of the limiting absorption principle in a pointwise topology. A justification of the limiting absorption principle in an H^1_{loc}-topology will be given in Section 4.

We proceed as follows. For $\omega\in(0,\omega_p)$, we define the pointwise limit

$$\mathcal{G}^+_\omega(\mathbf{x}):=\lim_{\delta\to 0+}\mathcal{G}_{\omega+i\delta}(\mathbf{x}),\quad \mathbf{x}\in\mathbb{R}^2, \tag{2.3}$$

and, correspondingly $u^+_\omega:=\mathcal{G}^+_\omega * f$, with a sufficiently smooth data f. We then prove that u^+_ω solves (1.8).

Similarly, let $\mathcal{G}^-_\omega(\mathbf{x}):=\lim_{\delta\to 0+}\mathcal{G}_{\omega-i\delta}$, (it holds that $\mathcal{G}^-_\omega\neq\mathcal{G}^+_\omega$). The corresponding solution u^-_ω also solves (1.8). We will refer to the solution u^+_ω as to the outgoing solution, and u^-_ω as to the incoming one (in analogy with the Helmholtz equation). We will concentrate on the construction of the outgoing solutions.

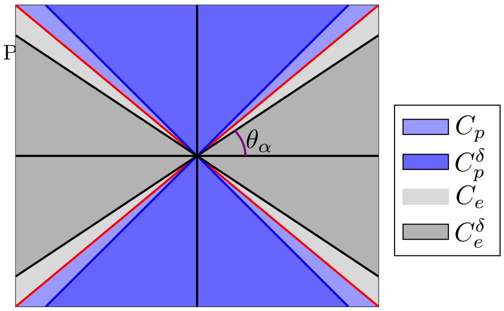

FIG. 1. *The domains $\mathcal{C}_p^\delta$, $\mathcal{C}_e^\delta$, $\mathcal{C}_p$, $\mathcal{C}_e$, with $\theta_\alpha = \operatorname{atan}\alpha^{-1}$.*

2.2.1 The outgoing fundamental solution and its properties. Let us fix $\omega \in (0, \omega_p)$ and introduce the following notation (recall that $\varepsilon(\omega) < 0$)

$$\alpha := (-\varepsilon(\omega))^{-\frac{1}{2}} > 0. \tag{2.4}$$

With this notation, (1.8) becomes

$$\omega^2 u - \alpha^2 \partial_x^2 u + \partial_y^2 u = f \qquad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{2.5}$$

and the outgoing fundamental solution (2.3) reads

$$\mathcal{G}_\omega^+(x,y) = \frac{1}{4\alpha} \begin{cases} H_0^{(1)}(\omega\sqrt{y^2 - \alpha^{-2}x^2}), & (x,y) \in \mathcal{C}_p, \\ \\ H_0^{(1)}(i\omega\sqrt{\alpha^{-2}x^2 - y^2}), & (x,y) \in \mathcal{C}_e, \end{cases} \tag{FS}$$

where

$$\begin{cases} \mathcal{C}_p = \{(x,y) \in \mathbb{R}^2 \setminus \{0\} : |y| > \alpha^{-1}|x|\}, \\ \mathcal{C}_e = \{(x,y) \in \mathbb{R}^2 \setminus \{0\} : |y| < \alpha^{-1}|x|\}. \end{cases} \tag{C}$$

The notations $\mathcal{C}_p$, $\mathcal{C}_e$ will be clarified later, in Lemma 2.2.

It is well-known that the fundamental solution for the initial-value problems for hyperbolic operators is causal and vanishes outside of the space-time cone, see e.g. [20, Chapter XII, Theorems 12.5.4, 12.5.1]. This latter property reflects the finite velocity of the wave propagation. The fundamental solution $\mathcal{G}_\omega^+$ possesses none of these features. This is one of the corollaries of Lemma 2.2, which we state in polar coordinates (r, ϕ): $x = r\cos\phi$, $y = r\sin\phi$. Let us introduce some auxiliary notations. Let $\gamma_\phi = \tan^2\phi - \alpha^{-2} \in \overline{\mathbb{R}}$. With this definition,

$$\mathcal{C}_p = \{(r,\phi) : \gamma_\phi > 0\}, \quad \mathcal{C}_e = \{(r,\phi) : \gamma_\phi < 0\}.$$

Let us also define, for all δ s.t. $0 < \delta < \alpha^{-2}$,

$$\mathcal{C}_p^\delta = \{(r,\phi) : \gamma_\phi > \delta\}, \quad \mathcal{C}_e^\delta = \{(r,\phi) : \gamma_\phi < -\delta\},$$

see Figure 1 for illustration. We then have the following result.

LEMMA 2.2 (Asymptotics of $\mathcal{G}_\omega^+$ at infinity). *Let $0 < \delta < \alpha^{-2}$. Then*

- *inside $\mathcal{C}_p^\delta$, as $r \to +\infty$,*

$$\mathcal{G}_\omega^+(r\cos\phi, r\sin\phi) = \frac{\mathrm{e}^{-i\frac{\pi}{4}}}{2\alpha\sqrt{2\pi\omega}} r^{-\frac{1}{2}} (\gamma_\phi \cos^2\phi)^{-\frac{1}{4}} \mathrm{e}^{i\omega r\sqrt{\gamma_\phi\cos^2\phi}} (1 + o(1)).$$

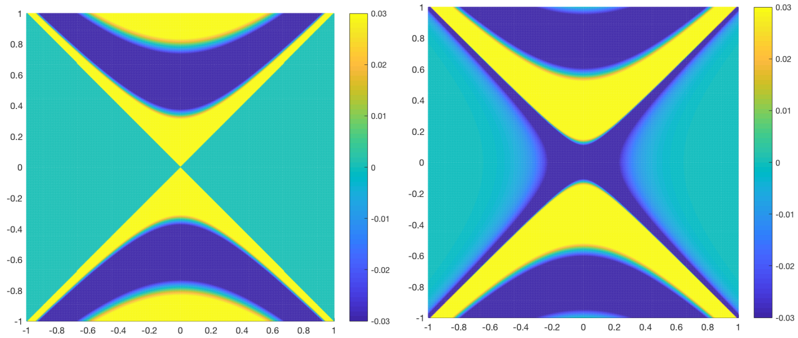

FIG. 2. *The real (left) and imaginary (right) parts of the fundamental solution $\mathcal{G}_\omega^+(\mathbf{x})$, with $\omega_p = 10$ and $\omega = 7.05$ (chosen so that $\varepsilon(\omega) \approx -1$).*

- *inside $\mathcal{C}_e^\delta$, as $r \to +\infty$,*

$$\mathcal{G}_\omega^+(r\cos\phi, r\sin\phi) = -\frac{i}{2\alpha\sqrt{2\pi\omega}} r^{-\frac{1}{2}} (-\gamma_\phi \cos^2\phi)^{-\frac{1}{4}} \mathrm{e}^{-\omega r\sqrt{-\gamma_\phi\cos^2\phi}} (1 + o(1)) .$$

The error terms in the asymptotic expansions depend on δ.

Proof. The proof is based on the following asymptotic expansion from [28, pp. 266-267]. Let $z \in \mathbb{C}$ be s.t. $0 \leq \operatorname{Arg} z \leq \frac{\pi}{2}$. Then, as $|z| \to +\infty$,

$$H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \mathrm{e}^{iz - i\frac{\pi}{4}} (1 + \eta(z)), \qquad |\eta(z)| \lesssim |z|^{-1}, \quad C > 0. \tag{2.6}$$

It remains to apply the above to $\mathcal{G}_\omega^+(\mathbf{x})$, with

$$z = \omega r\sqrt{\gamma_\phi \cos^2\phi}, \text{ in } \mathcal{C}_p^\delta, \qquad \text{and } z = i\omega r\sqrt{-\gamma_\phi\cos^2\phi}, \text{ in } \mathcal{C}_e^\delta.$$

The only statement that needs to be proven is that $\eta(z) = o(1)$, as $r \to +\infty$. From the expression for η (2.6), this amounts to showing that $\sqrt{\gamma_\phi\cos^2\phi}$ (resp. $\sqrt{-\gamma_\phi\cos^2\phi}$) is uniformly bounded from below away from zero when $(r, \phi) \in \mathcal{C}_p^\delta$ (resp. $\mathcal{C}_e^\delta$).

Let us consider the case $\mathcal{C}_p^\delta$. By evenness and periodicity, it suffices to study the case $\phi \in \left(\operatorname{atan}\sqrt{\alpha^{-2} + \delta}, \frac{\pi}{2}\right]$. The function $\phi \mapsto \gamma_\phi \cos^2\phi \equiv \sin^2\phi - \alpha^{-2}\cos^2\phi$ is non-negative and strictly monotonically increasing on $\left(\operatorname{atan}\alpha^{-1}, \frac{\pi}{2}\right]$; hence $\gamma_\phi\cos^2\phi \geq c_\delta > 0$, with $c_\delta > 0$, for all $(r, \phi) \in \mathcal{C}_p^\delta$.

The case $\mathcal{C}_e^\delta$ can be studied similarly. ∎

The above lemma justifies the notation $\mathcal{C}_p$ and $\mathcal{C}_e$: inside $\mathcal{C}_p$, the fundamental solution oscillates and decays at best as $O(r^{-\frac{1}{2}})$ (thus the index 'p' stands for 'propagative'), while inside $\mathcal{C}_e$, it decays exponentially fast (thus 'e' stands for 'evanescent').

An illustration to this result is shown in Figure 2.

2.2.2 Existence of classical solutions to (1.8). We start with proving the existence of classical solutions to (1.8). The results of this section will serve as a basis to prove the existence of the weak solutions.

THEOREM 2.3 (Existence of classical solutions to (1.8)). *Let $\omega \in (0, \omega_p)$ and $f \in C_0^2(\mathbb{R}^2)$. Then $u_\omega^+ = \mathcal{G}_\omega^+ * f \in C^2(\mathbb{R}^2)$ and satisfies (1.8) in a strong sense.*

The proof of this theorem relies on the following auxiliary proposition.

PROPOSITION 2.4. *Let $0<\omega<\omega_p$. Then*

1. $\mathcal{G}_{\omega+i\delta}\in L^1_{loc}(\mathbb{R}^2)$ *for all* $\delta>0$.
2. $\lim\limits_{\delta\to 0+}\mathcal{G}_{\omega+i\delta}=\mathcal{G}^+_\omega$ *in* $L^1_{loc}(\mathbb{R}^2)$.

Proof. **Proof of the statement 1.** To understand the behaviour of $\mathcal{G}_{\omega+i\delta}$, let us make use of the following expression for $H_0^{(1)}(z)$ stemming from [1, §9.1.3, §9.1.13]:

$$
\begin{aligned}
&H_0^{(1)}(z)=J_0(z)+iY_0(z),\\
&J_0(z)=1+g_J(z^2),\quad Y_0(z)=\frac{2}{\pi}J_0(z)\log\frac{z}{2}+g_Y(z^2),
\end{aligned}
\tag{2.7}
$$

where g_J, g_Y are entire[1] functions; moreover, $g_J(0)=0$, $g_J'(0)\neq 0$.
With $z_\delta=(\omega+i\delta)^2(\varepsilon(\omega+i\delta)x^2+y^2)$ and (2.7), we get

$$
\mathcal{G}_{\omega+i\delta}(\mathbf{x})=\mathcal{G}^{reg}_{\omega+i\delta}(\mathbf{x})+\frac{\sqrt{\varepsilon(\omega+i\delta)}}{2\pi}\log\sqrt{z_\delta},\qquad\text{where}
\tag{2.8}
$$

$$
\mathcal{G}^{reg}_{\omega+i\delta}=-i\frac{\sqrt{\varepsilon(\omega+i\delta)}}{4}\left(1-\frac{2i}{\pi}\log 2+g_J(z_\delta)\left(1+\frac{2i}{\pi}\log\frac{\sqrt{z_\delta}}{2}\right)+ig_Y(z_\delta)\right).
$$

The fact that $\mathcal{G}_{\omega+i\delta}\in L^1_{loc}(\mathbb{R}^2)$ follows from the above: indeed, as $z_\delta\neq 0$ on $\mathbb{R}^2\setminus\{0\}$, $\mathcal{G}_{\omega+i\delta}$ is continuous on $\mathbb{R}^2\setminus\{0\}$, and its only singularity is the logarithmic (thus, integrable) singularity in the origin.
Proof of the statement 2. See Appendix D. ∎

With the above result, the proof of Theorem 2.3 is almost immediate.

Proof of Theorem 2.3. Let us fix $\omega\in(0,\omega_p)$, $\delta>0$. Let $u_{\omega+i\delta}=\mathcal{G}_{\omega+i\delta}*f$. Because $f\in C^2(\mathbb{R}^2)$, by Proposition 2.4, Statement 1, $u_{\omega+i\delta}\in C^2(\mathbb{R}^2)$. It satisfies, cf. Section 2.1, in the strong sense: $\mathcal{L}_{\omega+i\delta}u_{\omega+i\delta}=f$. Proving that $\mathcal{L}_\omega u^+_\omega=f$ amounts to proving that the following holds in the topology of pointwise convergence:

$$
|\mathcal{L}_{\omega+i\delta}u_{\omega+i\delta}-\mathcal{L}_\omega u^+_\omega|\to 0,\qquad\text{as }\delta\to 0.
\tag{2.9}
$$

The above rewrites as

$$
\begin{aligned}
\mathcal{L}_{\omega+i\delta}u_{\omega+i\delta}-\mathcal{L}_\omega u^+_\omega&=\mathcal{L}_{\omega+i\delta}\mathcal{G}_{\omega+i\delta}*f-\mathcal{L}_\omega\mathcal{G}^+_\omega*f=\mathcal{G}_{\omega+i\delta}*\mathcal{L}_{\omega+i\delta}f-\mathcal{G}^+_\omega*\mathcal{L}_\omega f\\
&=(\mathcal{G}_{\omega+i\delta}-\mathcal{G}^+_\omega)*\mathcal{L}_{\omega+i\delta}f-\mathcal{G}^+_\omega*(\mathcal{L}_\omega-\mathcal{L}_{\omega+i\delta})f.
\end{aligned}
$$

Let us assume that $\operatorname{supp} f\subset B_R(0)$, $R>0$. Then the above yields

$$
\begin{aligned}
|\left(\mathcal{L}_{\omega+i\delta}u_{\omega+i\delta}-\mathcal{L}_\omega u^+_\omega\right)(\mathbf{x})|&\leq\|\mathcal{L}_{\omega+i\delta}f\|_{L_\infty(\mathcal{B}_R(0))}\|\left(\mathcal{G}_{\omega+i\delta}-\mathcal{G}^+_\omega\right)(\mathbf{x}-.)\|_{L^1(\mathcal{B}_R(0))}\\
&+\|\mathcal{L}_{\omega+i\delta}f-\mathcal{L}_\omega f\|_{L_\infty(\mathcal{B}_R(0))}\|\mathcal{G}^+_\omega(\mathbf{x}-.)\|_{L^1(\mathcal{B}_R(0))}.
\end{aligned}
$$

The analyticity of the coefficients of $\mathcal{L}_\omega$, and Proposition 2.4 yield (2.9). This shows that u^+_ω satisfies (1.8) in a strong sense. The fact that $u^+_\omega\in C^2(\mathbb{R}^2)$ follows from $\mathcal{G}^+_\omega\in L^1_{loc}(\mathbb{R}^2)$, cf. Proposition 2.4, Statement 2, and $f\in C^2_0(\mathbb{R}^2)$. ∎

[1] The fact that the series in [1, §9.1.10, §9.1.13] define entire functions can be validated by studying their radius of convergence

2.2.3 Existence and regularity of weak solutions. Let us extend the statement of Theorem 2.3 to more general data, as well as quantify the behavior of u_ω^+ at infinity. This will be of importance, in particular, when constructing an appropriate radiation condition. All over this section we assume that $0 < \omega < \omega_p$.

We start by defining the domain and the range of the solution operator, defined for $f \in C_0^\infty(\mathbb{R}^2)$ as the following Lebesgue integral:

$$(\mathcal{N}_\omega^+ f)(\mathbf{x}) := (\mathcal{G}_\omega^+ * f)(\mathbf{x}) = \int_{\mathbb{R}^2} \mathcal{G}_\omega^+(\mathbf{x}')f(\mathbf{x}-\mathbf{x}')d\mathbf{x}'. \tag{2.10}$$

For this we will use an appropriate Sobolev space framework. To do so, let us motivate the definitions that follow by describing an asymptotic behaviour of $\mathcal{N}_\omega^+ f$.

2.2.3.1 Behaviour of $\mathcal{N}_\omega^+ f$ at infinity. The asymptotic expansions of Lemma 2.2 yield $\mathcal{G}_\omega^+ \notin L^2(\mathbb{R}^2)$. However, this lack of decay at infinity concerns only one coordinate direction, namely y; it is possible to show that for fixed $y \in \mathbb{R}$, $\mathcal{G}_\omega^+(x, y)$ decays exponentially fast in x, see the result below.

LEMMA 2.5 (Decay in x-direction). *For all $\delta > 0$, there exists $C_{\alpha,\delta} > 0$, s.t. for all $(x, y) \in \mathbb{R}^2$ with $|x| > \alpha|y| + \delta$, $|\mathcal{G}_\omega^+(x,y)| \leq C_{\alpha,\delta} \mathrm{e}^{-\omega\sqrt{\alpha^{-2}x^2-y^2}}$.*

Proof. See Appendix E. ∎

For a fixed $x > 0$, as $y \to +\infty$, as seen from Lemma 2.2,

$$\left|\mathcal{G}_\omega^+(x,y)\right| = \frac{C}{(y^2-\alpha^2x^2)^{\frac{1}{4}}} + o(|y|^{-\frac{1}{2}}), \quad C > 0. \tag{2.11}$$

From Lemma 2.5 and (2.11) we can expect that, for $f \in C_0^\infty(\mathbb{R}^2)$, $\mathcal{N}_\omega^+ f(x, y)$ decays exponentially fast in the direction x and at most as $O(|y|^{-\frac{1}{2}})$ in the y-direction.

2.2.3.2 Definition of $\mathcal{N}_\omega^+$. The main result of this section provides the extension by density of the operator $\mathcal{N}_\omega^+$.

PROPOSITION 2.6. *Let $s, s' > \frac{1}{2}$. The operator $\mathcal{N}_\omega^+$ defined in (2.10) can be extended by density to a bounded linear operator $\mathcal{N}_\omega^+ : L^2_{s,\perp} \to H^1_{-s',\perp}$.*

Before proving the above proposition, let us recall several useful facts. First, the partial Fourier transform of $\mathcal{G}_\omega^+$ is given by, see Appendix C,

$$\left(\mathcal{F}_x\mathcal{G}_\omega^+(x,y)\right)(\xi_x, y) = \frac{\mathrm{e}^{i\kappa(\xi_x,\omega)|y|}}{2i\sqrt{2\pi}\kappa(\xi_x,\omega)}, \text{ with} \tag{2.12}$$

$$\kappa(\xi_x, \omega) = \sqrt{\alpha^2\xi_x^2 + \omega^2} > 0. \tag{2.13}$$

In particular, it holds that

$$\mathcal{F}_x u_\omega^+ = \mathcal{F}_x\left(\mathcal{N}_\omega^+ f\right)(\xi_x, y) = \int_{\mathbb{R}} \frac{\mathrm{e}^{i\kappa(\xi_x,\omega)|y-y'|}}{2i\sqrt{2\pi}\kappa(\xi_x,\omega)} \mathcal{F}_x f(\xi_x, y')dy'. \tag{2.14}$$

REMARK 3. *The motivation to work with the Fourier transform comes from the following observation: a formal application of $\mathcal{F}_x$ to (1.8) results in the 1D Helmholtz equation for almost all Fourier variables $\xi_x \in \mathbb{R}$:*

$$\left(\omega^2 + \xi_x^2\alpha^2\right)\mathcal{F}_x u_\omega(\xi_x, y) + \partial_y^2 \mathcal{F}_x u_\omega(\xi_x, y) = \mathcal{F}_x f(\xi_x, y) \qquad \text{in } \mathcal{D}'(\mathbb{R}). \tag{2.15}$$

Thus, H^{ℓ}-bounds for the solution of (1.8) can be obtained by considering the dependence on the frequency of the bounds on the solution to the 1D Helmholtz equation.

In particular, from the definition of $\kappa(\xi_x, \omega)$ (2.13), it follows that

$$\frac{1}{2}\left(\alpha|\xi_x| + \omega\right) \le \kappa(\xi_x, \omega) = \sqrt{\alpha^2 \xi_x^2 + \omega^2} \le \alpha|\xi_x| + \omega.$$

Therefore, by (1.10), (1.11), an equivalent norm in $H^1_{p,\perp}$ is given by

$$\|v\|^2_{H^1_{p,\perp}} \sim \|\kappa(\xi_x, \omega)\mathcal{F}_x v\|^2_{L^2_{p,\perp}} + \|\partial_y \mathcal{F}_x v\|^2_{L^2_{p,\perp}}. \tag{2.16}$$

The constants in norm-equivalence inequalities depend on ω only.

Proof of Proposition 2.6. Let $s, s' > \frac{1}{2}$ be fixed. To prove the statement, it suffices to show that there exists $C_{s,s'} > 0$, s.t. for any $\phi \in C_0^\infty(\mathbb{R}^2)$,

$$\|\mathcal{N}^+_\omega \phi\|_{H^1_{-s',\perp}} \le C_{s,s'} \|\phi\|_{L^2_{s,\perp}}. \tag{2.17}$$

We will use the equivalent norm (2.16) in the derivation of the above bound. For this let us remark that, cf. (2.14) and (2.12),

$$\kappa(\xi_x, \omega)\mathcal{F}_x \mathcal{N}^+_\omega \phi(\xi_x, y) = \frac{1}{2i\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{i\kappa(\xi_x,\omega)|y-y'|} \mathcal{F}_x \phi(\xi_x, y') dy',$$

$$\partial_y \mathcal{F}_x \mathcal{N}^+_\omega \phi(\xi_x, y) = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{i\kappa(\xi_x,\omega)|y-y'|} \,\mathrm{sign}(y - y') \mathcal{F}_x \phi(\xi_x, y') dy'.$$

Therefore, with (2.16), using $|\mathrm{e}^{i\kappa(\xi_x,\omega)|y-y'|}| = 1$, and defining

$$v(\xi_x, y) := \int_{\mathbb{R}} |\mathcal{F}_x \phi(\xi_x, y')| \, dy', \tag{2.18}$$

we have

$$\|\mathcal{N}^+_\omega \phi\|^2_{H^1_{-s',\perp}} \lesssim \|v\|^2_{L^2_{-s',\perp}}. \tag{2.19}$$

To bound the right hand side of (2.19), we start with the following L^∞-bound. An application of the Cauchy-Schwarz inequality yields: for all $(\xi_x, y) \in \mathbb{R}^2$,

$$\begin{aligned} |v(\xi_x, y)|^{2} &\le \int_{\mathbb{R}} (1 + y'^2)^{-s} dy' \int_{\mathbb{R}} (1 + y'^2)^s |\mathcal{F}_x \phi(\xi_x, y')|^2 dy' \\ &= c_s \int_{\mathbb{R}} (1 + y'^2)^s |\mathcal{F}_x \phi(\xi_x, y')|^2 dy', \quad c_s = \int_{\mathbb{R}} (1 + y'^2)^{-s} dy' < \infty, \end{aligned} \tag{2.20}$$

where we used $s > \frac{1}{2}$. The above bound implies, with $c_{s'}$ defined like above,

$$\begin{aligned} \|v\|^2_{L^2_{-s',\perp}} &\le c_s \int_{\mathbb{R}^2} (1 + y^2)^{-s'} \left(\int_{\mathbb{R}} (1 + y'^2)^s |\mathcal{F}_x \phi(\xi_x, y')|^2 dy' \right) dy d\xi_x \\ &= c_s c_{s'} \|\mathcal{F}_x \phi\|^2_{L^2_{s,\perp}} \overset{(1.10)}{=} c_s c_{s'} \|\phi\|^2_{L^2_{s,\perp}}. \end{aligned}$$

In the above $c_{s'}$ is finite because $s' > \frac{1}{2}$. Inserting the above bound into (2.19), cf. (2.18), yields $\|\mathcal{N}^+_\omega \phi\|_{H^1_{-s',\perp}} \le C_{s,s'} \|\phi\|_{L^2_{s,\perp}}$, i.e. (2.17). ∎

2.3 On the optimality of Proposition 2.6. The regularity result of Proposition 2.6 is not surprising, and had been shown for the so-called operators of the principal type (modulo the weights in the weighted spaces) by Agmon in [3, Appendix A]. Let us show that the result of Proposition 2.6 is in some sense optimal. For this we will need the following observation about the norm in $H^{\mu}_{s,\perp}$ space. By the Plancherel identity, $\|v\|^2_{H^{\mu}_{s,\perp}}$ can be expressed as follows:

$$\|v\|^2_{H^{\mu}_{s,\perp}} = \int\limits_{\mathbb{R}^2} (1+y^2)^s \left(|\mathcal{F}_x v|^2 \left(1+|\xi_x|^{2\mu}\right) + |\mathcal{F}_y^{-1}\left(|\xi_y|^{\mu}\mathcal{F}_y\mathcal{F}_x v\right)|^2\right) d\xi_x dy. \tag{2.21}$$

We then have the following result.

PROPOSITION 2.7. *Let $s, s' > \frac{1}{2}$. Then $\mathcal{N}^+_\omega \in \mathcal{B}\left(L^2_{s,\perp},\, H^{1+\sigma}_{-s',\perp}\right)$ iff $\sigma \le 0$.*

Proof. By Proposition 2.6, we know already that $\mathcal{N}^+_\omega \in \mathcal{B}\left(L^2_{s,\perp}, H^{1+\sigma}_{-s',\perp}\right)$ for $\sigma \le 0$. It thus remains to show that $\mathcal{N}^+_\omega \notin \mathcal{B}\left(L^2_{s,\perp}, H^{1+\sigma}_{-s',\perp}\right)$ for all $\sigma > 0$.

Let $s, s' > \frac{1}{2}$ be fixed. We will prove the result by showing that for every $\sigma > 0$, there exists $\phi \in L^2_{s,\perp}$ (that depends on σ), such that $v = \mathcal{N}^+_\omega \phi \notin H^{1+\sigma}_{-s,\perp}$.

Let us take $\phi \in L^2(\mathbb{R}^2)$, s.t. for all $x \in \mathbb{R}$, $\operatorname{supp}\phi(x,.) \subseteq [-a,a]$, for some $a > 0$. This in particular guarantees that $\phi \in L^2_{s,\perp}(\mathbb{R}^2)$ for any s. For $y < -a$, cf. (2.14),

$$\mathcal{F}_x v(\xi_x, y) = \frac{i\mathrm{e}^{-i\kappa(\xi_x,\omega)y}}{2\sqrt{2\pi}\kappa(\xi_x,\omega)} \int\limits_{-a}^{a} \mathrm{e}^{i\kappa(\xi_x,\omega)y'}\mathcal{F}_x\phi(\xi_x,y')dy'.$$

Since for all $\xi_x \in \mathbb{R}$, $\operatorname{supp}\mathcal{F}_x\phi(\xi_x,.) \subseteq [-a,a]$, the right-hand side of the above expression is nothing else than the Fourier transform of ϕ (where we used the Fubini theorem $(\mathcal{F}_y\mathcal{F}_x\phi = \mathcal{F}\phi)$):

$$\begin{aligned}\mathcal{F}_x v(\xi_x, y) &= \frac{i\mathrm{e}^{-i\kappa(\xi_x,\omega)y}}{2\sqrt{2\pi}\kappa(\xi_x,\omega)} \left(\mathcal{F}_y\mathcal{F}_x\phi\right)(\xi_x, \kappa(\xi_x,\omega)) \\ &= \frac{i\mathrm{e}^{-i\kappa(\xi_x,\omega)y}}{2\kappa(\xi_x,\omega)}\mathcal{F}\phi(\xi_x,\kappa(\xi_x,\omega)), \quad \text{for all } y < -a. \end{aligned} \tag{2.22}$$

Let us now bound from below the norm $\|v\|_{H^{1+\sigma}_{-s',\perp}}$. By (2.21):

$$\begin{aligned}\|v\|^2_{H^{1+\sigma}_{-s',\perp}} &\gtrsim \int\limits_{-\infty}^{\infty}(1+y^2)^{-s'}\int\limits_{-\infty}^{\infty}(1+\xi_x^2)^{1+\sigma}|\mathcal{F}_x v(\xi_x,y)|^2 d\xi_x dy \\ &\ge C_{\omega,\alpha}\int\limits_{-\infty}^{-a}(1+y^2)^{-s'}\int\limits_{-\infty}^{\infty}(\omega^2+\alpha^2\xi_x^2)^{1+\sigma}|\mathcal{F}_x v(\xi_x,y)|^2 d\xi_x dy, \end{aligned} \tag{2.23}$$

for some constant $C_{\omega,\alpha} > 0$. From (2.22) it follows that for any $\sigma \ge 0$, cf. the definition of $\kappa(\xi_x,\omega)$ in (2.13), it holds:

$$(\omega^2+\alpha^2\xi_x^2)^{1+\sigma}|\mathcal{F}_x v(\xi_x,y)|^2 = \frac{1}{2}(\omega^2+\alpha^2\xi_x^2)^{\sigma}|\mathcal{F}\phi(\xi_x,\kappa(\xi_x,\omega))|^2. \tag{2.24}$$

Using the above expression in (2.23) yields the lower bound on $\|v\|_{H^{1+\sigma}_{-s',\perp}}$ in terms of the right-hand side ϕ:

$$\|v\|^2_{H^{1+\sigma}_{-s',\perp}} \overset{(2.24)}{\geq} \frac{C_{\omega,\alpha}}{2}\int_{-\infty}^{-a}(1+y^2)^{-s'}\int_{-\infty}^{\infty}(\omega^2+\alpha^2\xi_x^2)^{\sigma}\left|\mathcal{F}\phi(\xi_x,\kappa(\xi_x,\omega))\right|^2 d\xi_x dy$$

$$= C^0(\omega,\alpha,s',a)I_\sigma(\phi), \quad \text{with } C^0(\omega,\alpha,s',a) = C_{\omega,\alpha}\int_{-\infty}^{-a}(1+y^2)^{-s'}dy > 0, \tag{2.25}$$

$$\text{and } I_\sigma(\phi) := \int_{-\infty}^{\infty}(\omega^2+\alpha^2\xi_x^2)^{\sigma}\left|\mathcal{F}\phi(\xi_x,\kappa(\xi_x,\omega))\right|^2 d\xi_x.$$

Let us now fix $\sigma > 0$. Let us show that we can choose $\phi = \phi_\sigma \in L^2_{s,\perp}(\mathbb{R}^2)$, s.t. $\operatorname{supp}\phi_\sigma(x,.) \subset (-a,a)$, for which $I_\sigma(\phi_\sigma)$ defined in (2.25) is not finite. The main idea is to choose ϕ_σ, so that $\mathcal{F}\phi_\sigma$ is supported in the vicinity of the line $(\xi_x,\kappa(\xi_x))$, however grows in ξ_x fast enough to ensure that $I_\sigma(\phi_\sigma)$ blows up.

Step 1. Let us define

$$\hat{g}_\sigma(\xi_x,\xi_y) := (\omega^2+\alpha^2\xi_x^2)^{-\frac{1}{4}-\delta}\mathbb{1}_{\{|\xi_y-\alpha|\xi_x||<\omega\}}, \text{ with some } 0<\delta\leq\frac{\sigma}{2}. \tag{2.26}$$

This function is in $L^2(\mathbb{R}^2)$; to see this we apply the Fubini theorem to compute

$$\|\hat{g}_\sigma\|^2 = \int_{\mathbb{R}^2}(\omega^2+\alpha^2\xi_x^2)^{-\frac{1}{2}-2\delta}\mathbb{1}_{\{|\xi_y-\alpha|\xi_x||<\omega\}}d\xi_x d\xi_y = 2\omega\int_{-\infty}^{\infty}(\omega^2+\alpha^2\xi_x^2)^{-\frac{1}{2}-2\delta}d\xi_x,$$

which is finite because $\delta > 0$. Therefore, $\mathcal{F}^{-1}\hat{g}_\sigma \in L^2(\mathbb{R}^2)$. The function $\hat{g}_\sigma$ has the following important property:

$$I_\sigma(\mathcal{F}^{-1}\hat{g}_\sigma) = \int_{-\infty}^{\infty}(\omega^2+\alpha^2\xi_x^2)^{\sigma-\frac{1}{2}-2\delta}\mathbb{1}_{\{|\sqrt{\omega^2+\alpha^2\xi_x^2}-\alpha|\xi_x||<\omega\}}d\xi_x$$

$$= \int_{-\infty}^{\infty}(\omega^2+\alpha^2\xi_x^2)^{\sigma-\frac{1}{2}-2\delta}d\xi_x = +\infty,$$

because $2\delta \leq \sigma$. Therefore, we could have chosen ϕ as $\mathcal{F}^{-1}g_\sigma$, had we not imposed that a.e. in $x\in\mathbb{R}$, $\phi(x,.)$ is supported in $(-a,a)$, $a>0$.

Step 2. To respect the constraint of the finiteness of the support in one of the directions, let us define

$$\phi_\sigma := \mathbb{1}_{\{y\in(-a,a)\}}\mathcal{F}^{-1}\hat{g}_\sigma \in L^2(\mathbb{R}^2). \tag{2.27}$$

Step 3. Let us show that $I_\sigma(\phi_\sigma) = \infty$. For this we will examine the behaviour of $\mathcal{F}\phi_\sigma(\xi,\sqrt{\omega^2+\alpha^2\xi^2})$ for large ξ. First of all,

$$\mathcal{F}\phi_\sigma(\xi_x,.) = \mathcal{F}_y\mathbb{1}_{\{y\in(-a,a)\}} * \hat{g}_\sigma(\xi_x,.), \quad \text{for all } \xi_x\in\mathbb{R},$$

and because $\mathcal{F}_y \mathbb{1}_{y\in(-a,a)}(\xi_y) = \sqrt{\frac{2}{\pi}}\frac{\sin(a\xi_y)}{\xi_y}$,

$$\mathcal{F}\phi_\sigma(\xi_x,\xi_y) = \sqrt{\frac{2}{\pi}}\int\limits_{-\infty}^{\infty}\frac{\sin(a(\xi_y-\xi_y'))}{\xi_y-\xi_y'}\hat{g}_\sigma(\xi_x,\xi_y')d\xi_y'$$

$$\overset{(2.26)}{=} \sqrt{\frac{2}{\pi}}\int\limits_{\alpha|\xi_x|-\omega}^{\alpha|\xi_x|+\omega}\frac{\sin(a(\xi_y-\xi_y'))}{\xi_y-\xi_y'}(\omega^2+\alpha^2\xi_x^2)^{-\frac{1}{4}-\delta}d\xi_y'.$$

Next, to estimate $I_\sigma(\phi_\sigma)$, cf. (2.25), let us consider the above expression evaluated on the curve

$$(\xi_x,\kappa(\xi_x)) = (\xi_x,\sqrt{\omega^2+\alpha^2\xi_x^2}),$$

namely

$$\mathcal{F}\phi_\sigma(\xi_x,\kappa(\xi_x,\omega)) = \sqrt{\frac{2}{\pi}}(\omega^2+\alpha^2\xi_x^2)^{-\frac{1}{4}-\delta}\int\limits_{\alpha|\xi_x|-\omega}^{\alpha|\xi_x|+\omega}\frac{\sin(a(\kappa(\xi_x,\omega)-\xi_y'))}{\kappa(\xi_x,\omega)-\xi_y'}d\xi_y'$$

$$= \sqrt{\frac{2}{\pi}}(\omega^2+\alpha^2\xi_x^2)^{-\frac{1}{4}-\delta}\int\limits_{-\omega}^{\omega}\frac{\sin(a(\kappa(\xi_x,\omega)-\alpha|\xi_x|-\xi_y'))}{\kappa(\xi_x,\omega)-\alpha|\xi_x|-\xi_y'}d\xi_y'. \tag{2.28}$$

The goal is to show that, for sufficiently large $|\xi_x|$, thanks to a properly chosen $a>0$, the quantity $|\mathcal{F}\phi_\sigma(\xi_x,\kappa(\xi_x,\omega))|$ is bounded from below by $|\xi_x|^{-\frac{1}{2}-\delta}$, so that $I(\phi_\sigma)=\infty$. Let us choose a so that the integral in the right-hand side is strictly positive and bounded from below. For this let us remark the following: there exists a sufficiently large $R>0$ and corresponding $h_R>0$, s.t. for all $|\xi_x|>R$,

$$\kappa(\xi_x,\omega)-\alpha|\xi_x| = \alpha|\xi_x|\left(\left(1+\frac{\omega^2}{\xi_x^2\alpha^2}\right)^{\frac{1}{2}}-1\right)\in(-h_R,h_R).$$

The value R in the above depends on ω,α only, and, evidently, $h_R = O\left(R^{-1}\right)$. Therefore, for all $\xi_y'\in(-\omega,\omega)$,

$$\kappa(\xi_x,\omega)-\alpha|\xi_x|-\xi_y'\in(-\omega-h_R,\omega+h_R).$$

Then, if we fix $0<a<\frac{\pi}{2|\omega+h_R|}$, we have, for all $|\xi_x|>R$ and $\xi_y'\in(-\omega,\omega)$,

$$\left|a\left(\kappa(\xi_x,\omega)-\alpha|\xi_x|-\xi_y'\right)\right|<\frac{\pi}{2},$$

and so, as $x^{-1}\sin x>\frac{2}{\pi}$ on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,

$$\frac{\sin(a(\kappa(\xi_x,\omega)-\alpha|\xi_x|-\xi_y'))}{\kappa(\xi_x,\omega)-\alpha|\xi_x|-\xi_y'}>\frac{2a}{\pi}.$$

Combining the above with (2.28), we conclude that there exists $c>0$, s.t. for all $|\xi_x|>R$,

$$\mathcal{F}\phi_\sigma(\xi_x,\kappa(\xi_x,\omega))>c|\xi_x|^{-\frac{1}{2}-2\delta}.$$

This implies that

$$I_\sigma(\phi_\sigma) \geq \int_R^\infty (\omega^2+\alpha^2\xi_x^2)^\sigma \xi_x^{-1-4\delta} d\xi_x = +\infty, \tag{2.29}$$

because $2\sigma - 4\delta \geq 0$, see (2.26).

Summary. For arbitrary $\sigma > 0$, with the choice of $\phi = \phi_\sigma$, by (2.25) and (2.29) yields $v = v_\sigma = \mathcal{N}_\omega^+ \phi_\sigma \notin H^{1+\sigma}_{-s',\perp}$, and hence the conclusion. ∎

In Section 3.4 we refine the above result to show that $\mathcal{N}_\omega^+ \in \mathcal{B}\left(L^2_{comp}, H^{1+\sigma}_{loc}\right)$ (where $L^2_{comp} = \{v \in L^2(\mathbb{R}^2) : \operatorname{supp} v \text{ is bounded}\}$) if and only if $\sigma \leq 0$.

2.4 Radiation condition for $0 < \omega < \omega_p$. Similarly to the Helmholtz equation, the solutions to (1.8) are, in general, not unique, see the discussion in the beginning of Section 2.2. The main idea in the derivation of the radiation condition to impose the uniqueness of the solution to (1.8) comes from Remark 3: the partial Fourier transform of u_ω, namely $\mathcal{F}_x u_\omega$, solves the Helmholtz equation (2.15). The outgoing solutions to (2.15) are given by (2.14), with the fundamental solution defined in (2.12). The uniqueness of the outgoing solutions is then assured by the classical Sommerfeld radiation condition. Hence, it remains to justify the application of the Fourier transform to (1.8), which enabled us to work with $\mathcal{F}_x u(\xi_x, .)$ defined for almost all $\xi_x \in \mathbb{R}$. For this it is sufficient that $u(., y) \in L^2(\mathbb{R})$ for all y. Combining all these reasonings, we formulate the following radiation condition.

DEFINITION 2.8 (Outgoing Fourier-domain radiation condition). *A function $\phi \in L^2_{loc}(\mathbb{R}^2)$ satisfies an outgoing Fourier-domain radiation condition if*

(RC1) a.e. in $y \in \mathbb{R}$, $\phi(., y) \in L^2(\mathbb{R})$.

(RC2) the partial Fourier transform of ϕ satisfies (recall that α is given by (2.4))

$$\lim_{|y|\to+\infty} \left| \partial_{|y|} \mathcal{F}_x \phi(\xi_x, y) - i\sqrt{\alpha^2\xi_x^2+\omega^2}\,\mathcal{F}_x\phi(\xi_x, y)\right| = 0 \text{ a.e. in } \xi_x \in \mathbb{R}.$$

Let us remark that this radiation condition resembles the radiation condition provided by the angular spectrum representation for the rough surface scattering [5]. Next we show that it indeed ensures the uniqueness of solutions to (2.5).

PROPOSITION 2.9 (Uniqueness). *Let $0 < \omega < \omega_p$. Let u_ω satisfy (1.8) with $f = 0$ and the outgoing Fourier-domain radiation condition from Definition 2.8. Then $u_\omega = 0$.*

Proof. Because of (RC1) from Definition 2.8, $\mathcal{F}_x u_\omega(\xi_x, y)$ is defined a.e. in ξ_x, $y \in \mathbb{R}$, and thus $\mathcal{F}_x u_\omega$ satisfies (2.15) with $f = 0$ a.e. in $\xi_x \in \mathbb{R}$:

$$\kappa^2(\xi_x, \omega)\mathcal{F}_x u(\xi_x, y) + \partial_y^2 \mathcal{F}_x u(\xi_x, y) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{2.30}$$

From (RC2), which is the radiation condition for the above 1D Helmholtz equation, it follows that $\mathcal{F}_x u(\xi_x, y) = 0$ a.e. in $\xi_x \in \mathbb{R}$. ∎

2.5 Existence and uniqueness of solutions in the hyperbolic regime $0 < \omega < \omega_p$. The principal result of Section 2 is summarized below.

THEOREM 2.10 (Existence and uniqueness). *Let $0 < \omega < \omega_p$ and $s, s' > \frac{1}{2}$. For all $f \in L^2_{s,\perp}(\mathbb{R}^2)$, there exists a unique solution $u_\omega \in L^2_{loc}(\mathbb{R}^2)$ to (2.5) that satisfies the radiation condition (RC1), (RC2). Moreover, $u_\omega = u_\omega^+ = \mathcal{N}_\omega^+ f$, $u_\omega \in H^1_{-s',\perp}$, and, with some $C_{s,s'}(\omega) > 0$,*

$$\|u_\omega\|_{H^1_{-s',\perp}} \leq C_{s,s'}(\omega)\|f\|_{L^2_{s,\perp}}. \tag{2.31}$$

Proof. The uniqueness of u_ω follows from Proposition 2.9.

By Theorem 2.3 and a classical density argument $u_\omega := u_\omega^+ = \mathcal{N}_\omega^+ f$ solves (2.5); the stability bound is from Proposition 2.6. It remains to show that u_ω^+ satisfies the radiation condition.

Obviously, $u_\omega^+ \in L^2_{loc}$ by the stability bound (2.31). Then, (RC1) follows from the fact that $u_\omega^+ \in H^1_{-s',\perp}$. The condition (RC2) follows from (2.14) by direct computation, using the partial Fourier transform (2.14) and the explicit form of the partial Fourier transform of the fundamental solution (2.12). Indeed, we have, for $y > 0$,

$$\partial_y \mathcal{F}_x u_\omega^+(\xi_x, y) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\kappa(\xi_x,\omega)|y-y'|}}{2\sqrt{2\pi}} \operatorname{sgn}(y-y') \mathcal{F}_x f(\xi_x, y') dy'$$

$$= i\kappa(\xi_x,\omega)\mathcal{F}_x u_\omega^+(\xi_x, y) - \int_{y}^{+\infty} \frac{\mathrm{e}^{i\kappa(\xi_x,\omega)|y-y'|}}{\sqrt{2\pi}} \mathcal{F}_x f(\xi_x, y') dy'.$$

It remains to use the Cauchy-Schwarz inequality to estimate

$$\left| \int_{y}^{+\infty} \frac{\mathrm{e}^{i\kappa(\xi_x,\omega)|y-y'|}}{\sqrt{2\pi}} \mathcal{F}_x f(\xi_x, y') dy' \right|^2 \lesssim \int_{y}^{+\infty} (1+y'^2)^{-s} dy' \int_{y}^{\infty} |\mathcal{F}_x f(\xi_x, y')|^2 (1+y'^2)^s dy'$$

$$\lesssim y^{-2s+1} \|\mathcal{F}_x f(\xi_x, .)\|^2_{L^2_s(\mathbb{R})} \to 0, \quad y \to +\infty.$$

A similar computation shows the validity of (RC2) for u_ω^+ when $y \to -\infty$. ∎

3 Regularity analysis in the hyperbolic regime. This section is dedicated to finer regularity estimates of the solution in the hyperbolic regime. We first provide a motivation to the regularity analysis, which takes the form of the numerical experiments: they indicate that the regularity of the solution depends on a certain directional regularity of the data. Then we provide a theoretical justification of the results of those numerical experiments: we demonstrate that if the singularities of the data f are not 'aligned' with characteristics, the solution is more regular than in the case when they are.

Recall that the result of Proposition 2.6 is somehow disappointing: it shows that, provided an $L^2_{s,\perp}$-right hand side data, we cannot expect the solution regularity to be better than $H^1_{-s',\perp}$. To discuss the numerical experiments, we need the following corollary of Proposition 2.6.

PROPOSITION 3.1. *$\mathcal{N}_\omega^+ \in \mathcal{B}(H^\lambda_{s,\perp}, H^{1+\lambda}_{-s',\perp})$, for all $\lambda \geq 0$, $s, s' > \frac{1}{2}$.*

Proof. It is straightforward to extend the proof of Proposition 2.6 to show that $\mathcal{N}_\omega^+ \in \mathcal{B}(H^m_{s,\perp}, H^{m+1}_{-s',\perp})$, $m \in \mathbb{N}$. The desired result than follows by the standard interpolation argument [24, p. 320, Theorem B.2] and the interpolation results for weighted Sobolev spaces obtained by Löfström [23, Theorem 4 and (5.3)]. ∎

Let us consider the following numerical experiment. We compute[2] the solution to the problem (2.5) with $\alpha = 1$ in the free space $\mathbb{R}^2$, using the perfectly matched layer method of [7] adapted to the frequency domain.[3] We take two right-hand side data

[2] For these simulations we used the XLife++ library [25].

[3] While for the moment we do not have a rigorous proof of the convergence of this perfectly matched layer method, neither in the frequency nor in time domain, our numerical experiments indicate that it does indeed converge.

$f = f_j = \mathbb{1}_{\mathcal{O}_j}$, $j = 1, 2$, with either

$$\mathcal{O}_1 = (-a, a) \times (-a, a), \quad \text{or} \quad \mathcal{O}_2 = \left\{ |x - y| < \sqrt{2}a, |x + y| < \sqrt{2}a \right\}, \quad a = 0.5.$$

In both cases, $f_j \in \bigcap_{\varepsilon > 0} H^{\frac{1}{2}-\varepsilon}_{comp}(\mathbb{R}^2)$, $j = 1, 2$, the only difference being that the singularities of f_2 (jumps) are aligned with the characteristics of the equation (2.5). In both cases, according to Proposition 3.1, we expect the corresponding solution $u_j, j = 1, 2$, to belong to $\bigcap_{s' > \frac{1}{2}, \varepsilon > 0} H^{\frac{3}{2}-\varepsilon}_{-s',\perp}(\mathbb{R}^2)$. Visually, cf. Figure 3, the solution u_1 seems to be smoother than the solution u_2. It appears that this phenomenon is not only numerical,

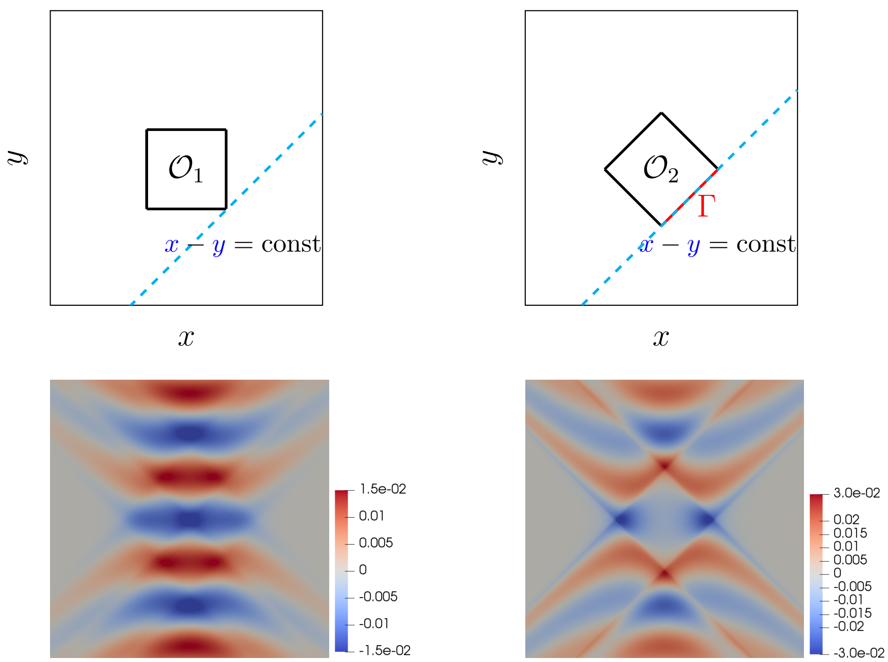

FIG. 3. *Top: the open sets $\mathcal{O}_j$ and one of the characteristic lines passing through their boundary. Bottom: the imaginary part of the solution to the problem (2.5) with the parameters described in the beginning of Section 3, restricted to the square $(-2, 2) \times (-2, 2)$. Left: $f = f_1$. Right: $f = f_2$.*

but occurs also at the continuous level: indeed, when the singularities of the source term are aligned with characteristics (we will give a precise mathematical definition of the 'alignment' in further sections), the solution is less regular than otherwise.

Another interesting phenomenon illustrated in Figure 3, left, is that unlike in the elliptic case, the singularities of the solution are no longer concentrated at the singularities of the data, but propagate along the characteristics, see [19, Theorem 4.4.1 and discussion afterwards] for the elliptic case and [19, Theorem 8.3.1] for the hyperbolic case.

In order to present the essential difficulties, rather than technicalities, in this section we examine the behavior of the solution in a particular case when the data f is s.t. $\operatorname{supp} f = \overline{\mathcal{O}}$, for a bounded convex open set $\mathcal{O}$ of $\mathbb{R}^2$, and $f \in C^{0,\alpha}(\overline{\mathcal{O}})$. In other words, the continuation of f outside of $\overline{\mathcal{O}}$ by zero may have discontinuities only on $\partial\mathcal{O}$. We will show that in this case the derivatives of the solution may have jump and logarithmic singularities, and show how these singularities are related to the characteristics passing through $\overline{\mathcal{O}}$. The estimates in the Sobolev spaces, which are in general better suited for the numerical analysis, are provided in Appendix F.

For convenience, we rewrite (2.5) by performing a rotational change of coordinates which transforms the characteristics of (1.8) governed by $y \pm \alpha^{-1}x = \text{const}$ into the lines $\xi = \text{const}$ and $\eta = \text{const}$, where

$$\xi = y + \alpha^{-1}x, \quad \eta = y - \alpha^{-1}x. \tag{3.1}$$

An open set $\mathcal{O}$ will be denoted by Ω in the coordinates (ξ, η). Given a function $v(x, y)$, we denote by $\widetilde{v}(\xi, \eta) := v\left(\frac{1}{2}\alpha(\xi - \eta), \frac{1}{2}(\xi + \eta)\right)$. It is readily checked that (2.5) transforms into

$$4\partial^2_{\xi\eta}\tilde{u}_\omega + \omega^2\tilde{u}_\omega = \widetilde{f} \text{ in } \mathcal{D}'(\mathbb{R}^2). \tag{3.2}$$

The solution that satisfies the outgoing Fourier-domain radiation condition, cf. (RC1), (RC2), is transformed to (with an abuse of notation in the definition of $\widetilde{\mathcal{G}}^+_\omega$):

$$\begin{aligned} &\widetilde{u}^+_\omega = \widetilde{\mathcal{N}}^+_\omega \widetilde{f} = \widetilde{\mathcal{G}}^+_\omega * \widetilde{f}, \\ &\widetilde{\mathcal{G}}^+_\omega(\xi, \eta) := \frac{1}{8}\begin{cases} H_0^{(1)}(\omega\sqrt{\xi\eta}), & \xi\eta > 0, \\ H_0^{(1)}(i\omega\sqrt{-\xi\eta}), & \xi\eta < 0. \end{cases} \end{aligned} \tag{3.3}$$

REMARK 4. *In this section we use the following notation:* $\widetilde{u} := \widetilde{u}^+_\omega$ *and* $\widetilde{\mathcal{G}} := \widetilde{\mathcal{G}}^+_\omega$.

3.1 Regularity results.

In the beginning of this section we will summarize the regularity results, while most of their proofs will be postponed to the later sections.

We start with the following proposition that states that the singularities of the solution to (3.2) lie inside the set of characteristics passing through the support of $\widetilde{f}$. To formulate this result, let us define two regions, given $a_+ > a_-$ and $b_+ > b_-$,

$$\Omega^\xi_{\boldsymbol{a}} := \{(\xi, \eta) : a_- < \xi < a_+\}, \quad \Omega^\eta_{\boldsymbol{b}} := \{(\xi, \eta) : b_- < \eta < b_+\}.$$

Then the region $\Omega_{\boldsymbol{a},\boldsymbol{b}} := \Omega^\xi_{\boldsymbol{a}} \cup \Omega^\eta_{\boldsymbol{b}}$ contains all the characteristics of (3.2) passing through the rectangle $(a_-, a_+) \times (b_-, b_+)$, see also Figure 4, left.

THEOREM 3.2 (Smoothness regions). *Let* $\widetilde{f} \in L^2(\mathbb{R}^2)$ *s.t.* $\operatorname{supp} \widetilde{f} \subseteq [a_-, a_+] \times [b_-, b_+]$. *Then the function* $\widetilde{u} = \widetilde{\mathcal{G}} * \widetilde{f} \in C^\infty(\mathbb{R}^2 \setminus \overline{\Omega}_{\boldsymbol{a},\boldsymbol{b}})$.

The next result shows that, even if $\widetilde{f}$ has jump singularities, the solution has continuous derivatives, if the jumps are not aligned with characteristics. In order to formulate the desired result, let us introduce the following assumption.

ASSUMPTION 1 (Assumption on the data). *Let* Ω *be a bounded convex (thus, Lipschitz, cf. [17, Corollary 1.2.2.3]) open set of* $\mathbb{R}^2$. *We define*

$$\begin{aligned} a_- &:= \inf\{\xi : (\xi, \eta) \in \Omega\}, \quad a_+ := \sup\{\xi : (\xi, \eta) \in \Omega\}, \\ b_- &:= \inf\{\eta : (\xi, \eta) \in \Omega\}, \quad b_+ := \sup\{\eta : (\xi, \eta) \in \Omega\}, \end{aligned}$$

so that the smallest rectangle containing Ω *is given by* $(a_-, a_+) \times (b_-, b_+)$. *Let*

$$\Gamma_{a_\pm} := \{(a_\pm, \eta), \eta \in \mathbb{R}\} \cap \partial\Omega, \quad \Gamma_{b_\pm} := \{(\xi, b_\pm), \xi \in \mathbb{R}\} \cap \partial\Omega,$$

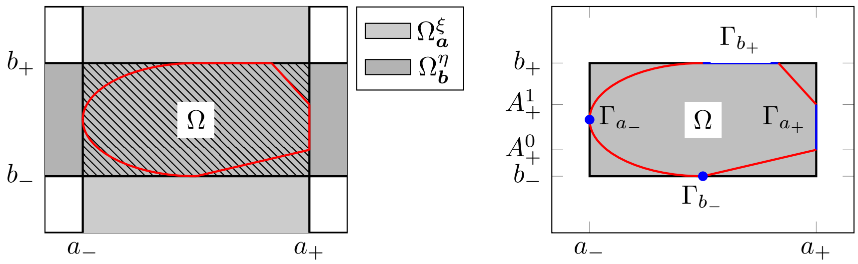

FIG. 4. *An illustration to the geometric configuration of Section 3. Left: open sets $\Omega^\xi_{\boldsymbol{a}}$ and $\Omega^\eta_{\boldsymbol{b}}$. Right: illustration to the notations of Assumption 1. In particular, in this case $A^0_- = A^1_-$ and $B^0_- = B^1_-$.*

so that, with some $A^0_\pm \leq A^1_\pm$, $B^0_\pm \leq B^1_\pm$,

$$\Gamma_{a_\pm} = \{(a_\pm, \eta) : A^0_\pm \leq \eta \leq A^1_\pm\}, \quad \Gamma_{b_\pm} = \{(\xi, b_\pm) : B^0_\pm \leq \xi \leq B^1_\pm\}.$$

Let $\widetilde{f}$ be defined as follows:

$$\widetilde{f} = \begin{cases} \widetilde{F} \text{ in } \overline{\Omega}, \\ 0 \text{ otherwise}, \end{cases} \qquad \text{with } \widetilde{F} \in C^{0,\alpha}(\overline{\Omega}).$$

An illustration to the above geometric configuration is given in Figure 4, right. As a matter of fact, the requirement of the convexity of Ω simplifies the presentation of the results. This condition ensures that the boundary is Lipschitz, and, moreover, that $\Gamma_{a_\pm}$ and $\Gamma_{b_\pm}$ are connected sets (intervals or points). For non-convex sets, the requirement that Ω is Lipschitz can be weakened to require that $\partial\Omega$ is $C^{0,\beta}$, for some $\beta > 0$. It appears naturally in the proof of the estimates, and it does not seem that it can be weakened to C^0.

In what follows, we will denote by $|\Gamma|$ the length of the curve Γ.

THEOREM 3.3 (Propagation of singularities). *Let $\widetilde{f}$ satisfy Assumption 1. Then the function $\widetilde{u} = \widetilde{\mathcal{G}} * \widetilde{f}$ satisfies $\widetilde{u} \in C^1\left(\mathbb{R}^2 \setminus (\partial\Omega^\xi_{\boldsymbol{a}} \cup \partial\Omega^\eta_{\boldsymbol{b}})\right)$. Moreover,*

1. *if $|\Gamma_{a_\pm}| = |\Gamma_{b_\pm}| = 0$, then $\widetilde{u} \in C^1(\mathbb{R}^2)$;*
2. *if $|\Gamma_{a_\pm}| = 0$ (resp. $|\Gamma_{b_\pm}| = 0$), then $\partial_\xi \tilde{u} \in C^0(\mathbb{R}^2)$ (resp. $\partial_\eta \tilde{u} \in C^0(\mathbb{R}^2)$);*
3. *if $|\Gamma_{a_+}| \neq 0$ (and/or $|\Gamma_{a_-}| \neq 0$), $\partial_\xi \widetilde{u} \in C^0(\mathbb{R}^2 \setminus \partial\Omega^\xi_{\boldsymbol{a}})$. Moreover, the following identities hold true:*

$$\begin{aligned} \partial_\xi \widetilde{u}(\xi, \eta) =& \frac{i}{8\pi}\left(F_{a_-} \log|\xi - a_-| - F_{a_+} \log|\xi - a_+|\right) \\ &- \frac{1}{8}\Lambda_a(\xi, \eta)\mathbb{1}_{\overline{\Omega}^\xi_{\boldsymbol{a}}}(\xi, \eta) + g(\xi, \eta), \end{aligned} \tag{3.4}$$

where

(a) *the constants $F_{a_\pm}$ are given by:*

$$F_{a_\pm} := \int\limits_{\Gamma_{a_\pm}} \widetilde{F}(a_\pm, \eta')d\eta',$$

(b) *the function* $\Lambda_a \in C^0(\overline{\Omega}^{\xi}_{\boldsymbol{a}})$ *is defined as*

$$\Lambda_a(\xi,\eta) = \frac{\xi - a_+}{a_- - a_+} f_{a_-}(\eta) + \frac{\xi - a_-}{a_+ - a_-} f_{a_+}(\eta),$$

where

$$f_{a_\pm}(\eta) = \begin{cases} F_{a_\pm}, & \eta \le A^0_\pm, \\ F_{a_\pm} - 2\int\limits_{A^0_\pm}^{\eta} \widetilde{F}(a_\pm, \eta')d\eta', & A^0_\pm < \eta < A^1_\pm, \\ -F_{a_\pm}, & \eta > A^1_\pm. \end{cases}$$

(c) $g \in C^0(\mathbb{R}^2)$.

Similar expressions hold for $\partial_\eta \widetilde{u}(\xi,\eta)$*, which, in general, has a logarithmic and jump singularities across the lines* $\eta = b_+$ *(resp.* $\eta = b_-$*) when* $|\Gamma_{b_+}| \neq 0$ *(resp.* $|\Gamma_{b_-}| \neq 0$*).*

REMARK 5. *Theorem 3.3 concerns the data that has jump singularities, and shows the following. If the intersection of the support of the singularity with one of the characteristics* $\{\xi = \text{const}\}$ *or* $\{\eta = \text{const}\}$ *is of non-zero Lebesgue measure, the solution has discontinuous derivatives in general, with discontinuities aligned along the respective characteristics. Otherwise, the solution has continuous derivatives.*

The above theorem leads to the following corollary. When the 'mean value' of the jump vanishes (i.e. $F_{a_\pm} = 0$, $F_{b_\pm} = 0$), the singularities no longer propagate along the characteristics but are concentrated along the jumps of the data lying on the characteristics, i.e. on $\Gamma_{a_\pm}$ ($\Gamma_{b_\pm}$).

COROLLARY 3.4 (Concentration of singularities). *Let* $\widetilde{f}$ *satisfy Assumption 1. Let additionally the following quantities vanish:*

$$F_{a_\pm} = \int\limits_{\Gamma_{a_\pm}} \widetilde{F}(a_\pm, \eta')d\eta' = 0 = \int\limits_{\Gamma_{b_\pm}} \widetilde{F}(\xi', b_\pm)d\xi' = F_{b_\pm}.$$

Then $\widetilde{u} \in C^1(\mathbb{R}^2 \setminus (\Gamma_{a_+} \cup \Gamma_{a_-} \cup \Gamma_{b_+} \cup \Gamma_{b_-}))$.

Proof. We will show the reasoning for $\partial_\xi \widetilde{u}$ only. According to (3.4), the discontinuities of $\partial_\xi \widetilde{u}$ are concentrated along the lines $\xi = a_\pm$. Additionally, it is clear that $\partial_\xi \widetilde{u} - \frac{1}{8}\Lambda_a(\xi,\eta)\mathbb{1}_{\overline{\Omega}^{\xi}_{\boldsymbol{a}}}$ is continuous on $\mathbb{R}^2$. On the other hand,

$$\Lambda_a(a_\pm, \eta) = 0, \text{ for } \eta > A^1_\pm \text{ and for } \eta < A^0_\pm.$$

Therefore, $\Lambda_a(\xi,\eta)\mathbb{1}_{\bar{\Omega}^{\xi}_{\boldsymbol{a}}}(\xi,\eta)$ is continuous on $\mathbb{R}^2 \setminus (\Gamma_{a_+} \cup \Gamma_{a_-})$, and so is $\partial_\xi \widetilde{u}$. ∎

REMARK 6. *The results of Theorem 3.3 and Corollary 3.4 can of course be improved to show that* $\widetilde{u} \in C^{1,\alpha}(\mathbb{R}^2 \setminus (\partial\Omega^{\xi}_{\boldsymbol{a}} \cup \partial\Omega^{\eta}_{\boldsymbol{b}}))$.

The following sections are dedicated to the proofs of Theorems 3.2, 3.3.

3.2 Proof of Theorem 3.2

Consider the explicit expression for $\widetilde{u}$:

$$\widetilde{u}(\xi,\eta) = \frac{1}{8}\int\limits_{a_-}^{a_+}\int\limits_{b_-}^{b_+} (K_1(\xi-\xi', \eta-\eta') + K_2(\xi-\xi', \eta-\eta'))\widetilde{f}(\xi', \eta')d\xi'\, d\eta',$$

$$K_1(\xi,\eta) := \mathbb{1}\{\xi\eta > 0\}H_0^{(1)}(\omega\sqrt{\xi\eta}), \quad K_2(\xi,\eta) := \mathbb{1}\{\xi\eta < 0\}H_0^{(1)}(i\omega\sqrt{-\xi\eta}).$$

It is then easy to verify that the function $(\xi,\eta)\mapsto K_1(\xi-\xi',\eta-\eta')$, provided arbitrary $(\xi',\eta')\in[a_-,a_+]\times[b_-,b_+]$, is C^∞ in the following open set:

$$\{(\xi,\eta):\ \xi>a_+ \text{ or } \xi<a_-,\ \text{ and } \eta>b_+ \text{ or } \eta<b_-\}=\mathbb{R}^2\setminus\overline{\Omega}_{\boldsymbol{a},\boldsymbol{b}}.$$

In the same way, $(\xi,\eta)\mapsto K_2(\xi-\xi',\eta-\eta')\in C^\infty\left(\mathbb{R}^2\setminus\overline{\Omega}_{\boldsymbol{a},\boldsymbol{b}}\right)$. The result follows by the Lebesgue dominated convergence theorem.

3.3 Proof of Theorem 3.3. Before proving Theorem 3.3, we start with the following observation.

LEMMA 3.5. *The fundamental solution can be split as $\widetilde{\mathcal{G}}=\widetilde{\mathcal{G}}_{sing}+\widetilde{\mathcal{G}}_{reg}$, where*

$$\widetilde{\mathcal{G}}_{sing}(\xi,\eta)=\frac{i}{8\pi}\log|\xi|+\frac{i}{8\pi}\log|\eta|-\frac{1}{8}\mathbb{1}\{\xi\eta<0\}, \tag{3.5}$$

$$\widetilde{\mathcal{G}}_{reg}(\xi,\eta)=\frac{i}{8\pi}g_J(\omega^2\xi\eta)\left(\log|\xi\eta|+i\pi\mathbb{1}\{\xi\eta<0\}\right)+g_H(\omega^2\xi\eta), \tag{3.6}$$

with g_J, g_H being entire functions, $g_J(0)=0$, $g_J'(0)\neq 0$.

Proof. The proof relies on the explicit decomposition of the fundamental solution (3.3), given by (2.7), (2.8). It remains to rewrite it in a form suggested by the statement of the lemma. In the notations of (2.7),

$$g_H(z):=\frac{1}{8}\left(\left(1+i\frac{2}{\pi}\log\frac{\omega}{2}\right)(1+g_J(z))+ig_Y(z)\right).$$

We leave the remaining details to the reader. ∎

We then split accordingly

$$\widetilde{u}=\widetilde{u}_{sing}+\widetilde{u}_{reg},\quad \widetilde{u}_{sing}=\widetilde{\mathcal{G}}_{sing}*\widetilde{f},\quad \widetilde{u}_{reg}=\widetilde{\mathcal{G}}_{reg}*\widetilde{f}. \tag{3.7}$$

The proof of Theorem 3.3 then relies on the simple observation that $\widetilde{u}_{reg}\in C^1(\mathbb{R}^2)$, while the singularities of the derivatives of $\widetilde{u}_{sing}$ can be computed explicitly.

LEMMA 3.6. *Let $\widetilde{f}$ satisfy Assumption 1. Then $\widetilde{u}_{reg}\in C^1(\mathbb{R}^2)$.*

Proof. Using the explicit expression of $\widetilde{\mathcal{G}}_{reg}$ (3.6), we introduce

$$\begin{aligned}
&\widetilde{u}^1_{reg}:=g_J(\omega^2\xi\eta)\log|\xi|*\widetilde{f},\quad \widetilde{u}^2_{reg}:=g_J(\omega^2\xi\eta)\log|\eta|*\widetilde{f},\\
&\widetilde{u}^3_{reg}:=g_J(\omega^2\xi\eta)\mathbb{1}\{\xi\eta<0\}*\widetilde{f},\quad \widetilde{u}^4_{reg}:=g_H(\omega^2\xi\eta)*\widetilde{f},
\end{aligned}$$

so that $\widetilde{u}_{reg}=\frac{i}{8\pi}(\widetilde{u}^1_{reg}+\widetilde{u}^2_{reg})-\frac{1}{8}\widetilde{u}^3_{reg}+\widetilde{u}^4_{reg}$. Evidently $\widetilde{u}^4_{reg}\in C^\infty(\mathbb{R}^2)$, and the rest of the functions are continuous in $\mathbb{R}^2$, by continuity of the respective convolution kernels and because $\widetilde{f}\in L^\infty(\mathbb{R}^2)$. Let us examine their derivatives.

Step 1. Proof that $\widetilde{u}^1_{reg}$, $\widetilde{u}^2_{reg}\in C^1(\mathbb{R}^2)$. By symmetry, it suffices to study only one of these functions. We first consider

$$\partial_\xi\widetilde{u}^1_{reg}=\frac{g_J(\omega^2\xi\eta)}{\xi}*\widetilde{f}+\omega^2(\eta g_J'(\omega^2\xi\eta)\log|\xi|)*\widetilde{f}.$$

Because $g_J\in C^\infty(\mathbb{R})$ and vanishes at zero, $\xi^{-1}g_J(\omega^2\xi\eta)$ is continuous and thus the first term in the above expression is continuous in $\mathbb{R}^2$. The remaining term is continuous as a convolution of an $L^1_{loc}(\mathbb{R}^2)$ function with $\widetilde{f}\in L^\infty_{comp}(\mathbb{R}^2)$.

Step 2. Proof that $\widetilde{u}^3_{reg}\in C^1(\mathbb{R}^2)$. Again by symmetry, it is sufficient to study $\partial_\xi\widetilde{u}^3_{reg}$:

$$\partial_\xi\widetilde{u}^3_{reg}=\omega^2\left(\eta\, g_J'(\omega^2\xi\eta)\,\mathbb{1}\{\xi\eta<0\}\right)*\widetilde{f},$$

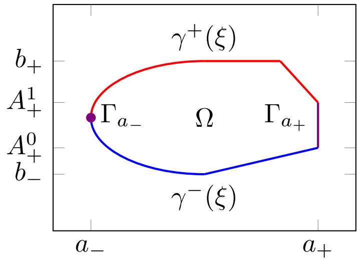

FIG. 5. *An illustration to the notations of the proof of Theorem 3.3.*

where we used $g_J(0) = 0$. The above is again continuous as a convolution of an $L^1_{loc}(\mathbb{R}^2)$ function with $\widetilde{f} \in L^\infty_{comp}(\mathbb{R}^2)$. ∎

We now have the necessary ingredients to prove Theorem 3.3. Before proving this result, let us remark the following. Because Ω is convex, the part of the boundary that lies between the vertical lines $\xi = a_\pm$ can be parametrized as follows:

$$\partial\Omega \setminus \Gamma_{a_\pm} = \Gamma^+ \cup \Gamma^-, \qquad \Gamma^\pm = \{(\xi,\eta) : \xi \in (a_-, a_+), \eta = \gamma^\pm(\xi)\}, \tag{3.8}$$

and $\gamma^\pm : (a_-, a_+) \to \mathbb{R}$ Lipschitz functions, s.t. $\gamma^+ > \gamma^-$. Moreover, they can be extended by continuity to $[a_-, a_+]$, with $\gamma^+(a_\pm) = A^1_\pm$ and $\gamma^-(a_\pm) = A^0_\pm$. We then have $|\Gamma_{a_\pm}| = \gamma^+(a_\pm) - \gamma^-(a_\pm)$. This is illustrated in Figure 5.

Proof of Theorem 3.3. We start with the decomposition (3.7). By Lemma 3.6, it suffices to consider only the derivatives of $\widetilde{u}_{sing}$. Based on (3.5), we split

$$\widetilde{u}_{sing} = \frac{i}{8\pi}\left(\widetilde{u}^1_{sing} + \widetilde{u}^2_{sing}\right) - \frac{1}{8}\widetilde{u}^3_{sing}, \tag{3.9}$$
$$\widetilde{u}^1_{sing} = \log|\xi| * \widetilde{f}, \quad \widetilde{u}^2_{sing} = \log|\eta| * \widetilde{f}, \quad \widetilde{u}^3_{sing} = \mathbb{1}\{\xi\eta < 0\} * \widetilde{f}.$$

Let us examine the derivatives of the above expressions.

Step 1. Derivatives of $\widetilde{u}^1_{sing}$, $\widetilde{u}^2_{sing}$. By symmetry it suffices to study only $\partial_\xi \widetilde{u}^1_{sing}$ and $\partial_\xi \widetilde{u}^2_{sing}$. Evidently,

$$\partial_\xi \widetilde{u}^2_{sing} = 0. \tag{3.10}$$

To study $\partial_\xi \widetilde{u}^1_{sing}$, let us introduce $\widetilde{F}_2(\xi) := \int_{\mathbb{R}} \widetilde{f}(\xi,\eta')d\eta' = \int_{\gamma^-(\xi)}^{\gamma^+(\xi)} \widetilde{f}(\xi,\eta')d\eta'$ (the notation indicates that we integrate in the second variable η). This function has the following properties:

- when $\xi \notin [a_-, a_+]$, $\widetilde{F}_2(\xi) = 0$, because $\operatorname{supp} \widetilde{f} \subset \overline{\Omega}^\xi_{\boldsymbol{a}}$;
- $\widetilde{F}_2\Big|_{[a_-,a_+]} \in C^{0,\alpha}([a_-, a_+])$, because $\widetilde{f} \in C^{0,\alpha}(\overline{\Omega})$ and $\gamma^\pm$ are Lipschitz.

By definition, $\widetilde{u}^1_{sing}(\xi,\eta) = \int_{\mathbb{R}} \log|\xi - \xi'|\widetilde{F}_2(\xi')d\xi'$, and does not depend on η. We consider two cases.

Step 1.1. $\partial_\xi \widetilde{u}^1_{sing}$ for $\xi \notin [a_-, a_+]$. A straightforward computation yields

$$\partial_\xi \widetilde{u}^1_{sing}(\xi,\eta) = \int_{a_-}^{a_+} \frac{\widetilde{F}_2(\xi')}{\xi - \xi'}d\xi' \in C^\infty(\mathbb{R}^2 \setminus \overline{\Omega}^\xi_{\boldsymbol{a}}). \tag{3.11}$$

Step 1.2. $\partial_\xi \widetilde{u}^1_{sing}$ *for* $\xi\in(a_-,a_+)$. An explicit computation gives

$$\partial_\xi \widetilde{u}^1_{sing}(\xi,\eta) = (P.V.\frac{1}{\xi} * \widetilde{F}_2)(\xi,\eta)$$
$$= \int_{a_-}^{a_+} \underbrace{\frac{\widetilde{F}_2(\xi')-\widetilde{F}_2(\xi)}{\xi-\xi'}}_{P(\xi,\xi')} d\xi' + \widetilde{F}_2(\xi) P.V.\int_{a_-}^{a_+} \frac{1}{\xi-\xi'}d\xi'$$
$$= \int_{a_-}^{a_+} P(\xi,\xi')d\xi' - \widetilde{F}_2(\xi)\left(\log|\xi-a_+| - \log|\xi-a_-|\right). \quad (3.12)$$

For all ξ, $P(\xi,.)\in L^1((a_-,a_+))$, because $\widetilde{F}_2\in C^{0,\alpha}([a_-,a_+])$. The first term above defines a continuous function on $[a_-,a_+]$. Indeed, given $h>0$, one has

$$\int_{a_-}^{a_+} P(\xi+h,\xi')d\xi' = \int_{a_- -h}^{a_+ -h} \frac{\widetilde{F}_2(\xi'+h)-\widetilde{F}_2(\xi+h)}{\xi-\xi'}d\xi',$$

and $\int_{a_-}^{a_+}\left(P(\xi+h,\xi')-P(\xi,\xi')\right)d\xi' \to 0$ as $h\to 0$, by the Lebesgue convergence theorem, again using $\widetilde{F}_2\in C^{0,\alpha}([a_-,a_+])$. Thus, $\partial_\xi\widetilde{u}^1_{sing}\in C^0(\Omega^\xi_{\mathbf{a}})$.

Step 1.3. Behaviour when $\xi\to a_\pm$. Let us define

$$F_{a_\pm} = \int_{\gamma_-(a_\pm)}^{\gamma_+(a_\pm)} \widetilde{F}(a_\pm,\eta')d\eta', \text{ so that } F_{a_+} = \lim_{\xi\uparrow a_+}\widetilde{F}_2(\xi), \quad F_{a_-} = \lim_{\xi\downarrow a_-}\widetilde{F}_2(\xi). \quad (3.13)$$

We claim that (3.12) and (3.11) imply that the following holds true:

$$G_0(\xi,\eta) := \partial_\xi\widetilde{u}^1_{sing}(\xi) + F_{a_+}\log|\xi-a_+| - F_{a_-}\log|\xi-a_-| \in C^0(\mathbb{R}^2). \quad (3.14)$$

The continuity of G_0 is evident for $(\xi,\eta)\in\mathbb{R}^2\setminus\partial\Omega^\xi_{\mathbf{a}}$, and it remains to prove it in the points $(a_\pm,\eta)$. We consider (a_+,η). For $\xi>a_+$, from (3.11) we have

$$G_0(\xi,\eta) = \int_{a_-}^{a_+}\frac{\widetilde{F}_2(\xi')-F_{a_+}}{\xi-\xi'}d\xi' + \left(F_{a_+}-F_{a_-}\right)\log|\xi-a_-|.$$

Since $\widetilde{F}_2\in C^{0,\alpha}([a_-,a_+])$, and using (3.13), the same argument as for $\int_{a_-}^{a_+}P(\xi,\xi')d\xi'$ before shows that the first term in the above expression is continuous in $\xi=a_+$, and

$$\lim_{\xi\downarrow a_+} G_0(\xi,\eta) = \int_{a_-}^{a_+}\frac{\widetilde{F}_2(\xi')-F_{a_+}}{a_+-\xi'}d\xi' + \left(F_{a_+}-F_{a_-}\right)\log|a_+-a_-|. \quad (3.15)$$

For $\xi<a_+$, (3.12) and left continuity of $\xi\mapsto P(\xi,\xi')$ in a_+ yield

$$\lim_{\xi\uparrow a_+} G_0(\xi,\eta) = \int_{a_-}^{a_+}\frac{\widetilde{F}_2(\xi')-F_{a_+}}{a_+-\xi'}d\xi' + \left(F_{a_+}-F_{a_-}\right)\log|a_+-a_-| = \lim_{\xi\downarrow a_+}G_0(\xi,\eta).$$

This shows that G_0 is continuous in $\xi = a_+$; similarly one shows that it is continuous in $\xi = a_-$.

Step 2. Derivatives of $\widetilde{u}^3_{sing}$. A straightforward computation yields

$$\partial_\xi \widetilde{u}^3_{sing}(\xi,\eta) = \int_\eta^\infty \widetilde{f}(\xi,\eta')d\eta' - \int_{-\infty}^\eta \widetilde{f}(\xi,\eta')d\eta'.$$

Because $\operatorname{supp} \widetilde{f} \subseteq \overline{\Omega}$,

$$\partial_\xi \widetilde{u}^3_{sing} = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega}^\xi_{\boldsymbol{a}}. \tag{3.16}$$

With (3.8), we have, for $\xi \in [a_-, a_+]$,

$$\partial_\xi \widetilde{u}^3_{sing}(\xi,\eta) = \begin{cases} \displaystyle\int_{\gamma^-(\xi)}^{\gamma^+(\xi)} \widetilde{F}(\xi,\eta')d\eta', & \eta \le \gamma^-(\xi), \\ \displaystyle\int_\eta^{\gamma^+(\xi)} \widetilde{F}(\xi,\eta')d\eta' - \int_{\gamma^-(\xi)}^\eta \widetilde{F}(\xi,\eta')d\eta', & \gamma^-(\xi) < \eta < \gamma^+(\xi), \\ \displaystyle -\int_{\gamma^-(\xi)}^{\gamma^+(\xi)} \widetilde{F}(\xi,\eta')d\eta', & \eta \ge \gamma^+(\xi). \end{cases} \tag{3.17}$$

Because $\gamma^\pm$ are continuous and $\widetilde{F} \in C^{0,\alpha}(\overline{\Omega})$, the above function is $C^0(\overline{\Omega}^\xi_{\boldsymbol{a}})$. Let

$$f_{a_+}(\eta) := \lim_{\xi \uparrow a_+} \partial_\xi \widetilde{u}^3_{sing}(\xi,\eta), \quad f_{a_-}(\eta) := \lim_{\xi \downarrow a_-} \partial_\xi \widetilde{u}^3_{sing}(\xi,\eta).$$

In particular, from (3.16), it follows that

$$\lim_{\xi \uparrow a_+} \partial_\xi \widetilde{u}^3_{sing}(\xi,\eta) - \lim_{\xi \downarrow a_+} \partial_\xi \widetilde{u}^3_{sing}(\xi,\eta) = f_{a_+}(\eta).$$

Let us introduce the following function:

$$\Lambda(\xi,\eta) := \frac{\xi - a_-}{a_+ - a_-} f_{a_+}(\eta) + \frac{\xi - a_+}{a_- - a_+} f_{a_-}(\eta),$$

so that $\Lambda(\xi,\eta)\mathbb{1}_{\overline{\Omega}^\xi_{\boldsymbol{a}}}$ has the same jumps as $\partial_\xi \widetilde{u}^3_{sing}$. Therefore, from (3.16) we have

$$G_1(\xi,\eta) := \partial_\xi \widetilde{u}^3_{sing} - \Lambda(\xi,\eta)\mathbb{1}_{\overline{\Omega}^\xi_{\boldsymbol{a}}} \in C^0(\mathbb{R}^2). \tag{3.18}$$

Similar expressions can be obtained for $\partial_\eta \widetilde{u}^3_{sing}(\xi,\eta)$.

Summary of the results. Combining (3.9), (3.10), Steps 1 and 2, we obtain the desired statement. ∎

3.4 Revisiting numerical results. Let us consider the problem described in the beginning of Section 3. We aim to apply Theorem 3.3. The open sets $\mathcal{O}_j$ (Ω_j in the coordinates ξ,η) are shown in Figure 6. For $\widetilde{f}_1$, $|\Gamma_{a_\pm}| = 0$, $|\Gamma_{b_\pm}| = 0$, and therefore $\partial_\xi \widetilde{u}_1$, $\partial_\eta \widetilde{u}_1 \in C^0(\mathbb{R}^2)$. This is not the case for $\widetilde{f}_2$: as seen from Figure 6, $|\Gamma_{a_\pm}| \neq 0$,

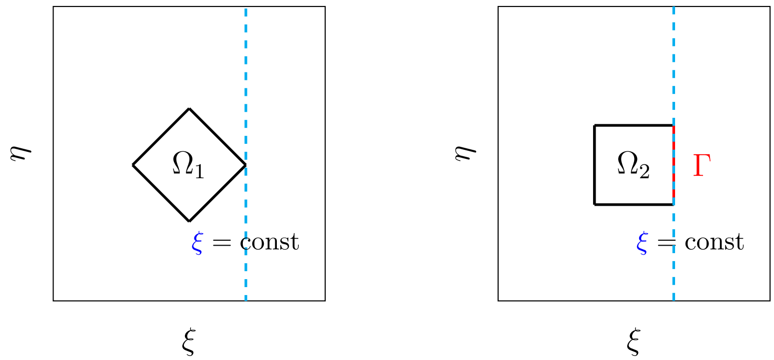

FIG. 6. *Open sets Ω_j and the characteristics touching their boundaries.*

$|\Gamma_{b_\pm}| \neq 0$. Moreover, $F_{a_\pm} := \int_{b_-}^{b_+} \widetilde{F}_2(a_\pm, \eta)d\eta = 2\sqrt{2}a > 0$. This shows in particular that across the lines $\xi = a_\pm$, $\partial_\xi \widetilde{u}_2$ has jump and logarithmic singularities (while $\partial_\eta \widetilde{u}_2$ stays continuous). This example allows to improve the result of Proposition 2.7.

COROLLARY 3.7. *The operator $\mathcal{N}_\omega^+ \in \mathcal{B}(L^2_{comp}(\mathbb{R}^2), H^{1+\sigma}_{loc}(\mathbb{R}^2))$ iff $\sigma \leq 0$.*

Proof. Assume that $\mathcal{N}_\omega^+ \in \mathcal{B}(L^2_{comp}(\mathbb{R}^2), H^{1+\sigma}_{loc}(\mathbb{R}^2))$ for some $\sigma > 0$. Then, since it is a convolution operator, one deduces that $\mathcal{N}_\omega^+ \in \mathcal{B}(H^1_{comp}(\mathbb{R}^2), H^{2+\sigma}_{loc}(\mathbb{R}^2))$. By interpolation, in particular, $\mathcal{N}_\omega^+ \in \mathcal{B}(H^\delta_{comp}(\mathbb{R}^2), H^{1+\sigma+\delta}_{loc}(\mathbb{R}^2))$, for $\delta \in (0,1)$. Consider the function f_2, defined like in the beginning of Section 3, which belongs in particular, to $H^{\frac{1}{2}-\sigma}_{comp}(\mathbb{R}^2)$. This would mean that $u_2 := \mathcal{N}_\omega^+ f_2 \in H^{\frac{3}{2}}(\mathbb{R}^2)$, which is impossible since $\partial_x u_2$, $\partial_y u_2$ have jump singularities. ∎

4 Limiting absorption and limiting amplitude principles. Finally, let us formulate the limiting absorption principle in a strong operator topology.

THEOREM 4.1. *Let $s, s' > \frac{3}{2}$, $0 < \omega < \omega_p$. Let $\omega_n \in \mathbb{C}^+$, $\operatorname{Re}\omega_n > 0$, and $\omega_n \to \omega$ as $n \to +\infty$. Then, for all $f \in L^2_{s,\perp}$,*

$$\mathcal{N}_{\omega_n} f \to \mathcal{N}_\omega^+ f \text{ in } H^1_{-s',\perp}(\mathbb{R}^2).$$

Proof. The proof is quite easy and is based on the explicit representation of the operator $\mathcal{N}_\omega$. Let us fix $s, s' > \frac{3}{2}$ and set $r_n := \mathcal{N}_{\omega_n} f - \mathcal{N}_\omega^+ f$, $\kappa_n := \sqrt{-\varepsilon^{-1}(\omega_n)\xi_x^2 + \omega_n^2}$. Using (2.14), we obtain

$$\kappa \mathcal{F}_x r_n(\xi_x, y) = \frac{1}{2i\sqrt{2\pi}} \int_{\mathbb{R}} \left(\frac{\kappa}{\kappa_n} \mathrm{e}^{i\kappa_n |y-y'|} - \mathrm{e}^{i\kappa|y-y'|} \right) \mathcal{F}_x f(\xi_x, y') dy', \tag{4.1}$$

$$\partial_y \mathcal{F}_x r_n(\xi_x, y) = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \left(\mathrm{e}^{i\kappa_n |y-y'|} - \mathrm{e}^{i\kappa|y-y'|} \right) \mathcal{F}_x f(\xi_x, y') \operatorname{sign}(y-y') dy'. \tag{4.2}$$

Recall the norm equivalence (2.16). We will show that $\lim_{n\to+\infty} \|\kappa \mathcal{F}_x r_n\|_{L^2_{-s',\perp}} = 0$; the analogous result for $\partial_y \mathcal{F}_x r_n$ will follow in the same way.

Step 1. A few auxiliary bounds. First, remark that, as $\operatorname{Im}\kappa_n \geq 0$,

$$\left| \frac{\kappa}{\kappa_n} \mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|} \right| \lesssim \left| \frac{\kappa}{\kappa_n} - 1 \right| + \left| \mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|} \right|. \tag{4.3}$$

Evidently, we have in particular

$$\left|\frac{\kappa}{\kappa_n}\mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|}\right| \lesssim 1. \tag{4.4}$$

A finer bound can be obtained by remarking that the function

$$\omega \to \kappa(\omega) := \sqrt{\omega^2 - \varepsilon^{-1}(\omega)\xi_x^2}$$

is uniformly Lipschitz on all compact subsets of $\{z : 0 < \operatorname{Re} z < \omega_p\}$. Let $\delta > 0$ be sufficiently small. With $B_\delta^+(\omega) = \mathbb{C}^+ \cap B_\delta(\omega)$, for all n sufficiently large, it holds that

$$|\kappa - \kappa_n| \lesssim \sup_{z \in B_\delta^+(\omega)} \left|\frac{\partial \kappa}{\partial \omega}(z)\right| |\omega - \omega_n|, \quad \left|\frac{\partial \kappa}{\partial \omega}(z)\right| = \left|\frac{2z - (\varepsilon^{-1}(z))'\xi_x^2}{2\sqrt{z^2 - \varepsilon^{-1}(z)\xi_x^2}}\right|.$$

Therefore,

$$|\kappa - \kappa_n| \lesssim \max(|\xi_x|, 1)|\omega_n - \omega|. \tag{4.5}$$

Similarly, since for $|\omega_n - \omega| \to 0$, $|\kappa_n| \gtrsim |\xi_x| + 1$, we conclude from the above that

$$\left|\frac{\kappa}{\kappa_n} - 1\right| \lesssim |\omega_n - \omega|. \tag{4.6}$$

As for the second term in (4.3), since $\operatorname{Im} \kappa_n > 0$, the same argument as above gives

$$\left|\mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|}\right| \lesssim |y-y'||\kappa_n - \kappa| \overset{(4.5)}{\lesssim} |\omega_n - \omega||y - y'| \max(|\xi_x|, 1). \tag{4.7}$$

Combining (4.6) and (4.7), and using the fact that all the quantities in the left-hand-side of (4.3) are bounded uniformly in y, ξ_x and for all ω_n sufficiently close to ω (cf. (4.4)), we obtain the following bound valid for all n sufficiently large:

$$\left|\frac{\kappa_n}{\kappa}\mathrm{e}^{i\kappa_n|y-y'|} - \mathrm{e}^{i\kappa|y-y'|}\right| \lesssim \min(1, |\omega_n - \omega||y - y'| \max(|\xi_x|, 1)). \tag{4.8}$$

Step 2. Splitting in high and low frequencies. Next, let us split

$$\mathcal{F}_x r_n(\xi_x, y) = \hat{r}_n^{lf}(\xi_x, y) + \hat{r}_n^{hf}(\xi_x, y),$$
$$\hat{r}_n^{lf}(\xi_x, y) = \mathbb{1}_{|\xi_x|<A}\hat{r}_n(\xi_x, y), \quad \hat{r}_n^{hf}(\xi_x, y) = \mathbb{1}_{|\xi_x|\geq A}\hat{r}_n(\xi_x, y),$$

where $A > 1$ will be chosen later. We will estimate these two quantities separately.

Step 2.1. Estimating $\hat{r}_n^{hf}(\xi_x, y)$. We use a uniform bound (4.4) in (4.1), which yields

$$|\kappa \hat{r}_n^{hf}(\xi_x, y)| \lesssim \int_{\mathbb{R}} |\mathcal{F}_x(\xi_x, y')| dy' \lesssim \left(\int_{\mathbb{R}} (1 + y'^2)^s |\mathcal{F}_x(\xi_x, y')|^2 dy'\right)^{\frac{1}{2}},$$

where the last bound follows from the Cauchy-Schwarz inequality and $s > \frac{1}{2}$. From the definition of $\hat{r}_n^{hf}(\xi_x, y)$ and $s' > \frac{1}{2}$ it follows that

$$\|\kappa \hat{r}_n^{hf}\|^2_{L^2_{-s',\perp}} \lesssim \int_{|\xi_x|>A} \int_{\mathbb{R}} (1 + y'^2)^s |\mathcal{F}_x(\xi_x, y')|^2 dy' d\xi_x. \tag{4.9}$$

Step 2.2. Estimating $\hat{r}_n^{lf}(\xi_x, y)$. To estimate $\hat{r}_n^{lf}(\xi_x, y)$, we use the estimate (4.8) for small $|\omega - \omega_n|$ in (4.1) which results in

$$\left|\kappa \hat{r}_n^{lf}(\xi_x, y)\right| \lesssim A|\omega_n - \omega| \int_{\mathbb{R}} (|y| + |y'|)|\mathcal{F}_x f(\xi_x, y')|dy',$$

and using the Cauchy-Schwarz inequality ($s > \frac{3}{2}$) yields

$$\left|\kappa \hat{r}_n^{lf}(\xi_x, y)\right| \lesssim A|\omega_n - \omega| \left(|y| + 1\right) \|\mathcal{F}_x f(\xi_x, .)\|_{L^2_s(\mathbb{R})}.$$

Finally, we obtain ($s' > \frac{3}{2}$)

$$\|\kappa \hat{r}_n^{lf}\|^2_{L^2_{-s',\perp}} \lesssim A^2|\omega_n - \omega|^2 \|\mathcal{F}_x f\|^2_{L^2_{s,\perp}}. \tag{4.10}$$

Step 2.3. Summary. Combining (4.9), (4.10) yields

$$\|\kappa \hat{r}_n\|^2_{L^2_{-s',\perp}} \lesssim A^2|\omega_n - \omega|^2 \|\mathcal{F}_x f\|^2_{L^2_{s,\perp}} + \int_{|\xi_x|>A} \int_{\mathbb{R}} (1 + y'^2)^s |\mathcal{F}_x(\xi_x, y')|^2 dy' d\xi_x.$$

For any $\varepsilon > 0$, we can choose $A := A_\varepsilon$ so that the last term of the above expression does not exceed $\varepsilon^2/2$; next we choose n so that $A_\varepsilon^2|\omega_n - \omega|^2 \|\mathcal{F}_x f\|^2_{L^2_{s,\perp}} < \frac{\varepsilon^2}{2}$, which allows us to conclude that $\|\kappa \hat{r}_n\|_{L^2_{-s',\perp}} \to 0$, as $n \to +\infty$. ∎

It is seen in the above proof that to obtain (4.10), it is necessary to have the constraints on the weights $s, s' > \frac{3}{2}$ in the scale of the weighted Sobolev spaces with polynomial weights. A finer result could be obtained by using Hörmander (Fourier transforms of Besov) spaces.

Using the classical techniques of Eidus, cf. [15], it is possible to prove the limiting amplitude principle. The proof of this result can be found in the technical report [21].

THEOREM 4.2. *Let $s > \frac{3}{2}$, $f \in L^2_s(\mathbb{R}^2)$, and $0 < \omega < \omega_p$. Let $(\boldsymbol{E}, H_z, j)$ solve*

$$\begin{aligned}
&\partial_t E_x - \partial_y H_z = 0,\\
&\partial_t E_y + \partial_x H_z + j = 0, \quad \partial_t j - \omega_p^2 E_y = 0,\\
&\partial_t H_z + \partial_x E_y - \partial_y E_x = f\mathrm{e}^{i\omega t},\\
&H_z(0) = E_x(0) = E_y(0) = j(0) = 0.
\end{aligned}$$

Then, for all $s' > \frac{3}{2}$, $\lim_{t\to+\infty} \|H_z(t, .) - h_z(.)\mathrm{e}^{i\omega t}\|_{L^2_{-s'}} = 0$, where $h_z = -i\omega\mathcal{N}^+_\omega f$, cf. (2.10). In other words, $h_z \in H^1_{-s',\perp}$ is the unique solution to

$$\omega^2 h_z - \alpha^2 \partial_x^2 h_z + \partial_y^2 h_z = -i\omega f,$$

equipped with the radiation condition (RC1), (RC2).

5 Conclusions. In this work we have studied a model for wave propagation in a hyperbolic metamaterial in the free space, described by the Klein-Gordon equation. With the help of a suitable radiation condition, we have shown its well-posedness; a detailed regularity analysis is presented. Our future efforts are directed towards the study of a more mathematically involved case of propagation in the exterior domains, as well as the design of numerical methods for this kind of problems.

Appendix A. Derivation of (1.2). Electromagnetic wave propagation in a three-dimensional cold collisionless plasma under a background magnetic field $\mathbf{B}_0 = (0,\ B_0,\ 0)$ is described by Maxwell's equations

$$\partial_t \mathbf{D} - \operatorname{curl} \mathbf{H} = 0, \qquad \partial_t \mathbf{B} + \operatorname{curl} \mathbf{E} = 0. \tag{A.1}$$

Here $\mathbf{B} = \mu_0 \mathbf{H}$, and the relation between $\mathbf{D}$ and $\mathbf{E}$ is given in the frequency domain by $\widehat{\mathbf{D}} = \underline{\underline{\varepsilon_{cp}}}(\omega)\widehat{\mathbf{E}}$, where $\underline{\underline{\varepsilon_{cp}}}(\omega)$ is the cold plasma dielectric tensor, see [31, (18), (25)] or [16, Chapter 15.5]. In the simplest case when the plasma is comprised of particles of a single species with mass m and charge q, and whose number density is $N = N(\mathbf{x})$, this tensor reads

$$\underline{\underline{\varepsilon_{cp}}}(\omega) = \varepsilon_0 \begin{pmatrix} 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 & -i\frac{\omega_p^2 \omega_c}{\omega(\omega^2 - \omega_c^2)} \\ 0 & 1 - \frac{\omega_p^2}{\omega^2} & 0 \\ i\frac{\omega_p^2 \omega_c}{\omega(\omega^2 - \omega_c^2)} & 0 & 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \end{pmatrix}, \tag{A.2}$$

where $\omega_p = \sqrt{\frac{Nq^2}{m\varepsilon_0}}$ is the plasma frequency and $\omega_c = \frac{qB_0}{m}$ is the cyclotron frequency. In what follows we will assume that the density N is uniform in space, i.e. $\omega_p = \text{const}$.

In the strong magnetic field limit ($|B_0| \to +\infty$, or $|\omega_c| \to +\infty$), the cold plasma dielectric tensor reduces to a diagonal matrix

$$\underline{\underline{\varepsilon}}(\omega) = \varepsilon_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{\omega_p^2}{\omega^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.3}$$

In order to rewrite the Maxwell system in the time domain, we first consider the relation between D_y and E_y

$$\hat{D}_y = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \hat{E}_y \implies -i\omega \hat{D}_y = -i\omega\varepsilon_0 \hat{E}_y + \varepsilon_0 \frac{\omega_p^2}{(-i\omega)} \hat{E}_y. \tag{A.4}$$

Let us define an auxiliary unknown (a current), so that, in the frequency domain $\hat{j} = \varepsilon_0 \frac{\omega_p^2}{(-i\omega)} \hat{E}_y$, or, in the time domain,

$$\partial_t j - \varepsilon_0 \omega_p^2 E_y = 0.$$

This allows to express

$$\partial_t D_y = \varepsilon_0 \partial_t E_y + j.$$

With this notation (A.1) reads (where $\mathbf{e}_y = (0, 1, 0)^T$)

$$\begin{aligned} &\varepsilon_0 \partial_t \mathbf{E} - \operatorname{curl} \mathbf{H} + j\mathbf{e}_y = 0, \qquad \partial_t j - \varepsilon_0 \omega_p^2 E_y = 0, \\ &\mu_0 \partial_t \mathbf{H} + \operatorname{curl} \mathbf{E} = 0. \end{aligned}$$

In the case when the fields do not depend on the space variable z, the above system is decoupled into the TE system (with respect to E_x, E_y, H_z, j) and the TM system (with respect to H_x, H_y, E_z). While the TM system is the same as in the vacuum (this is left as an easy exercise to the reader), the TE system reads

$$\begin{aligned}&\varepsilon_0\partial_t E_x - \partial_y H_z = 0,\\ &\varepsilon_0\partial_t E_y + \partial_x H_z + j = 0, \qquad \partial_t j - \varepsilon_0\omega_p^2 E_y = 0,\\ &\mu_0\partial_t H_z + \partial_x E_y - \partial_y E_x = 0.\end{aligned} \tag{A.5}$$

Appendix B. Computation of the fundamental solution $\mathcal{G}_\omega$. Recall that we choose $\sqrt{z}$ as the branch of the square root, with the branch cut along $(-\infty, 0]$. By $\operatorname{Arg} z \in (-\pi, \pi]$ we denote the principal argument of z. Before studying the fundamental solution for the equation (1.8), we first consider the following problem. Let us assume that $\operatorname{Im}\omega \neq 0$, and $a > 0$. Consider the fundamental solution for a scaled Helmholtz equation with the frequency ω, i.e. the unique $G^a_\omega \in \mathcal{S}'$ solving

$$\omega^2 G^a_\omega(\mathbf{x}) + a^{-1}\partial_x^2 G^a_\omega(\mathbf{x}) + \partial_y^2 G^a_\omega(\mathbf{x}) = \delta(\mathbf{x}). \tag{B.1}$$

It can be verified that the fundamental solution G^a_ω is defined by

$$G^a_\omega(\mathbf{x}) = -\frac{i\sqrt{a}}{4}\begin{cases} H_0^{(1)}(\omega\sqrt{ax^2+y^2}), & \operatorname{Im}\omega > 0,\\ H_0^{(2)}(\omega\sqrt{ax^2+y^2}), & \operatorname{Im}\omega < 0,\end{cases} \tag{B.2}$$

where $H_0^{(1)}(z)$ ($H_0^{(2)}(z)$) is the Hankel function of the first (second) kind (see [1, Chapter 9]). It is analytic in $\mathbb{C}\setminus\mathbb{R}_-$, where $\mathbb{R}_- = \{z : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$.

Performing a partial Fourier transform of (B.1) in x, we can obtain explicitly $\mathcal{F}_x G^a_\omega$ as the fundamental solution of a 1D Helmholtz equation. After a series of elementary computations, we obtain

$$G^a_\omega(x,y) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\xi_x x}\frac{\mathrm{e}^{-\sqrt{a^{-1}\xi_x^2-\omega^2}|y|}}{\sqrt{a^{-1}\xi_x^2-\omega^2}}d\xi_x, \qquad a > 0. \tag{B.3}$$

Let us now obtain the fundamental solution for (1.8), i.e. the solution of

$$\omega^2\mathcal{G}_\omega(\mathbf{x}) + \varepsilon(\omega)^{-1}\partial_x^2\mathcal{G}_\omega(\mathbf{x}) + \partial_y^2\mathcal{G}_\omega(\mathbf{x}) = \delta(\mathbf{x}). \tag{B.4}$$

We cannot immediately write $\mathcal{G}_\omega$ using (B.2), because $\varepsilon(\omega)$ in the above is complex, and, in general, a slightly stronger argument is needed. For this we will use (B.3), which we will rewrite in an appropriate form that will allow to use an analytic continuation argument.

Performing the partial Fourier transform of (B.4) in x yields

$$\partial_y^2\left(\mathcal{F}_x\mathcal{G}_\omega\right) - (\varepsilon(\omega)^{-1}\xi_x^2 - \omega^2)\mathcal{F}_x\mathcal{G}_\omega = \frac{\delta(y)}{\sqrt{2\pi}}. \tag{B.5}$$

By definition, $\mathcal{F}_x\mathcal{G}_\omega$ is the fundamental solution of a 1D Helmholtz equation with absorption. To see this we remark that

$$(\varepsilon(\omega)^{-1}\xi_x^2 - \omega^2) \notin \mathbb{R}^-. \tag{B.6}$$

The justification of the above follows by a direct computation. In particular,

$$\operatorname{Im}(\varepsilon(\omega)^{-1}\xi_x^2 - \omega^2) = \operatorname{Im}\varepsilon(\omega)^{-1}\xi_x^2 - \operatorname{Im}\omega^2, \text{ and}$$

$$\operatorname{sign}\operatorname{Im}\varepsilon(\omega)^{-1} = -\operatorname{sign}\operatorname{Im}\varepsilon(\omega) = \operatorname{sign}\operatorname{Im}\frac{\omega_p^2}{\omega^2} = -\operatorname{sign}\operatorname{Im}\omega^2. \tag{B.7}$$

Therefore, for $\omega = \omega_r + i\omega_i$, with ω_i, $\omega_r \neq 0$,

$$\operatorname{sign}\operatorname{Im}(\varepsilon(\omega)^{-1}\xi_x^2 - \omega^2) = -\operatorname{sign}\omega_i\omega_r \neq 0, \tag{B.8}$$

while when $\omega_r = 0$, $\varepsilon(\omega)^{-1}\xi_x^2 - \omega^2 > 0$. This shows (B.6). Let us define

$$s(\xi_x, \omega) = \sqrt{\varepsilon(\omega)^{-1}\xi_x^2 - \omega^2}.$$

By the above considerations, the function $\omega \mapsto s(\xi_x, \omega)$ is analytic for all $\omega \in \mathbb{C}^+$.

Next, the fundamental solution $\mathcal{F}_x\mathcal{G}_\omega$ is defined as follows:

$$\mathcal{F}_x\mathcal{G}_\omega(\xi_x, y) = -\frac{1}{2\sqrt{2\pi}}\frac{\mathrm{e}^{-\sqrt{\varepsilon(\omega)^{-1}\xi_x^2-\omega^2}|y|}}{\sqrt{\varepsilon(\omega)^{-1}\xi_x^2 - \omega^2}}. \tag{B.9}$$

For $y \neq 0$, $\mathcal{F}_x\mathcal{G}_\omega(.,y) \in L^1(\mathbb{R})$; we also have

$$\mathcal{G}_\omega(x,y) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\xi_x x}\frac{\mathrm{e}^{-\sqrt{\varepsilon(\omega)^{-1}\xi_x^2-\omega^2}|y|}}{\sqrt{\varepsilon(\omega)^{-1}\xi_x^2 - \omega^2}}d\xi_x. \tag{B.10}$$

To compute the inverse Fourier transform, we remark the following:

- for $y \neq 0$, $\omega \mapsto \mathcal{G}_\omega(x,y)$ defined as above is analytic in $\mathbb{C}^+$. This follows from the analyticity of $\omega \mapsto \frac{\mathrm{e}^{-s(\xi_x,\omega)}}{s(\xi_x,\omega)}$ in $\mathbb{C}^+$ and uniform boundedness of its derivatives by an L^1-function of ξ_x on compact subsets of $\mathbb{C}^+$.
  The same can be said about the analyticity of $\omega \mapsto \mathcal{G}_\omega(x,y)$ in $\mathbb{C}^-$.
- for $\omega \in i\mathbb{R}^*$, we have $\varepsilon(\omega) > 0$. We thus reduce to the case (B.3), for which the inverse Fourier transform is known and given by

$$\mathcal{G}_\omega(\mathbf{x}) = -\frac{i\sqrt{\varepsilon(\omega)}}{4}\begin{cases} H_0^{(1)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2}), & \operatorname{Im}\omega > 0, \\ H_0^{(2)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2}), & \operatorname{Im}\omega < 0. \end{cases} \tag{B.11}$$

- for $(x,y) \neq 0$, the function $\omega \mapsto -\frac{i\sqrt{\varepsilon(\omega)}}{4}H_0^{(1)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2})$ is analytic in $\mathbb{C}^+$. To verify this, it suffices to check that $\omega\sqrt{\varepsilon(\omega)x^2+y^2} \notin \mathbb{R}^-$ (the branch cut of $H_0^{(1)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2})$). This being obvious for $\omega \in i\mathbb{R}^*$, let us consider the case $\operatorname{Re}\omega \neq 0$. Then

$$\operatorname{Im}\left(\omega\sqrt{\varepsilon(\omega)x^2+y^2}\right) = \operatorname{Im}\omega\operatorname{Re}\sqrt{\varepsilon(\omega)x^2+y^2} + \operatorname{Re}\omega\operatorname{Im}\sqrt{\varepsilon(\omega)x^2+y^2}.$$

  For $\operatorname{Im}\omega > 0$, the first term above is positive; the second term, cf. (B.7), as $\operatorname{sign}\operatorname{Im}\varepsilon(\omega) = \operatorname{sign}\operatorname{Im}\omega^2 = \operatorname{sign}\operatorname{Re}\omega$ is positive as well.
  Therefore, $\omega \mapsto -\frac{i\sqrt{\varepsilon(\omega)}}{4}H_0^{(1)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2})$ is analytic in $\mathbb{C}^+$.
  In the same way we check that $\omega \mapsto -\frac{i\sqrt{\varepsilon(\omega)}}{4}H_0^{(2)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2})$ is analytic in $\mathbb{C}^-$.

Using the analytic continuation argument, (B.10) being equal to (B.11) on $i\mathbb{R}^+$, and analyticity of both functions, we conclude that, for $|y| \neq 0$, (B.10) coincides with (B.11). For $|y| = 0$, the result follows immediately by noticing that $\mathcal{F}_x\mathcal{G}_\omega \in L^2(\mathbb{R}^2)$. Thus

$$\mathcal{G}_\omega(\mathbf{x}) = -\frac{i\sqrt{\varepsilon(\omega)}}{4}\begin{cases} H_0^{(1)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2}), & \operatorname{Im}\omega > 0, \\ H_0^{(2)}(\omega\sqrt{\varepsilon(\omega)x^2+y^2}), & \operatorname{Im}\omega < 0. \end{cases} \tag{B.12}$$

Appendix C. Proof of (2.12). By definition, $\mathcal{G}_\omega^+ = \lim\limits_{\operatorname{Im}\omega\to 0+} \mathcal{G}_\omega$.

Let us assume that $\operatorname{Im}\omega > 0$. Starting with (B.9), let us consider the case when $\omega = \omega_r + i\epsilon$, with $0 < \omega_r < \omega_p$, and take $\epsilon \to 0+$. In this case, cf. (B.8), $\lim\limits_{\epsilon\to 0+} \sqrt{\varepsilon(\omega)^{-1}\xi_x^2 - \omega^2} = -i\sqrt{-\varepsilon(\omega_r)^{-1}\xi_x^2 + \omega^2}$, hence the conclusion.

Appendix D. Proof of Statement 2 in Proposition 2.4. In the proof, we will extensively use the following. Because for all $\delta > 0$, we have

$$\begin{aligned} &\operatorname{Im}\left((\omega+i\delta)^2(\varepsilon(\omega+i\delta)x^2+y^2)\right) > 0, \text{ and}\\ &\operatorname{Im}(\omega+i\delta)^2 > 0, \quad \operatorname{Im}(\varepsilon(\omega+i\delta)x^2+y^2) > 0, \end{aligned} \tag{D.1}$$

it follows that

$$\sqrt{(\omega+i\delta)^2(\varepsilon(\omega+i\delta)x^2+y^2)} = (\omega+i\delta)\sqrt{\varepsilon(\omega+i\delta)x^2+y^2}, \tag{D.2}$$

and

$$\log\sqrt{(\omega+i\delta)^2(\varepsilon(\omega+i\delta)x^2+y^2)} = \log(\omega+i\delta) + \frac{1}{2}\log\left(\varepsilon(\omega+i\delta)x^2+y^2\right). \tag{D.3}$$

Let us fix $R > 0$, and show that $\mathcal{G}_{\omega+i\delta} \to \mathcal{G}_\omega^+$ in $L^1(B_R(0))$. The pointwise convergence of $\mathcal{G}_{\omega+i\delta} \to \mathcal{G}_\omega^+$ being obvious, one would want to apply the Lebesgue dominated convergence theorem. This is however not possible, because the logarithmic term above cannot be bounded uniformly in δ by an L^1_{loc}-function. To see this it suffices to notice that $\operatorname{Im}\left(\varepsilon(\omega+i\delta)x^2+y^2\right) = O(\delta)$, and in the points where $\left|\operatorname{Re}\varepsilon(\omega+i\delta)x^2+y^2\right| \le \delta$ (this set is of non-zero measure) one has $\left|\log\left(\varepsilon(\omega+i\delta)x^2+y^2\right)\right| \gtrsim |\log\delta|$.

Let us thus prove the L^1-convergence of the two terms in (2.8) separately. Let

$$l_\delta(\mathbf{x}) := \log(y^2+\varepsilon(\omega+i\delta)x^2), \text{ so that} \tag{D.4}$$

$$\mathcal{G}_{\omega+i\delta} = \frac{\sqrt{\varepsilon(\omega+i\delta)}}{4\pi} l_\delta + \mathcal{G}^{reg}_{\omega+i\delta} + \frac{\sqrt{\varepsilon(\omega+i\delta)}}{2\pi}\log(\omega+i\delta). \tag{D.5}$$

Step 1. L_1-convergence of l_δ. The pointwise limit of $l_\delta(\mathbf{x})$ is the function $l(\mathbf{x})$ defined by (recall that $\alpha = (-\varepsilon(\omega))^{\frac{1}{2}}$, see (2.4)):

$$l(\mathbf{x}) := \begin{cases} \log\left(y^2 - \alpha^{-2}x^2\right), & |y| > \alpha^{-1}|x|,\\ \log(-y^2-\alpha^{-2}x^2) + i\pi, & |y| < \alpha^{-1}|x|. \end{cases}$$

We will study the L^1-convergence separately on the following two domains:

$$\begin{aligned} B_R(0) = K^+ \cup K^-, \quad &K^+ := \{\mathbf{x} \in B_R(0),\, |y| \ge \alpha^{-1}|x|\},\\ &K^- := \{\mathbf{x} \in B_R(0),\, |y| < \alpha^{-1}|x|\}. \end{aligned} \tag{D.6}$$

Step 1.1. Convergence in K^-. Our goal is to show that

$$\lim_{\delta\to 0+} \int_{K^-} |l_\delta(\mathbf{x}) - l(\mathbf{x})|\, d\mathbf{x} = 0.$$

For this we rewrite the above in a more convenient form.

First, we remark that there exists $C > 0$, s.t.

$$|\varepsilon(\omega+i\delta) - \varepsilon(\omega)| \le C\delta, \text{ for all } \delta > 0 \text{ sufficiently small.} \tag{D.7}$$

Choosing δ so that the above holds true and such that $\alpha^{-2} - C\sqrt{\delta} > 0$, we split $K^- = K^-_{sing,\delta} \cup K^-_{reg,\delta}$ (with the constant C as above) defined as follows:

$$\begin{aligned} K^-_{reg,\delta} &= \{\mathbf{x} \in B_R(0) : \quad 0 < y^2 \leq (\alpha^{-2} - C\sqrt{\delta})x^2\}, \\ K^-_{sing,\delta} &= \{\mathbf{x} \in B_R(0) : \quad (\alpha^{-2} - C\sqrt{\delta})x^2 < y^2 < \alpha^{-2}x^2\}. \end{aligned} \tag{D.8}$$

The choice $\sqrt{\delta}$ in the above will be motivated later, cf. (D.11), (D.12).

Step 1.1.1. Convergence on $K^-_{reg,\delta}$. An explicit computation yields

$$\begin{aligned} l_\delta(x,y) - l(x,y) &= \log\left(-\frac{\varepsilon(\omega+i\delta)x^2 + y^2}{\varepsilon(\omega)x^2 + y^2}\right) - i\pi \\ &= \log\left(-1 - \frac{\varepsilon(\omega+i\delta) - \varepsilon(\omega)}{\varepsilon(\omega)x^2 + y^2}x^2\right) - i\pi \\ &= I^{abs}_\delta(x,y) + iI^{arg}_\delta(x,y), \end{aligned} \tag{D.9}$$

$$I^{abs}_\delta(x,y) = \log\left|1 + \frac{\varepsilon(\omega+i\delta) - \varepsilon(\omega)}{y^2 - \alpha^{-2}x^2}x^2\right|,$$

$$I^{arg}_\delta(x,y) = \mathrm{Arg}\left(-1 - \frac{\varepsilon(\omega+i\delta) - \varepsilon(\omega)}{y^2 - \alpha^{-2}x^2}x^2\right) - \pi. \tag{D.10}$$

Let us show that the above converges to zero in $L^1(K^-_{reg,\delta})$.

Convergence of $\|I^{abs}_\delta\|_{L^1(K^-_{reg,\delta})}$. Using the bound (D.7) and the definition of $K^-_{reg,\delta}$ (D.8), where we have $-\alpha^{-2}x^2 < y^2 - \alpha^{-2}x^2 \leq -C\sqrt{\delta}x^2$, we obtain

$$\left|\frac{\varepsilon(\omega+i\delta) - \varepsilon(\omega)}{y^2 - \alpha^{-2}x^2}x^2\right| \leq \sqrt{\delta}, \quad \forall \mathbf{x} \in K^-_{reg,\delta}. \tag{D.11}$$

Therefore, for all δ sufficiently small, we have that $\|I^{abs}_\delta \mathbb{1}_{K^-_{reg,\delta}}\|_{L^1(K^-)} \lesssim \sqrt{\delta}$, thus

$$\lim_{\delta\to 0+} \|I^{abs}_\delta\|_{L^1(K^-_{reg,\delta})} = 0. \tag{D.12}$$

Convergence of $\|I^{arg}_\delta\|_{L^1(K^-_{reg,\delta})}$. Let us examine the real and imaginary parts of the argument of Arg in (D.10). With (D.11) we have that

$$\mathrm{Re}\left(-1 - \frac{\varepsilon(\omega+i\delta) - \varepsilon(\omega)}{y^2 - \alpha^{-2}x^2}x^2\right) = -1 + O(\sqrt{\delta}). \tag{D.13}$$

Using the definition of $K^-_{reg,\delta}$ in (D.8) and the fact that $\mathrm{Im}\,\varepsilon(\omega+i\delta) > 0$ (this follows by a direct computation), we obtain the following inequality:

$$\mathrm{Im}\left(-1 - \frac{\varepsilon(\omega+i\delta) - \varepsilon(\omega)}{y^2 - \alpha^{-2}x^2}x^2\right) = \mathrm{Im}\,\frac{\varepsilon(\omega+i\delta)x^2}{\alpha^{-2}x^2 - y^2} > 0 \text{ in } K^-_{reg,\delta}. \tag{D.14}$$

With $\mathrm{Im}\,\varepsilon(\omega+i\delta) = O(\delta)$ and the definition of $K^-_{reg,\delta}$ in (D.8), we also have

$$\mathrm{Im}\left(-1 - \frac{\varepsilon(\omega+i\delta) - \varepsilon(\omega)}{y^2 - \alpha^{-2}x^2}x^2\right) = O(\sqrt{\delta}). \tag{D.15}$$

Combining (D.13), (D.14), (D.15), we conclude that inside $K^-_{reg,\delta}$, it holds that:

$$\lim_{\delta\to 0} I^{arg}_\delta(\mathbf{x}) = 0, \quad \forall \mathbf{x}\in K^-_{reg,\delta}, \text{ thus}$$

$$\lim_{\delta\to 0} \|I^{arg}_\delta\|_{L^1(K^-_{reg,\delta})} = 0. \tag{D.16}$$

Summary. Combination of (D.12), (D.16) and (D.9) yields

$$\lim_{\delta\to 0}\|l_\delta - l\|_{L^1(K^-_{reg,\delta})} = 0. \tag{D.17}$$

Step 1.1.2. Convergence on $K^-_{sing,\delta}$. We will prove the following:

$$\lim_{\delta\to 0}\|l_\delta\|_{L^1(K^-_{sing,\delta})} = \lim_{\delta\to 0}\|l\|_{L^1(K^-_{sing,\delta})} = 0. \tag{D.18}$$

The result is obvious for $l\in L^1(B_R(0))$, by the Lebesgue's dominated convergence theorem. Let us prove it for l_δ by a direct computation. First of all, we remark that

$$\|l_\delta\|_{L^1(K^-_{sing,\delta})} \le \|\operatorname{Re} l_\delta\|_{L^1(K^-_{sing,\delta})} + \|\operatorname{Im} l_\delta\|_{L^1(K^-_{sing,\delta})}, \tag{D.19}$$

and from (D.4), because $|\operatorname{Im} l_\delta| \le \pi$, with the Lebesgue's dominated convergence theorem it follows that

$$\lim_{\delta\to 0}\|\operatorname{Im} l_\delta\|_{L^1(K^-_{sing,\delta})} = 0. \tag{D.20}$$

It remains to prove the result for $\operatorname{Re} l_\delta = \log|\varepsilon(\omega+i\delta)x^2+y^2|$. We rewrite

$$\varepsilon(\omega+i\delta)x^2+y^2 = (-\alpha^{-2}x^2+y^2) + x^2\left(\varepsilon(\omega+i\delta)-\varepsilon(\omega)\right),$$

and by definition of $K^-_{sing,\delta}$ (applied to estimate the first term above), as well as analyticity of ε, we conclude that the above quantity is $O(\sqrt{\delta})$, and thus

$$|\operatorname{Re} l_\delta| = \left|\log|\varepsilon(\omega+i\delta)x^2+y^2|\right| \lesssim |\log\delta|.$$

By definition of $K^-_{sing,\delta}$,

$$\|\operatorname{Re} l_\delta\|_{L^1(K^-_{sing,\delta})} \lesssim \int\limits_{K^-_{sing,\delta}} |\log\delta| d\mathbf{x} \lesssim \sqrt{\delta}|\log\delta|. \tag{D.21}$$

This, combined with (D.19), proves (D.18).

Step 1.1.3. Convergence in K^-. Combining (D.18), (D.17) and (D.8), we conclude that

$$\|l_\delta - l\|_{L^1(K^-)} \to 0. \tag{D.22}$$

Step 1.2. Convergence $\|l_\delta - l\|_{L^1(K^+)}\to 0$. The proof mimics the proof of the analogous result for K^-, hence we omit it here.

Step 1.3. Conclusion. Combination of the results of Steps 1.1 and 1.2, together with (D.8) results in the desired statement

$$\lim_{\delta\to 0}\|l_\delta - l\|_{L^1(B_R(0))} = 0. \tag{D.23}$$

Step 2. Proof of convergence of $\mathcal{G}^{reg}_{\omega+i\delta}$ to its pointwise limit in $L^1(B_R(0))$. To prove the result, we show that the following bound holds for $\mathcal{G}^{reg}_{\omega+i\delta}$ and all $\delta > 0$ sufficiently small:

$$\left\|\mathcal{G}^{reg}_{\omega+i\delta}\right\|_{L^\infty(B_R(0))} \lesssim 1. \tag{D.24}$$

To show this bound, it suffices to prove two bounds, cf. the explicit expression for $\mathcal{G}_{\omega+i\delta}$ in (2.8),

$$\sup_{(x,y,\delta)\in B_R(0)\times(0,1)} |g_J(z_\delta)|, \quad \sup_{(x,y,\delta)\in B_R(0)\times(0,1)} |g_Y(z_\delta)| \lesssim 1, \tag{D.25}$$

$$\sup_{(x,y,\delta)\in B_R(0)\times(0,1)} |g_J(z_\delta)\log z_\delta| \lesssim 1. \tag{D.26}$$

To prove the above we remark that the application

$$Z_\delta : (x,y,\delta) \to z_\delta \tag{D.27}$$

maps $B_R(0)\times(0,1)$ into a bounded subset $\mathcal{C}$ of $\mathbb{C}^+$. Then

- (D.25) follows from the analyticity of $g_J(z)$, $g_Y(z)$.
- (D.26) can be obtained using the following argument. The function $z \to g_J(z)\log z$ is analytic in $\mathbb{C}\setminus(-\infty,0)$. Also,

  $$\sup_{(x,y,\delta)\in B_R(0)\times(0,1)} |g_J(z_\delta)\log z_\delta| = \sup_{z\in\mathcal{C}} |g_J(z)\log z| = \sup_{z\in\bar{\mathcal{C}}} |g_J(z)\log z|,$$

  which is bounded because 1) $\bar{\mathcal{C}} \subset \mathbb{C}^+\cup\mathbb{R}$ and $\bar{\mathcal{C}}$ is bounded; 2) as $g_J(0)=0$ and is analytic, the function $z \to g_J(z)\log z,\ z\in\mathbb{C}^+$, can be defined by continuity up to $\mathbb{R}$, and is bounded on compact subsets of $\mathbb{C}^+\cup\mathbb{R}$.

With the bound (D.24), and Lebesgue's dominated convergence theorem, we deduce that as $\delta\to 0$, $\mathcal{G}^{reg}_{\omega+i\delta}$ converges to its pointwise limit in L^1.

Step 3. Conclusion. Combining the results of Steps 1 and 2, together with the splitting (2.8), we deduce that $\mathcal{G}_{\omega+i\delta} \to \mathcal{G}^+_\omega$ in $L^1(B_R(0))$, as $\delta\to 0$.

Appendix E. Proof of Lemma 2.5.

For $|x| > \alpha|y|$, by (FS) on page 6, we have

$$\mathcal{G}^+_\omega(x,y) = \frac{1}{4\alpha} H_0^{(1)}(i\omega\sqrt{\alpha^{-2}x^2-y^2}). \tag{E.1}$$

By [1, formulas 9.6.4, 9.6.23],

$$H_0^{(1)}(i\omega\sqrt{\alpha^{-2}x^2-y^2}) = \frac{2}{i\pi}\int_1^\infty \mathrm{e}^{-\omega\sqrt{\alpha^{-2}x^2-y^2}\,t}(t^2-1)^{-\frac{1}{2}}\,dt$$

$$= \frac{2}{i\pi}\int_0^\infty \frac{\mathrm{e}^{-\omega\sqrt{\alpha^{-2}x^2-y^2}(\eta+1)}}{\sqrt{\eta}\sqrt{\eta+2}}\,d\eta.$$

Because $|x| > \alpha|y|+\delta$, $\sqrt{\alpha^{-2}x^2-y^2} > \sqrt{\alpha^{-2}(\alpha|y|+\delta)^2-y^2} \geq \alpha^{-1}\delta$. Therefore,

$$\left|H_0^{(1)}(i\omega\sqrt{\alpha^{-2}x^2-y^2})\right| \lesssim \mathrm{e}^{-\omega\sqrt{\alpha^{-2}x^2-y^2}}\int_0^\infty \frac{\mathrm{e}^{-\omega\alpha^{-1}\delta\eta}}{\sqrt{\eta}\sqrt{\eta+2}}\,d\eta$$

$$= c_{\alpha,\delta}\mathrm{e}^{-\omega\sqrt{\alpha^{-2}x^2-y^2}}, \quad c_{\alpha,\delta} > 0.$$

Combining the above bound with (E.1) results in the desired statement of the lemma.

Appendix F. Sobolev style regularity results. Let us introduce the following norm and function spaces tailored to meet the requirements of Lemma 3.5:

$$\|\phi\|^2_{X^0} := \|\phi\|^2 + \left\| \int_{-\infty}^{\infty} \phi(.,\eta')d\eta' \right\|^2_{H^1(\mathbb{R})} + \left\| \int_{-\infty}^{\infty} \phi(\xi',.)d\xi' \right\|^2_{H^1(\mathbb{R})}$$

$$+ \left\| \partial_\xi \int_{-\infty}^{\eta} \phi(\xi,\eta')d\eta' \right\|^2 + \left\| \partial_\eta \int_{-\infty}^{\xi} \phi(\xi',\eta)d\xi' \right\|^2,$$

$$X^0(\mathbb{R}^2) := \overline{C_0^\infty(\mathbb{R}^2)}^{X^0},$$

$$X^0_{comp}(\mathbb{R}^2) := \{f \in X^0(\mathbb{R}^2) : \operatorname{supp} f \text{ is bounded}\}.$$

We then have the following result.

THEOREM F.1. *The operator* $\mathcal{N}^+_\omega \in \mathcal{B}\left(X^0_{comp}(\mathbb{R}^2), H^2_{loc}(\mathbb{R}^2)\right)$.

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