Energy Environment:
Science Technology and Management
(STEEM)

Fluid-structure interaction problems in marine renewable energies

Lesson 1 : Definition of the fluid-structure interaction problem

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Outline of the course

Block 1 - Fluid-structure interactions *(Olivier Doaré, ENSTA Paristech, Clément Grouthier, WGK)*

- Definition of the problem
- Structural dynamics
- Dimensional analysis

- Structural vibrations in still fluid
- Structural vibrations in flows
- Positive ineractions : Energy harvesting using flow-structure instabilities
- Negative interaction : design methods in offshore wind

- Floating structures

3 sessions
September/October

Block 2 - Marine energies

- Offshore wind introduction and ocean thermal energies *(Vincent de Laleu, EDF)*

- Ocean waves energy harvesting *(Jean-Christophe Gilloteaux, IFPEN)*

- Offshore wind installation : case studies *(Christophe Peyrard, EDF)*

- Marine currents *(Nicolas Relun, EDF)*

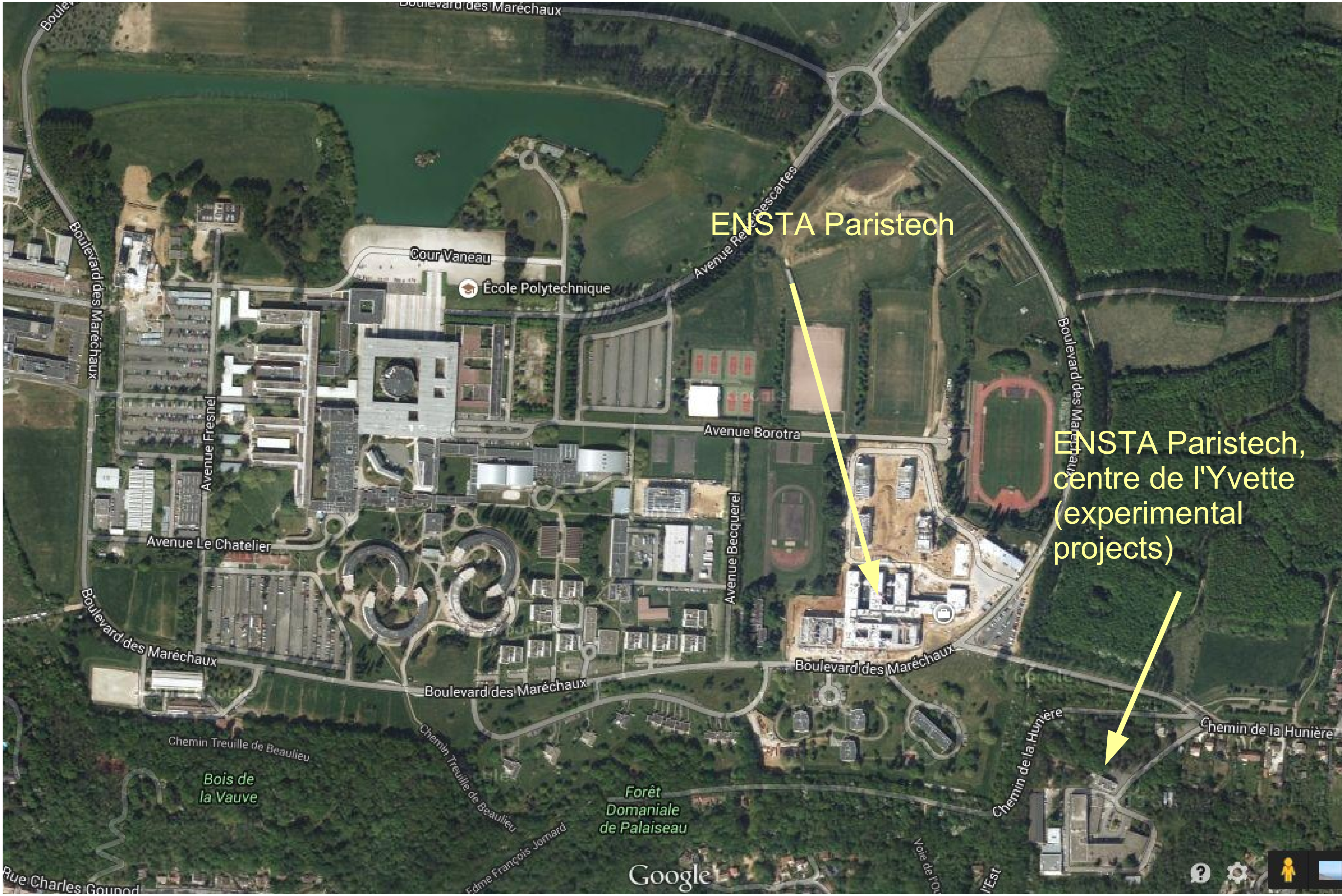
4 sessions
October/november

Block 3 – Numerical projects *(Olivier Doaré, ENSTA Paristech)*

- Offshore wind turbine submitted to wind, current and waves : matlab project

2 sessions
November

Where will it take place ?



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Evaluation

On table exam
(small questions, calculations,
proposed by each professor)

+

Matlab numerical project
(statics and dynamics of
an offshore wind turbine)

2

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Block 1

- **Day 1 (Today)**
  - Definition of fluid-structure interaction
  - Structural mechanics
  - Dimensional analysis

- **Day 2 (Next week)**
  - Fluid-structure interaction phenomena
  - Negative interactions : Dammage of offshore structures
  - Positive interactions : New concepts of energy harveting using fluid-structure interaction

- **Day 3 (October, 19)**
  - Floating structures

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Fluid-structure interaction

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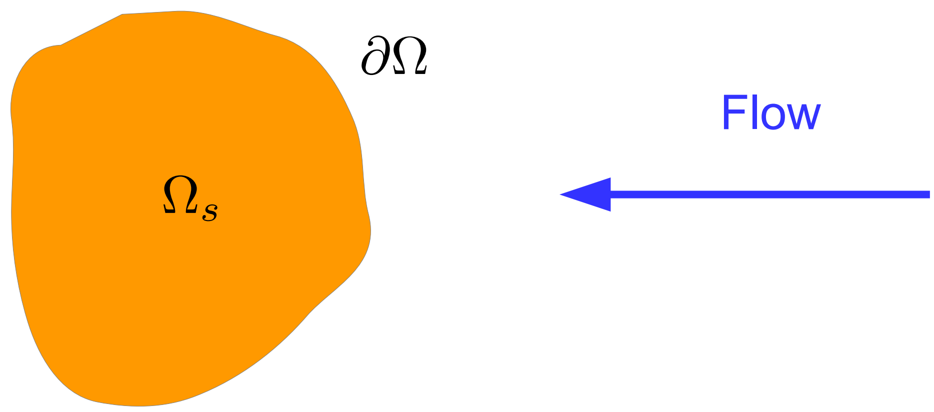

Un-coupled problems

Solid-mechanics problem

The dynamics of the solid in vacuum is studied.

But what happens if there is a flow generated by the solid's displacement ?

What is the influence of the flow on the solid's dynamics ?

Fluid-mechanics problem

The solid is viewed as a perfectly rigid boundary by the flow.

But what happens if the structures deforms after being stressed by the flow ?

How the flow properties change after solid's deformation ?

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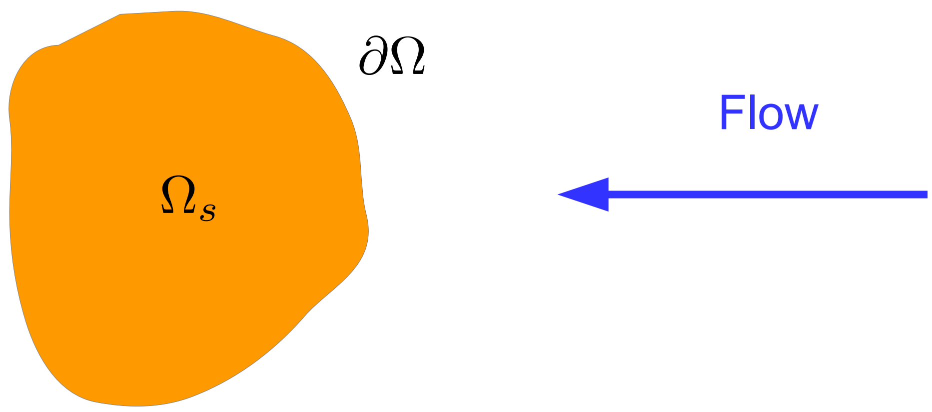

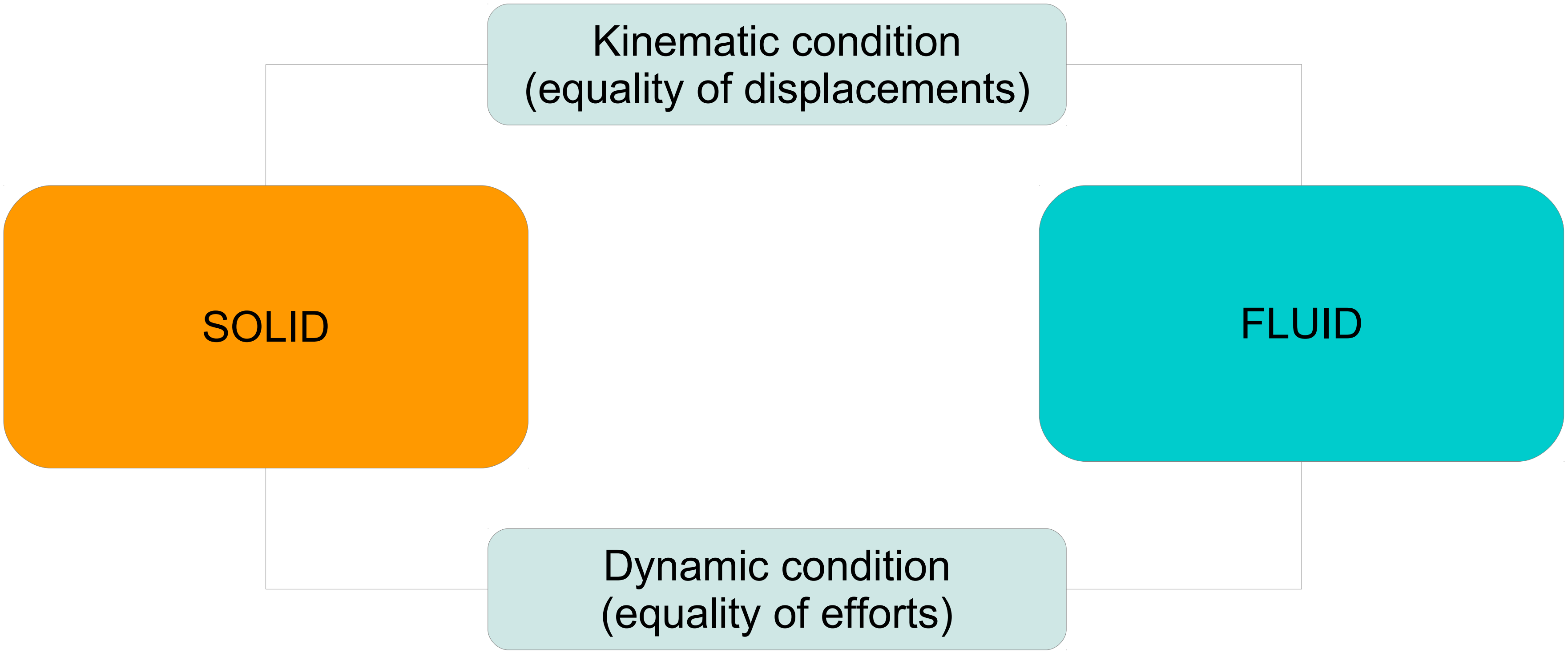

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Prerequisities of the course

Although some reminders will be given, some knowledge in the following topics are necessary :

- Fluid mechanics
- Solid mechanics
- Tensor algebra

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Negative interactions

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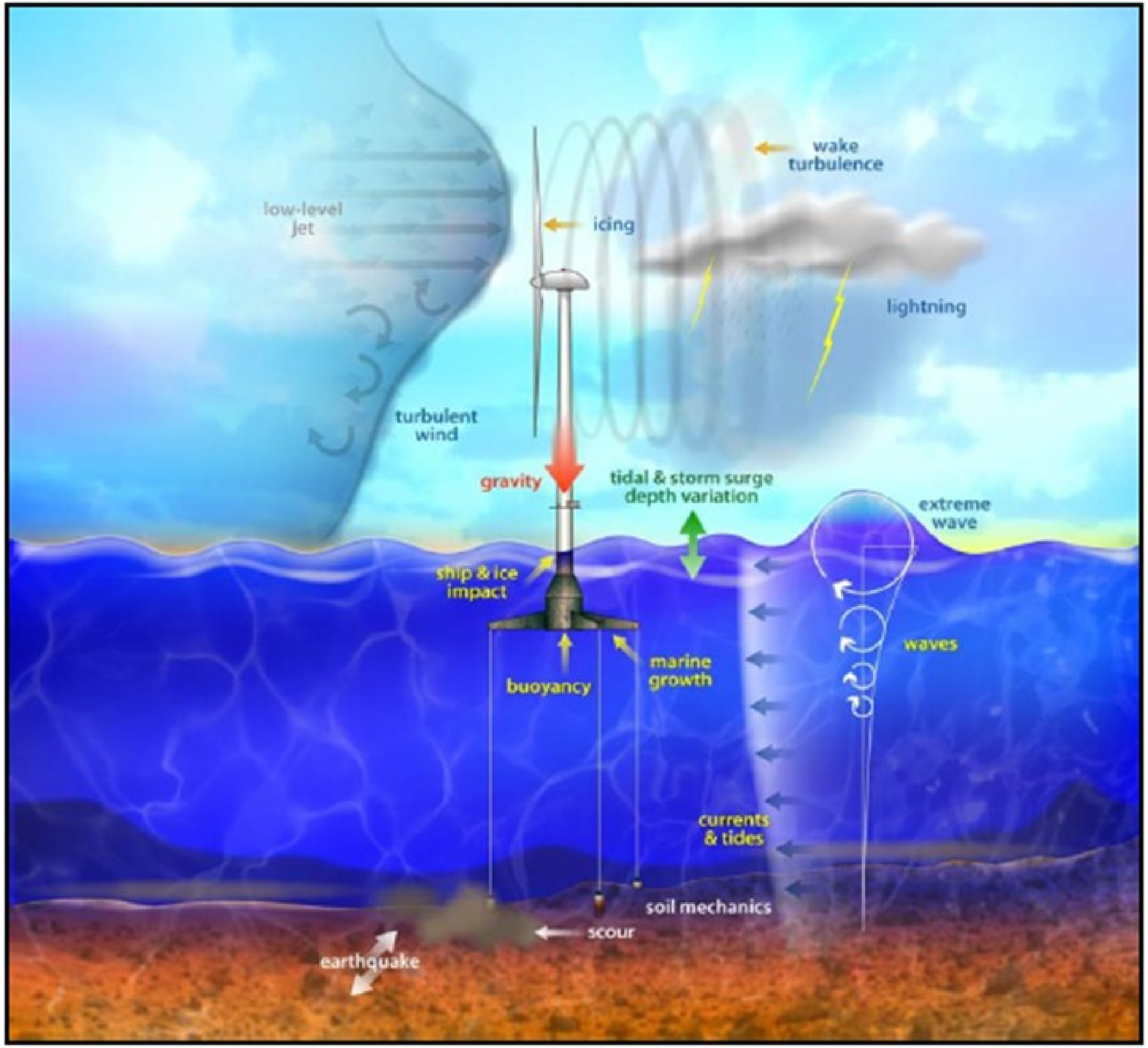
LOADS ON OFFSHORE STRUCTURES

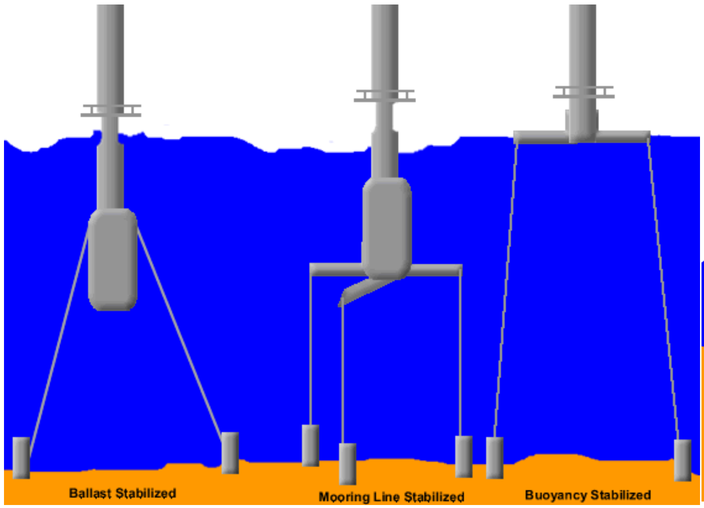
Dynamics associated with the design of advanced offshore wind energy systems with floating platforms (Musial 2010)



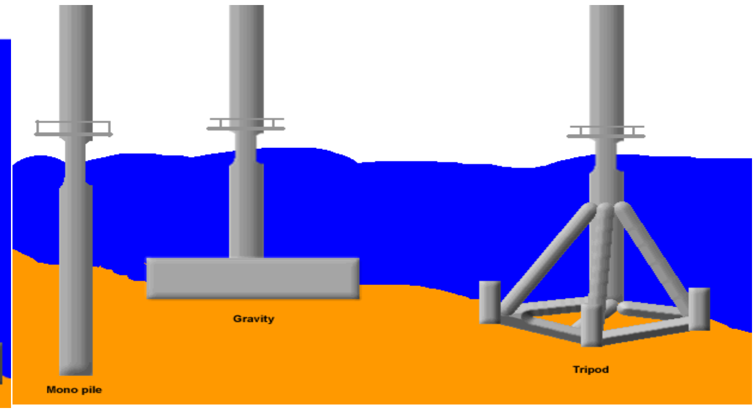
(from the NREL paper on wind system design)

Problematic interations I : Current-structure interactions

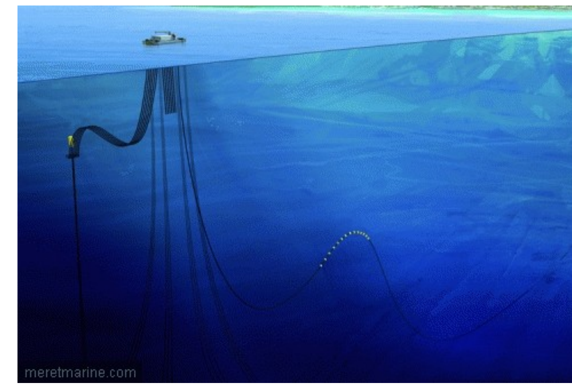
Floating wind turbines



Offshore wind turbines

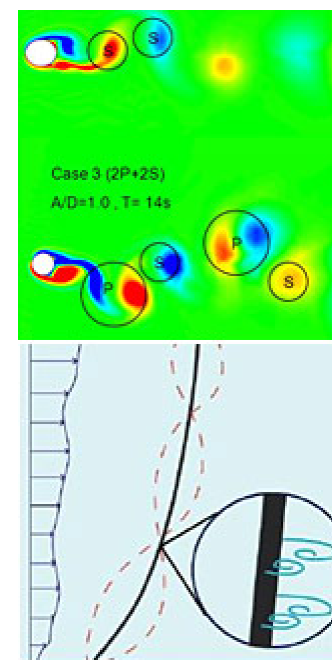


Floating ocean thermal energy converters



Instability
Takoma Narrows bridge
(1940)

Vortex induced vibrations (VIV)



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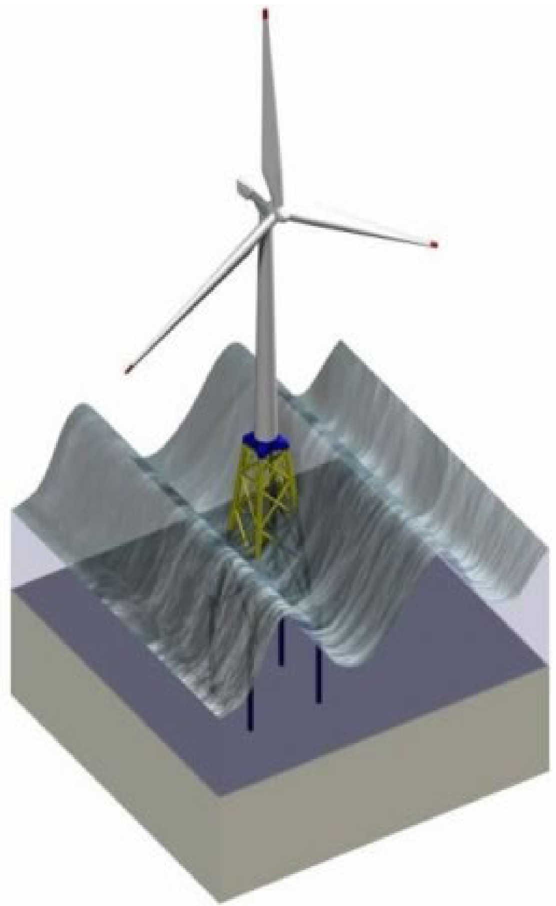
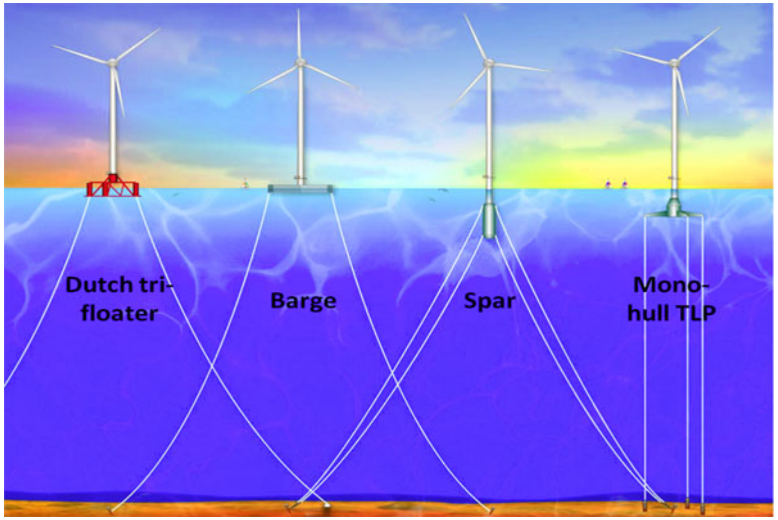
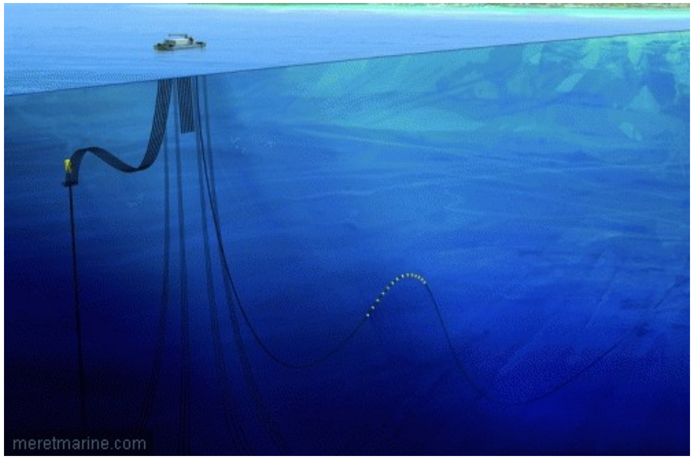
Problematic interations II : Wave-structure interaction



Offshore turbine
on a jacket type base



Floating turbines



Floating OTEC

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Dramatic consequences

Dammage / Fatigue of offshore structures

- Offshore structures under extreme conditions :
  DAMAGE
- Offshore structure submitted to standard load case during years : FATIGUE → DAMMAGE

There is a growing number of industrials that want to respond to tenders on offshore windfarm installation

Lifetime prediction and guarantee of structures is mandatory → IFS

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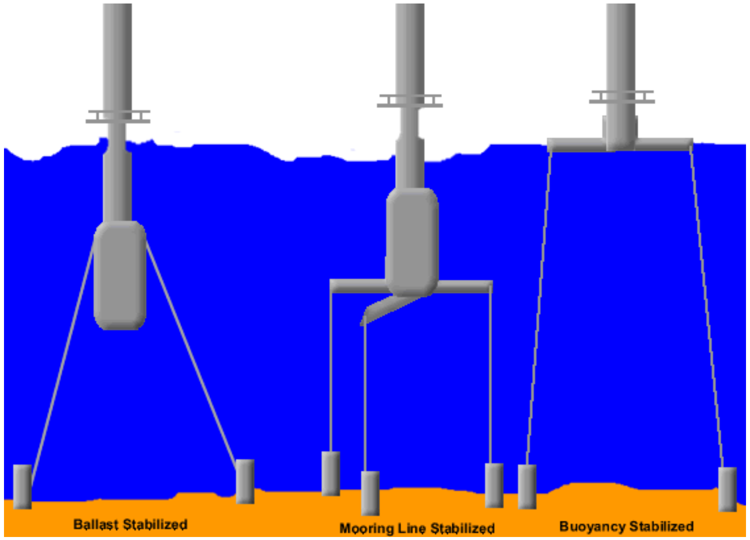
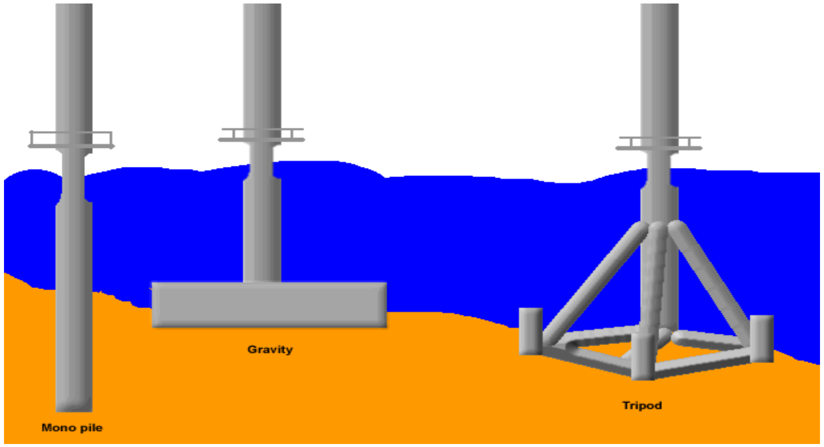
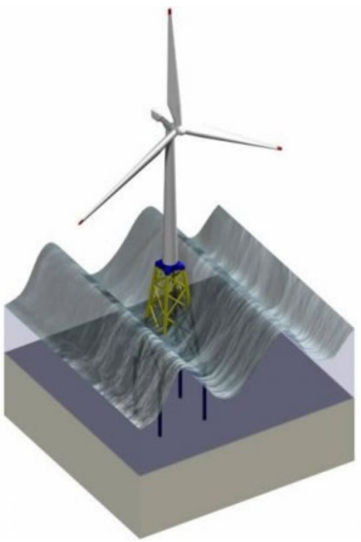
Useful experience from oil and gas industry

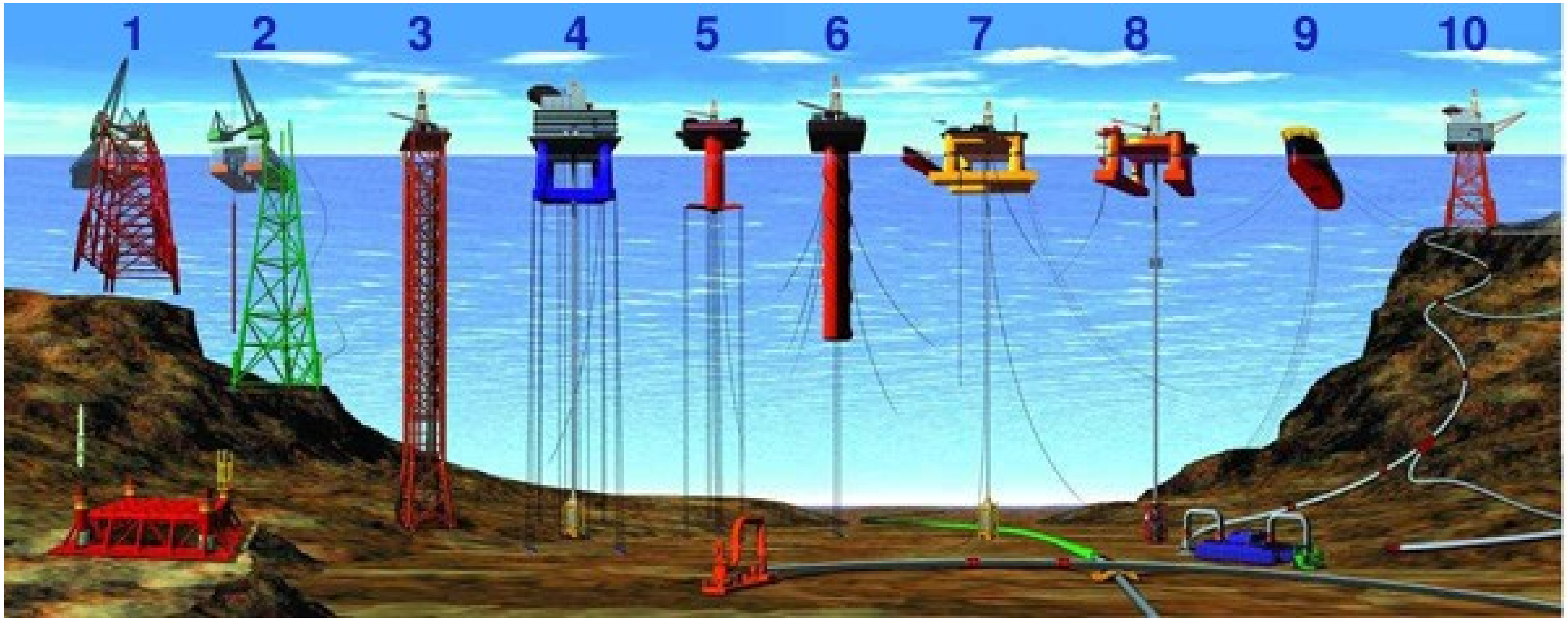
Offshore wind







Offshore oil&gas



Wikipedia.org

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Positive interactions

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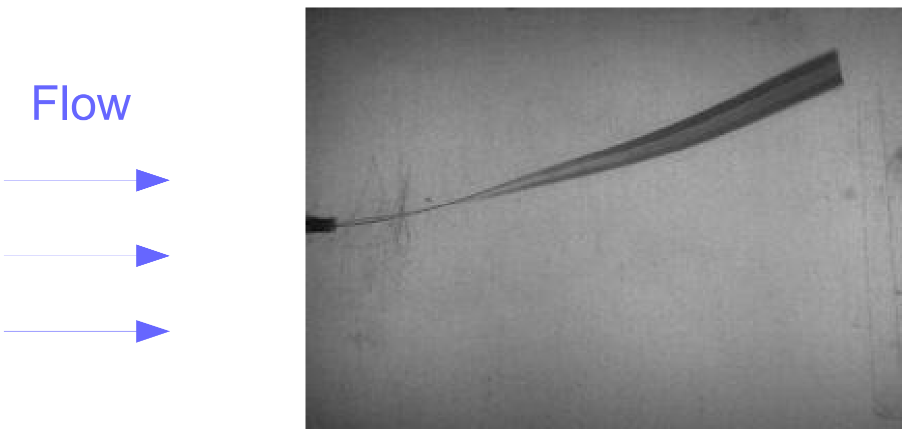
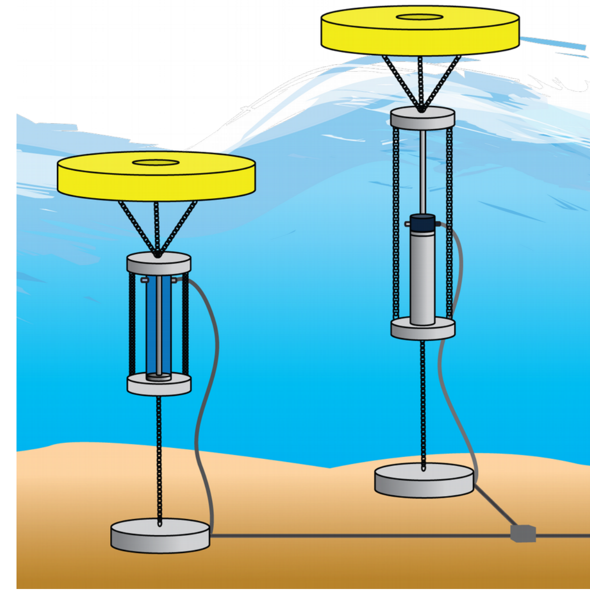
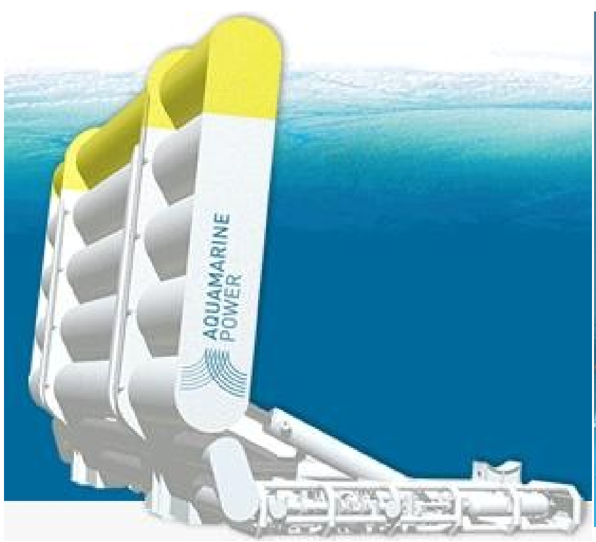
Energy harvesting : using energy transfer from a flow to a structure to convert kinetic energy of a flow into electrical energy

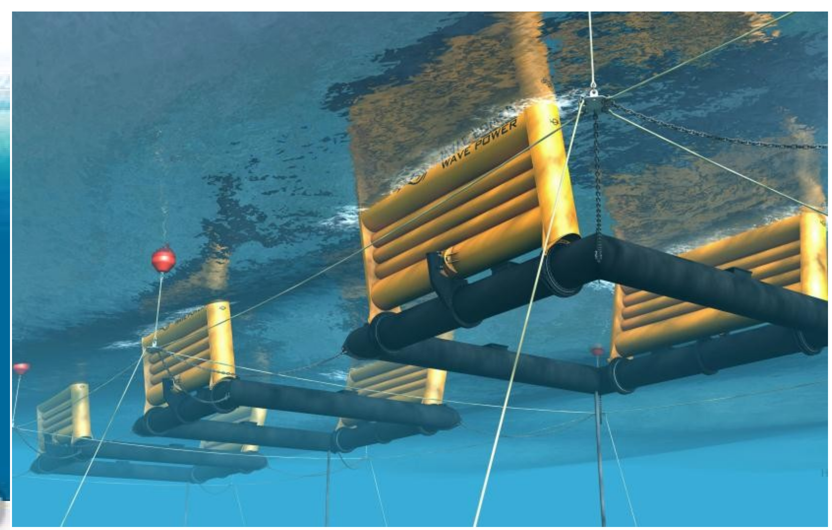


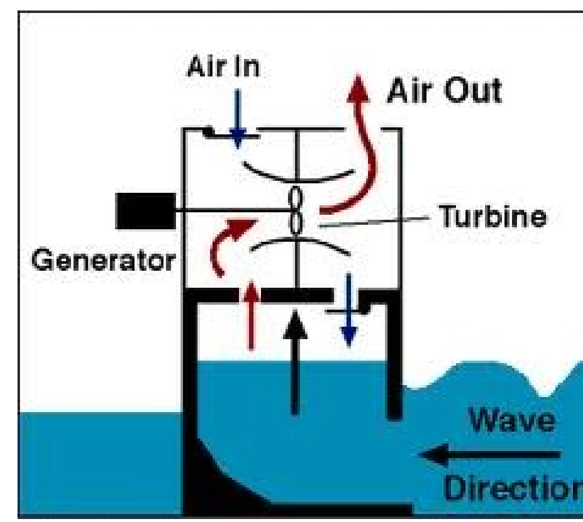
Using induction, piezoelectric coupling, dielectric materials

Energy harvesting : Incident wave energy conversion









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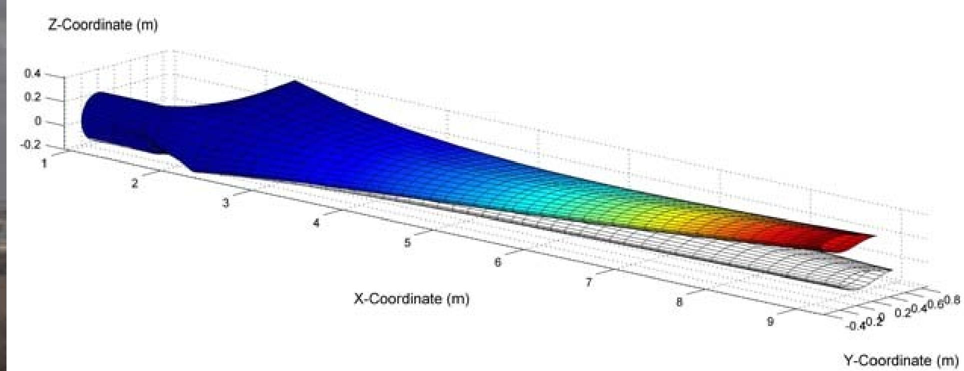
Using flexibility of blades to improve efficiency of wind turbines



http://eolos.umn.edu

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Fluid-mechanics and Solid-mechanics problems

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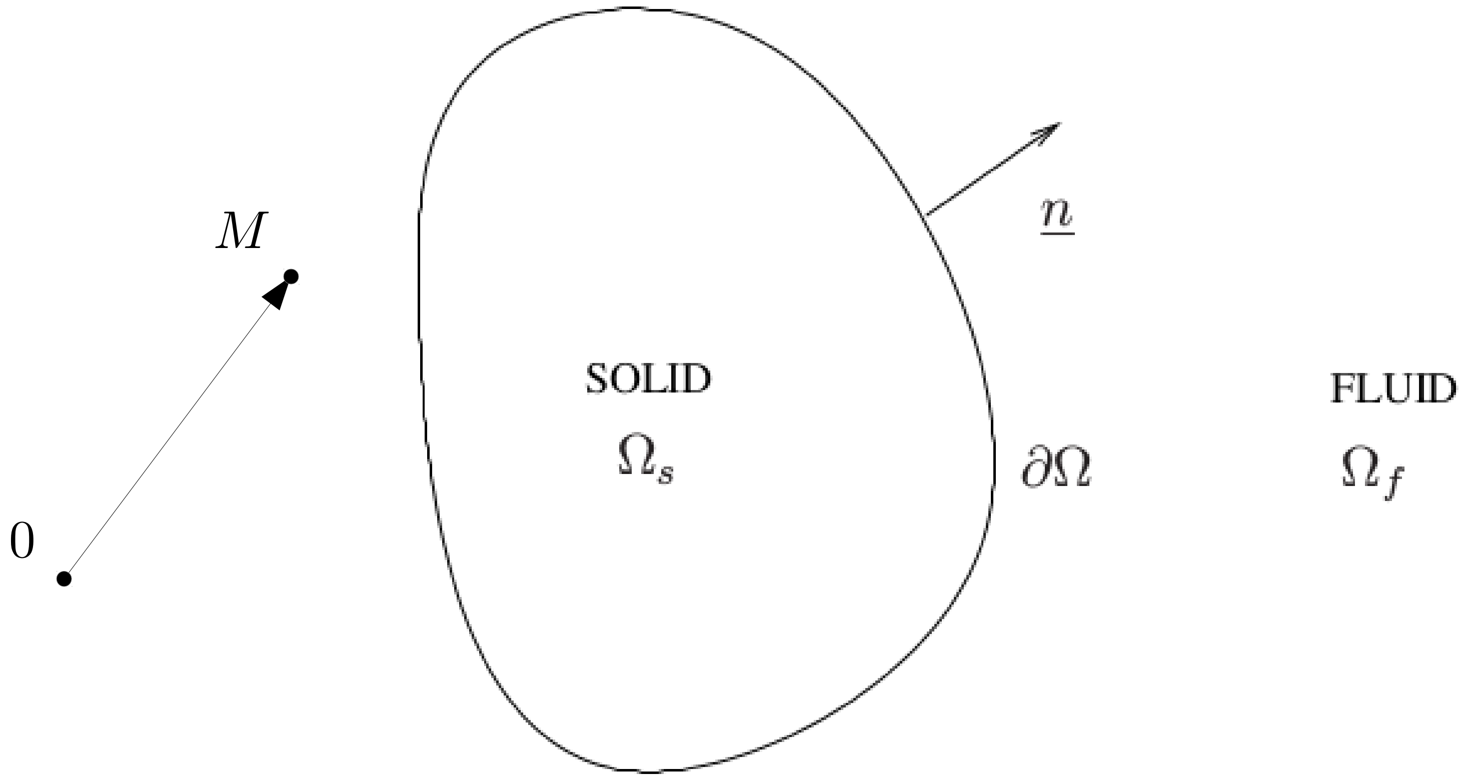
Domains and field variables



$$\underline{OM} = \underline{X} = X\underline{e}_x + Y\underline{e}_y + Z\underline{e}_z$$

$$\underline{n} = n_x\underline{e}_z + n_y\underline{e}_y + n_z\underline{e}_z$$

Oriented by convention from the solid to the fluid

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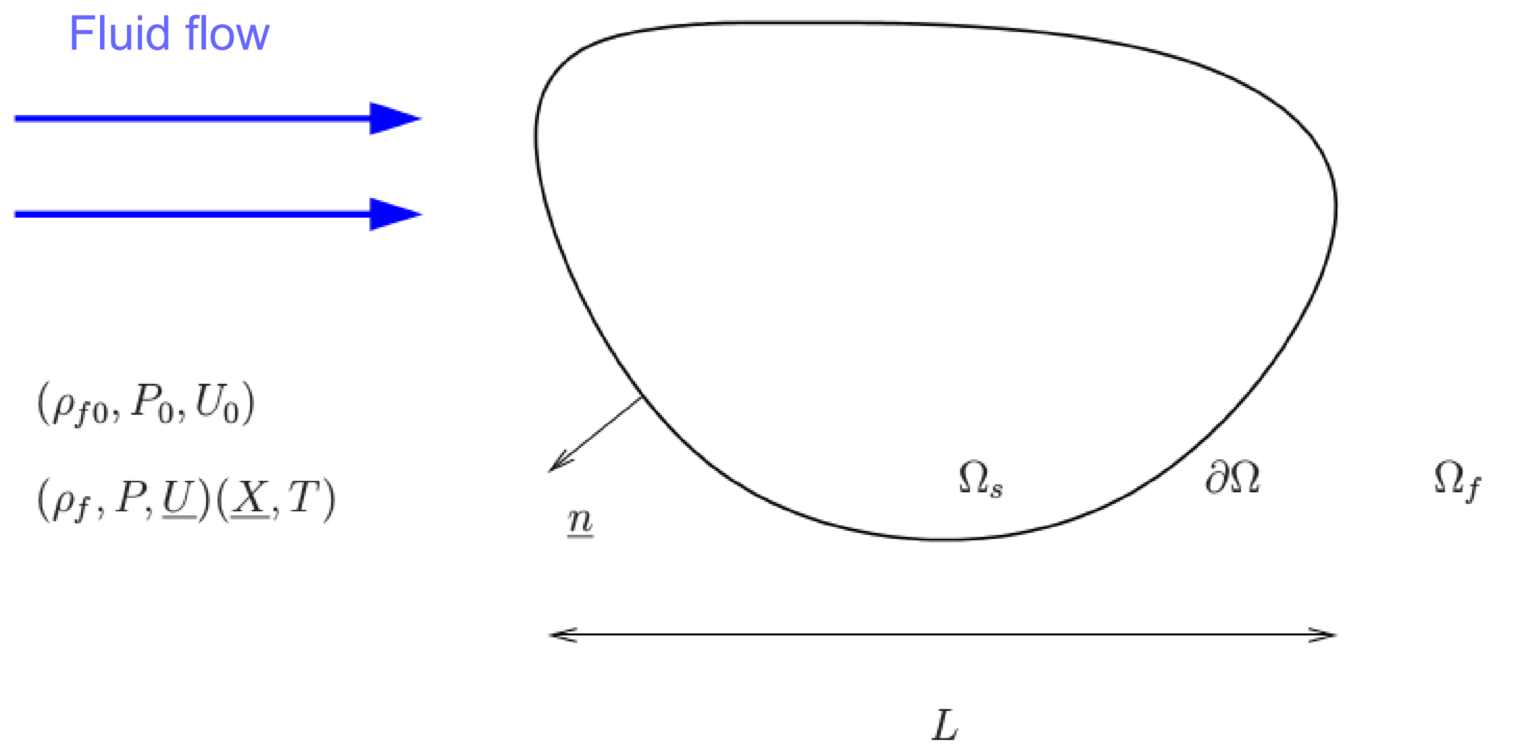
The fluid-mechanics problem



Parameters :

$\lambda\ , \quad \mu\ , \quad \rho_{f0}\ , \quad P_0\ , \quad U_0$: Lamé coefficients, reference density, pressure, velocity

Field variables :

$\rho_f\ , \quad P\ , \quad \underline{U}$: Density, pressure and velocity field

In a fluid mechanics problem, the solid is perfectly rigid and imposes boundary conditions for the fluid.

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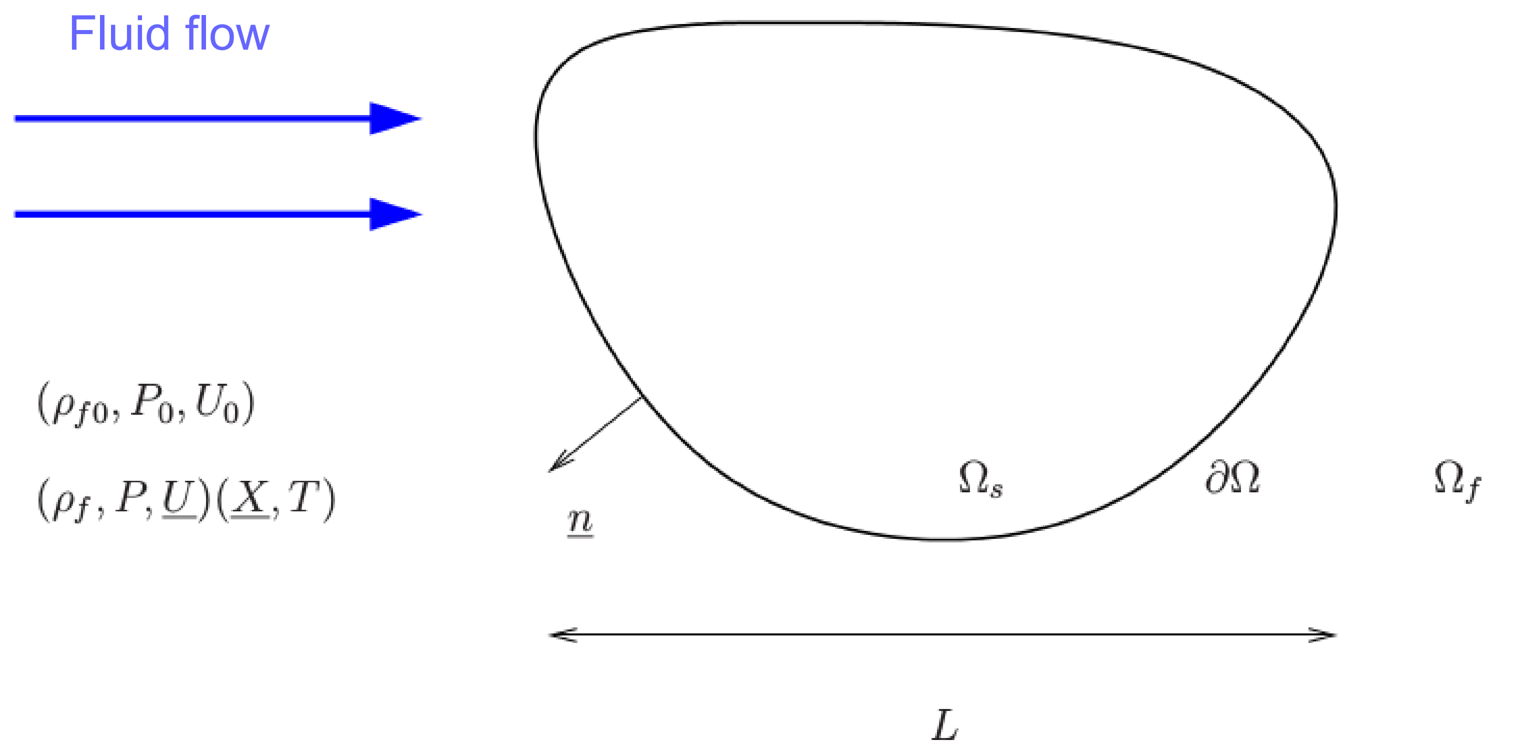

Mass conservation

$$\frac{\partial \rho_f}{\partial T} + \mathrm{div}\,(\rho_f \underline{U}) = 0$$

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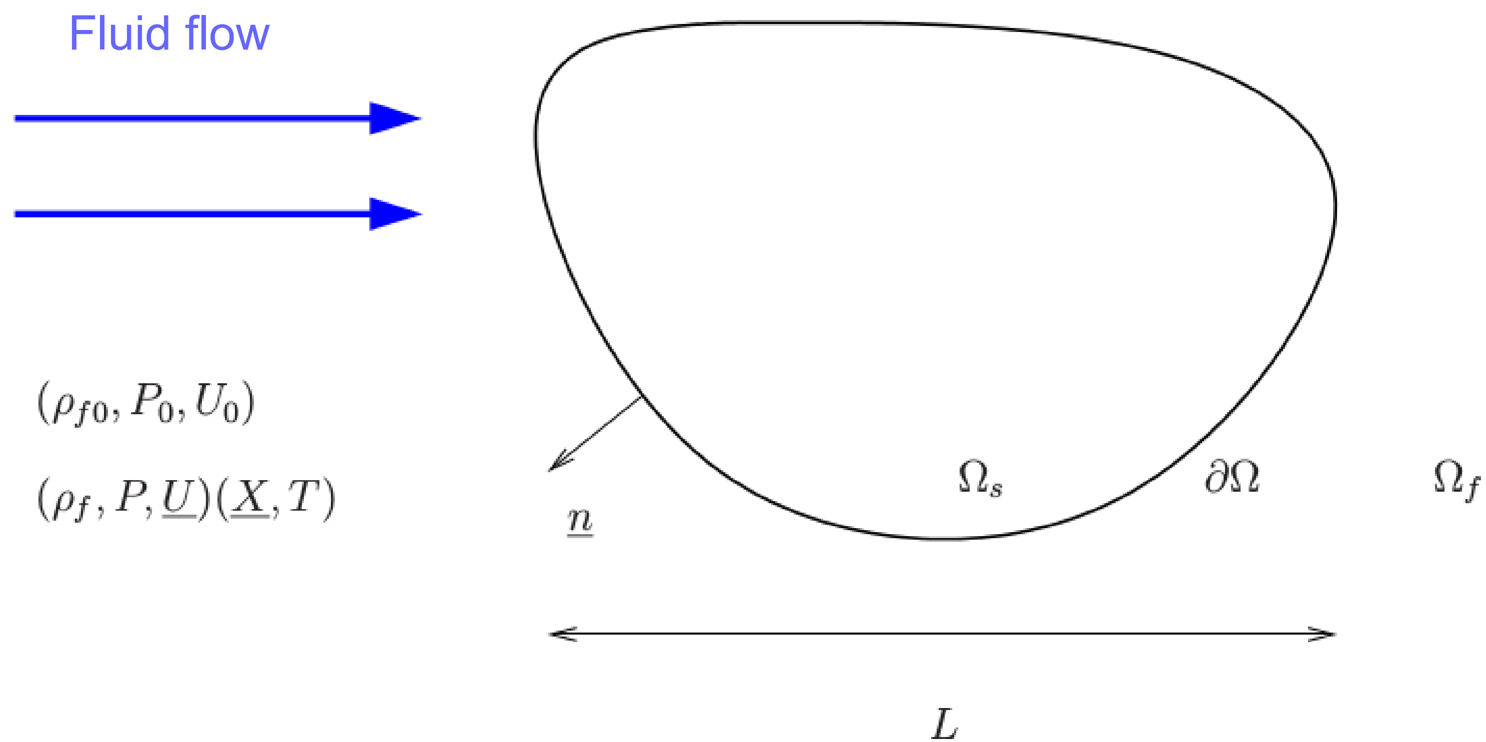

Momentum conservation

$$\rho_f \frac{\mathrm{d}\underline{U}}{\mathrm{d}T} = \underline{F} + \mathrm{div}\, \underline{\underline{\Sigma}}_f$$

with

$$\underline{\underline{\Sigma}}_f = (-P + \lambda\, \mathrm{div}\, \underline{U})\, \underline{\underline{1}} + 2\mu \underline{\underline{D}}$$ Stress deformation relationship

$$\underline{\underline{D}} = \frac{1}{2}({}^t\nabla\underline{U} + \nabla\underline{U})$$ Deformation rate tensor

$$\underline{F} = -\rho_f g \underline{e}_Z$$ Volumic forces are due to gravity

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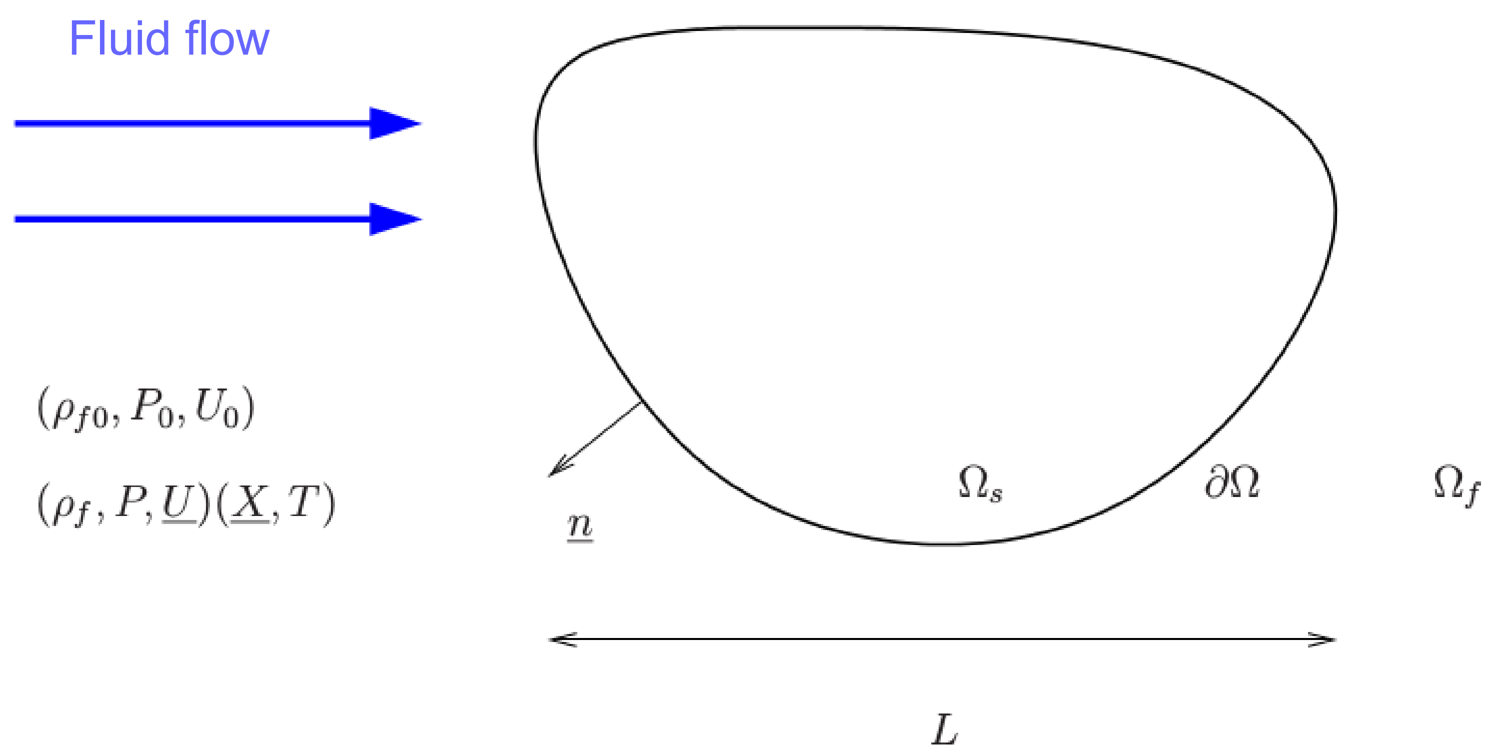

Momentum conservation

$$\rho_f \frac{\partial \underline{U}}{\partial T} + \rho_f \left(\underline{\underline{\mathrm{grad}}}\ \underline{U} \right) . \underline{U} =$$
$$-\rho_f\ g\ \underline{e}_Z - \underline{\mathrm{grad}}\ P + (\lambda + \mu) \underline{\mathrm{grad}}\ \mathrm{div}\ \underline{U} + \mu\ \Delta\ \underline{U}$$

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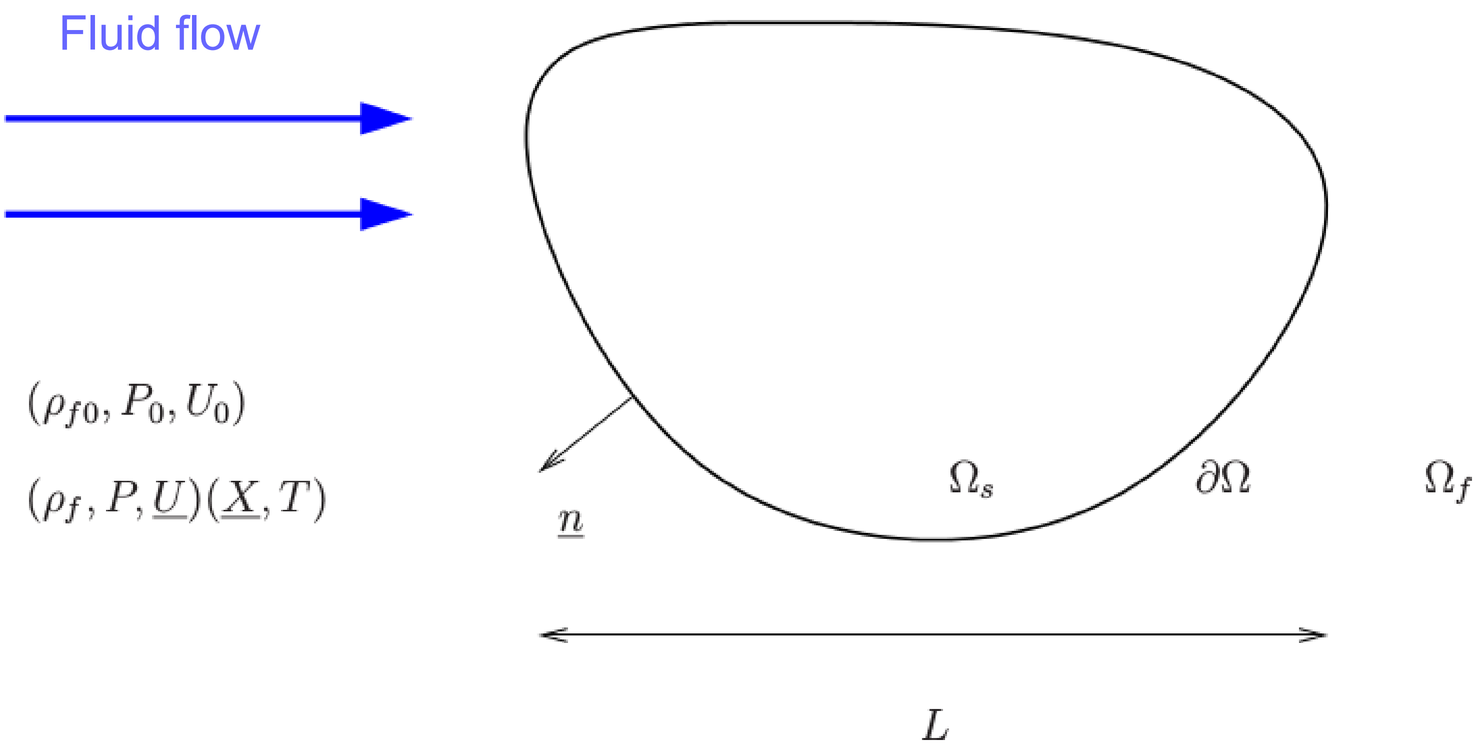

Boundary conditions

$$\underline{U} = 0 \text{ on } \partial\Omega$$

+ inlet and/or outlet conditions

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Navier-Stokes equations

Mass conservation

$$\frac{\partial \rho_f}{\partial T} + \text{div}\,(\rho_f \underline{U}) = 0$$

Momentum conservation

$$\rho_f \frac{\partial \underline{U}}{\partial T} + \rho_f \left(\underline{\underline{\text{grad}}}\ \underline{U}\right) . \underline{U} = -\rho_f\ g\ \underline{e}_Z - \underline{\text{grad}}\ P + (\lambda + \mu)\underline{\text{grad}}\ \text{div}\ \underline{U} + \mu\ \Delta\ \underline{U}$$

Boundary conditions

$$\underline{U} = 0 \text{ on } \partial\Omega$$

+ inlet and/or outlet conditions

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Fluid mechanics problem

Incompressible flow

Mass conservation

$$\frac{\partial \rho_f}{\partial T} + \mathrm{div}\,(\rho_f \underline{U}) = 0 \quad \longrightarrow \quad \mathrm{div}\,\underline{U} = 0$$

Momentum conservation

$$\rho_{f0} \frac{\partial \underline{U}}{\partial T} + \rho_{f0} \left(\underline{\underline{\mathrm{grad}}}\ \underline{U} \right) . \underline{U} =$$
$$-\rho_{f0}\ g\ \underline{e}_Z - \underline{\mathrm{grad}}\ P + \cancel{(\lambda + \mu)\underline{\mathrm{grad}}\ \mathrm{div}\ \underline{U}} + \mu\ \Delta\ \underline{U}$$

Boundary conditions

$$\underline{U} = 0 \text{ on } \partial\Omega$$

+ inlet and/or outlet conditions

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Fluid mechanics problem

Incompressible flow — Inviscid fluid

Mass conservation

$$\frac{\partial \rho_f}{\partial T} + \mathrm{div}\,(\rho_f \underline{U}) = 0 \quad \longrightarrow \quad \mathrm{div}\underline{U} = 0$$

Momentum conservation

$$\rho_{f0} \frac{\partial \underline{U}}{\partial T} + \rho_{f0} \left(\underline{\underline{\mathrm{grad}}}\ \underline{U} \right) . \underline{U} =$$

$$-\rho_{f0}\ g\ \underline{e}_Z - \underline{\mathrm{grad}}\ P + \cancel{(\lambda + \mu)\underline{\mathrm{grad}}\ \mathrm{div}\ \underline{U}} + \cancel{\mu\ \Delta\ \underline{U}}$$

Boundary conditions

$$\underline{U}\ .\ \underline{n} = 0 \text{ on } \partial\Omega$$

+ inlet and/or outlet conditions

Fluid mechanics problem – Incompressible flow and inviscid fluid

Mass conservation

$$\text{div } \underline{U} = 0$$

Momentum conservation

$$\rho_f \frac{\partial \underline{U}}{\partial T} + \rho_f \left(\underline{\underline{\text{grad}}} \; \underline{U} \right) . \underline{U} = -\rho_f \; g \; \underline{e}_Z - \underline{\text{grad}} \; P$$

Boundary conditions

$$\underline{U} \; . \; \underline{n} = 0 \text{ on } \partial\Omega$$

+ inlet and/or outlet conditions

Incompressible flow and inviscid fluid = Potential flow

Scalar potential

$$\underline{U} = \underline{\mathrm{grad}}\phi$$

Mass conservation

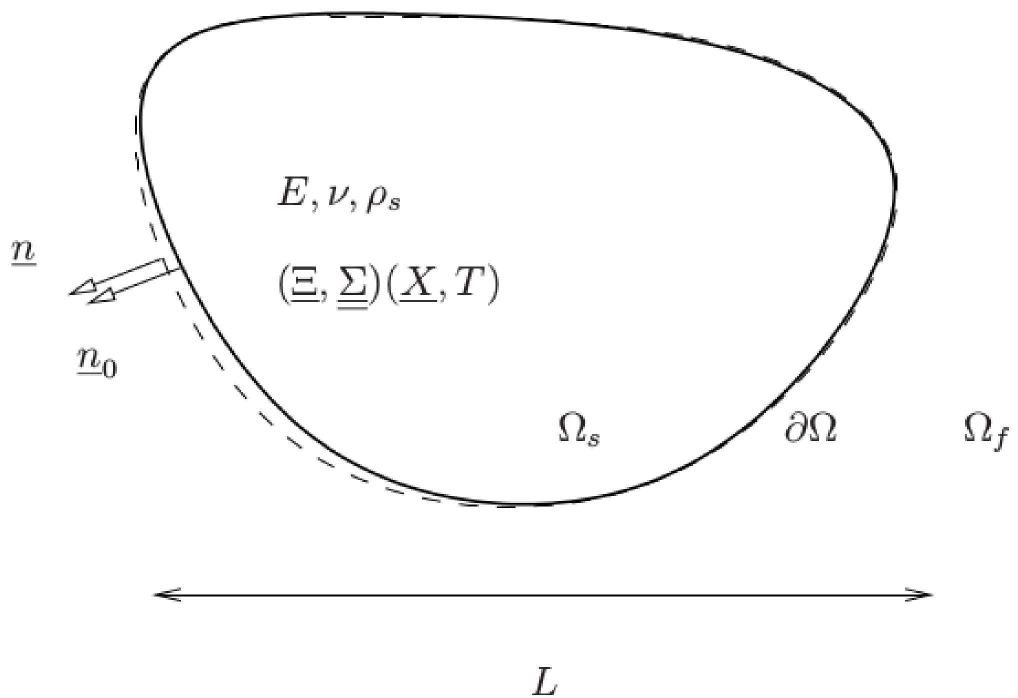
$$\Delta\phi = 0$$

Momentum conservation

$$\rho_f \frac{\partial \phi}{\partial T} + \rho_f \frac{1}{2}(\underline{\mathrm{grad}}\phi)^2 + \rho_f g + P = cte$$

Bernoulli equation

Solid mechanics problem



Parameters :

ρ_s , E , ν : Density, Young's modulus, Poisson's coefficient

Field variables :

$\underline{\Xi}$: Displacement field of the solid

In a solid mechanics problem, the fluid is neglected. The solid is in vacuum.

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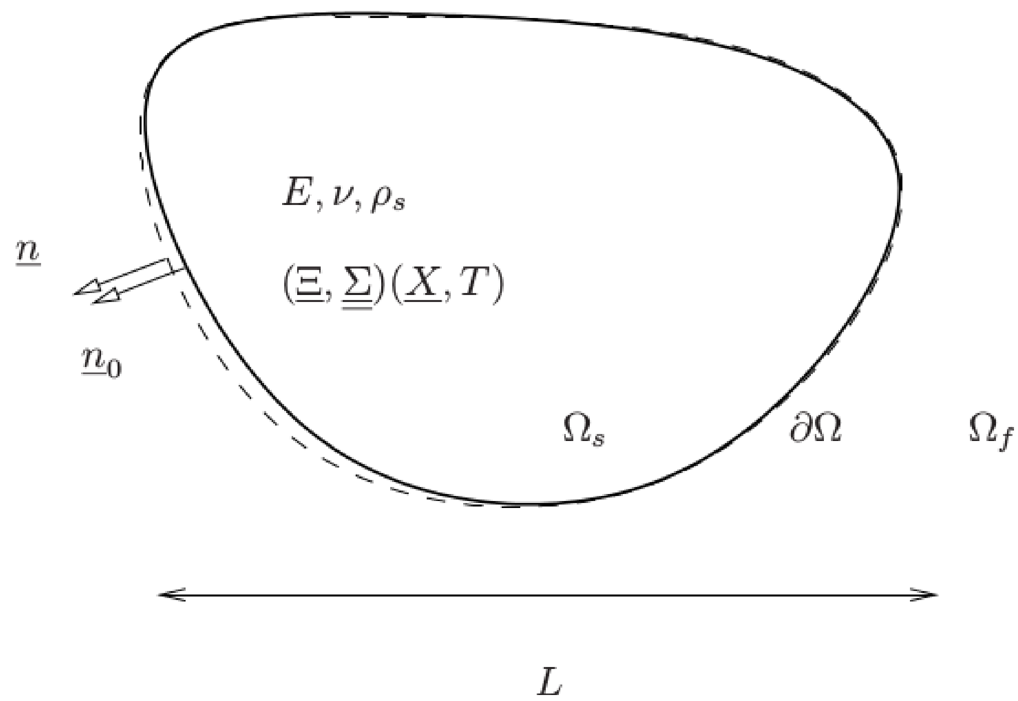
Solid mechanics problem



Momentum conservation

$$\rho_s \frac{\partial^2 \underline{\Xi}}{\partial T^2} = \underline{F} + \mathrm{div}\, \underline{\underline{\Sigma}}_s$$

$$\underline{\underline{\Sigma}}_s = \lambda \mathrm{tr}\left(\underline{\underline{\epsilon}}\right) \underline{\underline{1}} + 2\mu \underline{\underline{\epsilon}}$$ Stress deformation relationship

with $$\underline{\underline{\epsilon}} = \frac{1}{2}({}^t\nabla\underline{\Xi} + \nabla\underline{\Xi})$$ Deformation tensor

$$\underline{F} = -\rho_s g \underline{e}_Z$$ Volumic forces are due to gravity

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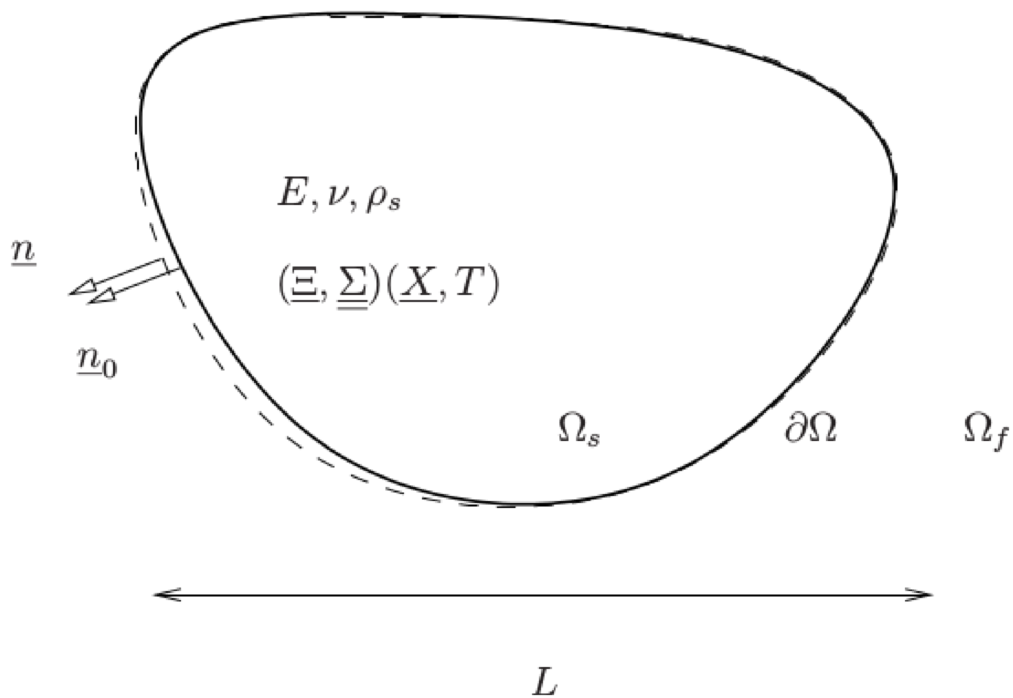
Solid mechanics problem



Momentum conservation

$$\rho_s \frac{\partial^2 \underline{\Xi}}{\partial T^2} = -\rho_s g \underline{e}_Z + \frac{E}{2(1+\nu)(1-2\nu)} \underline{\mathrm{grad}}(\mathrm{div}\underline{\Xi}) + \frac{E}{2(1+\nu)} \mathrm{div}(\underline{\underline{\mathrm{grad}}}\underline{\Xi})$$

Navier's equation

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$$\rho_s \frac{\partial^2 \underline{\Xi}}{\partial T^2} + \frac{E}{2(1+\nu)(1-2\nu)} \underline{\mathrm{grad}}(\mathrm{div}\underline{\Xi}) + \frac{E}{2(1+\nu)} \mathrm{div}(\underline{\underline{\mathrm{grad}}\Xi}) = -\rho_s g \underline{e}_Z$$

Inertia — Stiffness — External force

(+ Boundary conditions)

General form of a dynamical equation :

$$\mathcal{M}(\ddot{X}) + \mathcal{K}(X) = f$$ + BC

Inertia operator — Stiffness operator — Forcing

→ Tool to analyse these equations : Modal analysis

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Structural dynamics
&
Modal analysis

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Infinite media

- Modelization : PDE
- Wave propagation analysis
- *Local* approach

Finite dimension systems

- Modelization : PDE + Boundary conditions
- Modal analysis
- *Global* approach

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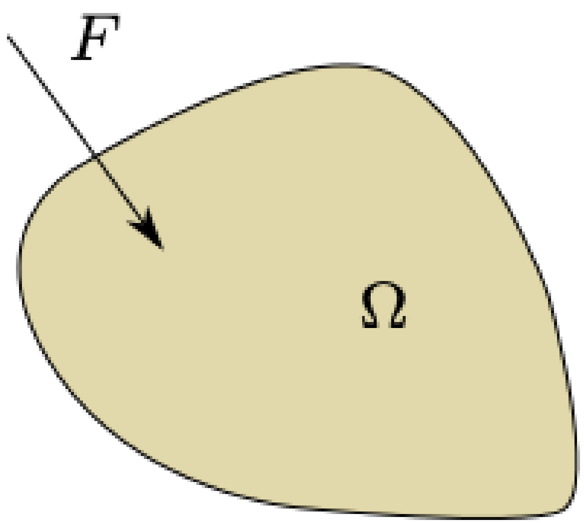
A mechanical system



- Local equation governing the displacement w :

$$\forall \, \underline{x} \in \Omega, \; \forall \, t \; : \quad \frac{\partial^2}{\partial t^2}\mathcal{M}[w(\underline{x},t)] + \mathcal{K}[w(\underline{x},t)] \; = \; F(\underline{x},t).$$

- $\mathcal{M}$: Mass operator $\mathcal{K}$: Stiffness operator
- Free problem : $F = 0$ Forced problem : $F \neq 0$
- Global approach : Boundary conditions

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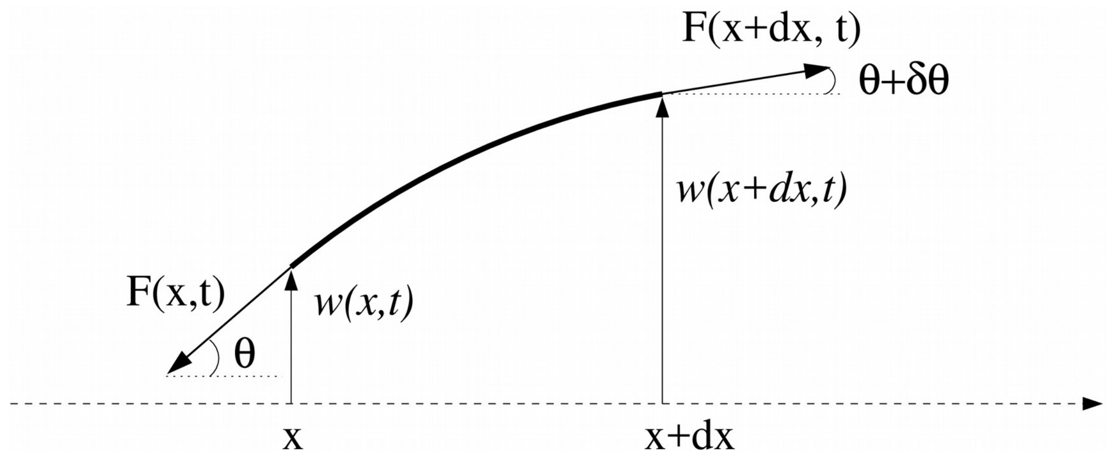
A first basic example : the vibrating string



- Tension : $T(X)$
- Lineic density : $\mu(X) = \rho(X)A(X)$
- Vertical displacement : $W(X,T)$

$$\mu \frac{\partial^2 W}{\partial T^2} = \frac{\partial}{\partial X}\left(T(X)\frac{\partial W}{\partial X}\right)$$

+ Boundary conditions

Mass operator :

$$\mathcal{M} = \mu(X)$$

Stiffness operator :

$$\mathcal{K}(W) = \frac{\partial}{\partial X}\left(T(X)\frac{\partial W}{\partial X}\right)$$

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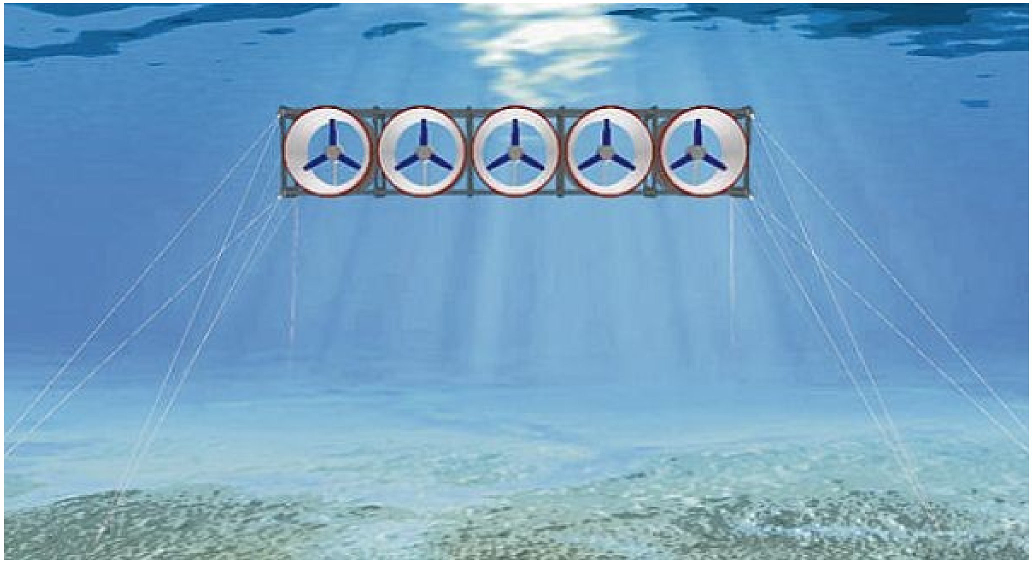
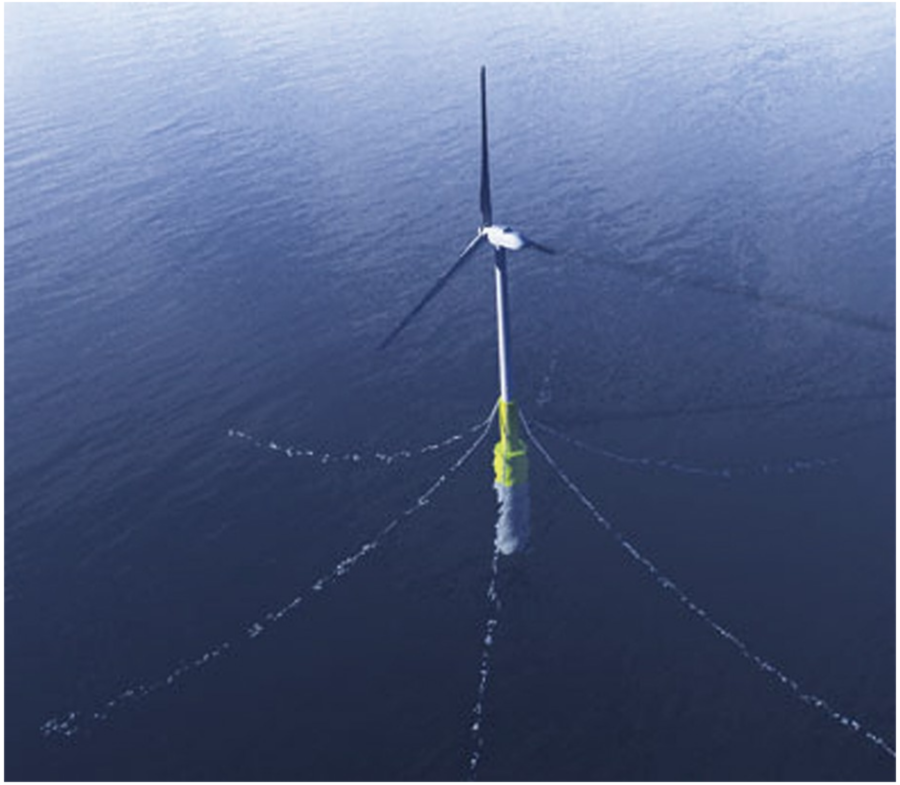
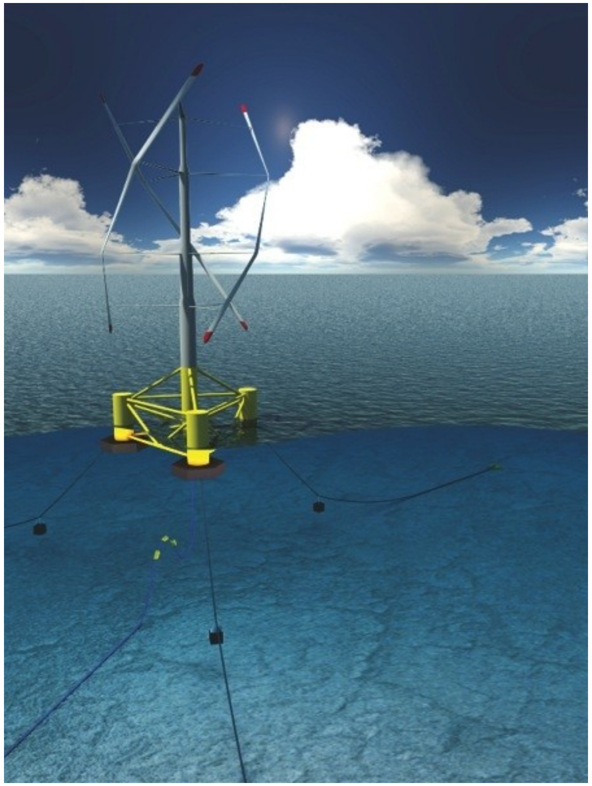
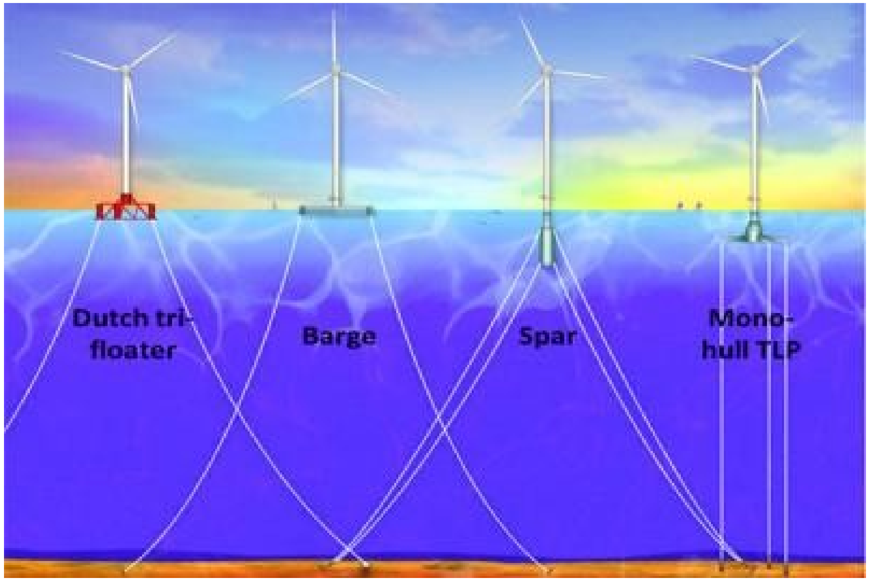




Mooring lines can be modelled as strings





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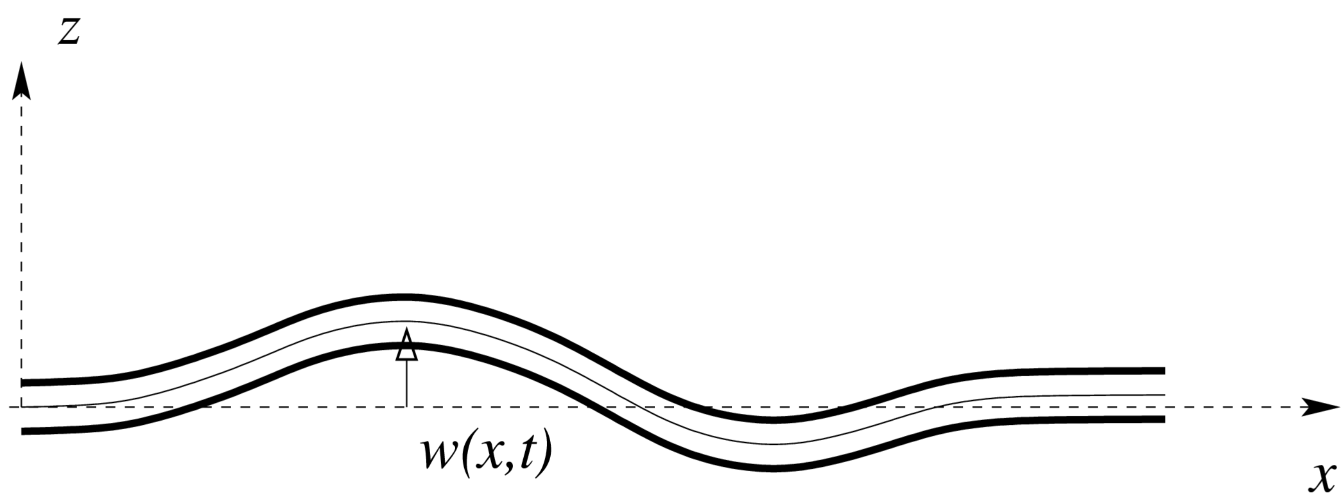
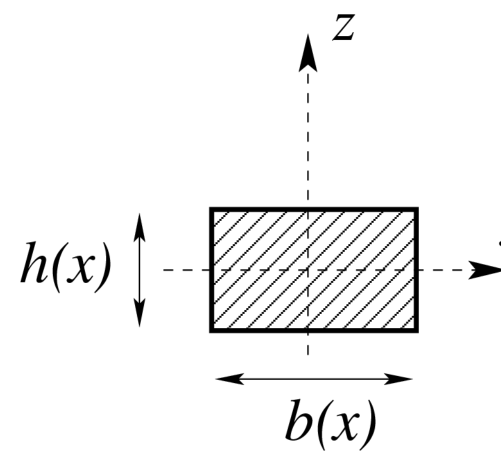
The beam



Area : S(x)=b(x)h(x)



- Young's modulus : E
- Lineic mass : μ
- Moment of inertia : I

For an homogeneous beam (constant section) :

$$EI\frac{\partial^4 W}{\partial X^4} + \mu\frac{\partial^2 W}{\partial T^2} = 0$$

Mass operator :

$$\mathcal{M} = \mu$$

Stiffness operator :

$$\mathcal{K} = EI\frac{\partial^4}{\partial X^4}$$

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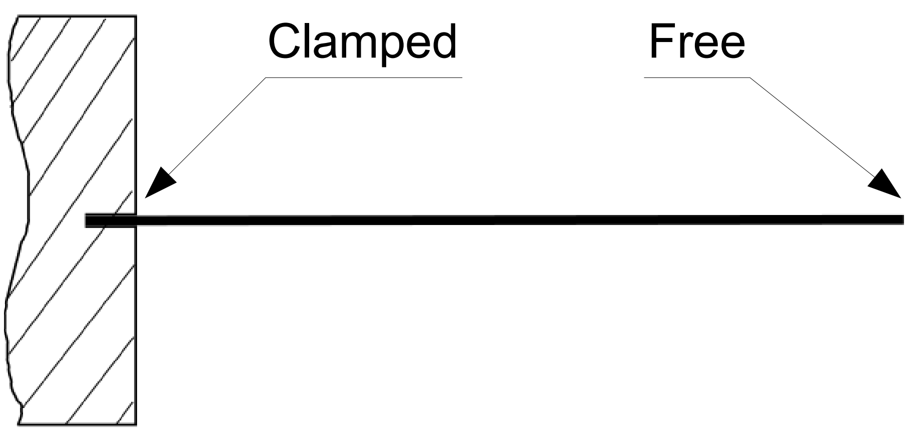
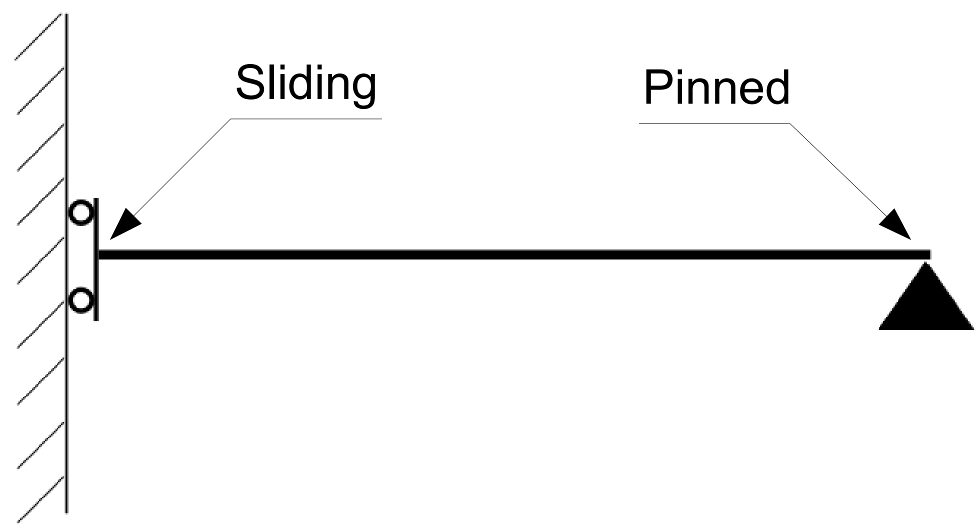
Boundary conditions of the beam

- Clamped boundary : $Y = \dfrac{\partial W}{\partial X} = 0$
- Sliding boundary : $\dfrac{\partial W}{\partial X} = \dfrac{\partial^2 W}{\partial X^2} = 0$
- Free boundary : $\dfrac{\partial^2 W}{\partial X^2} = \dfrac{\partial^3 W}{\partial X^3} = 0$
- Pinned boundary : $W = \dfrac{\partial^2 W}{\partial X^2} = 0$





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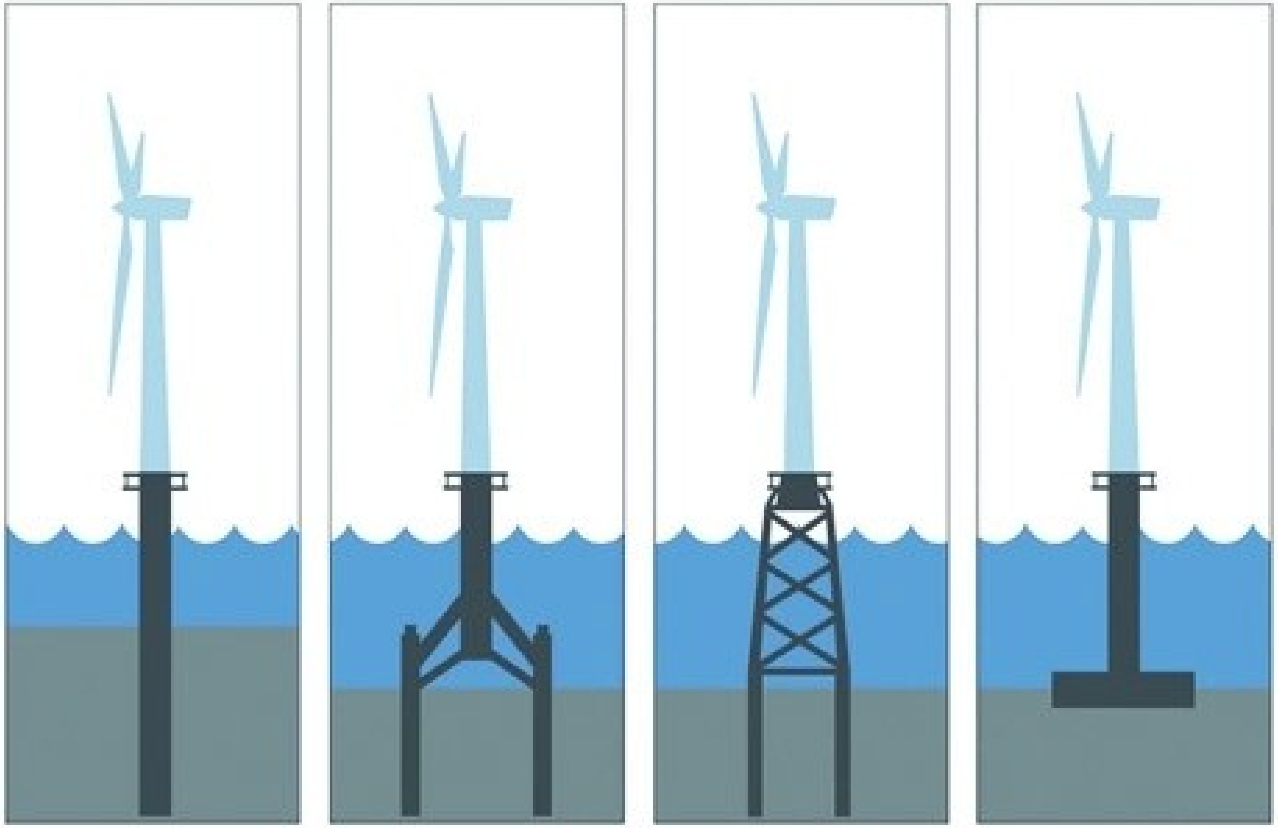

Offshore wind turbines are complex structures composed by beam elements

Eigenmodes and eigenfrequencies of mechanical systems

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The string case

- One wants to solve : $\dfrac{\partial^2 W}{\partial T^2} = c^2 \dfrac{\partial^2 W}{\partial X^2}$ with $W(X=0) = W(X=L) = 0 \quad \forall T$

- A solution to separate variables is sought for :

$$W(X,T) = f(X)g(T)$$

- General solution found :

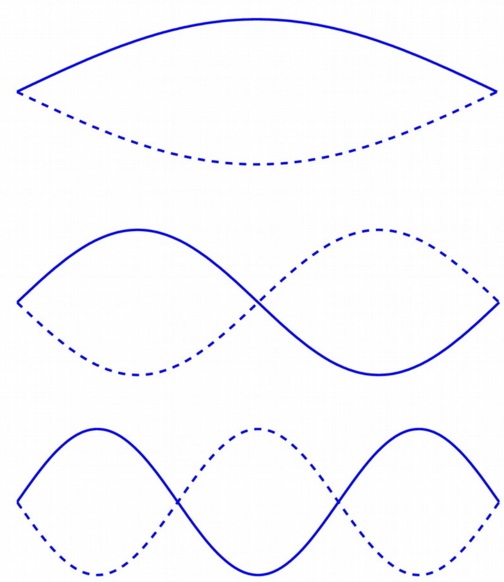
$$W(X,T) = \sin \frac{n\pi X}{L} \left[C\cos(\omega_n T) + D\sin(\omega_n T)\right]$$

- An infinite set of eigenfrequencies is selected :

$$\omega_n = \frac{n\pi c}{L}$$

- Associated to an eigenfunction

$$\phi_n(X)$$



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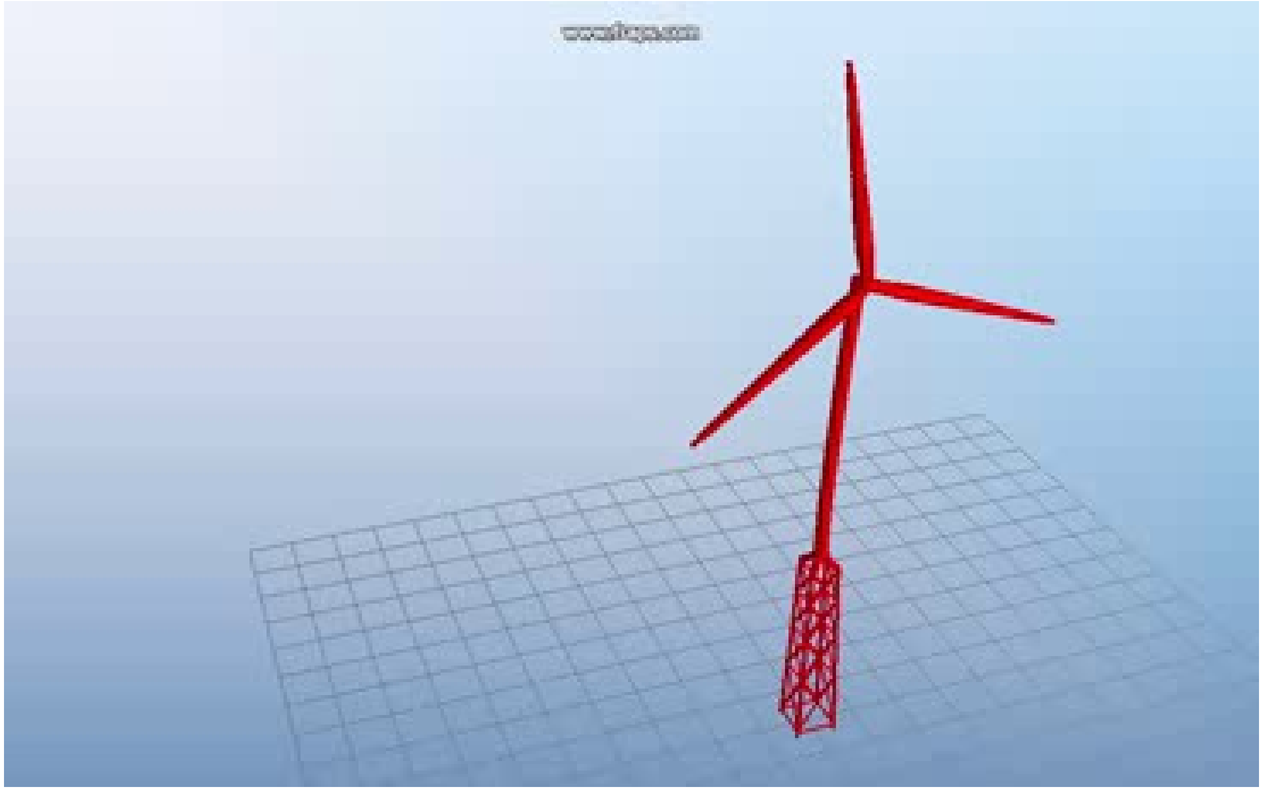
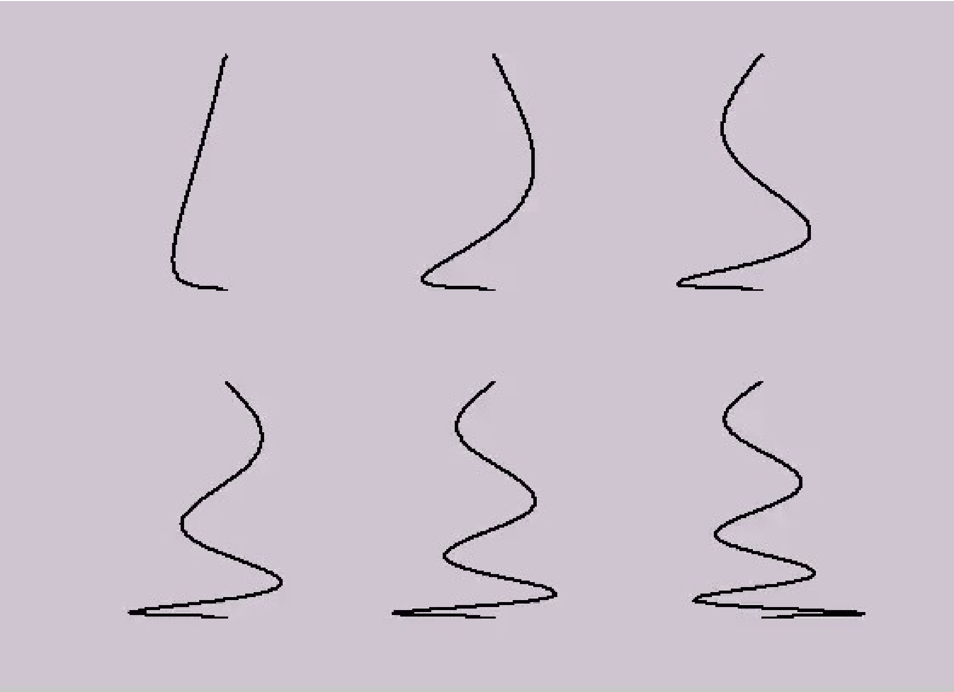
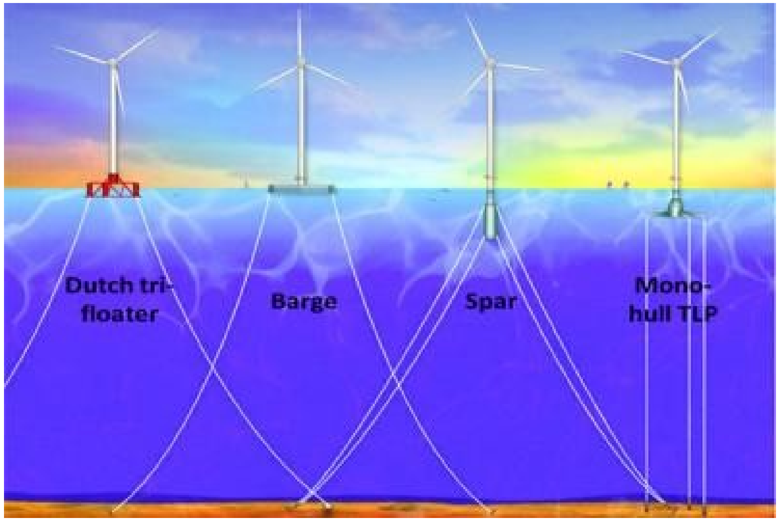
Engineering cases

Eigenmodes of a wind turbine on a jacket type fundation



Eigenmodes of mooring cables





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Orthogonality

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- General form of a mechanical system :

$$\forall \underline{x} \in \Omega, \qquad \frac{\partial^2}{\partial t^2}\left[\mathcal{M}(w(\underline{x},t))\right] + \mathcal{K}(w(\underline{x},t)) = 0$$

- Example for the beam :
  - Domain : $\Omega = x \in [0, L]$
  - Mass operator : $\mathcal{M} = \mu$
  - Stiffness operator : $\mathcal{K} = EI\dfrac{\partial^4}{\partial x^4}$
- Eigenmodes and eigenfrequencies :

$$\phi_n \quad , \quad \omega_n \quad , \quad n \in \mathbb{N}$$

- The problem writes for each ϕ_n

$$\forall\, x \in \Omega, \quad \mathcal{K}(\phi(x)) \;=\; \omega^2\mathcal{M}(\phi(x)),$$
$$\forall\, x \in \partial\Omega, \quad \mathcal{B}_i(\phi(x)) \;=\; 0, \quad i = 1...p.$$

- p equals the maximum order of the spatial derivatives in $\mathcal{K}$

Note : we switch to lowercase in this part $\rightarrow \quad w, x, t$

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Scalar product

- Introduction of a scalar product for all admissible functions (functions that satify the boundary conditions)

$$< f, g >= \int_\Omega f g \mathrm{d}\Omega.$$

- Bilinear, symmetric, positive definite
→ IT IS a scalar product

Scalar product
with respect to the inertia
and stiffness operators

- Operators are self-adjoint if :

$$< f|\mathcal{K}(g) > \; =< \mathcal{K}(f)|g >,$$
$$< f|\mathcal{M}(g) > \; =< \mathcal{M}(f)|g > .$$

- They are positive definite if :

$$< f|\,\mathcal{K}(f) > \;\geq 0, \qquad \text{et} \qquad < f|\,\mathcal{M}(f) > \;\geq 0.$$
$$< f|\,\mathcal{K}(f) > \;= 0 \quad \Longrightarrow \quad f = 0.$$

- If positive definite, these are also scalar products :

$$< f|g >_\mathcal{K} \; = \int_\Omega f\mathcal{K}(g)\mathrm{d}\Omega,$$
$$< f|g >_\mathcal{M} \; = \int_\Omega f\mathcal{M}(g)\mathrm{d}\Omega.$$

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Orthogonality with respect to mass and stiffness operators

- Let ϕ_p and ϕ_q two eigenfunctions, with associated eigenfrequencies ω_p and ω_q

$$\mathcal{K}(\phi_p) = \omega_p^2 \mathcal{M}(\phi_p),$$
$$\mathcal{K}(\phi_q) = \omega_q^2 \mathcal{M}(\phi_q).$$

- If operators are self-adjoints, we show

$$\left(\omega_p^2 - \omega_q^2\right) \int_\Omega \phi_q \mathcal{M}(\phi_p) \mathrm{d}\Omega \; = \; 0.$$

- For different eigenfrequencies, the functions are orthogonal with respect to the inertia operator :

$$< f|g >_{\mathcal{M}} = \; 0$$

- Direct consequence : the functions are orthogonal with respect to the stiffness operator :

$$< f|g >_{\mathcal{K}} = \; 0$$

- When p=q :

$$< \phi_p \,|\, \phi_p >_{\mathcal{M}} \; = \; m_p,$$
$$< \phi_p \,|\, \phi_p >_{\mathcal{K}} \; = \; k_p,$$

Modal mass and modal stiffness

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The family of eigenmodes form a projection basis
IDEA : Projection of the PDE on this basis

- Step one : Modal expansion

$$w(x,t) = \sum_{p=1}^{+\infty} X_p(t)\phi_p(x)$$

Xp(t) : modal amplitude of mode p

- Step two : Insert this development in the PDE

$$\forall \underline{x} \in \Omega, \qquad \frac{\partial^2}{\partial t^2}\left[\mathcal{M}(w(\underline{x},t))\right] + \mathcal{K}(w(\underline{x},t)) = f(x,t)$$

where f are external forces

- Step three : projection of the PDE on a mode ϕ_n

$$\forall n \geq 1, \qquad m_n \ddot{X}_n + k_n X_n = F_n(t)$$

- Fn is the modal force :

$$F_n(t) = \int_\Omega f(\underline{x},t)\phi_n(\underline{x})\mathrm{d}\Omega$$

ONE PDE

↓

AN INFINITE SET OF ODEs

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Physical space

- Unknown : displacement
- Equations : PDE + Boundary conditions + initial displacement

Modal space

- Unknowns : modal displacements
- Equations : ODEs + initial conditions

$$\mathcal{K}(w) + \frac{\partial^2}{\partial t^2}\mathcal{M}(w) = f$$

Projection

$$\begin{aligned} < \phi_n \,|\, \phi_n >_{\mathcal{M}} \; &= \; m_n \\ < \phi_n \,|\, \phi_n >_{\mathcal{K}} \; &= \; k_n \\ < \phi_n \,|\, f > \; &= \; F_n \\ < \phi_n \,|\, w(x, t=0) > \; &= \; X_n(t=0) \end{aligned}$$

$$m_n \ddot{X}_n + k_n X_n = F_n$$

$$n \in \mathbb{N}$$

$$X_1(t) \;,\quad X_2(t) \;,\quad \ldots$$

Modal recomposition

$$w(x,t) = \sum_{n=1}^{+\infty} X_n(t)\phi_n(x)$$

SOLUTION
$w(x,t)$

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The harmonic oscillator

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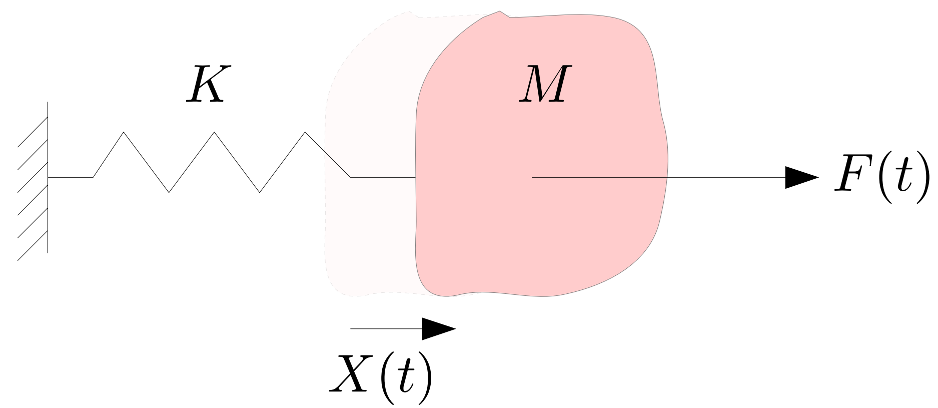
The undamped oscillator

$$M\ddot{X} + KX = F(t)$$



Free oscillations

$$F = 0$$

Solution

$$X(t) = X_0 \cos \omega t + \frac{\dot{X}_0}{\omega} \sin \omega t$$

with

$$\omega = \sqrt{\frac{k}{m}}$$

Forced vibrations

$$F = F_0 \sin \Omega t$$

Solution

$$x(t) = x_0 \sin \Omega t$$

with

$$\frac{x_0}{F_0} = \frac{1}{M(\omega^2 - \Omega^2)}$$

Impulse response

$$F = P\delta(t - t_0)$$

Equivalent to a velocity jump

$$[\dot{X}]_{t_0^-}^{t_0^+} = \frac{P}{M}$$

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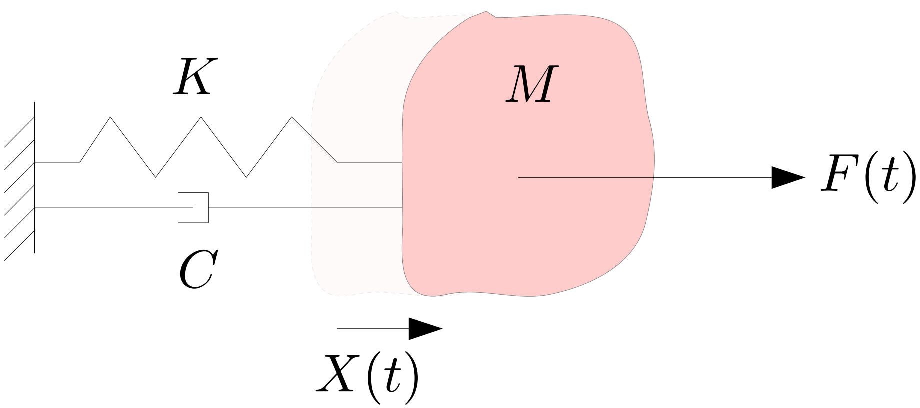
The damped oscillator

$$M\ddot{X} + C\dot{X} + KX = F(t)$$



Free oscillations

$$F = 0$$

Solution

$$X(t) = \mathrm{e}^{-\frac{C}{2M}t} \times \left[X_0 \cos\omega t + \frac{\dot{X}_0}{\omega} \sin\omega t \right]$$

with

$$\omega = \sqrt{\frac{k}{m}}$$

Forced vibrations

$$F = F_0 \sin\Omega t$$

Solution

$$X(t) = \mathrm{Re}\left(X_0 \mathrm{e}^{\mathrm{i}\Omega t} \right)$$

with

$$\frac{X_0}{F_0} = \frac{1}{M(\omega^2 - \Omega^2 + \mathrm{i}\Omega\frac{C}{M})}$$

Impulse response

$$F = P\delta(t - t_0)$$

Equivalent to a velocity jump

$$[\dot{X}]_{t_0^-}^{t_0^+} = \frac{P}{M}$$

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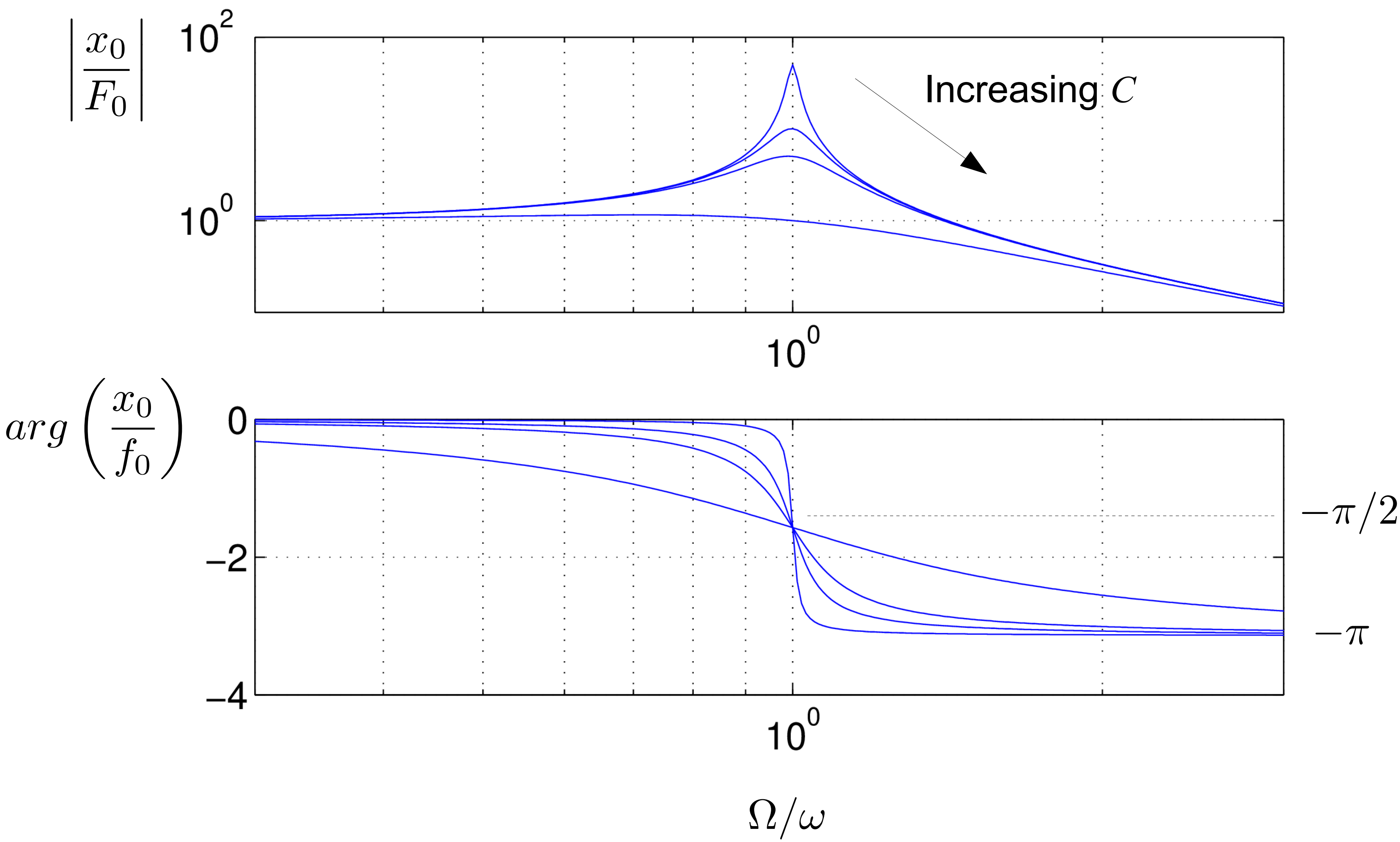
Forced vibrations – Transfer function



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Examples

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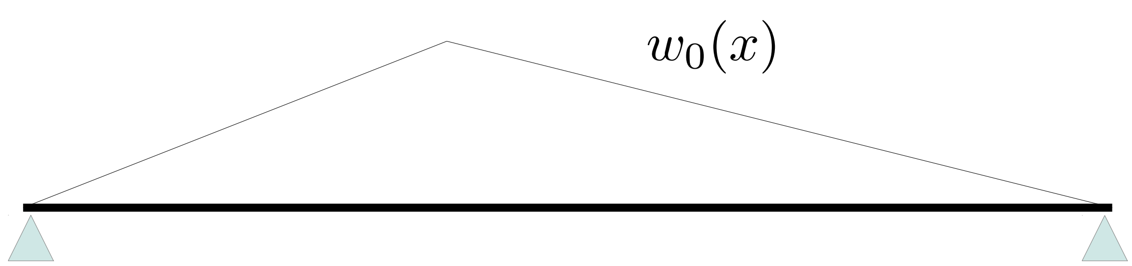
The plucked string

How to solve $$-T\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = f(x,t)$$

with $$x(0,t) = x(L,t) = 0 \quad \forall t$$

$$w(x,t=0) = w_0(x) \quad , \quad \dot{w}(x,t=0) = \dot{w}_0(x)$$

using modal analysis ?



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Physical space

- Unknown : displacement
- Equations : PDE + Boundary conditions + initial displacement

Modal space

- Unknowns : modal displacements
- Equations : ODEs + initial conditions

$$\mathcal{K}(w) + \frac{\partial^2}{\partial t^2}\mathcal{M}(w) = f$$

Projection

$$\begin{aligned} < \phi_n \,|\, \phi_n >_{\mathcal{M}} &= m_n \\ < \phi_n \,|\, \phi_n >_{\mathcal{K}} &= k_n \\ < \phi_n \,|\, f > &= F_n \\ < \phi_n \,|\, w(x, t=0) > &= X_n(t=0) \end{aligned}$$

$$m_n \ddot{X}_n + k_n X_n = F_n$$

$$n \in \mathbb{N}$$

$$X_1(t) \;,\;\; X_2(t) \;,\;\; \ldots$$

Modal recomposition

$$w(x,t) = \sum_{n=1}^{+\infty} X_n(t)\phi_n(x)$$

SOLUTION
$w(x,t)$

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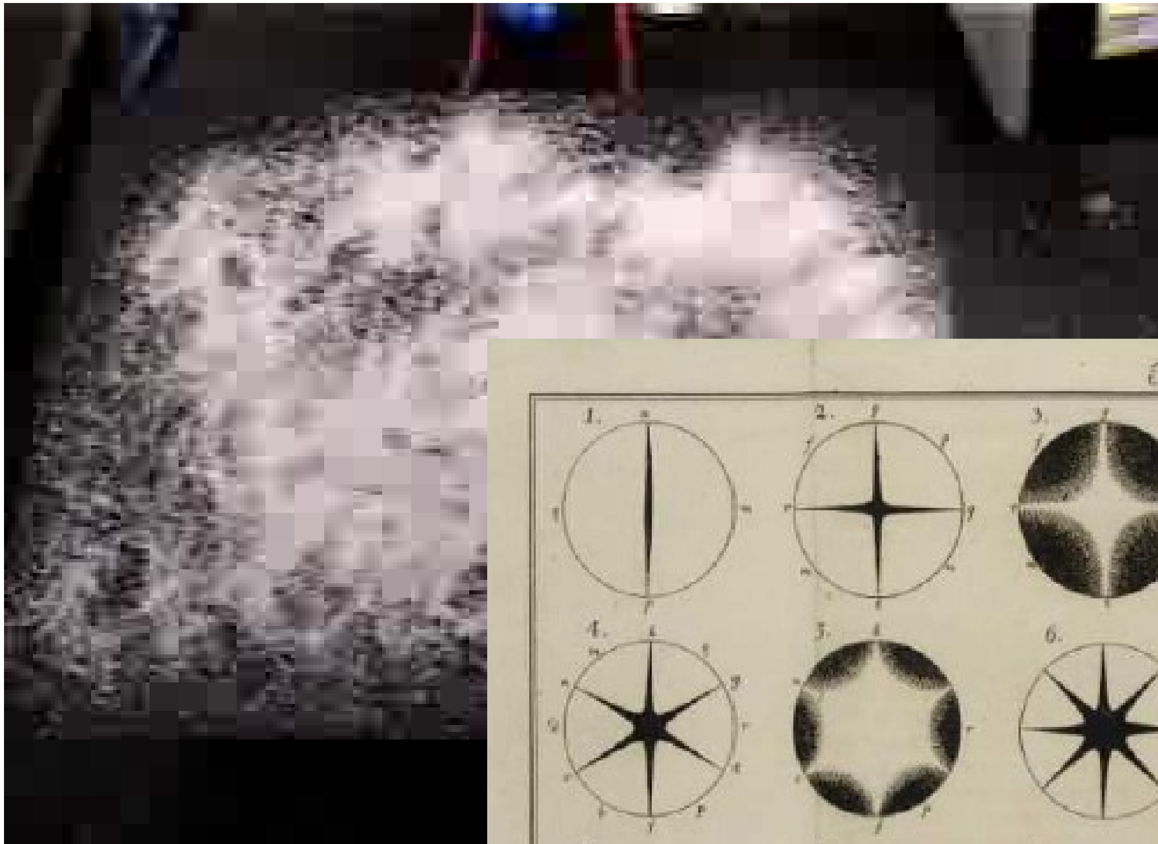
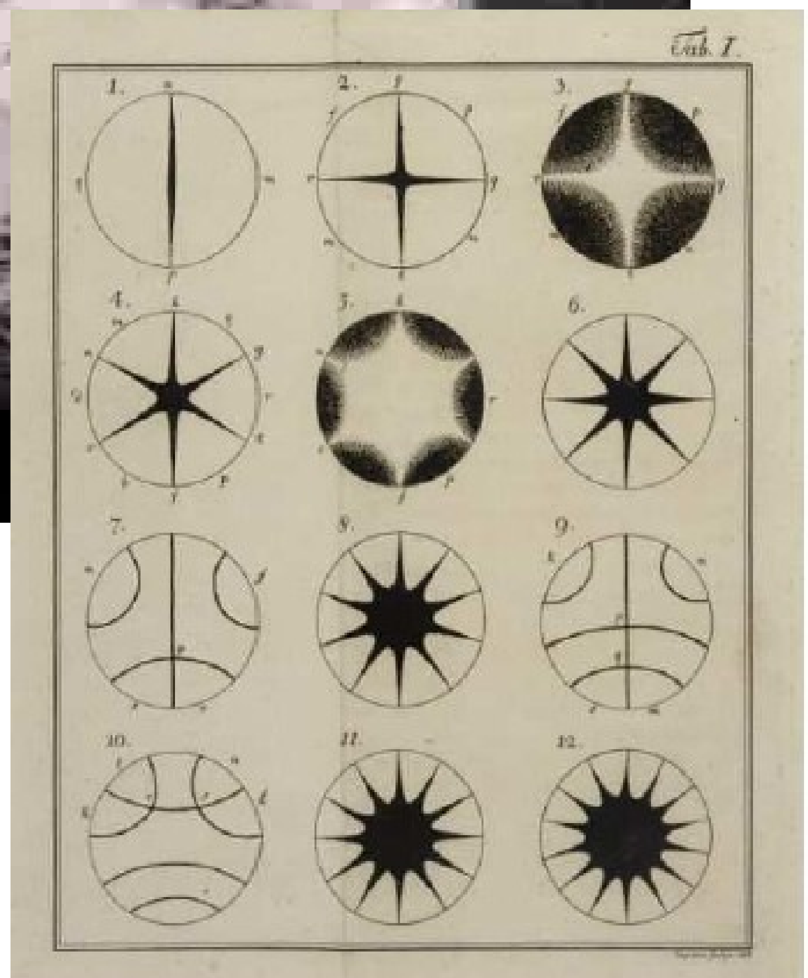
Chladni figures





The case of discrete systems

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- N degrees of freedom system

$$\vec{X} = [X_1, ...X_N]^t$$

- System of N equations governing the temporal evolution of

$$M\ddot{\vec{X}} + K\vec{X} = 0$$

- For a general discrete mechanical system, M and K are full matrices

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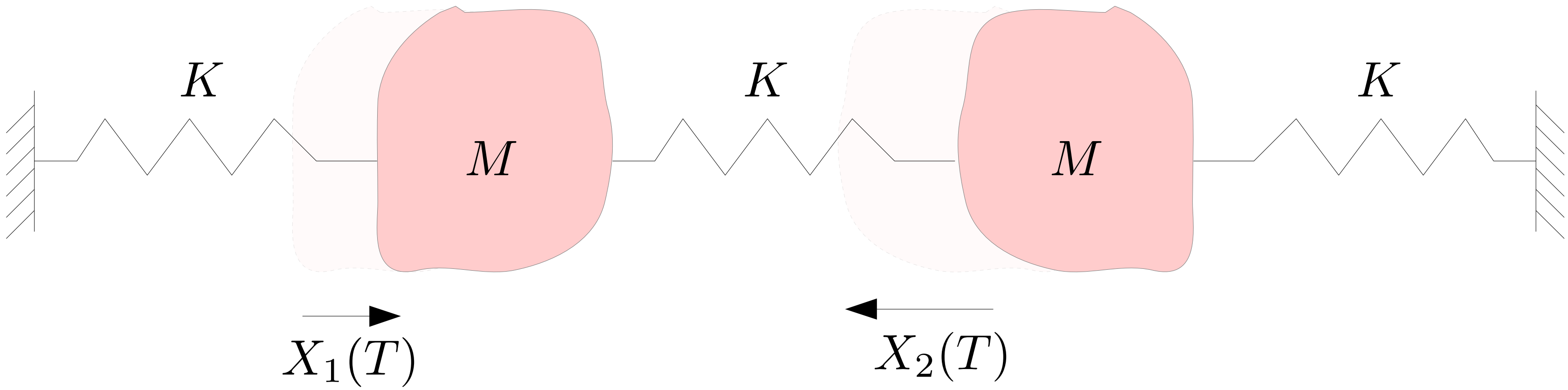
Example 1 : Coupled oscillators



$$\begin{cases} M\ddot{X}_1 + KX_1 - K(X_2 - X_1) = 0 \\ M\ddot{X}_2 + KX_2 - K(X_1 - X_2) = 0 \end{cases}$$

Matrix form :

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \ddot{\vec{X}} + \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix} \vec{X} = \vec{0}$$

Mass matrix Stiffness matrix

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Example 2 : Discretization of continuous systems

- Discretization of the vibrating string equation :

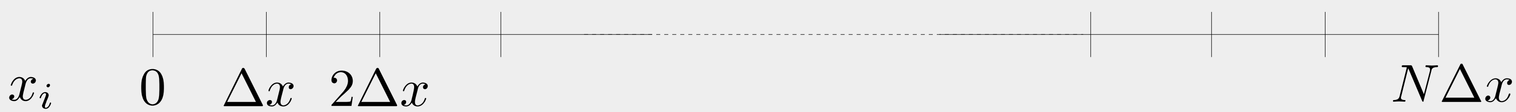
$$\ddot{w} \; - \; c^2 w'' = 0$$

- Space discretization :

$$x_i = (i-1)\Delta x \quad i \in [1, N+1] \qquad \text{with} \qquad \Delta x = L/N$$

- Spatial discretization of all quantities :

$$\begin{aligned} w_i &= w(x_i, t), \\ \ddot{w}_i &= \ddot{w}(x_i, t) \\ w''_i &= w''(x_i, t) \end{aligned}$$



Taylor expansion :

$$w_{i-1} = w_i - \Delta x \left.\frac{\partial w(x,t)}{\partial x}\right|_{x_i} + \frac{\Delta x^2}{2}\left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x_i} - \frac{\Delta x^3}{6}\left.\frac{\partial^3 w(x,t)}{\partial x^3}\right|_{x_i} + O(\Delta x^4)$$

$$w_{i+1} = w_i + \Delta x \left.\frac{\partial w(x,t)}{\partial x}\right|_{x_i} + \frac{\Delta x^2}{2}\left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x_i} + \frac{\Delta x^3}{6}\left.\frac{\partial^3 w(x,t)}{\partial x^3}\right|_{x_i} + O(\Delta x^4)$$

Sum of the two expansions → order two approximation of the the spatial derivative :

$$\left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x_i} = w_i'' = \frac{w_{i-1} - 2w_i + w_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

Discretized version of the string equation :

$$\forall i \in [2, N-1], \quad \ddot{w}_i - c^2 \frac{w_{i-1} - 2w_i + w_{i-1}}{\Delta x^2} = O(\Delta x^2)$$

Boundary conditions :

$$w_1 = 0 \ , \quad w_{N+1} = 0.$$

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- Matrix form of the coupled equations :

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} \ddot{w}_2 \\ \ddot{w}_3 \\ \vdots \\ \ddot{w}_{N-1} \\ \ddot{w}_N \end{bmatrix} - c^2 \frac{1}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \\ \vdots \\ w_{N-1} \\ w_N \end{bmatrix} = 0$$

Mass matrix

Stiffness matrix

$$M\ddot{\vec{w}} + K\vec{w} = 0$$

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Eigenmodes and eigenfrequencies of discrete mechanical systems

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Eigenvalue problem

- General form of the mechanical problem :

$$M\ddot{\vec{X}} + K\vec{X} = 0$$

- Harmonic solutions are sought for :

$$\vec{X}(t) = \vec{\phi}e^{i\omega t}$$

- The mechanical equation becomes :

$$(K - \omega^2 M)\vec{\phi} = 0$$

- Non trivial solution if :

$$\det(K - \lambda M) = 0 \qquad (\lambda \equiv \omega^2)$$

- Eigenmodes and eigenfrequencies :

$$\vec{\phi}_n e^{i\omega_n t} \qquad \text{with} \;\; \omega_n = \pm\sqrt{\lambda_n}$$

$\vec{\phi}_n$ Eigenmode

ω_n Associated eigenfrequency

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Orthogonality

- Orthogonality with respect to the mass matrix :

$$\vec{\phi_n}^t M \vec{\phi_m} = \tilde{M}_n \delta_{mn}$$

- Orthogonality with respect to the stiffness matrix :

$$\vec{\phi_n}^t K \vec{\phi_m} = \tilde{K}_n \delta_{mn}$$

- Projection of the equation on the eigenmodes :

$$\vec{X}(t) = \sum_n Q_n(t) \vec{\phi_n}$$

$$\sum_n \ddot{Q} M \vec{\phi_n} + \sum_n Q K \vec{\phi_n} = \vec{F}(t)$$

$$\vec{\phi_m}^t \sum_n \ddot{Q} M \vec{\phi_n} + \vec{\phi_m}^t \sum_n Q K \vec{\phi_n} = \vec{\phi_m}^t \vec{F}(t)$$

$$\boxed{\tilde{M}_m \ddot{Q}_m + \tilde{K}_m Q_m = \tilde{F}_m}$$

→ N harmonic oscillator equations

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- In the « physical » space mass and stiffness matrices are full matrices :

$$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}\begin{bmatrix} \ddot{\vec{X}} \end{bmatrix} + \begin{bmatrix} & & \\ & K & \\ & & \end{bmatrix}\begin{bmatrix} \vec{X} \end{bmatrix} = \vec{F}(t)$$

- In the modal space, matrices are diagonal :

$$\begin{bmatrix} \ddots & & \\ & \tilde{M} & \\ & & \ddots \end{bmatrix}\begin{bmatrix} \ddot{\vec{Q}} \end{bmatrix} + \begin{bmatrix} \ddots & & \\ & \tilde{K} & \\ & & \ddots \end{bmatrix}\begin{bmatrix} \vec{Q} \end{bmatrix} = \tilde{\vec{F}}(t)$$

- Change of basis :

$$P = \begin{bmatrix} \vec{\phi}_1 & \cdots & \vec{\phi}_N \end{bmatrix} \qquad \vec{x} = P\vec{q} \qquad \vec{q} = P^{-1}\vec{x}$$

Example of the coupled oscillators

- Mechanical equations :

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \ddot{\vec{X}} + \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix} \vec{X} = \vec{0}$$

- Solutions sought in the form :

$$\vec{X}(t) = \vec{\phi} e^{i\omega t}$$

- Eigenvalue problem :

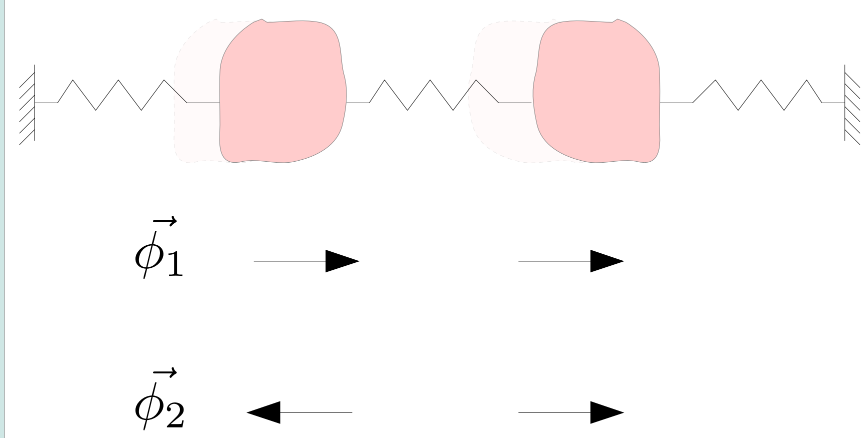
$$\begin{bmatrix} 2K - \omega^2 M & -K \\ -K & 2K - \omega^2 M \end{bmatrix} \vec{\phi} = 0$$

- Non-zero solution if the determinant vanishes :

$$\omega_1 = \pm\sqrt{\frac{K}{M}} \quad \omega_2 = \pm\sqrt{\frac{3K}{M}}.$$

- Eigenvalues and eigenvectors :

$$\vec{\phi}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{\phi}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



One mode where oscillators are in phase, one mode where they are in opposite phase.

Example of the string discretized by finite differences

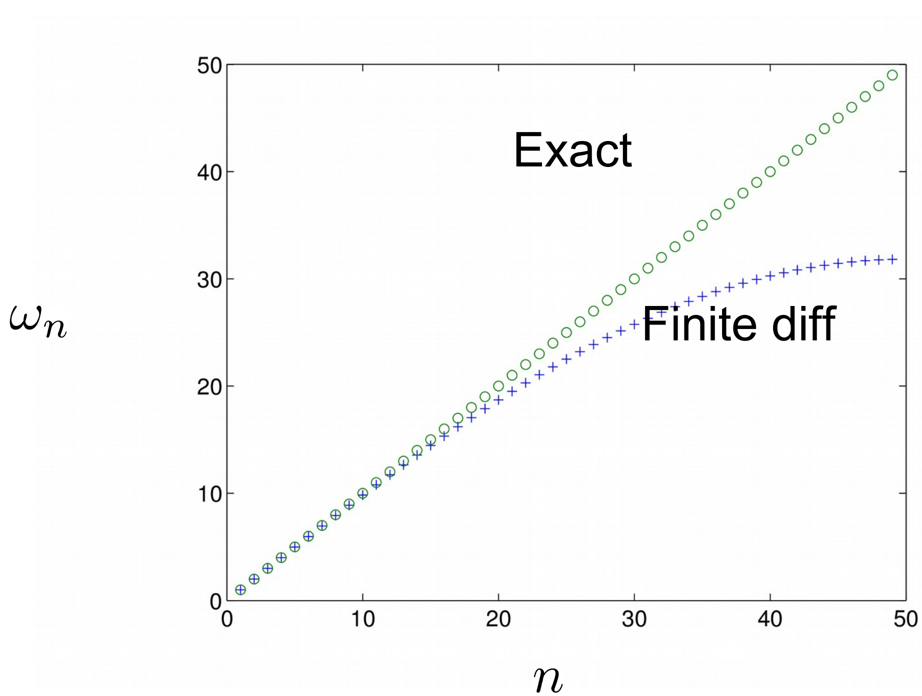
- Mechanical equations :

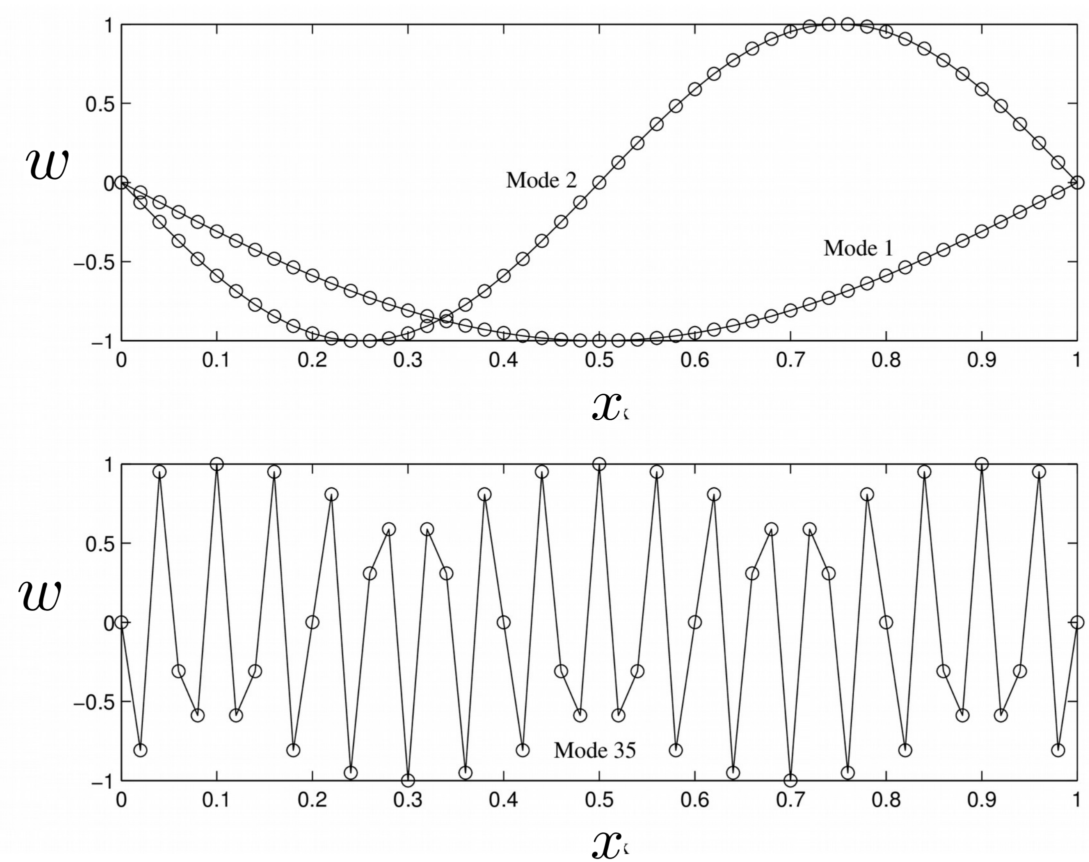
$$M\ddot{\vec{w}} + K\vec{w} = 0$$

where $\vec{w}$ is the vector containing the values of the displacement at discrete points x_i

- Solutions sought of the form : $\vec{w} = \vec{V}\mathrm{e}^{\mathrm{i}\omega t}$

- Eigenvalue problem : $(K - \omega^2 M)\vec{V} = 0$





Conclusion on structural dynamics

	Continuous systems	Discrete systems
Dynamical equation :	$\frac{\partial^2}{\partial t^2}\mathcal{M}(w) + \mathcal{K}(w) = F(\underline{x}, t)$	$M\ddot{\vec{w}} + K\vec{w} = \vec{F}(t)$
Eigenmodes and eigenfrequencies :	$\phi_n(\underline{x}) \quad , \quad \omega_n \quad , \quad n \in \mathbb{N}$	$\vec{\phi}_n \; , \quad \omega_n \; , \quad n \in [1..N]$
Projection of the dynamics on one eigenmode n :	$m_n \ddot{X}_n + k_n X_n = F_n$	$m_n \ddot{X}_n + k_n X_n = F_n$

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Back to the fluid-structure interaction problem

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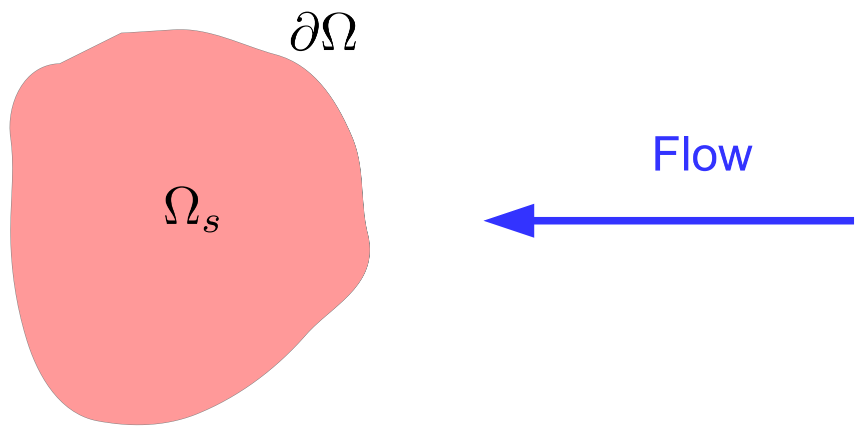

The structure dynamics can be now considered to be solved by modal analysis

$$m_n \ddot{X}_n + k_n X_n = F_n$$
$$n \in \mathbb{N}$$

$$X_1(t)\ ,\quad X_2(t)\ ,\quad \ldots$$

The flow dynamics has still to be analysed, as well as its effect on the structure

$$F_n$$

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Kinematic condition (equality of displacements)

$$\underline{u} = \sum_n \underline{\phi}_n(\underline{x})\dot{X}_n(t) \quad \text{on } \partial\Omega$$

$$m_n \ddot{X}_n + k_n X_n = F_n$$

$$n \in \mathbb{N}$$

$$X_1(t)\ , \quad X_2(t)\ , \quad \ldots$$

FLUID

Dynamic condition (equality of efforts)

$$F_n = \int_{\partial\Omega} (\underline{\underline{\Sigma}}_f \cdot \underline{n}) \cdot \underline{\phi}_n \, dS$$

The fluid has still to be treated

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Dimensional analysis

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Variables and parameters of a coupled fluid-solid problem

	Fluid	Fluid & solid	Solid
Variables	$\underline{U}, P, \rho_f$		$\underline{\Xi}\ ,\ \underline{\underline{\Sigma}}$
Parameters	$\mu\ ,\ \rho_{f0},\ U_0\ ,\ c_0$	g, L	$\rho_s\ ,\ E\ ,\ \nu,\ \Xi_0$

If the structural problem has already been decomposed on its eigenmodes :

	Fluid	Fluid & solid	Structure
Variables	$\underline{U}, P, \rho_f$		X_n
Parameters	$\mu\ ,\ \rho_{f0},\ U_0\ ,\ c_0$	g, L	M_n, K_n, ϕ_n

Many parameters → dimensional analysis is necessary to simplify the problem for e.g. a parametric study

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Washy-Buckingam theorem (Pi theorem)

If we have a physically meaningful equation such as

$$f(q_1, q_2, ...q_n) = 0$$

where the qi are the n physical variables, and they are expressed in terms of k independent physical units, then the above equation can be restated as

$$F(\pi_1, \pi_2, ..., \pi_p) = 0$$

where the πi are dimensionless parameters constructed from the qi by p = n − k dimensionless equations —the so-called Pi groups— of the form

$$\pi_i = q_1^{a_1} q_2^{a_2} ...q_n^{a_n}$$

where the exponents ai are rational numbers (they can always be taken to be integers: just raise it to a power to clear denominators).

Dimensionless form of the Navier-Stokes equations

Mass conservation

$$\frac{\partial \rho_f}{\partial T} + \mathrm{div}\,(\rho_f \underline{U}) = 0$$

Momentum conservation

$$\rho_f \frac{\partial \underline{U}}{\partial T} + \rho_f \left(\underline{\underline{\mathrm{grad}}}\ \underline{U} \right) . \underline{U} =$$
$$-\rho_f\ g\ \underline{e}_Z - \underline{\mathrm{grad}}\ P + (\lambda + \mu)\underline{\mathrm{grad}}\ \mathrm{div}\ \underline{U} + \mu\ \Delta\ \underline{U}$$

- This equation set is not complete, one need an additional law to link pressure and density. The most simple one is :

$$\frac{\delta p}{\delta \rho_f} = c_0^2$$

where c_0^2 is the speed of sound in the fluid

- We then introduce the following dimensionless quantities :

$$\underline{u} = \frac{\underline{U}}{U_0} \qquad r = \frac{\rho_f}{\rho_{f0}} \qquad \underline{x} = \frac{\underline{X}}{L}$$
$$p = \frac{P}{\rho_{f0} U_0^2} \qquad t = \frac{T U_0}{L}$$

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Dimensionless form to the Navier-Stokes equations

Mass conservation

$$M^2 \frac{dp}{dt} + r \, \mathrm{div} \, \underline{u} = 0$$

Momentum conservation

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\mathrm{grad}}} \, \underline{u} \right) . \underline{u}$$

$$= -\frac{1}{F_r^2} \, \underline{e}_z - \frac{1}{r} \underline{\mathrm{grad}} \, p + \frac{1}{r} \frac{1}{R_e} \left(\Delta \underline{u} + \frac{1}{3} \mathrm{grad}(\mathrm{div} \, \underline{u}) \right)$$

Dimensionless parameters :

$$M = \frac{U_0}{c_0}$$ Mach number

$$F_r = \frac{U_0}{\sqrt{gL}}$$ Froude number

$$R_e = \frac{\rho_{f0} U_0 L}{\mu}$$ Reynolds number

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If $M^2 \ll 1$ the equations become :

Mass conservation

$$\text{div}\ \underline{u} = 0$$

(incompressibility condition)

Momentum conservation

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right).\underline{u} = -\frac{1}{F_r^2}\ \underline{e}_z - \frac{1}{r}\underline{\text{grad}}\ p + \frac{1}{R_e}\Delta \underline{u}$$

Where $r \sim 1 + M^2$ has been considered

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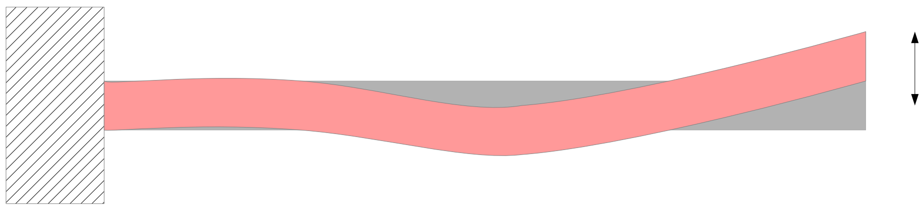
An example in solid mechanics : The beam



- Beam modelled under the Euler-Bernoulli approximations.
- Parameters : $EI,\ \mu,\ L$
- Variables : X, Y, T
- For an homogeneous beam (constant section) :

$$EI\frac{\partial^4 Y}{\partial X^4} + \mu\frac{\partial^2 Y}{\partial T^2} = 0$$ with boundary conditions

- General form of the solution $Y(X,T)$:

$$F(Y, X, T, L, EI, \mu) = 0 \longrightarrow n = 6\ , \quad k = 3$$ (space,time,mass)

- Dimensionless variables :

$$z = Z/L \qquad y = Y/L \qquad t = \frac{T}{\tau} = T\left(\frac{1}{L^2}\sqrt{\frac{EI}{\mu}}\right)$$

- Dimensionless equilibrium equation :

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial z^4} = 0 \ , \quad z \in [0,1]$$

+ boundary conditions at $z = 0, z = 1$

- General form of the solution is now :

$$f\,(y, z, t) = 0$$

→ No more parameter dependency !

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Application to the fluid-structure problem

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Application to the fluid-structure problem

Variables and parameters of a coupled fluid structure problem

	Fluid	Fluid & solid	Structure
Variables	$\underline{U}, P, \rho_f$		X_n
Parameters	$\mu\ ,\ \rho_{f0}, U_0\ ,\ c_0$	g, L	M_n, K_n, ϕ_n

Examples of questions that dimensional analysis can answer :

- Can the main flow be neglected when studying the dynamics of the structure ?
- How influent is the pressure gradient ?
- What is the role of viscosity ?
- How big is the influence the inertia given to the fluid when a structure oscillates ?

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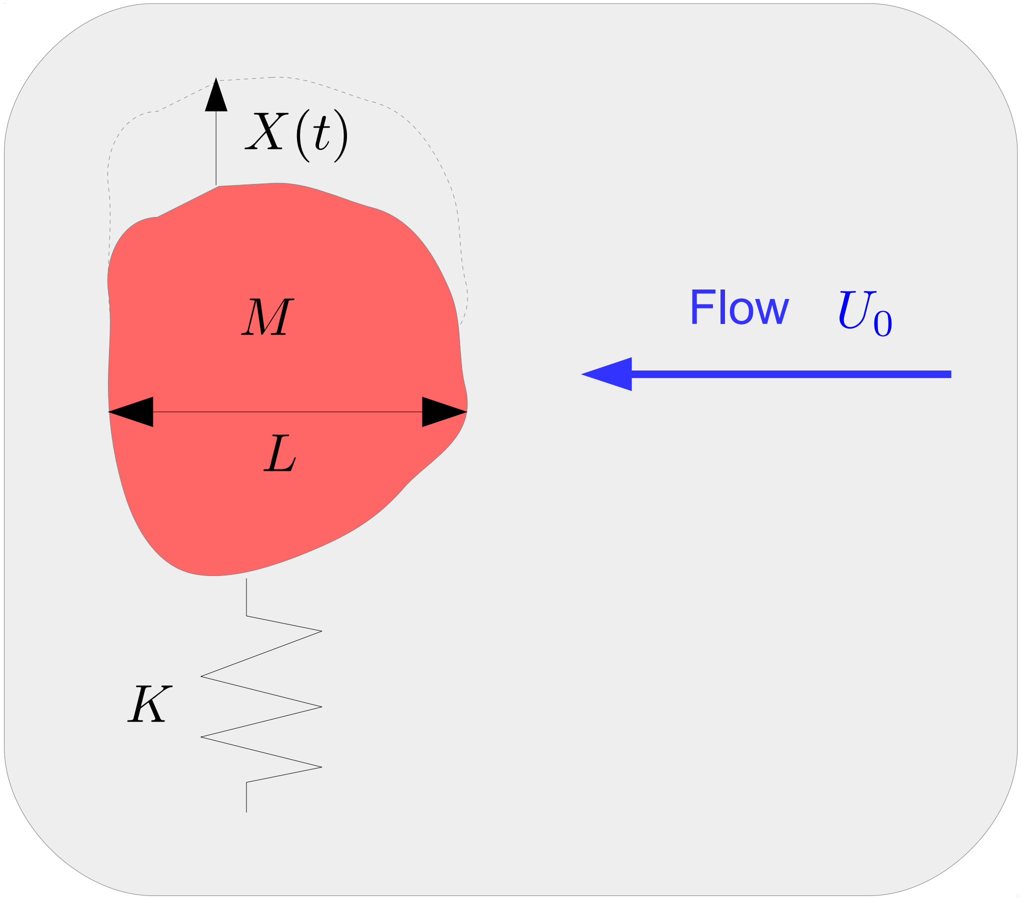
Oscilator in flow: a basic problem



On the contact surface, the kinematic boundary condition links the velocity of the fluid and the velocity of the structure :

$$\underline{U} = \frac{\partial \underline{\Xi}}{\partial T} \quad \text{on } \partial\Omega$$

One oscillatory mode approximation :

$$\underline{\Xi}(\underline{X}, t) = X(t)\underline{\phi}$$

$$\Longrightarrow \underline{U} = \underline{\phi}\frac{\partial X}{\partial T} \quad \text{on } \partial\Omega$$

Rigid body mode : $\underline{\phi} = \underline{i}$

Characteristic time for the solid : $\tau = 1/\Omega_0 \quad (\Omega_0 = \sqrt{K/M})$

Characteristic length : L

Characeristic velocity for the fluid : U_0

With or without main flow

Non-dimensional form of the kinematic boundary condition :

$$\underline{U} = \frac{\partial \underline{\Xi}}{\partial T} \qquad \longrightarrow \qquad \underline{u} = \frac{1}{\mathcal{U}} \frac{\partial \underline{\xi}}{\partial t} = \mathcal{O}\left(\frac{\mathcal{D}}{\mathcal{U}}\right)$$

with $\mathcal{U} = \frac{U_0}{\Omega_0 L} \qquad \mathcal{D} = \frac{\Xi_0}{L}$

$$\mathcal{U} \ll \mathcal{D} \implies \mathcal{O}(\underline{u}) \gg 1 \implies \mathcal{O}(\underline{U}) \gg U_0$$

The order of the velocity is imposed by the velocity of the structure. The fluid is seen as still from the point of view of the solid

$$\mathcal{U} \gg \mathcal{D} \implies \underline{u} \sim 0$$

The solid is seen as immobile from the point of view of the structure

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Non dimensional form of the fluid's equations

Choice of characteristic space and time differs if the fluid is still or not...

- **If still fluid** : no characteristic velocity imposed by the flow

  Characteristic time : $1/\Omega_0$ Characteristic length : L Characteristic velocity : $\Omega_0 L$

- **If flow** : a characteristic velocity exists, it can be used

  Characteristic time : L/U_0 Characteristic length : L Characteristic velocity : U_0

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Full equations' set for the fluid part

No main flow

- Time, length and density :

$$1/\Omega_0\ , \quad L\ , \quad \rho_f$$

- Mass conservation :

$$\text{div}\underline{u} = 0$$

- Momentum conservation :

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) . \underline{u} = -\frac{1}{F_D^2}\underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{S_t}\Delta \underline{u}$$

- Kinematic boundary condition :

$$\underline{u} = \frac{\partial \underline{\xi}}{\partial t}$$

$$F_D = \Omega_0 \sqrt{\frac{L}{g}} \qquad S_t = \frac{\Omega_0 L^2}{\nu}$$

Main flow U_0

- Time, length and density :

$$L/U_0\ , \quad L\ , \quad \rho_f$$

- Mass conservation :

$$\text{div}\underline{u} = 0$$

- Momentum conservation :

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) . \underline{u} = -\frac{1}{F_R^2}\underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{R_E}\Delta \underline{u}$$

- Kinematic boundary condition :

$$\underline{u} = \frac{1}{\mathcal{U}} \frac{\partial \underline{\xi}}{\partial t}$$

$$F_R = \frac{U_0}{\sqrt{gL}} \qquad R_e = \frac{U_0 L}{\nu} \qquad \mathcal{U} = \frac{U_0}{\Omega_0 L}$$

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Influence of viscosity

Stokes number

$$S_t = \frac{\Omega_0 L^2}{\nu}$$

Reynolds number

$$R_e = \frac{U_0 L}{\nu}$$

It is known that viscosity influences the flow dynamics, even if $1/S_t$ or $1/R_e$ are small.

However, the stress tensor in the fluid,

$$\underline{\underline{\Sigma}}_f = -P\,\underline{\underline{1}} + 2\mu\underline{\underline{D}} \quad \longrightarrow \quad \underline{\underline{\sigma}}_f = -p\,\underline{\underline{1}} + \frac{1}{S_t}\underline{\underline{d}} \quad \text{or} \quad \underline{\underline{\sigma}}_f = -p\,\underline{\underline{1}} + \frac{1}{R_e}\underline{\underline{d}}$$

is dominated by the pressure. Wall friction can be neglected.

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Pressure gradient effects

Dynamical Froude number

$$F_D = \Omega_0 \sqrt{\frac{L}{g}}$$

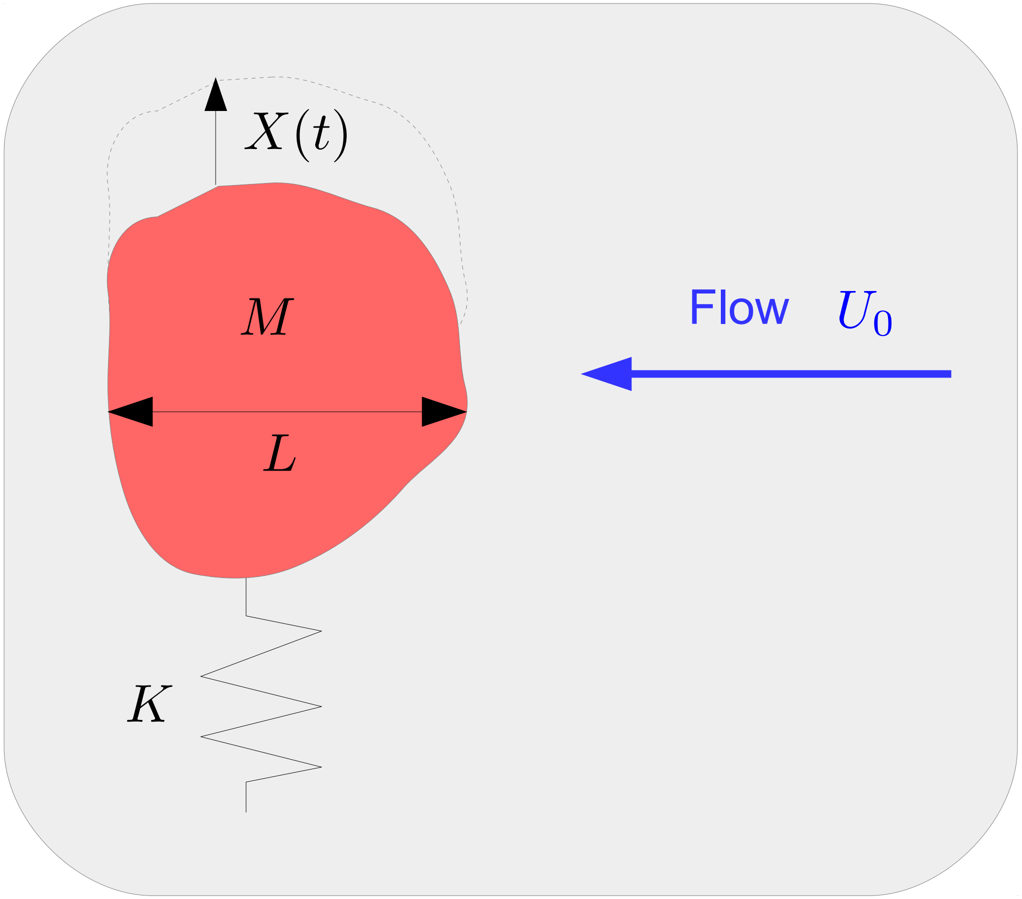
Froude number

$$F_R = \frac{U_0}{\sqrt{gL}}$$

These numbers quantify the influence of the pressure gradient on the fluid-structure dynamics

Back to the structure

Consider the fluid problem has been solved



$$M\ddot{X} + KX = F$$

where

$$F = \int_{\partial\Omega} (\underline{\underline{\Sigma}}_f \cdot \underline{n}) \cdot \underline{\phi} \, \mathrm{d}S$$

$$\underline{\underline{\Sigma}}_f = -P \, \underline{\underline{1}} + 2\mu \underline{\underline{D}}$$

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Full equations' set for the structural part

No main flow

- Time, length and density :

$$1/\Omega_0\ , \quad L\ , \quad \rho_f$$

- Oscillator's equation :

$$\ddot{x} + x = \mathcal{M} \int_{\partial\Omega} (\underline{\underline{\sigma}}_f \ . \ \underline{n}) \ . \ \underline{\phi} \, \mathrm{d}s$$

- Stress tensor :

$$\underline{\underline{\sigma}}_f = -p \, \underline{\underline{1}} + \frac{1}{S_t} \underline{\underline{d}}$$

$$\mathcal{M} = \frac{\rho_f L^3}{M}$$

Main flow U_0

- Time, length and density :

$$L/U_0\ , \quad L\ , \quad \rho_f$$

- Oscillator's equation :

$$\ddot{x} + x = \mathcal{C}_\mathcal{Y} \int_{\partial\Omega} (\underline{\underline{\sigma}}_f \ . \ \underline{n}) \ . \ \underline{\phi} \, \mathrm{d}s$$

- Stress tensor :

$$\underline{\underline{\sigma}}_f = -p \, \underline{\underline{1}} + \frac{1}{R_e} \underline{\underline{d}}$$

$$\mathcal{C}_\mathcal{Y} = \frac{\rho_f U_0^2 L}{K}$$

The STILL FLUID-structure problem : dimensionless version

Kinematic condition (equality of displacements)

$$\underline{u} = \sum_n \underline{\phi}_n(\underline{x})\dot{x}_n(t) \quad \text{on } \partial\Omega$$

$$m_n \ddot{x}_n + k_n x_n = \mathcal{M} f_n$$

$$n \in \mathbb{N}$$

$$\text{div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) . \underline{u} = -\frac{1}{F_D^2}\underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{S_t}\Delta \underline{u}$$

Dynamic condition (equality of efforts)

$$f_n = \int_{\partial\Omega} (\underline{\underline{\sigma}}_f . \underline{n}) . \underline{\phi}_n\, ds$$

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The FLOW-structure problem : dimensionless version

Kinematic condition (equality of displacements)

$$\underline{u} = \frac{1}{\mathcal{U}} \sum_n \underline{\phi}_n(\underline{x}) \dot{x}_n(t) \quad \text{on } \partial\Omega$$

$$\ddot{x}_n + \omega_n^2 x_n = \mathcal{C_Y} f_n$$

$$n \in \mathbb{N}$$

$$\text{div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) . \underline{u} = -\frac{1}{F_R^2}\underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{R_e}\Delta \underline{u}$$

Dynamic condition (equality of efforts)

$$f_n = \int_{\partial\Omega} (\underline{\underline{\sigma}}_f \cdot \underline{n}) \cdot \underline{\phi}_n \, ds$$

Linearization of the fluid-dynamics problems

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Kinematic condition (equality of displacements)

$$\underline{u} = \sum_n \underline{\phi}_n(\underline{x})\dot{x}_n(t) \quad \text{on } \partial\Omega$$

$$\ddot{x}_n + \omega_n^2 x_n = \mathcal{M} f_n$$

$$n \in \mathbb{N}$$

Non linearities

$$\text{div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) . \underline{u} = -\frac{1}{F_D^2}\underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{S_t}\Delta \underline{u}$$

Dynamic condition (equality of efforts)

$$f_n = \int_{\partial\Omega} (\underline{\underline{\sigma}}_f . \underline{n}) . \underline{\phi}_n \, ds$$

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The FLOW-structure problem : dimensionless version

Kinematic condition (equality of displacements)

$$\underline{u} = \frac{1}{\mathcal{U}} \sum_n \underline{\phi}_n(\underline{x}) \dot{x}_n(t) \quad \text{on } \partial\Omega$$

$$\ddot{x}_n + \omega_n^2 x_n = \mathcal{C_Y} f_n$$

$$n \in \mathbb{N}$$

Non linearities

$$\text{div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) \cdot \underline{u} = -\frac{1}{F_R^2} \underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{R_e} \Delta \underline{u}$$

Dynamic condition (equality of efforts)

$$f_n = \int_{\partial\Omega} (\underline{\underline{\sigma}}_f \cdot \underline{n}) \cdot \underline{\phi}_n \, ds$$

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Followed next week...

What we have done today

- Necessity of fluid-structure interaction studies in marine renewable energies
- The fluid-mechanics and solid mechanics problems
- Solid mechanics → structural dynamics → modal analysis
- Modal analysis : The dynamics is expressed in the form of un-coupled oscillators
- Fluid-structure interaction → Fluid dynamics coupled with these oscillators
- Fluid-structure problem : complex problem with many parameters → necessity of reducing the number of variables and parameters
- Dimensional analysis
- Application of dimensional analysis in the fluid-structure interaction case

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Next week...

- Linearization of the fluid-structure problem
- Still fluid effects : Added mass, added damping, added rigidity
- Flow effects : Added rigidity, damping
- Mode coupling
- Flow induced vibrations
- Flow energy harvesting using flow induced vibrations

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