Fluid-structure interaction problems in marine renewable energies

Lesson 2 : Vibrations of a structure in a flow

Olivier Doaré
UME, ENSTA-Paristech
Olivier.doare (@t) ensta-paristech.fr

Kinematic condition
(equality of displacements)

$$\underline{U} = \frac{\partial \underline{\Xi}}{\partial T} \quad \text{on } \partial\Omega$$

SOLID

$\underline{\Xi}$

FLUID

$\underline{U}\ , \quad P$

Dynamic condition
(equality of efforts)

$$\underline{\underline{\Sigma}}_s \cdot \underline{n} = \underline{\underline{\Sigma}}_f \cdot \underline{n}$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Kinematic condition
(equality of displacements)

$$\underline{U} = \sum_n \underline{\phi}_n(\underline{X})\dot{X}_n(t) \quad \text{on } \partial\Omega$$

STRUCTURE

X_n

$n \in \mathbb{N}$

FLUID

$\underline{U}\ , \quad P$

Dynamic condition
(equality of efforts)

$$F_n = \int_{\partial\Omega} (\underline{\underline{\Sigma}}_f \cdot \underline{n}) \cdot \underline{\phi}_n \, dS$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

The reduced velocity and the displacement number

Reduced velocity $\mathcal{U} = \dfrac{U_0}{\Omega_0 L}$

Displacement number $\mathcal{D} = \dfrac{\Xi_0}{L}$

Ω_0 Typical frequency

Ξ_0 Typical amplitude of displacement

$$\frac{\mathcal{U}}{\mathcal{D}}$$

- $\ll 1$: Oscillations of structures in still fluid
- ~ 1 : Strong coupling
- $\gg 1$: Quasi-static approach : the solid is at rest from the point of view of he fluid

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

The STILL FLUID-structure problem : dimensionless version

$$\frac{\mathcal{U}}{\mathcal{D}} \ll 1$$

Kinematic condition
(equality of displacements)

$$\underline{u} = \sum_n \underline{\phi}_n(\underline{x})\dot{x}_n(t) \quad \text{on } \partial\Omega$$

$$m_n \ddot{x}_n + k_n x_n = \mathcal{M} f_n$$

$$n \in \mathbb{N}$$

Non linearities

$$\text{div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) . \underline{u} = -\frac{1}{F_D^2}\underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{S_t}\Delta \underline{u}$$

Dynamic condition
(equality of efforts)

$$f_n = \int_{\partial\Omega} (\underline{\underline{\sigma}}_f . \underline{n}) . \underline{\phi}_n \, ds$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

The FLOW-structure problem : dimensionless version

Kinematic condition (equality of displacements)

$$\underline{u} = \frac{1}{\mathcal{U}} \sum_n \underline{\phi}_n(\underline{x}) \dot{x}_n(t) \quad \text{on } \partial\Omega$$

$$m_n \ddot{x}_n + k_n x_n = \mathcal{C}_{\mathcal{Y}} f_n$$

$$n \in \mathbb{N}$$

Non linearities

$$\text{div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) \cdot \underline{u} = -\frac{1}{F_R^2} \underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{R_e} \Delta \underline{u}$$

Dynamic condition (equality of efforts)

$$f_n = \int_{\partial\Omega} (\underline{\underline{\sigma}}_f \cdot \underline{n}) \cdot \underline{\phi}_n \, ds$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

I - Oscillations in a still fluid

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

The STILL FLUID-structure problem : dimensionless version

Kinematic condition (equality of displacements)

$$\underline{u} = \frac{\partial \underline{\xi}}{\partial t} = \sum_n \underline{\phi}_n(\underline{x}) \dot{x}_n(t) \quad \text{on } \partial\Omega$$

$\mathcal{O}(\mathcal{D})$

$$m_n \ddot{x}_n + k_n x_n = \mathcal{M} f_n$$

$$n \in \mathbb{N}$$

$$\text{div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) \cdot \underline{u} = -\frac{1}{F_D^2} \underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{S_t} \Delta \underline{u}$$

Non linear term

If $\mathcal{D} \ll 1$, the convective term can be neglected

Dynamic condition (equality of efforts)

$$f_n = \int_{\partial\Omega} \left(\underline{\underline{\sigma}}_f \cdot \underline{n}\right) \cdot \underline{\phi}_n \, ds$$

These equations are still nonlinear because they are evaluated at the current point

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

ONE MODE VERSION

Kinematic condition (equality of displacements)

$$\underline{u} = \frac{\partial \underline{\xi}}{\partial t} = \underline{\phi}(\underline{x})\dot{x}(t) \quad \text{on } \partial\Omega$$

$$\ddot{x} + x = \mathcal{M} f$$

$$\mathrm{div}\, \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\mathrm{grad}}}\, \underline{u}\right) \cdot \underline{u} = -\frac{1}{F_D^2}\underline{e}_z - \underline{\mathrm{grad}}\, p + \frac{1}{S_t}\Delta \underline{u}$$

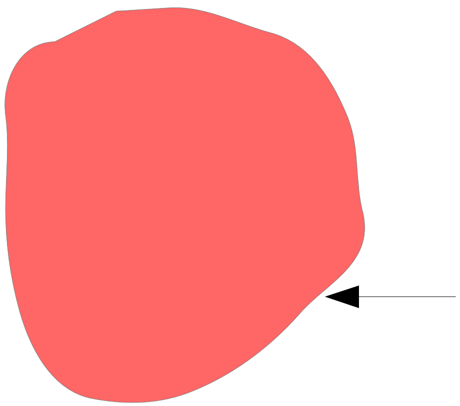
Dynamic condition (equality of efforts)

$$f = \int_{\partial\Omega} (\underline{\underline{\sigma}}_f \cdot \underline{n}) \cdot \underline{\phi}\, ds$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

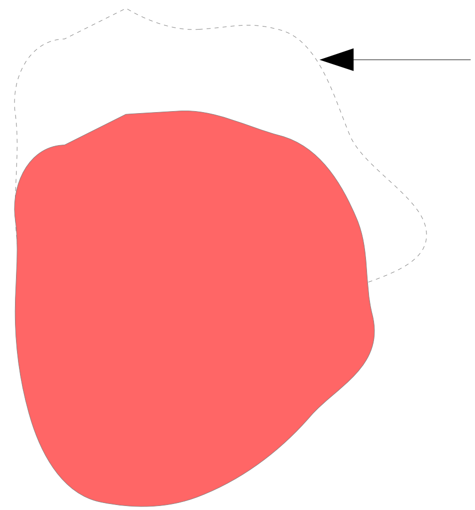
Perturbation technique

Equilibrium state



- Displacement : $\underline{\xi} = 0$
- Velocity in fluid : $\underline{u} = 0$
- Stress in fluid : $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_0 = -p_0\underline{\underline{1}}$

Perturbed state



- Displacement : $\underline{\xi} = \epsilon\underline{\xi}'$ $\quad \epsilon = \mathcal{O}(\mathcal{D}) \ll 1$
- Velocity in fluid : $\underline{u} = \epsilon\underline{u}'$
- Stress in fluid : $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_0 + \epsilon\underline{\underline{\sigma}}'$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Linearization of the fluid's equations

$$\mathrm{div}\ \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\mathrm{grad}}}\ \underline{u}\right) . \ \underline{u} = -\frac{1}{F_D^2}\underline{e}_z - \underline{\mathrm{grad}}\ p + \frac{1}{S_t}\Delta \underline{u}$$

Order ϵ^0

$$\mathrm{div}\ \underline{0} = 0$$

$$-\frac{1}{F_D^2} e_z = \underline{\mathrm{grad}}\ p_0$$

$$p_0 = -\frac{1}{F_D^2} z + \mathrm{const.}$$

Hydrostatic pressure

Order ϵ^1

$$\mathrm{div}\ \underline{u}' = 0$$

$$\frac{\partial u'}{\partial t} = -\underline{\mathrm{grad}}\ p' + \frac{1}{S_t}\Delta \underline{u}'$$

$S_T \gg 1$

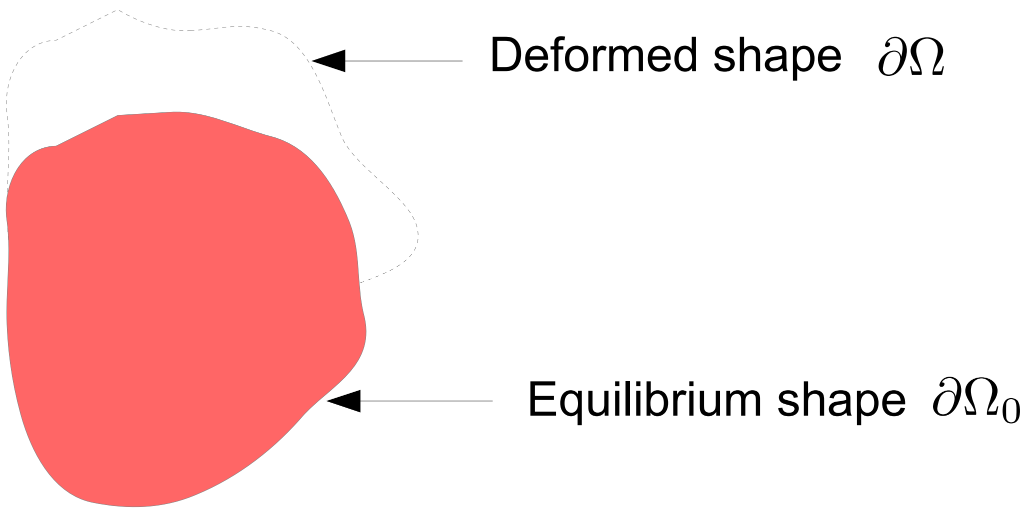
$$\frac{\partial \underline{u}'}{\partial t} = -\underline{\mathrm{grad}}\ p'$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Linearization of the boundary conditions

$$\underline{u} = \frac{\partial \underline{\xi}}{\partial t} = \underline{\phi}(\underline{x})\dot{x}(t) \quad \text{on } \partial\Omega$$

$$f = \int_{\partial\Omega} (\underline{\underline{\sigma}}_f \cdot \underline{n}) \cdot \underline{\phi} \, ds$$



Both boundary conditions are non linear because they are evaluated on a surface that depends on the deformation of the solid.

The linearized version of these expressions are obtained by a Taylor expansion of all quantities. It involves tensor algebra and… patience… It is not done in the present course.

- Linearized kinematic boundary condition :

$$\underline{u} = \frac{\partial \underline{\xi}}{\partial t} = \underline{\phi}(\underline{x})\dot{x}(t) \quad \text{on } \partial\Omega_0$$

- Linearized dynamic condition (projection of the stress on the mode) :

$$\begin{aligned} f = & - \int_{\partial\Omega_0} p_0 \underline{\phi}\,\underline{n}_0 \,\mathrm{d}s \\ & + \epsilon \int_{\partial\Omega_0} \left(\underline{\phi} \,.\, \underline{\underline{\sigma}}'\right) . \,\underline{n}_0 \mathrm{d}s \\ & + \epsilon x \int_{\partial\Omega_0} \Big(-\underline{\underline{\mathrm{grad}}}\underline{\phi}[\underline{\phi} - (\underline{\phi}.\underline{n}_0)\underline{n}_0]p_0 \\ & \quad + \underline{\phi} \,.\, \left[-\underline{\mathrm{grad}}\, p_0 \,.\, \underline{\phi}\,\underline{\underline{1}} - p_0 \left(\mathrm{div}\underline{\phi}\,\underline{\underline{1}} -^t \underline{\underline{\nabla \phi}}\right)\right]\Big) \,.\, \underline{n}_0 \,\mathrm{d}s \\ & + O(\varepsilon^2) \end{aligned}$$

Static pressure

Effect of stress fluctuation in th fluid

Added stiffness due to the deformation of the solid in a static pressure field

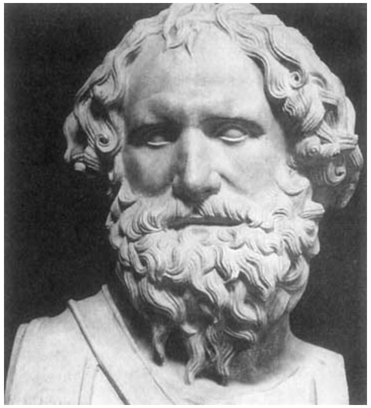
(not proven in the present course)

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Archimedes' force

We consider here the effect of the static pressure :

$$\underline{f}_0 = \int_{\partial\Omega_0} p_0 \underline{\phi}\,\underline{n}_0 \,\mathrm{d}s = \underline{\phi} \cdot \frac{v}{F_D^2}\underline{e}_z = \underline{\phi} \cdot \underline{f}_A$$

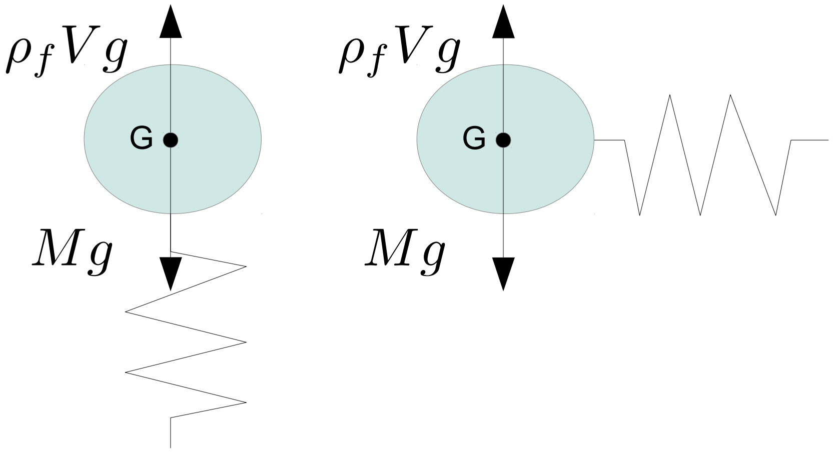


Archimedes of Syracuse
287 BC - 212 BC

$\underline{f}_A = \frac{v}{F_D^2}\underline{e}_z$ is the dimensionless expression of the Archimedes' force $\underline{F}_A = \rho_f V g \underline{e}_z$

As well as the weight of the solid, Archimedes' force affects the equilibrium position of the system only if it is oriented in the same direction as the displacement.

$$\ddot{x} + x = \mathcal{M} f_0$$



Influence No influence

The added mass effect

- We are now interested in the contribution of stress fluctuations in the fluid :

$$f_s = \epsilon \int_{\partial\Omega_0} \left(\underline{\phi} \,.\, \underline{\underline{\sigma}}'\right) . \,\underline{n}_0 \mathrm{d}s$$

- We consider the regime of large Stokes numbers :

$$S_T \gg 1$$

- The problem to solve is the following :

$$\mathrm{div}\ \underline{u}' = 0$$

$$\frac{\partial u'}{\partial t} = -\mathrm{grad}\ p'$$

Boundary conditions : $\underline{u} \,.\, \underline{n} = \dfrac{\partial \underline{\xi}}{\partial t} \,.\, \underline{n} = \dot{x}(t)\underline{\phi}(\underline{x}) \,.\, \underline{n} \quad \text{on } \partial\Omega_0$

- It can be put in the following form :

$$\Delta p' = 0$$

$$-\underline{\mathrm{grad}} p' \,.\, \underline{n} = \frac{\partial^2 \underline{\xi}}{\partial t^2} . \underline{n} = \ddot{x}(t)\underline{\phi}(\underline{x}) \,.\, \underline{n} \quad \text{on } \partial\Omega_0$$

Solution

- the pressure is looked for in the form of a solution to separate variables :

$$p' = x_p(t)\phi_p(\underline{x})$$

- Because of the form of the boundary condition, the solution is of the form :

$$p' = \ddot{x}\phi_p(\underline{x})$$

- Where ϕ_p satisfies :

$$\Delta\phi_p = 0$$

$$-\underline{\mathrm{grad}}\phi_p \cdot \underline{n}_0 = \underline{\phi} \cdot \underline{n}_0$$

- Consider the solution is known, the modal force has then for expression :

$$f = -\ddot{x}\int_{\partial\Omega_0} (\phi_p \cdot \underline{n}_0) \cdot \underline{\phi}\,\mathrm{d}s$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Added mass effect

- Modal force :

$$f = -m_a \ddot{x} \qquad m_a = \int_{\partial\Omega_0} (\phi_p \, . \, \underline{n}_0) \, . \, \underline{\phi} \, \mathrm{d}s$$

- In the oscillator equation :

$$(1 + \mathcal{M} m_a)\ddot{x} + x = 0$$

- The quantity $\mathcal{M} m_a$ is referred to as the **added mass** coefficient

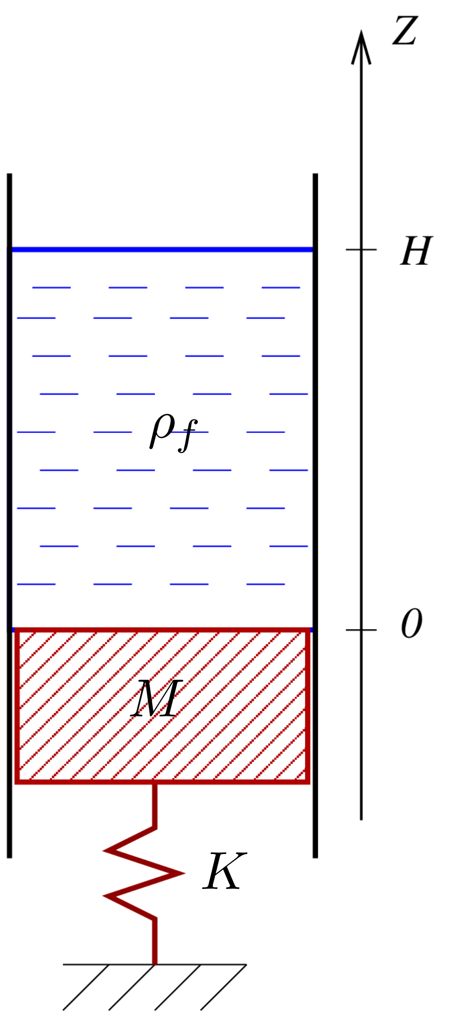
- This coefficient depends on :
  - The geometry $\Omega_0 \, , \; \partial\Omega_0$
  - The mode shape $\underline{\phi}$
  - The mass ratio $\mathcal{M} = \rho_f L^3 / M$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Examples of added mass calculations

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

The piston



Hypothesis :
The problem is independent of the *X* and *Y* coordinates.

- Characteristic length and time for the dimensionless eqs :

$$\tau = 1/\Omega_0 = \sqrt{M/K} \quad \eta = \sqrt{S}$$

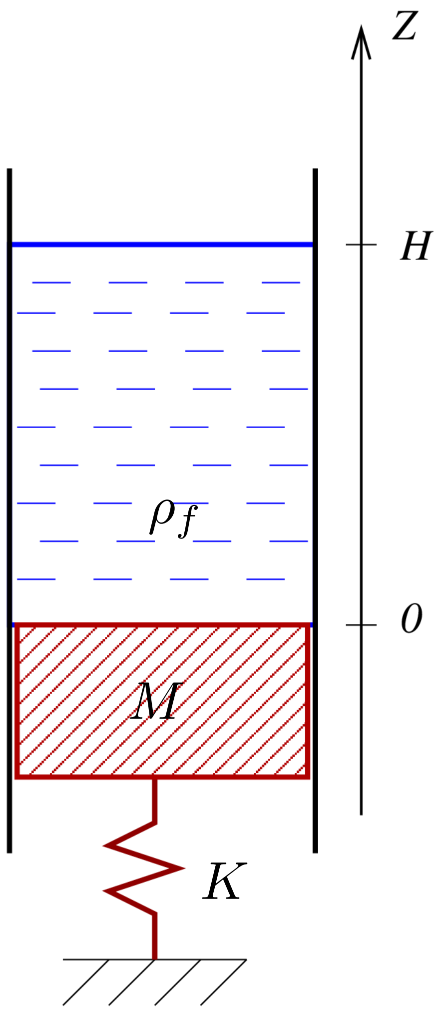
- Local equation : $\Delta\phi_p = 0 \quad \longrightarrow \quad \frac{\partial^2 \phi_p}{\partial z^2} = 0$
- Boundary conditions :
  - Atmospheric pressure : $p'(z = h, t) = 0 \rightarrow \phi_p(h) = 0$
  - Kinematic BC : $-\underline{\text{grad}}\phi_p \,.\, \underline{n}_0 = \underline{\phi} \,.\, \underline{n}_0 \quad \text{at} \quad z = 0$
- Solution : $\phi_p = z - h$

- Projection of the stress tensor :

$$f = -\ddot{x} \int_{\partial\Omega_0} (\phi_p \underline{e}_z).\underline{\phi} \text{d}s = -h\ddot{x}$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

The piston



- Dynamical equation :

$$(1+\mathcal{M}m_a)\ddot{x}+x=0$$

- Added mass :

$$\mathcal{M}m_a=\frac{\rho_f SH}{M}$$

- Dimensional dynamical equation :

$$M_a\ddot{X}+X=0$$

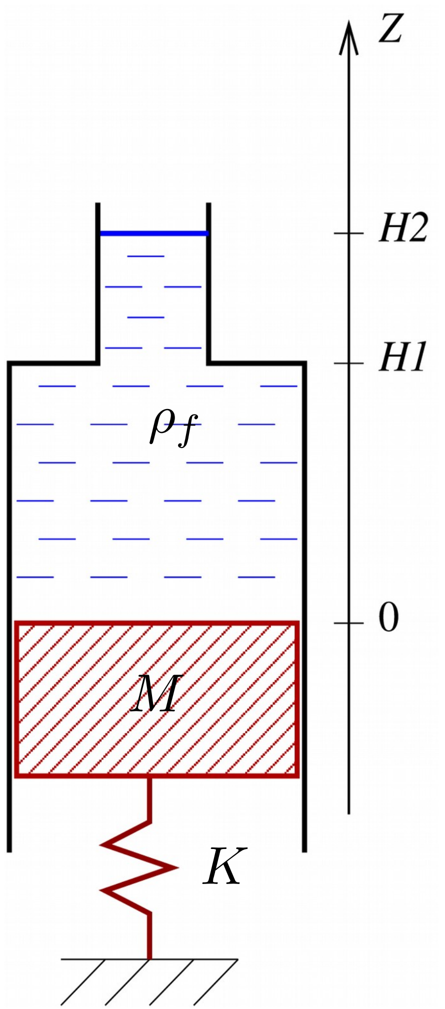
- Added mass :

$$M_a=\rho_f SH$$

The added mass is equal to the mass of the fluid !

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Piston with narrowing



Hypothesis :
The problem is independent of the *X* and *Y* coordinates.

- Characteristic length and time for the dimensionless eqs :

$$\tau = 1/\Omega_0 = \sqrt{M/K} \quad \eta = \sqrt{S}$$

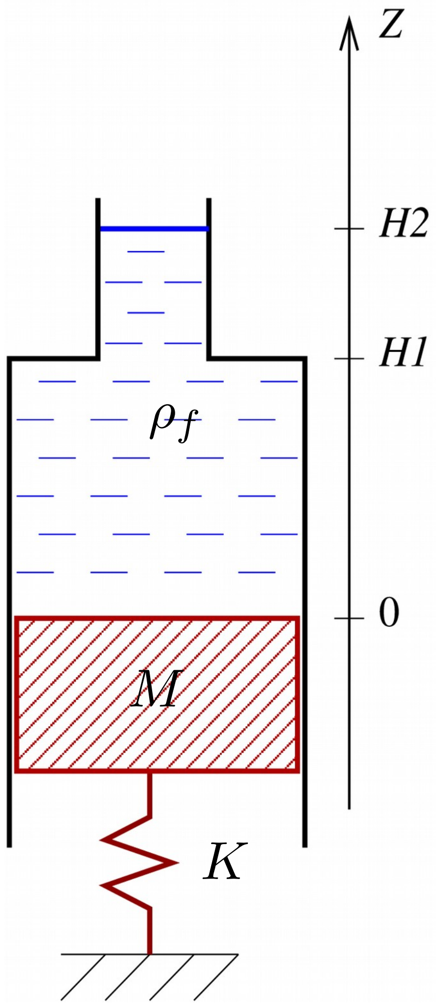
- Local equation : $\Delta\phi_p = 0 \quad \longrightarrow \quad \dfrac{\partial^2 \phi_p}{\partial z^2} = 0$
- Boundary conditions :
  - Atmospheric pressure : $p'(z = h_2, t) = 0 \rightarrow \phi_p(h_2) = 0$
  - Kinematic BC : $-\underline{\mathrm{grad}}\phi_p \,.\, \underline{n}_0 = \underline{\phi} \,.\, \underline{n}_0 \quad \text{at} \quad z = 0$
  - Continuity of pressure and flowrate at $z = h_1$
- Solution :

$$\phi_p(z \in [0, h_1]) = -z + h_1 + \frac{h_1 - h_2}{s_2}$$

$$\phi_p(z \in [h_1, h_2]) = \frac{1}{s_2}(z - h_2)$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Confinement effect



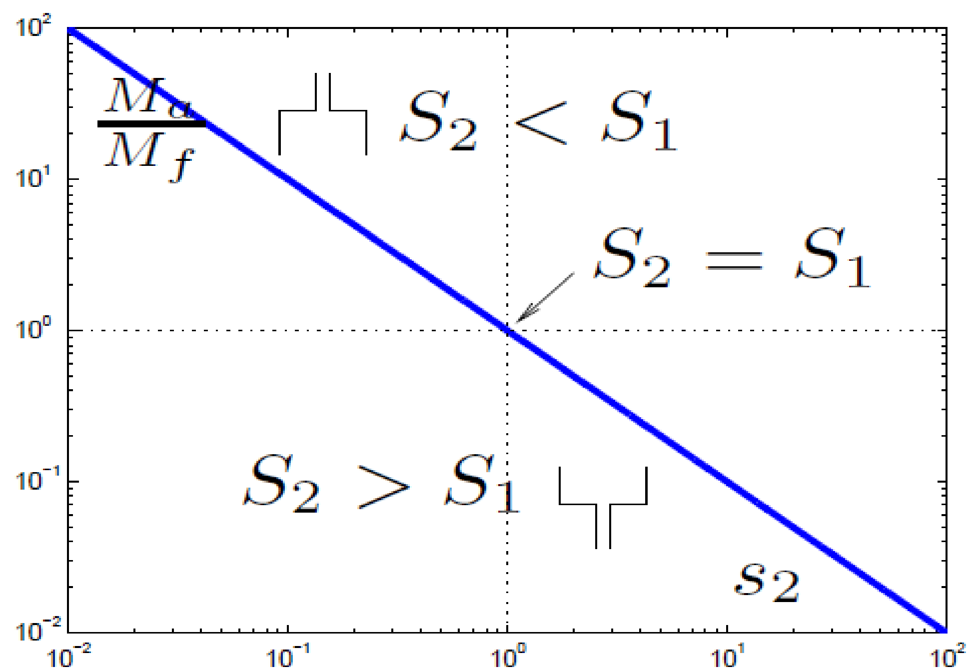
- Added mass :

$$M_a = \rho_f S_1 H_1 \left[1 + \frac{h_2 - h_1}{h_1 s_2}\right],$$

- Mass of the displaced fluid :

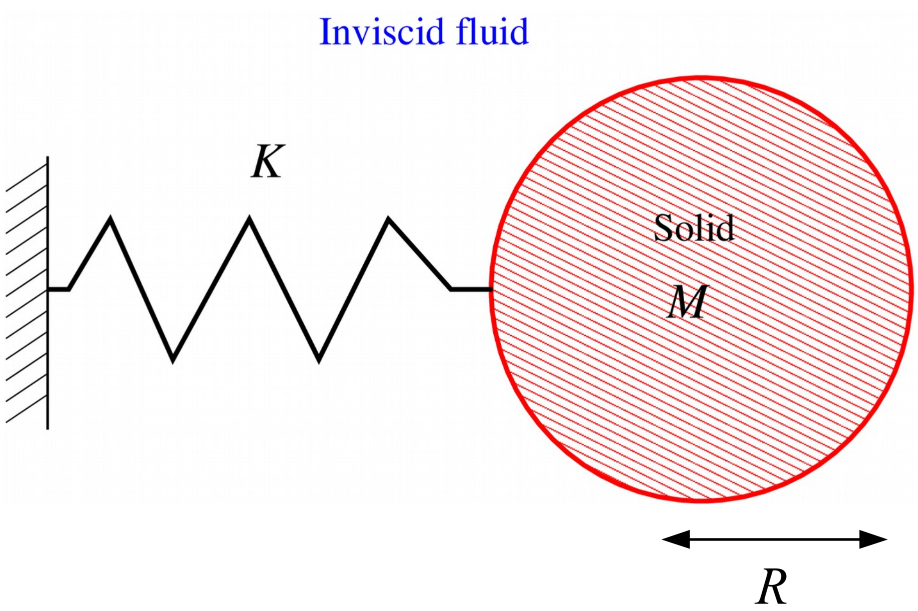
$$M_f = \rho_f S_1 H_1 \left[1 + s_2 \frac{h_2 - h_1}{h_1}\right].$$

$$M_a \neq M_f$$



Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Added mass of a cylinder



- Characteristic length and time :

$$\tau = \Omega_0^{-1} = \sqrt{M/K} \quad \eta = R$$

- Problem independent of axial coordinate : => M, K and the added mass are quantities per unit length

Problem to solve :

$$\Delta\phi_p(r,\theta) = 0 \quad \text{in } \Omega_f$$

$$\underline{\text{grad}}\phi_p \cdot \underline{n}_0 = -\phi \cdot \underline{n}_0 \quad \text{on } \partial\Omega_0$$

$$\text{with} \quad \underline{\phi} = \underline{e}_x = \cos\theta \underline{e}_r + \sin\theta \underline{e}_\theta$$

Solution :

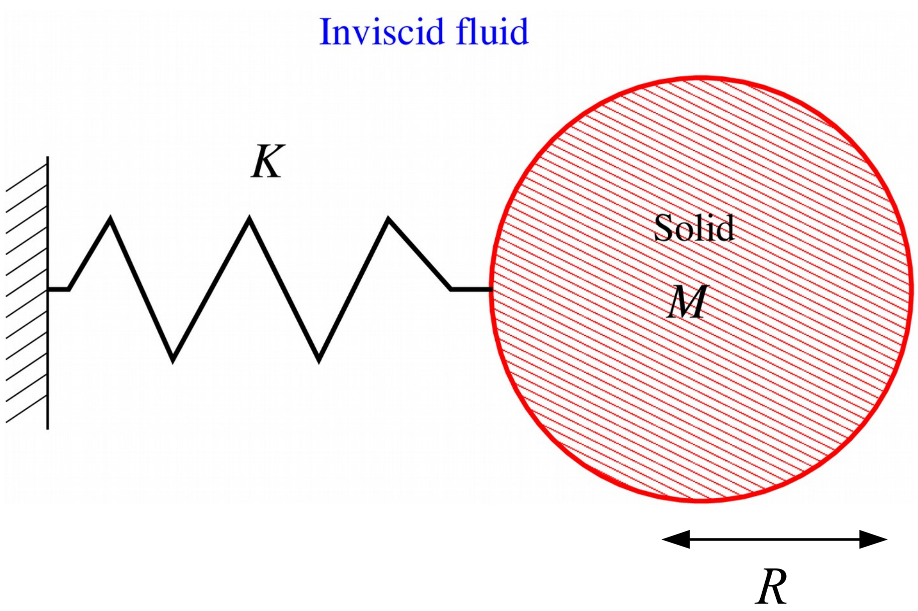
$$\phi_p = \frac{\cos\theta}{r}$$

Projection :

$$\begin{aligned} f' &= \int_0^{2\pi} -p'(r=1)\, \underline{n} \cdot \underline{\phi} d\theta \\ &= -\pi\ddot{x} \end{aligned}$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Added mass of a cylinder



Dimensional added mass (per unit length) :

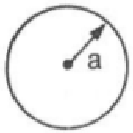
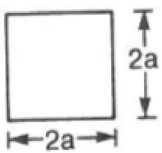
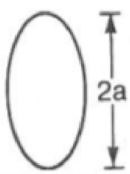
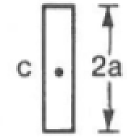
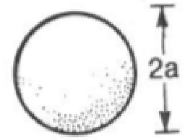
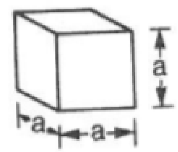
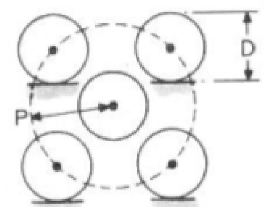
$$M_a = \rho_f \pi R^2$$

Equal to mass of fluid conained in the same volume !

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Other examples of added mass

Experimental or theoretical results

Geometry	Added mass	C_m
1. Circular cylinder of radius *a* 	$\rho\pi a^2 b$	1
2. Square section of side 2*a* 	$1.51\rho\pi a^2 b$	1.51
3. Elliptical section with major radius *a* 	$\rho\pi a^2 b$	0<Cm<∞
4. Flat plate of height 2*a* 	$\rho\pi a^2 b$ Added mass moment of inertia for rotation about centroid *c*, $\rho(\pi/8)a^4$.	0<Cm<∞
5. Sphere of radius *a* 	$\frac{2}{3}\rho\pi a^3$	0.5
6. Cube of side *a* 	$0.7\rho a^3$	0.7
7. Cylinder in array of fixed cylinders 	$\frac{\rho D^2 b}{4}\left[\frac{(D_e/D)^2+1}{(D_e/D)^2-1}\right]$ where $D_e/D=(1+\frac{1}{2}P/D)P/D$	

Definition of the added mass coefficient :

$$C_m = \frac{M_a}{\rho_f V}$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

General linear problem

- The linearized 2D problem of an oscillating cylinder in a viscous fluid can be solved analytically (Chen...)

- The approximate solutions for the added mass and added damping are :

$$M_a = \rho_f R^2 \left(\pi + 4\sqrt{\frac{\pi}{S_T}} + \mathcal{O}\left(\frac{1}{S_T}\right)\right)$$

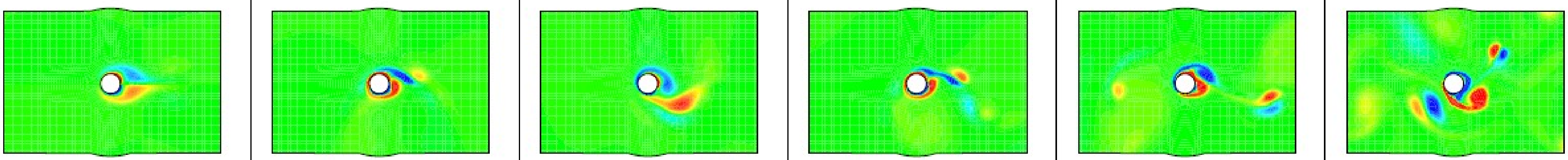
$$C_a = \rho_f \nu \left(2\pi^{3/2}\sqrt{S_T} + 2\pi + \mathcal{O}\left(\frac{1}{\sqrt{S_T}}\right)\right)$$

- For large values of the Stokes number, the inviscid added mass coefficient is recovered.

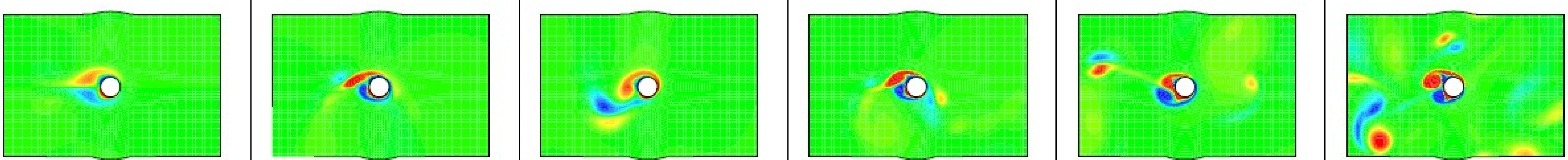
Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Non linear problem

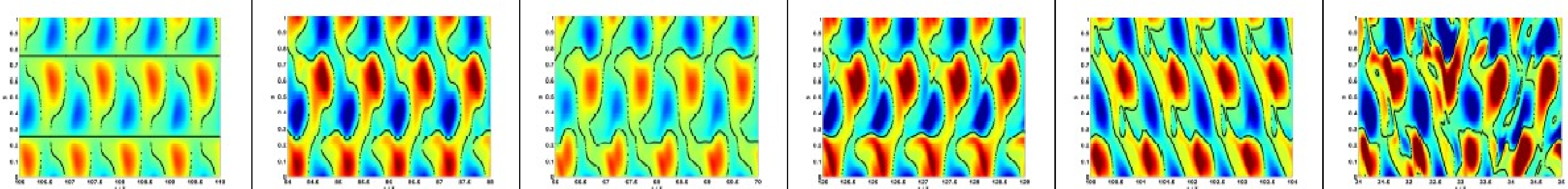
mode symétrique	mode en V	mode transverse	mode oblique	mode diagonal	mode chaotique
KC = 10,	KC = 6, 25,	KC = 10,	KC = 6, 25,	KC = 10,	KC = 10,
Re = 60	Re = 150	Re = 80	Re = 180	Re = 200	Re = 360



(a) $t = 109,45T$ (b) $t = 84,48T$ (c) $t = 68,45T$ (d) $t = 126,48T$ (e) $t = 102,45T$ (f) $t = 34,45T$



(g) $t = 109,95T$ (h) $t = 84,96T$ (i) $t = 68,95T$ (j) $t = 126,96T$ (k) $t = 102,95T$ (l) $t = 34,95T$



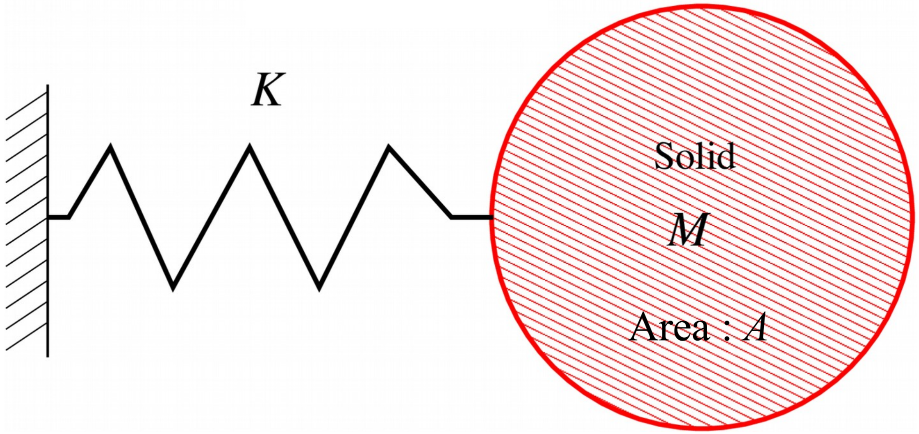
(m) (n) (o) (p) (q) (r)

$$R_e = \frac{U_0 L}{\nu} \quad KC = 2\pi \frac{\Xi_0}{L}$$

$$R_e = \mathcal{D} S_t \quad KC = 2\pi \mathcal{D}$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Vibrations induced by oscillating flow



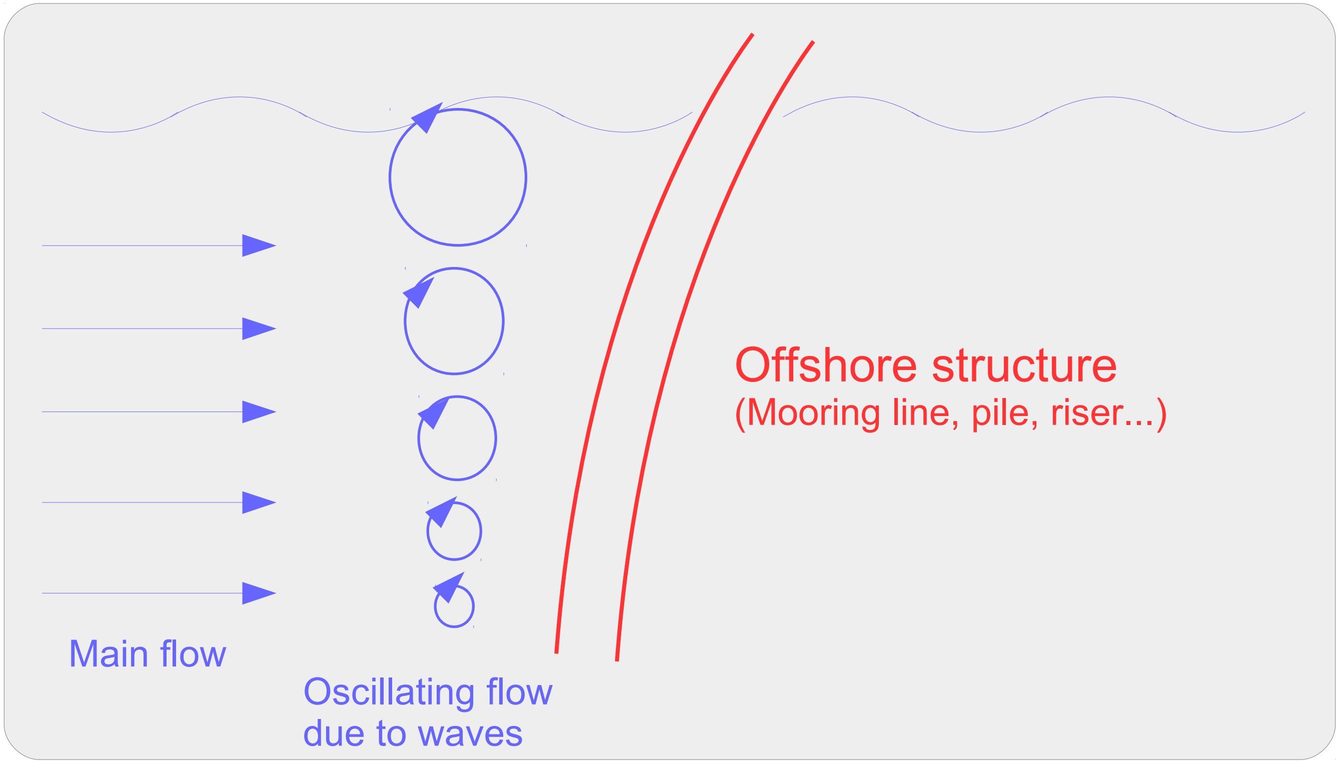
Flow

$$U = U_0 + U_m \cos(\Omega_m)t$$

Oscillations and flow are considered aligned

This case has many applications

Here, it is treated by analysing experimental results and propose empirical models



Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

What is the difference between a steady cylinder in an oscillating flow and an ocsillating cylinder in a fluid a rest ?

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

The Morison equation

- In the 1950's Morison proposed a general formulation for the efforts exerted on a vibrating body in an oscillating flow :

$$F = \rho_f A\dot{U} + \rho_f A C_A(\dot{U} - \ddot{X}) + \frac{1}{2}\rho_f |U - \dot{X}|(U - \dot{X}) D C_D$$

Accelerating flow
→ pressure gradient
→ Archimede's force

Added mass force,
Function of the relative acceleration

Drag force,
Function of the relative velocity

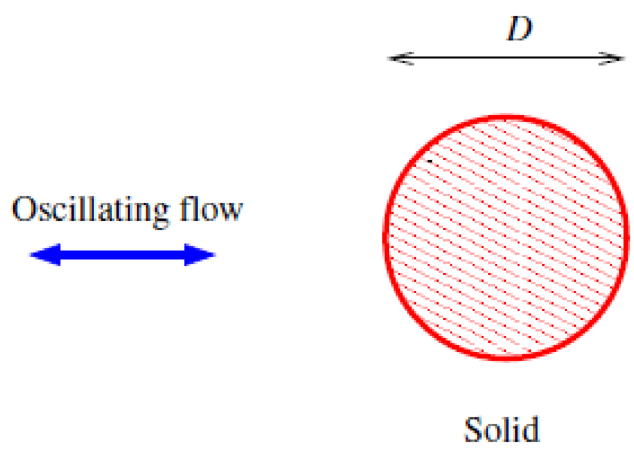
Added mass coeff.

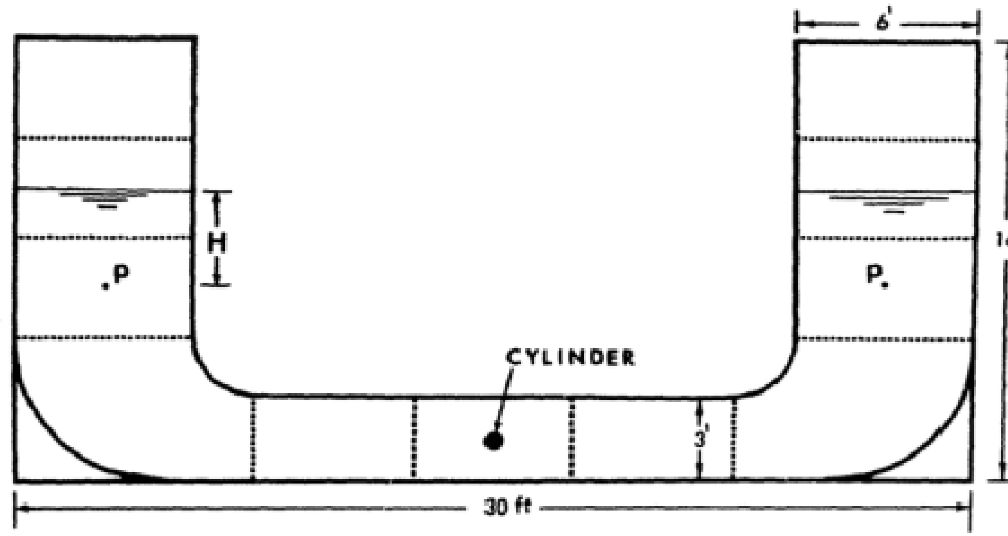
Drag coeff.

- This equation is exact for :
  - An inviscid fluid
  - Steady velocity at high Reynolds numbers (typically > 1000)
- It is an approximation for all other cases
- The work consists in evaluating the two coefficients as function of the parameters of the problem

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

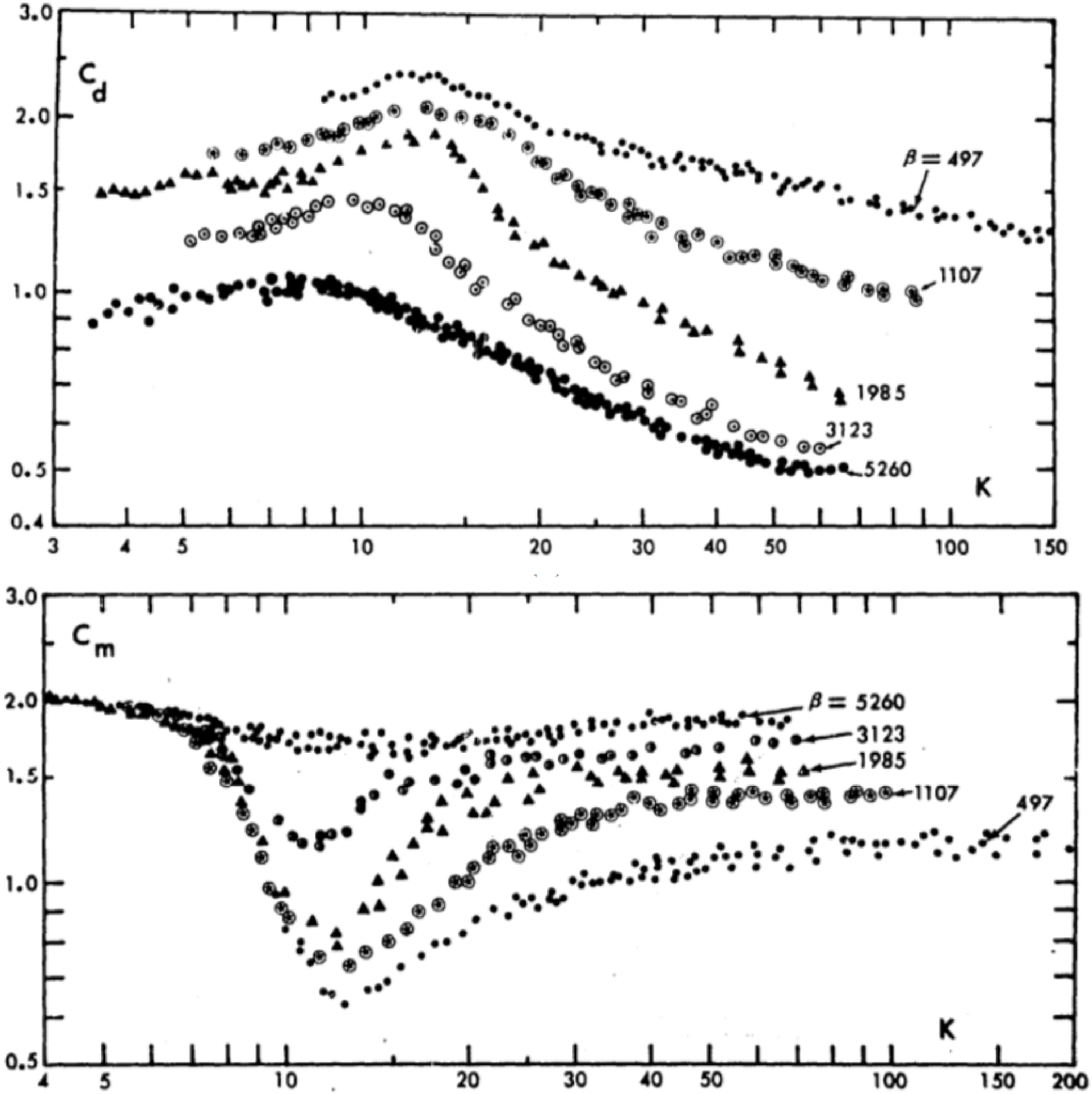
Sarpkaya experiment 1976





$$F = \rho_f A\dot{U} + \rho_f A C_A \dot{U} + \frac{1}{2}\rho_f U|U| D C_D$$
$$= C_m \rho_f A\dot{U} + \frac{1}{2}\rho_f U|U| D C_D$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr



C_d
β = 497
1107
1985
3123
5260
K
C_m
β = 5260
3123
1985
1107
497
K

Conclusion on oscillations in still fluid

Added mass :

Intertia effect that can been evidenced with a potential flow approximation

Added damping :

Viscous effects

Added stiffness :

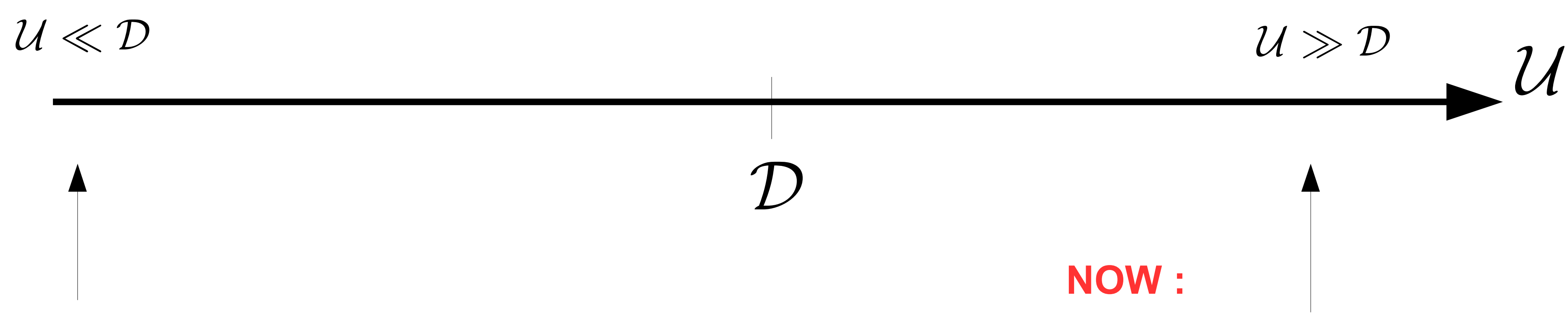
Pressure or pressure gradient effects (not addressed in details in the present course)

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

II - Oscillations in a flow

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

The reduced velocity



Still fluid
(added mass)
(added damping due to viscosity)

Quasi-static approach
The solid is seen at rest from the point of view of the solid

Non-dimensional form of the kinematic boundary condition :

$$\underline{U} = \frac{\partial \underline{\Xi}}{\partial T} \quad \longrightarrow \quad \underline{u} = \frac{1}{\mathcal{U}} \frac{\partial \underline{\xi}}{\partial t} = \mathcal{O}\left(\frac{\mathcal{D}}{\mathcal{U}}\right)$$

$$\mathcal{U} \gg \mathcal{D} \implies \underline{u} \sim 0$$

At each time, the fluid dynamics problem is a classical fluid dynamics problem (zero velocity at the boundaries)

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Quasi static approach

$$\mathcal{U} \gg \mathcal{D} \implies \underline{u} \sim 0$$

At each time, the fluid dynamics problem is a classical fluid dynamics problem (zero velocity at the boundaries)

- **Consequence :** The velocity, the pressure, the stress tensor in the fluid depends only on the position of the solid.
- **Case of a structure :** The forces exerted by the fluid on the structure depend only on the modal displacements.
- Hence, for each static configuration of the structure a fluid mechanics problems has to be solved.
- In many cases, **basic results of aerodynamics can be re-used.**

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Pseudo static approach

- If the solid's velocity is not negligible with respect to the fluid's velocity, $(\mathcal{U} \text{ not } \gg \mathcal{D})$ another approximation can be done.

- Time derivative of the kinematic boundary condition :

$$\frac{\partial \underline{U}}{\partial T} = \frac{\partial^2 \underline{\Xi}}{\partial T^2}$$

- Non dimensional version :

$$\frac{\partial \underline{u}}{\partial t} = \frac{1}{U_R^2} \frac{\partial^2 \underline{\xi}}{\partial t^2} \simeq O\left(\frac{\mathcal{D}}{U_R^2}\right)$$

$$\mathcal{U}^2 \gg \mathcal{D} \implies \frac{\partial \underline{u}}{\partial t} \sim 0$$

At each time, the fluid dynamics problem is a classical fluid dynamics problem (zero velocity at the boundaries) **in the referential of the moving solid.**

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

$$\mathcal{U} \gg \mathcal{D} \implies \underline{u} \sim 0$$

At each time, the fluid dynamics problem is a classical fluid dynamics problem (zero velocity at the boundaries)

Forces exerted by the flow on the structure depend only on the modal displacements

QUASI-STATIC

$$\mathcal{U}^2 \gg \mathcal{D} \implies \frac{\partial \underline{u}}{\partial t} \sim 0$$

At each time, the fluid dynamics problem is a classical fluid dynamics problem (zero velocity at the boundaries) **in the referential of the moving solid.**

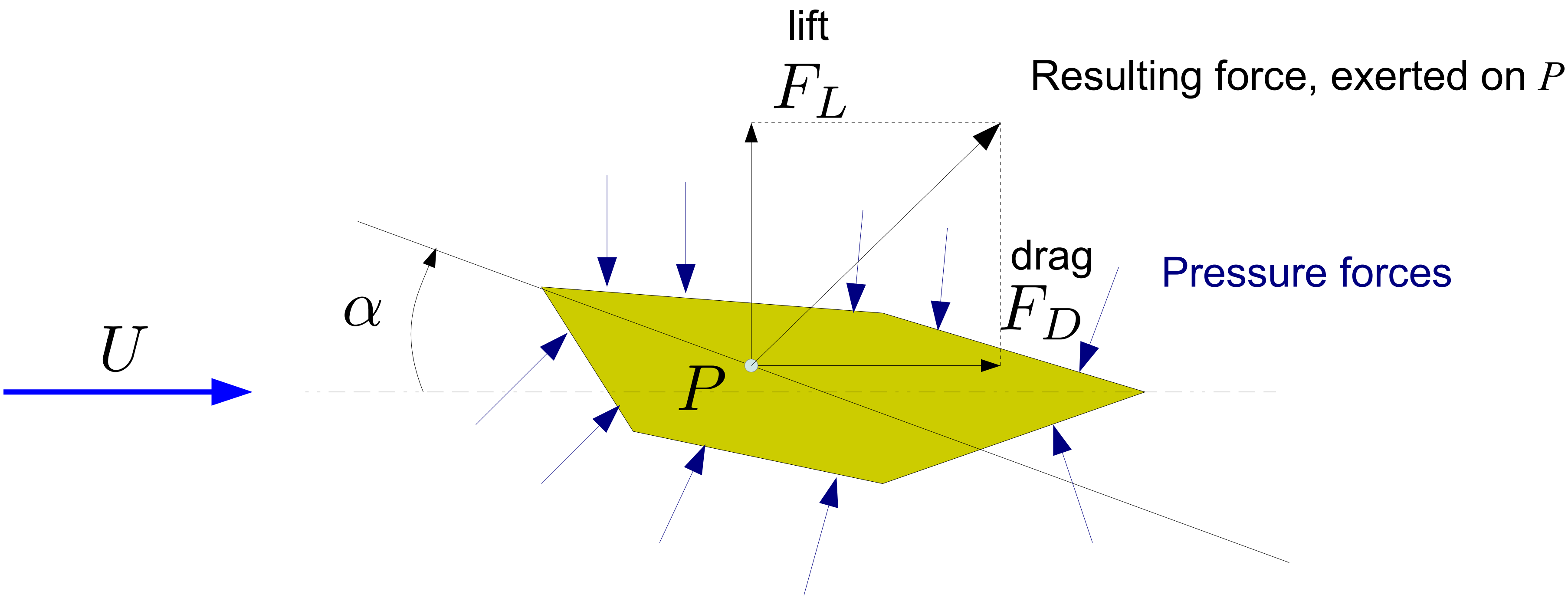
Forces exerted by the flow on the structure depend only on the modal displacements and velocities

PSEUDO-STATIC

Basics of aerodynamics

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Aerodynamic efforts acting on a solid (2D)



- Forces per unit length, exerted on the center of forces P
- Definition of non-dimensional coefficients :

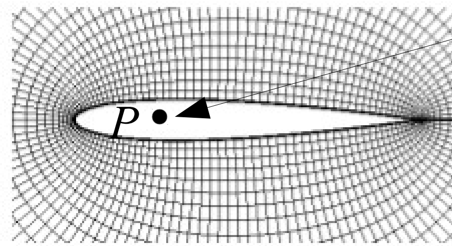
$$C_L = \frac{F_L}{\frac{1}{2}\rho U^2 L} \qquad C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L}$$

Remember ! $\mathcal{U} \gg \mathcal{D}$

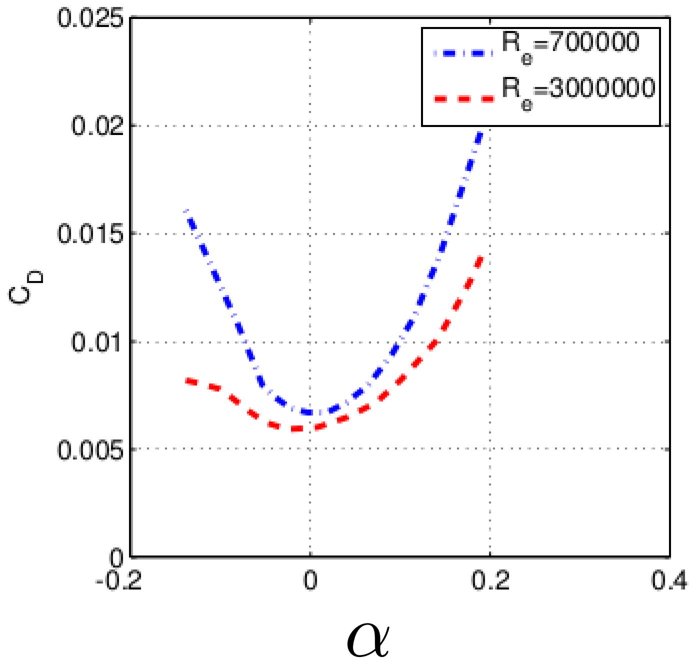
Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

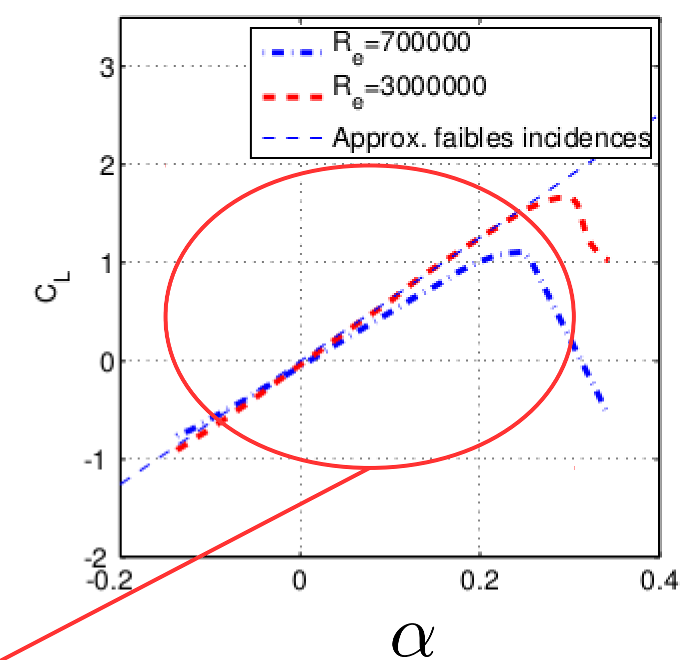
The case of typical thin profiles

NACA0012



For thin profiles, the center of forces is located around 1/4th of the length

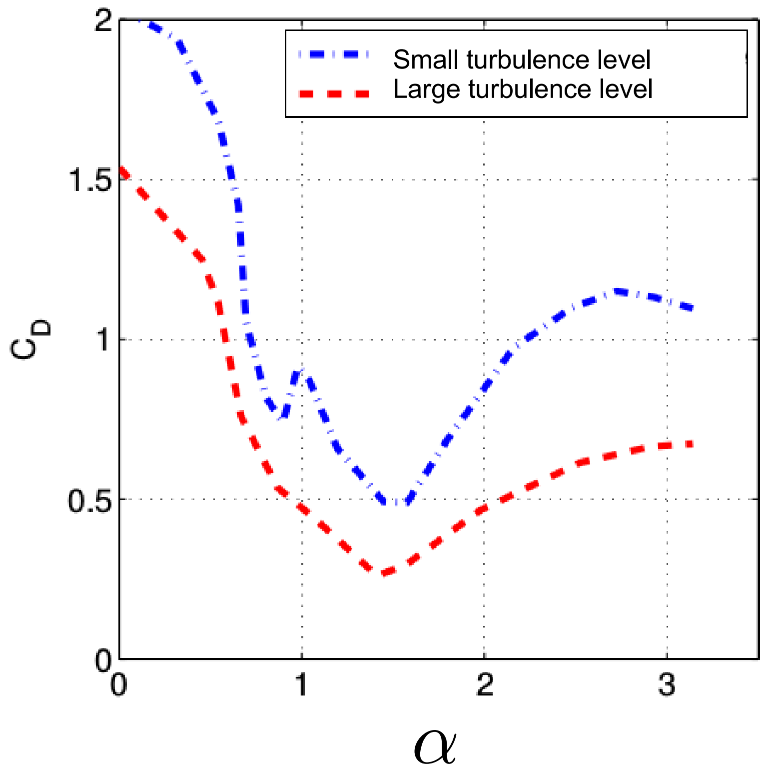


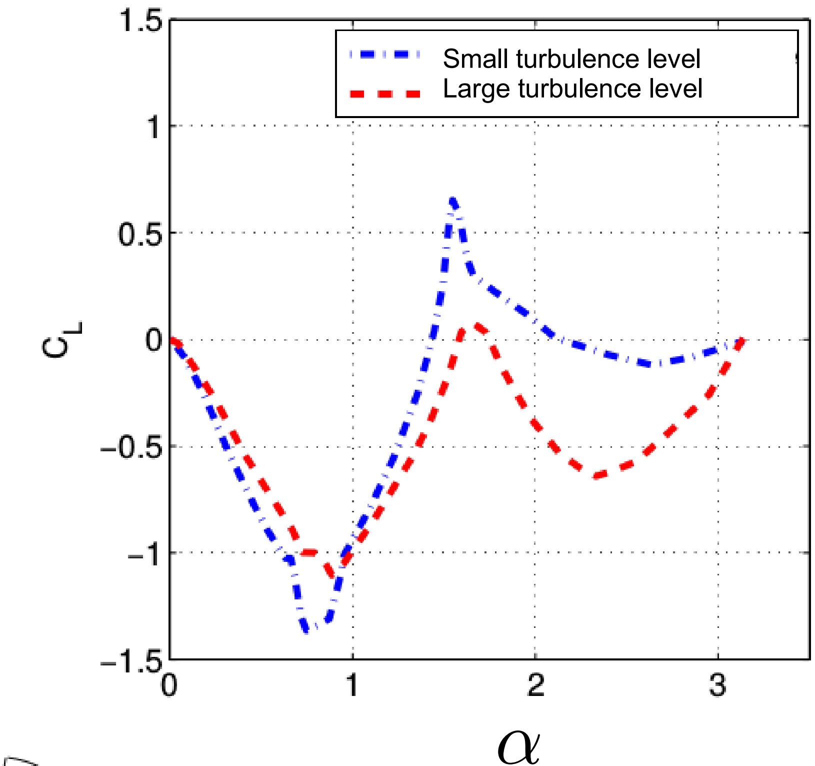


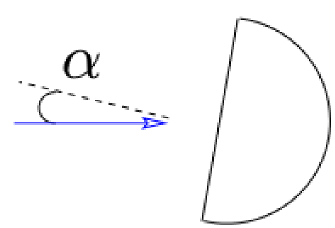
At low angles and high Reynolds numbers, $C_L \sim 2\pi\alpha$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

D profile

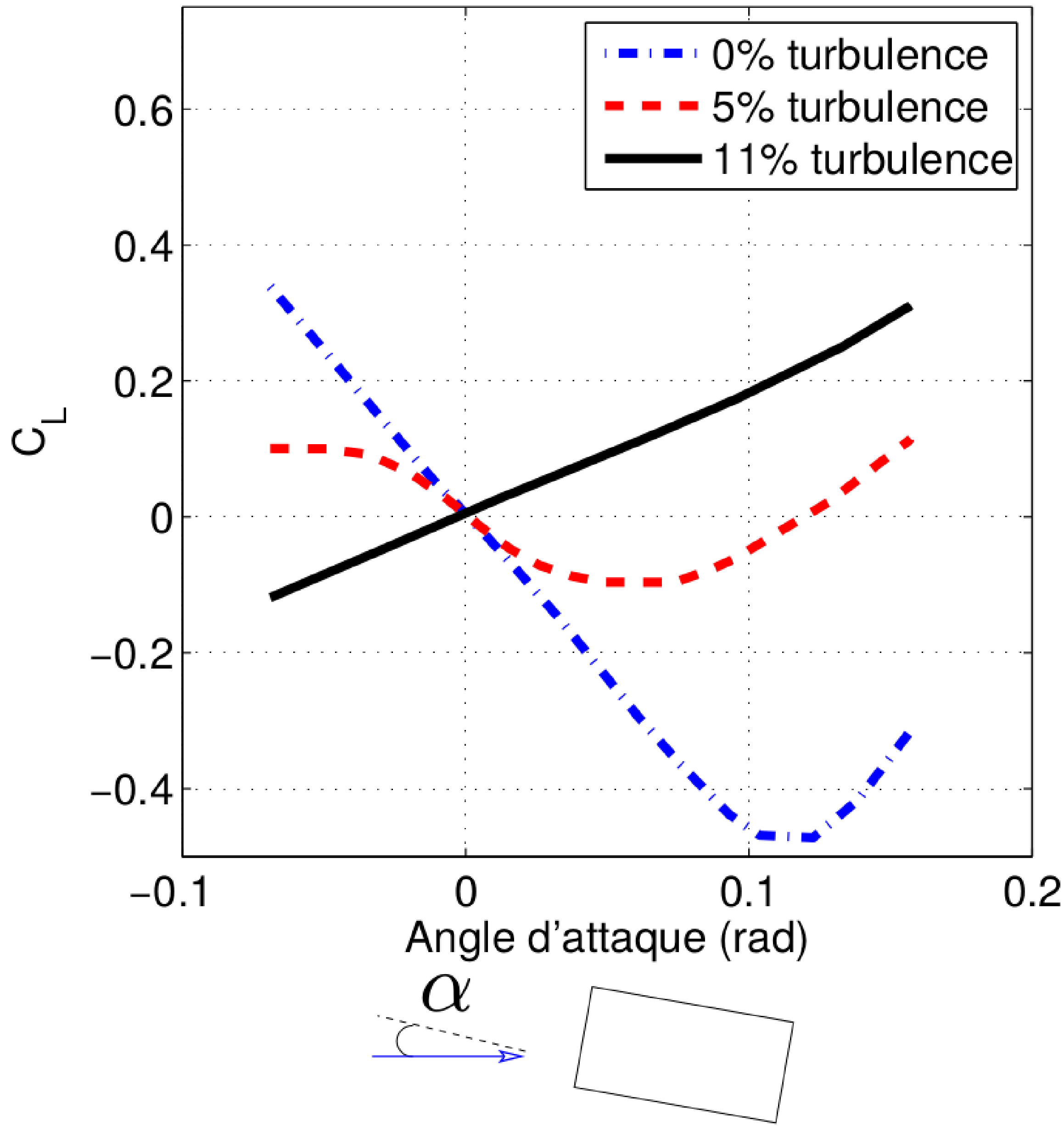




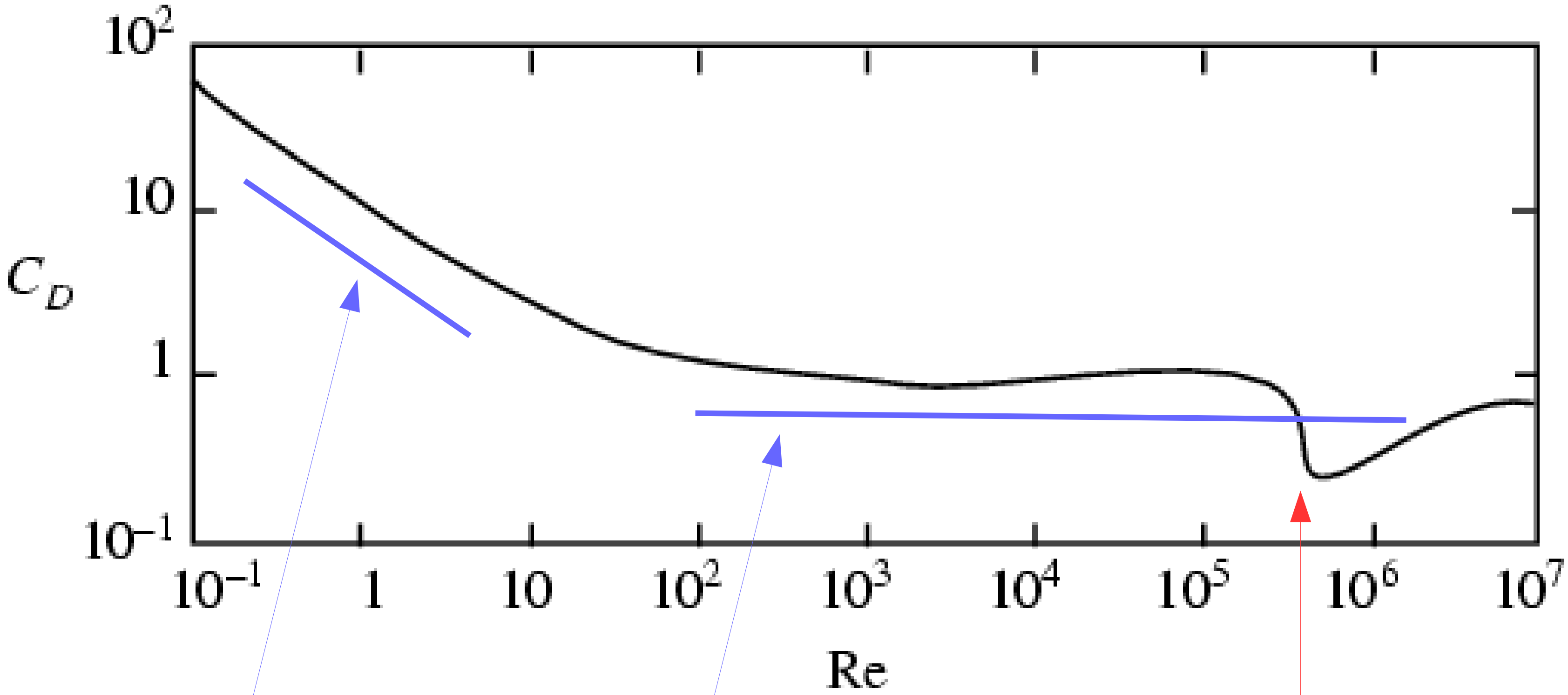


Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Rectangle 2:1



Mean drag of a cylinder



Dominated by friction
(~ 1/U)

Dominated by pressure
(Cst)

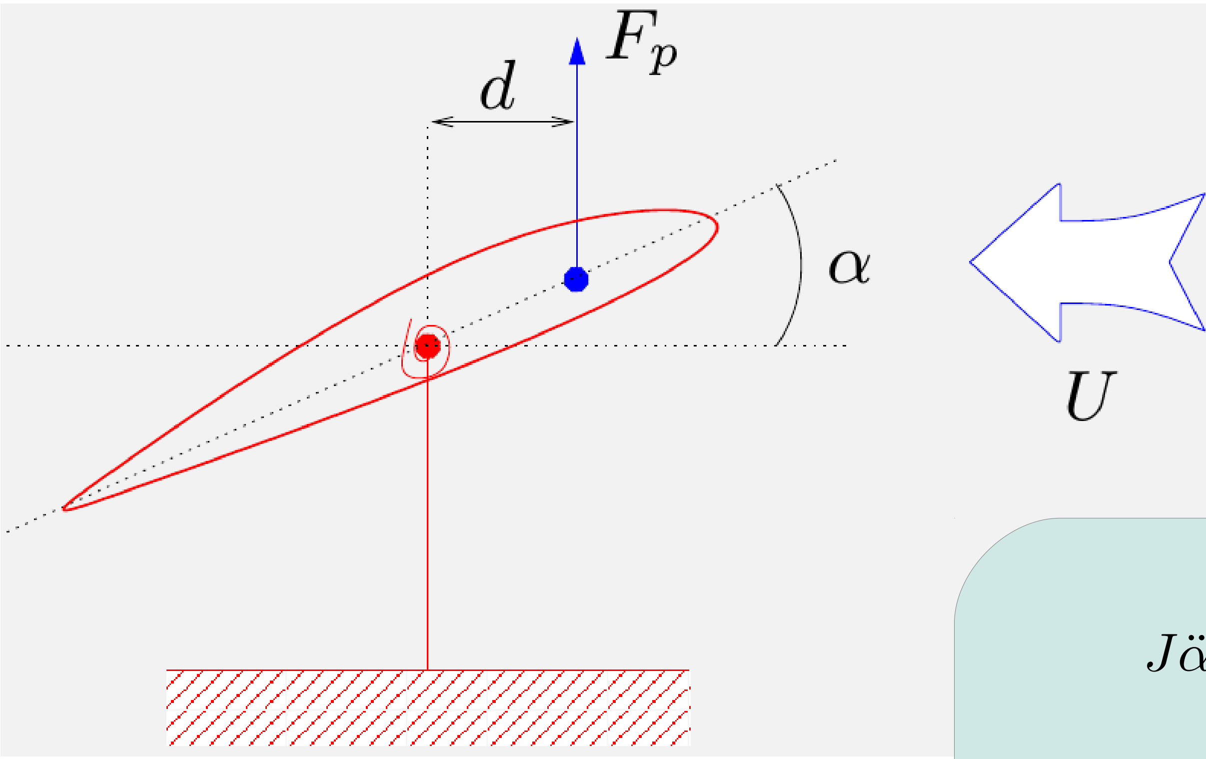
Drag crisis

$$\underline{\underline{\Sigma}}_f = -P\,\underline{\underline{1}} + 2\mu\underline{\underline{D}} \quad \longrightarrow \quad \underline{\underline{\sigma}}_f = -p\,\underline{\underline{1}} + \frac{1}{R_e}\underline{\underline{d}}$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Static instability of a rotating body

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr



Weathercock

$$J\ddot{\alpha} + C\alpha = \frac{1}{2}\rho_f U_0^2 L C_L(\alpha) d$$

$$C_L(\alpha) \sim 2\pi\alpha$$

$$J\ddot{\alpha} + (C - \rho_f U_0^2 L \pi d)\alpha = 0$$

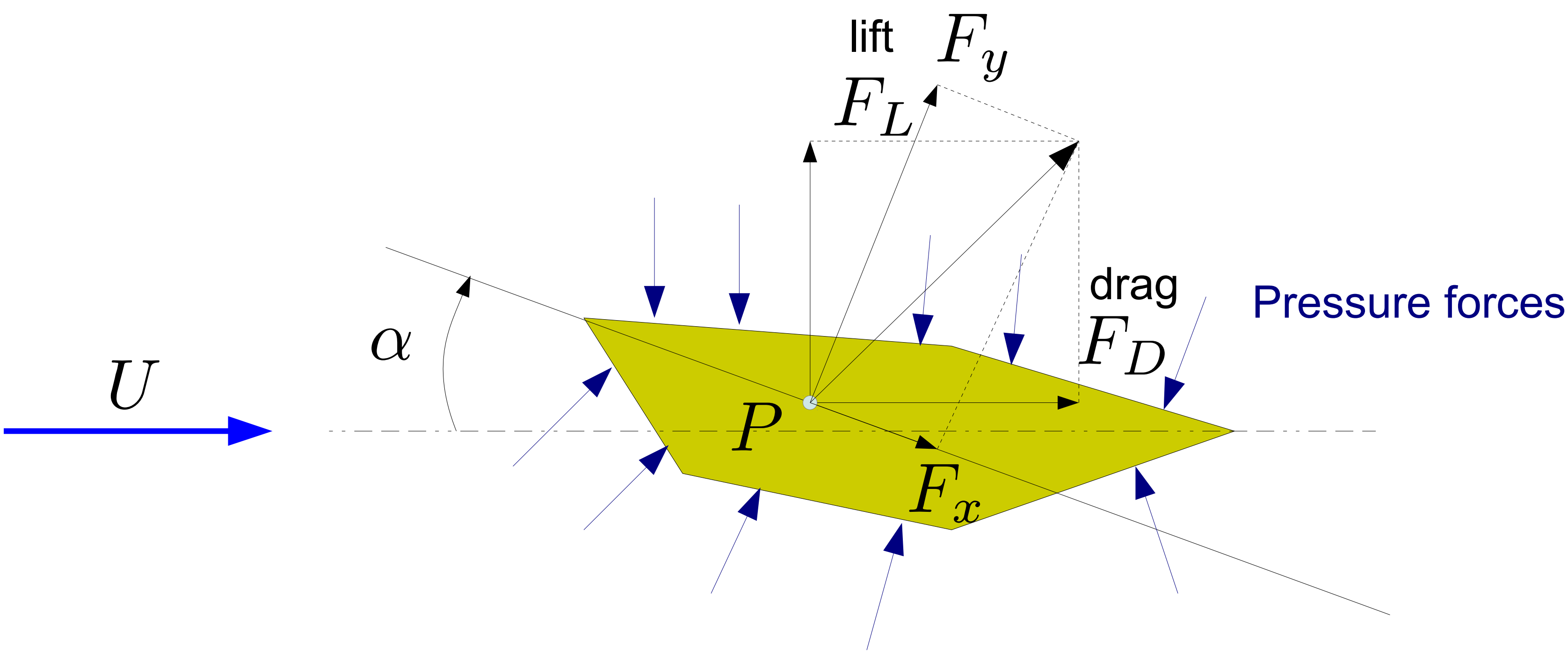
Possible negative stiffness !

- Rotational moment of inertia : J
- Rotational stiffness : C
- Small incidence angle : $\alpha \ll 1$
- Distance between elastic center and center of pressure : d
- Moment exerted on the profile : $m = \frac{1}{2}\rho_f U_0^2 L C_L(\alpha) d$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Basics of aerodynamics
(for translating bodies)

Aerodynamic efforts acting on a solid (2D)



- Forces per unit length
- Definition of non-dimensional coefficients :

$$C_L = \frac{F_L}{\frac{1}{2}\rho U^2 L} \qquad C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L} \qquad C_x = \frac{F_x}{\frac{1}{2}\rho U^2 L} \qquad C_y = \frac{F_y}{\frac{1}{2}\rho U^2 L}$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

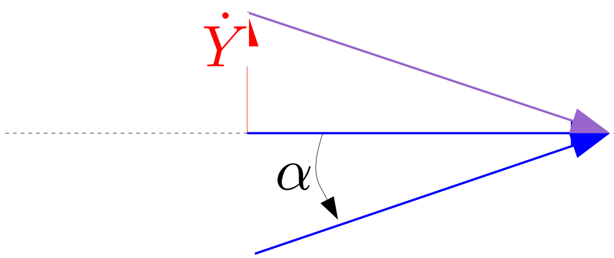
Forces acting on a translating profile

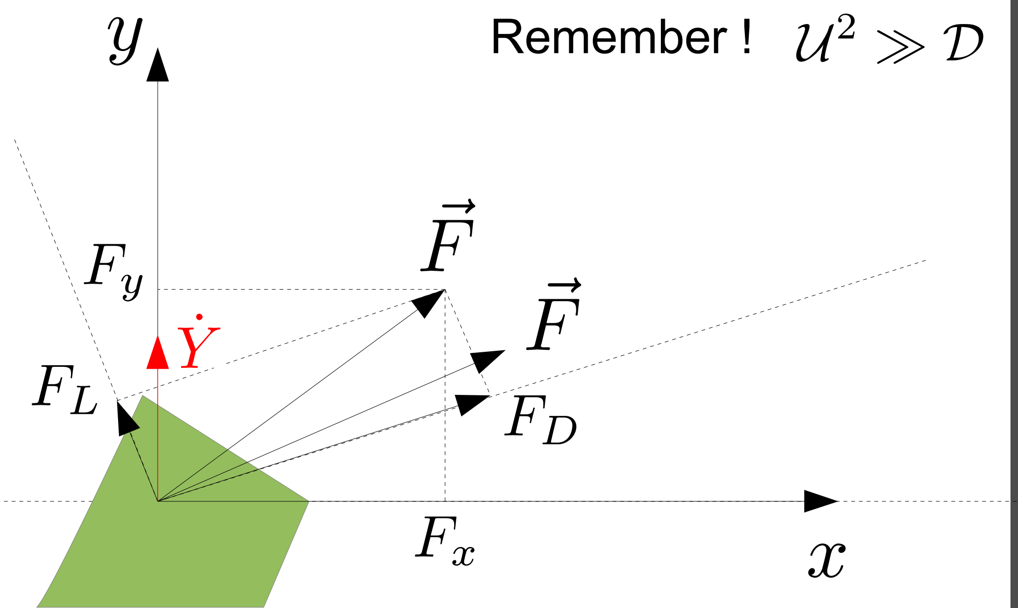
F_x and F_y forces in the solid's axes

Remember ! $\mathcal{U}^2 \gg \mathcal{D}$

F_D and F_L forces in the fluid's axes (Drag and Lift)

$$F_y(\alpha) = F_L(\alpha)\cos\alpha + F_D(\alpha)\sin\alpha$$





- Vertically translating solid in an horizontal flow
- Equivalent problem : Still solid in a flow with an angle of attack

$$\alpha = -\tan^{-1}\left(\frac{\dot{Y}}{U}\right) \sim -\frac{\dot{Y}}{U}$$

- Taylor expansion of the vertical force, considered to be known function of the angle of attack :

$$F_y(\alpha) = F_0 + \alpha\frac{\partial F_y}{\partial \alpha} + \mathcal{O}(\alpha^2) \quad \Rightarrow \quad F_y(\alpha) \sim -\frac{\dot{Y}}{U}\frac{\partial F_y}{\partial \alpha} = -\frac{\dot{Y}}{U}\left(\frac{\partial C_L}{\partial \alpha} + C_D\right)$$

- Introduction of the aerodynamic coefficient :

$$F_y = \frac{1}{2}\rho U^2 C_y(\alpha) \quad \Rightarrow \quad F_y(\alpha) \sim -\frac{\rho U}{2}\dot{Y}\left(\frac{\partial C_L}{\partial \alpha} + C_D\right)$$

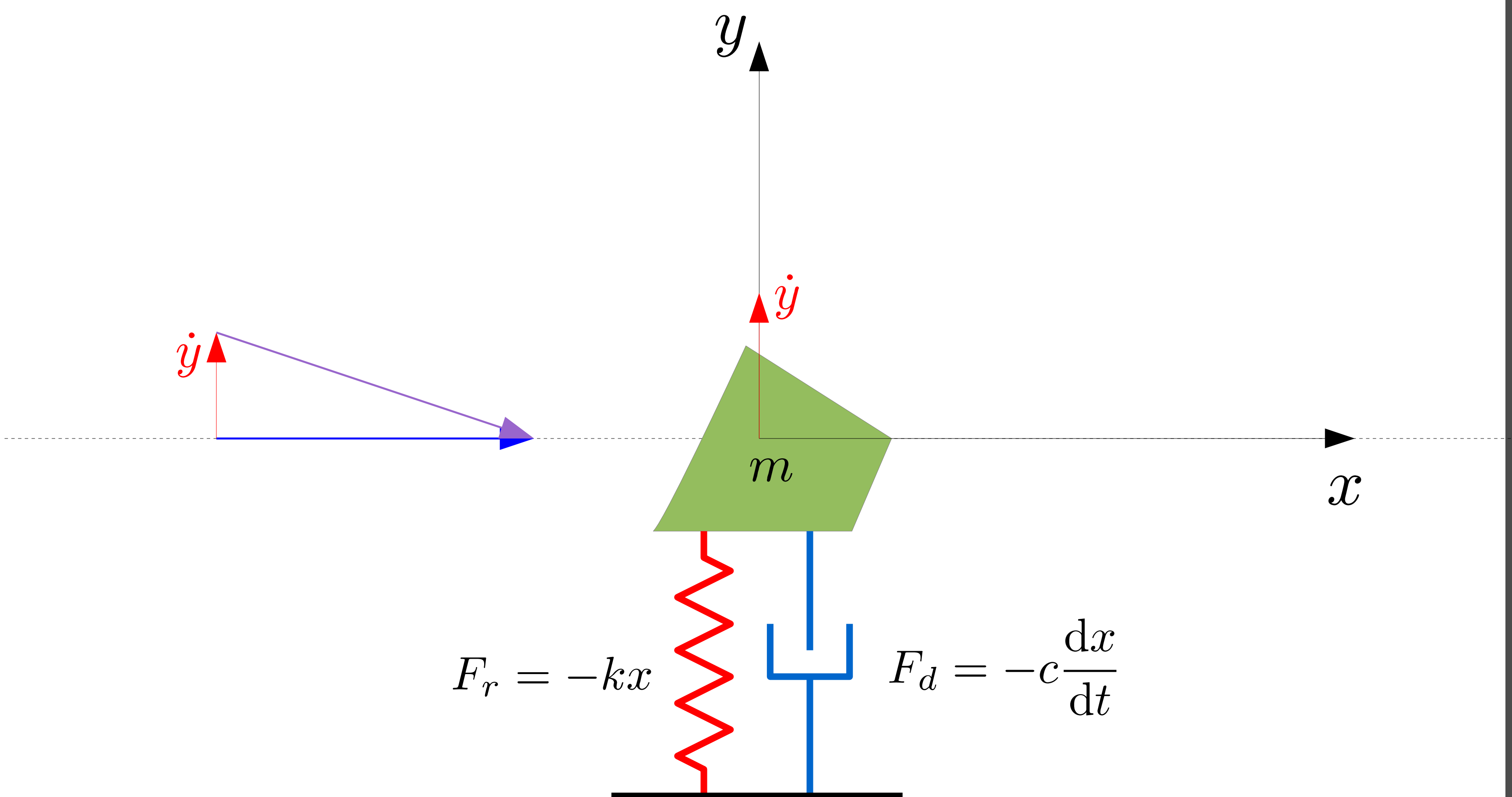
Possibility of negative damping ! Scales as U

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Instability by negative damping

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Galloping = negative damping



Oscillator equation : $$m\ddot{x} + \left(c + \frac{\rho U}{2}\frac{\partial C_y}{\partial \alpha}\right)\dot{y} + ky = 0$$

Criterion for instability : $$\frac{\partial C_y}{\partial \alpha} < 0 \quad \& \quad U > -\frac{2c}{\rho \frac{\partial C_y}{\partial \alpha}}$$ *(den Artog, 1932)*

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

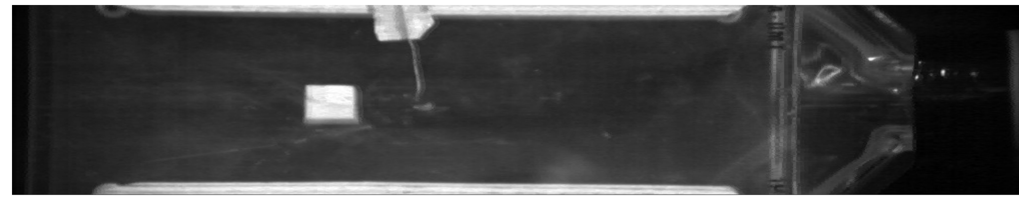
Stability criterion

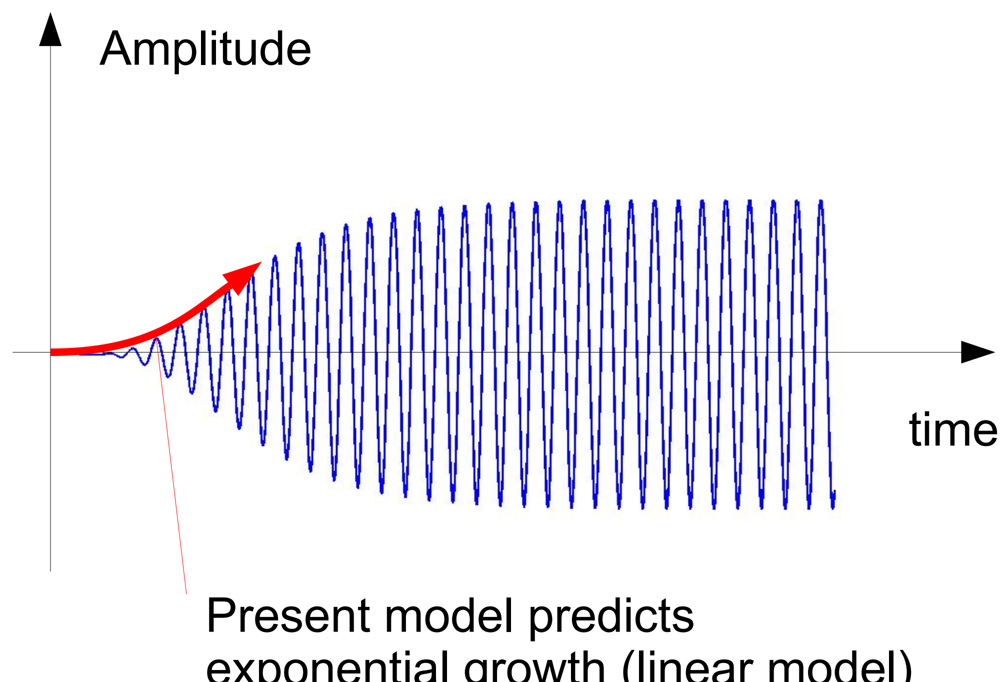
$$-\frac{\partial C_L}{\partial \alpha}$$

Section	Smooth flow	Turbulent flow[b]	Reynolds number
D, D	3.0	3.5	10^5
D, $\frac{2}{3}$D	0.	−0.7	10^5
D, D/2	−0.5	0.2	10^5
D, D/4	−0.15	0.	10^5
$\frac{2}{3}$D, D	1.3	1.2	66 000
D/2, D	2.8	−2.0	33 000
D/4, D	−10.	—	2 000–20 000
D	−6.3	−6.3	$>10^3$
D	−6.3	−6.3	$>10^3$
D	−0.1	0.	66 000
	−0.5	2.9	51 000
D	0.66	—	75 000

(Blevins, 1990)

Square section

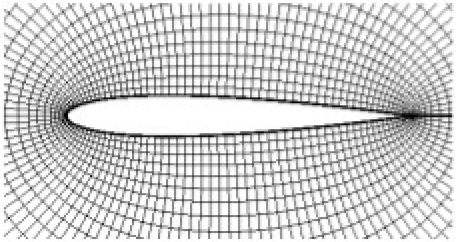


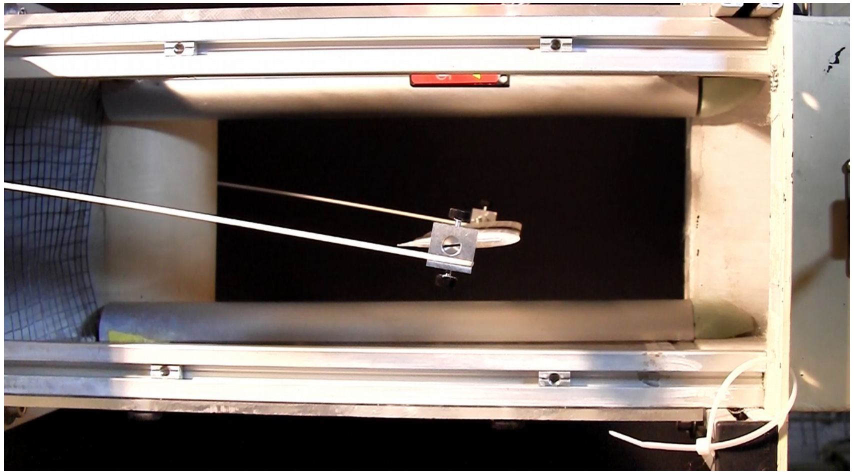


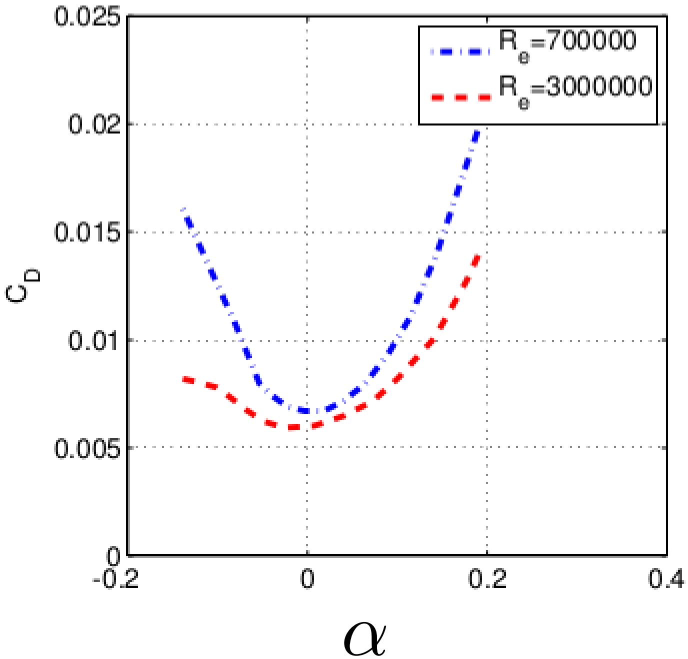
Observation : there is a saturation (nonlinearities)

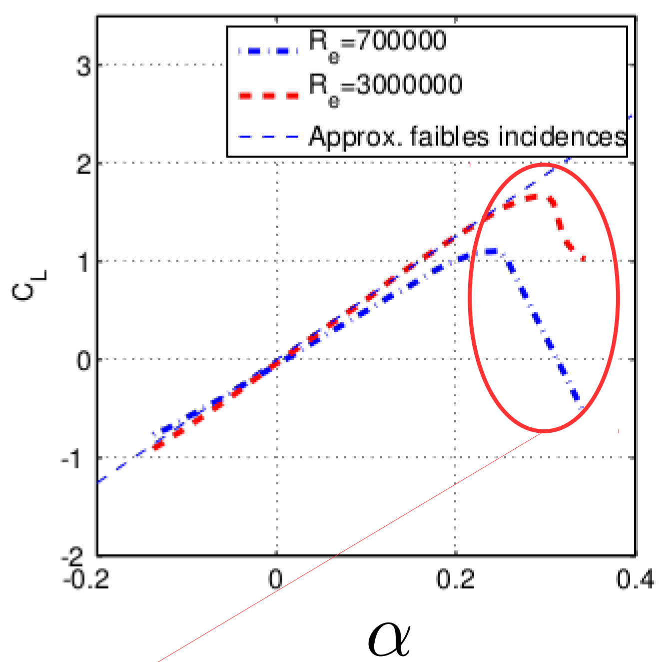
Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

NACA 0012









At large incidence angles, lift of wing profiles have a negative derivative, hence can induce negative damping flutter.

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

III - Modes coupling

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Fluid-structure problem : ONE MODE VERSION

Kinematic condition (equality of displacements)

$$\underline{u} = \frac{\partial \underline{\xi}}{\partial t} = \underline{\phi}(\underline{x})\dot{x}(t) \quad \text{on } \partial\Omega_0$$

$$\ddot{x} + x = \mathcal{M} f$$

$$\text{div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) . \underline{u} = -\frac{1}{F_D^2}\underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{S_t}\Delta \underline{u}$$

Dynamic condition (equality of efforts)

$$f = \int_{\partial\Omega_0} (\underline{\underline{\sigma}}_f \cdot \underline{n}) \cdot \underline{\phi}\ ds$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Fluid-structure problem : N modes version

Kinematic condition (equality of displacements)

$$\underline{u} = \frac{\partial \underline{\xi}}{\partial t} = \sum_n \underline{\phi}_n(\underline{x})\dot{x}_n(t) \quad \text{on } \partial\Omega$$

$$\ddot{x}_n + \omega_n^2 x_n = \mathcal{M} f_n$$

$$n \in \mathbb{N}$$

$$\text{div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \left(\underline{\underline{\text{grad}}}\ \underline{u}\right) . \underline{u} = -\frac{1}{F_D^2}\underline{e}_z - \underline{\text{grad}}\ p + \frac{1}{S_t}\Delta\underline{u}$$

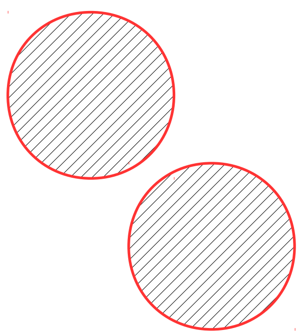
Dynamic condition (equality of efforts)

$$f_n = \int_{\partial\Omega} (\underline{\underline{\sigma}}_f . \underline{n}) . \underline{\phi}_n \, ds$$

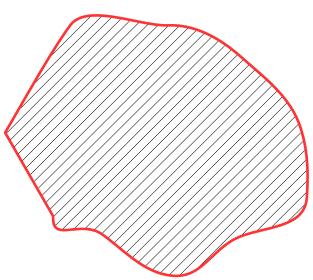
Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Inertial coupling :
Oscillations in a still fluid

$$S_T \gg 1 \quad \mathcal{U} \ll \mathcal{D} \ll 1$$



Proximity effect
The oscillations of one structure induce a pressure field on the other

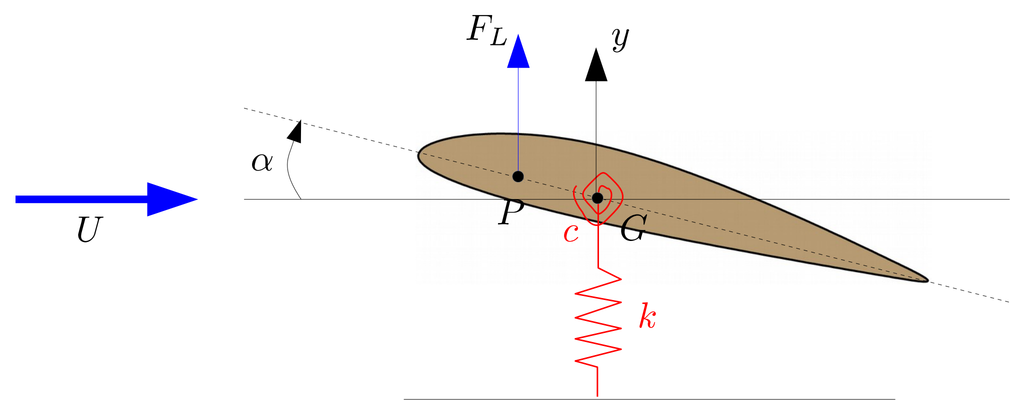


Shape effect
Asymetries of the solid may induce coupling between its eigenmodes

Consequence :
The eigenfrequencies **AND** eigenmodes are modified by the presence of the fluid.

Aerodynamic coupling :
Oscillations in high velocity flows

$$R_e \gg 1 \quad \mathcal{U} \gg \mathcal{D}$$



Coupling of fluexural and torsional modes

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Inertial coupling

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Linearization

Kinematic condition (equality of displacements)

$$\underline{u} = \frac{\partial \underline{\xi}}{\partial t} = \sum_n \underline{\phi}_n(\underline{x})\dot{x}_n(t) \quad \text{on } \partial\Omega_0$$

Hypotheses :

- Large Stokes number
- Large Froude number
- Small displacements
- Deformations such that added rigidity effects are absent

$$\ddot{x}_n + \omega_n^2 x_n = \mathcal{M} f_n$$

$$n \in \mathbb{N}$$

$$\text{div } \underline{u} = 0$$

$$\frac{\partial \underline{u}}{\partial t} + \cancel{\left(\underline{\underline{\text{grad}}}\ \underline{u}\right) . \underline{u}} = \cancel{-\frac{1}{F_D^2}\underline{e}_z} - \underline{\text{grad}}\ p + \cancel{\frac{1}{S_t}\Delta \underline{u}}$$

Dynamic condition (equality of efforts)

$$f_n = \int_{\partial\Omega_0} (\underline{\underline{\sigma}}_f . \underline{n}) . \underline{\phi}_n \, ds$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

- Linearized kinematic boundary condition :

$$\underline{u} = \frac{\partial \underline{\xi}}{\partial t} = \underline{\phi}(\underline{x})\dot{x}(t) \quad \text{on } \partial\Omega_0$$

- Linearized dynamic condition (projection of the stress on the mode) :

$$\begin{aligned} f = & - \int_{\partial\Omega_0} p_0 \underline{\phi}\,\underline{n}_0 \,\mathrm{d}s \\ & + \epsilon \int_{\partial\Omega_0} \left(\underline{\phi} \,.\, \underline{\underline{\sigma}}'\right) . \underline{n}_0 \mathrm{d}s \\ & + \epsilon x \int_{\partial\Omega_0} \Big(-\underline{\underline{\mathrm{grad}}}\underline{\phi}[\underline{\phi} - (\underline{\phi}.\underline{n}_0)\underline{n}_0] p_0 \\ & \quad + \underline{\phi} \,.\, \left[-\underline{\mathrm{grad}}\, p_0 \,.\, \underline{\phi}\, \underline{\underline{1}} - p_0 \left(\mathrm{div}\underline{\phi}\, \underline{\underline{1}} - {}^t\, \underline{\underline{\nabla\phi}} \right) \right] \Big) \,.\, \underline{n}_0 \,\mathrm{d}s \\ & + O(\varepsilon^2) \end{aligned}$$

Static pressure (STATIC FORCE)

Effect of stress fluctuation in th fluid

Only term to consider in the present approach

Added rigidity due to the deformation of the solid in a static pressure field NOT CONSIDERED HERE

(not proven in the present course)

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

- The fluid mechanics problem is linear, hence solution for the boundary condition :

$$\underline{u}\,.\,\underline{n} = \frac{\partial \underline{\xi}}{\partial t}\,.\,\underline{n} = \left(\sum_n \underline{\phi}_n(\underline{x}) \dot{x}_n(t) \right)\,.\,\underline{n} \quad \text{on } \partial\Omega_0$$

- Can be expressed as the sum of solutions :

$$\underline{u}(\underline{x},t) = \sum_n \underline{u}_n(\underline{x},t) \qquad p(\underline{x},t) = \sum_n p_n(\underline{x},t)$$

- Each $\underline{u}_n$ being the solution of the fluid mechanics problem with,

$$\underline{u}_n\,.\,\underline{n} = \frac{\partial \underline{\xi}_n}{\partial t}\,.\,\underline{n} = \dot{x}_n(t)\underline{\phi}_n(\underline{x})\,.\,\underline{n} \quad \text{on } \partial\Omega_0$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Problem expressed in terms of pressure

- The problems to solve are the following :

$$\text{div } \underline{u}'_n = 0$$
$$\frac{\partial u'_n}{\partial t} = -\text{grad } p'_n$$

Boundary conditions : $\underline{u}'_n \cdot \underline{n} = \frac{\partial \underline{\xi}'}{\partial t} \cdot \underline{n} = \dot{x}(t)\underline{\phi}(\underline{x}) \cdot \underline{n} \quad \text{on } \partial\Omega_0$

- They can be put in the following form :

$$\Delta p'_n = 0 \qquad -\underline{\text{grad}} p'_n \cdot \underline{n} = \frac{\partial \underline{\xi}'}{\partial t} \cdot \underline{n} = \ddot{x}'(t)\underline{\phi}(\underline{x}) \cdot \underline{n} \quad \text{on } \partial\Omega_0$$

- Because of the form of the boundary condition, the pressure is looked for in the form of a solution to separate variables :

$$p'_n = f_n(t)\phi_{pn}(\underline{x})$$

- The solution is then of the form :

$$p'_n = \ddot{x}'_n \phi_{pn}(\underline{x})$$

- Where ϕ_{pn} satisfies :

$$\Delta \phi_{pn} = 0 \qquad -\underline{\text{grad}}\phi_{pn} \cdot \underline{n}_0 = \underline{\phi}_n \cdot \underline{n}_0$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Solution

- Consider the solution is known, the modal force m has then for expression :

$$f_m = \sum_n -\ddot{x}'_n \int_{\partial\Omega_0} (\phi_{pn} \cdot \underline{n}_0) \cdot \underline{\phi}_m \, \mathrm{d}s$$

- Hence, the modal force m contains terms proportional to the accelerations n=1..N

- The oscillator equations, describing the dynamics in the modal basis is

$$\begin{aligned} \ddot{x}'_1 + x'_1 &= \mathcal{M} f_1(\ddot{x}'_1, ...\ddot{x}'_N) \\ m_2\ddot{x}'_2 + k_2 x'_2 &= \mathcal{M} f_2(\ddot{x}'_1, ...\ddot{x}'_N) \\ &\cdots \\ m_N\ddot{x}'_N + k_N x'_N &= \mathcal{M} f_N(\ddot{x}'_1, ...\ddot{x}'_N) \end{aligned}$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Coupled problem in matrix form

State vector : $\vec{x} = \begin{pmatrix} x'_1 \\ x'_2 \\ ... \\ x'_N \end{pmatrix}$ → $(M + \mathcal{M}A)\ddot{\vec{x}} + K\vec{x} = 0$

$$M = \begin{pmatrix} 1 & & & & \\ & m_2 & & & \\ & & \ddots & & \\ & & & m_{N-1} & \\ & & & & m_N \end{pmatrix} \qquad K = \begin{pmatrix} 1 & & & & \\ & k_2 & & & \\ & & \ddots & & \\ & & & k_{N-1} & \\ & & & & k_N \end{pmatrix}$$

$$A_{ij} = \int_{\partial\Omega_0} (\phi_{pj} \,.\, \underline{n}_0) \,.\, \underline{\phi}_i \,\mathrm{d}s$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Aerodynamic coupling

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Quasi static approach

$$\mathcal{U} \gg \mathcal{D} \implies \underline{u} \sim 0$$

At each time, the fluid dynamics problem is a classical fluid dynamics problem (zero velocity at the boundaries)

- **Consequence :** The velocity, the pressure, the stress tensor in the fluid depends only on the position of the solid.
- **Case of a structure :** The forces exerted by the fluid on the structure depend only on the modal displacements.

→ Coupling though the stiffness matrix

How to explain wing flutter ?

$$-\frac{\partial C_y}{\partial \alpha}$$

Section	Smooth flow	Turbulent flow[b]	Reynolds number
D, D	3.0	3.5	10^5
D, $\frac{2}{3}$D	0.	−0.7	10^5
D, D/2	−0.5	0.2	10^5
D, D/4	−0.15	0.	10^5
$\frac{2}{3}$D, D	1.3	1.2	66 000
D/2, D	2.8	−2.0	33 000
D/4, D	−10.	—	2 000–20 000
D	−6.3	−6.3	$>10^3$
D	−6.3	−6.3	$>10^3$
D	−0.1	0.	66 000
	−0.5	2.9	51 000
D	0.66	—	75 000

(Blevins, 1990)

But ...

Observation : the instability mechanism should involve flexural and torsional deformations.

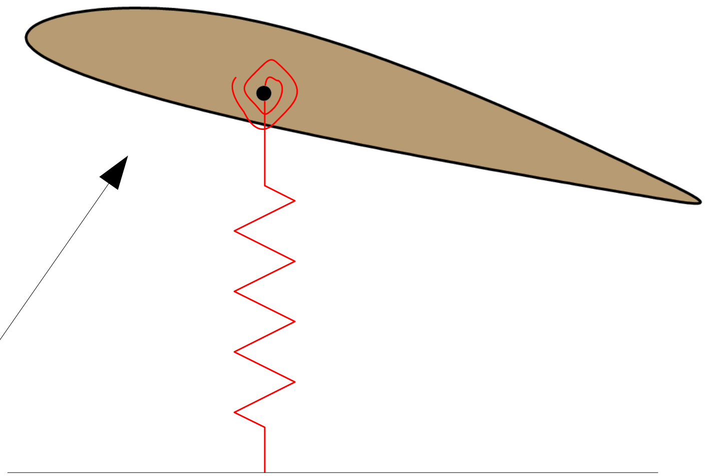
Thin profiles are stable with respect to galloping

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Example : flutter of a wing profile

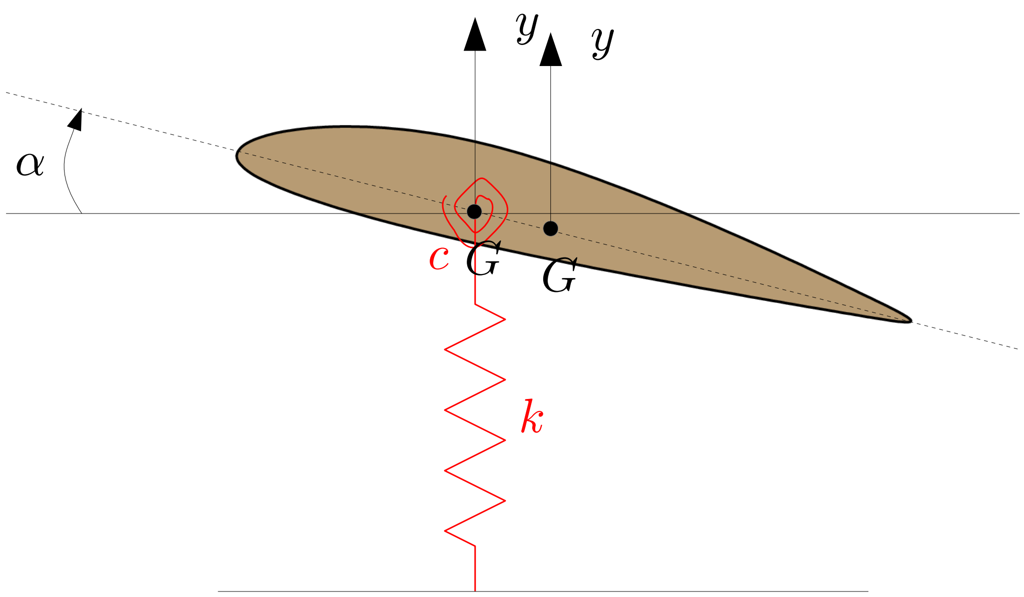
Coupled torsional and flexural modes of wing

Equivalent 2D profile in translation and rotation



Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

The model



G center of gravity

G is at the elastic center, decoupled flexural and torsional modes :

$$J\ddot{\alpha} + c\alpha = 0$$
$$my + ky = 0$$

G is at a distance x of the elastic center, coupled flexural and torsional modes :

$$J\ddot{\alpha} + (c + kx^2)\alpha + kxy = 0$$
$$m\ddot{y} + ky + kx\alpha = 0$$

- Dynamic problem of the airfoil without flow in matrix form :

$$\begin{bmatrix} J & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} c + kx^2 & kx \\ kx & k \end{bmatrix} \begin{bmatrix} \alpha \\ y \end{bmatrix} = 0$$

$$M\ddot{\vec{q}} + K\vec{q} = \vec{0}$$

- Solutions of the form :

$$\vec{q}(t) = \vec{V} e^{i\omega t}$$

- Eigenvalue problem :

$$(K - \omega^2)\vec{V} = \vec{0}$$

$\omega^2 \equiv$ eigenvalue $\quad \vec{V} \equiv$ eigenvector

- If the center of gravity and elastic center are the same : $x = 0$

$$\begin{bmatrix} J & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \alpha \\ y \end{bmatrix} = 0$$

- Eigenvalue problem :

$$\begin{bmatrix} -\omega^2 J + c & 0 \\ 0 & -\omega^2 m + k \end{bmatrix} \mathrm{V} = 0$$

$$\omega_1 = \sqrt{c/J}$$

$$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Torsionnal only oscillation

$$\omega_2 = \sqrt{k/m}$$

$$V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Flexural only oscillation

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Eigenfrequencies and eigenmodes of a linear dynamical system

- If the center of gravity and elastic center are different : $x \neq 0$

$$\begin{bmatrix} J & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{\alpha} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} c + kx^2 & kx \\ kx & k \end{bmatrix}\begin{bmatrix} \alpha \\ y \end{bmatrix} = 0$$

- Eigenvalue problem :

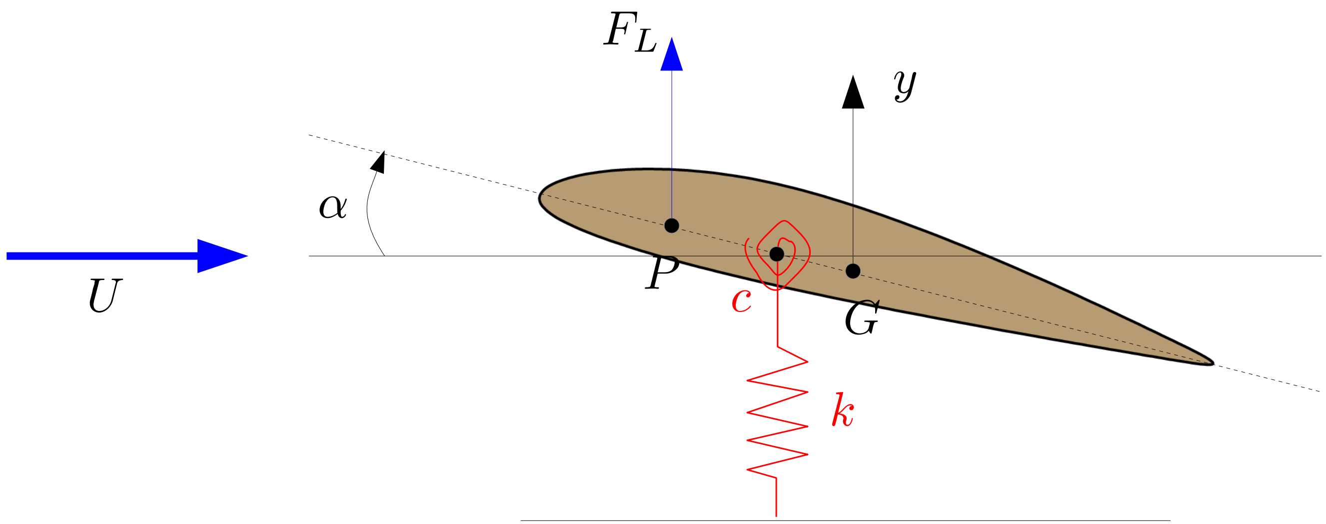
$$\begin{bmatrix} -\omega^2 J + c + kx^2 & kx \\ kx & -\omega^2 m + k \end{bmatrix} \mathrm{V} = 0$$

$$\omega_1 = \ldots \qquad V_1 = \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix}$$

$$\omega_2 = \ldots \qquad V_2 = \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix}$$

Coupled torsional and flexural motions

Introduction of flow effects in the model



G center of gravity

P is the aerodynamic center (at ¼ of the length for a thin profile)

- A lift force is exerted on the profile

$$F_L = \frac{1}{2}\rho U^2 L C_L$$

- Taylor expansion of the lift force for low values of the angle of attack :

$$F_L(\alpha) = F_0 + \alpha \frac{\partial F_L}{\partial \alpha} + \mathcal{O}(\alpha^2)$$

- Consider that there is static equilibrium and introduce the lift coefficient :

$$F_L(\alpha) \sim \alpha \frac{1}{2}\rho U^2 \frac{\partial C_L}{\partial \alpha} = \alpha F'$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Coupled mode flutter

- Results in a force and a momentum exerted on the profile :

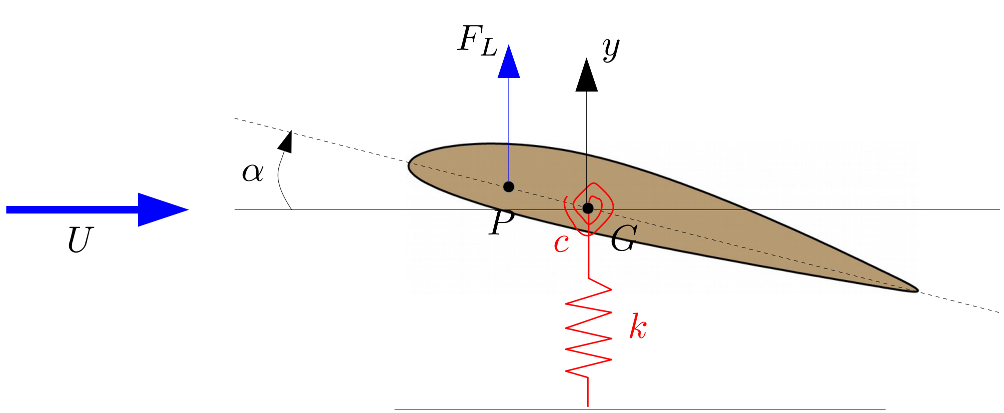
$$F_y \sim \alpha F' \qquad M_\alpha \sim (x+d)\alpha F' \qquad \text{with} \quad F' = \frac{1}{2}\rho U^2 \frac{\partial C_L}{\partial \alpha}$$

- F' is positive for thin profiles.

- Full coupled dynamical equation :

$$\begin{bmatrix} J & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} c + kx^2 - (x+d)F' & kx \\ kx - F' & k \end{bmatrix} \begin{bmatrix} \alpha \\ y \end{bmatrix} = 0$$

- Case 1 : The center of gravity is at the elastic center (x=0) :



Equation for torsion :

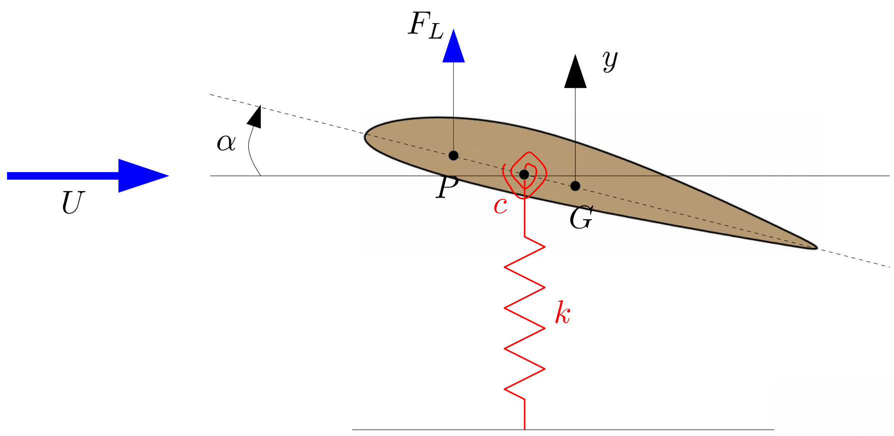
$$J\alpha + (c - dF')\alpha = 0$$

Rigidity can become negative !

Divergence instability, buckling

Introduction of flow effects in the model

- Case 2 : Center of efforts upstream of the elastic center, center of gravity downstream



- Solutions of the form :

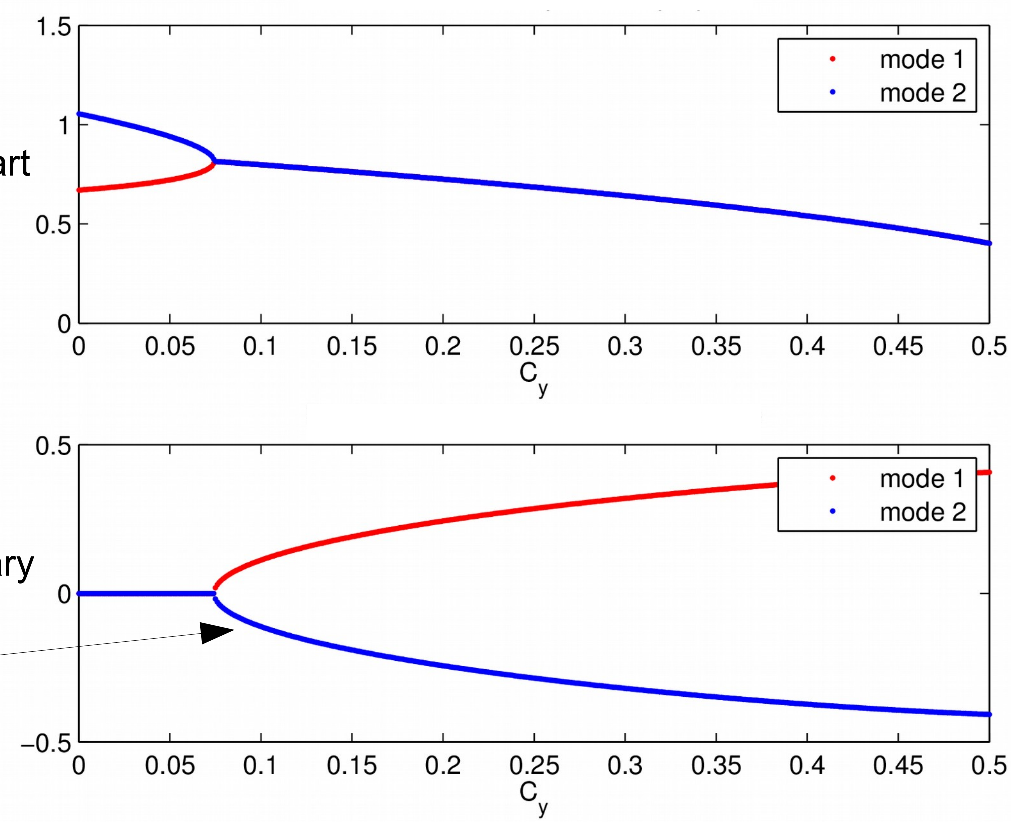
$$
\begin{aligned}
\vec{q}(t) &= \vec{V} e^{i\omega t} \\
&= \vec{V} e^{i\omega_r t} e^{-\omega_i t}
\end{aligned}
$$

$$\omega_i < 0$$

Instability !

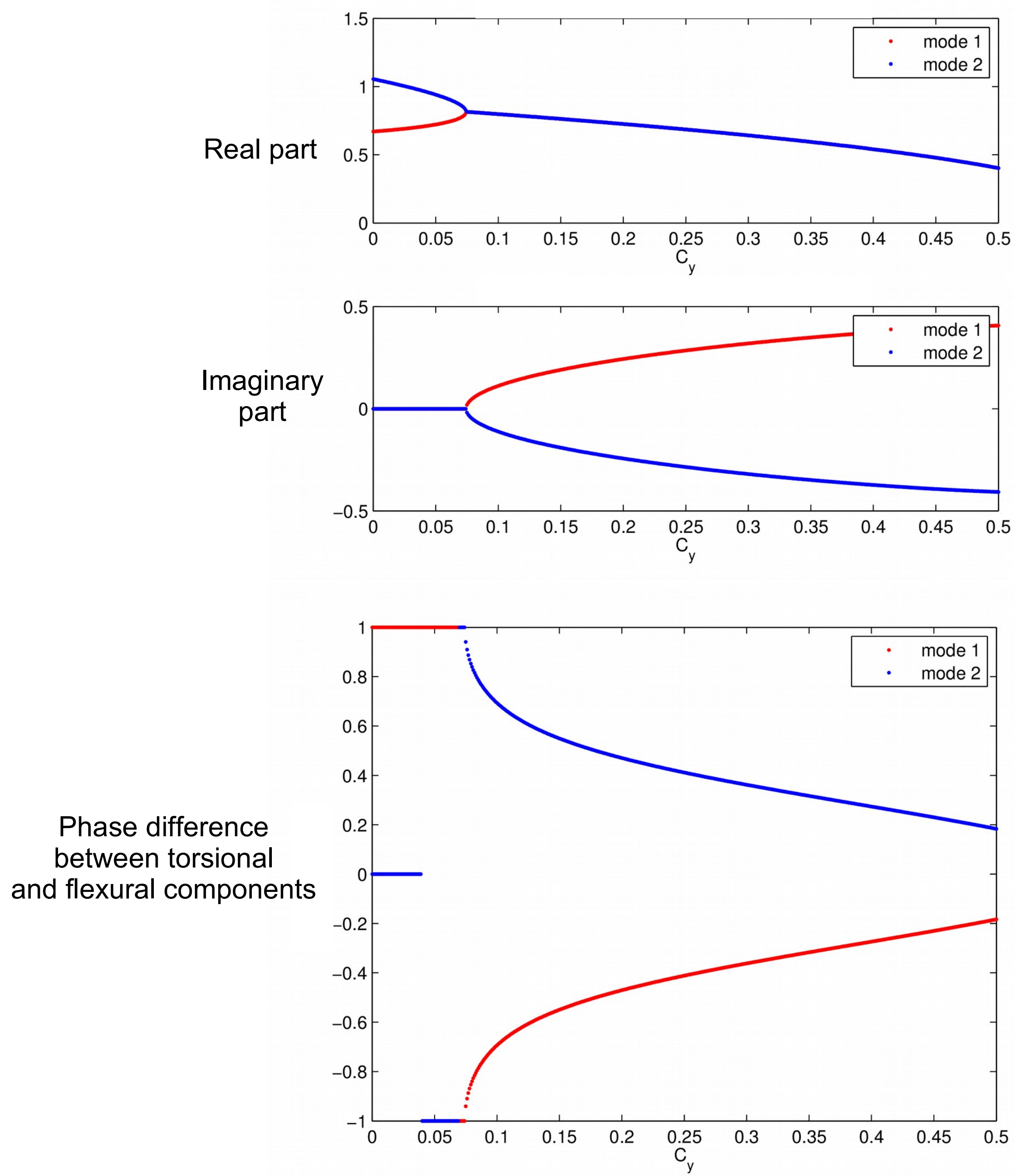
(Coupled-mode flutter)

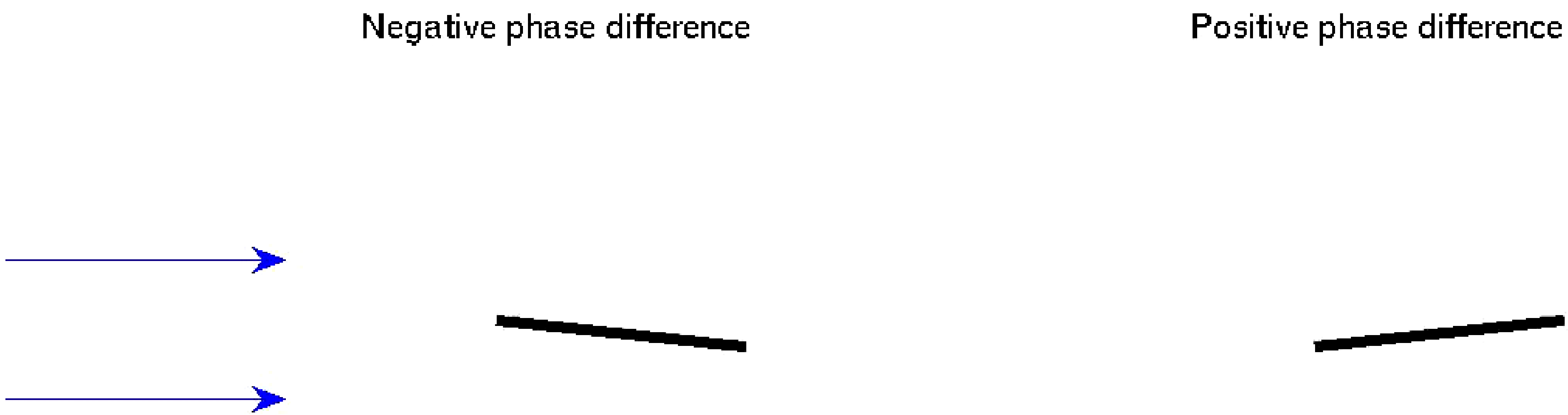
Typical evolution of the eigenfrequencies



Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Typical evolution of the eigenfrequencies





- When phase difference is negative, the work done by the fluid on the structure is negative → damped oscillations

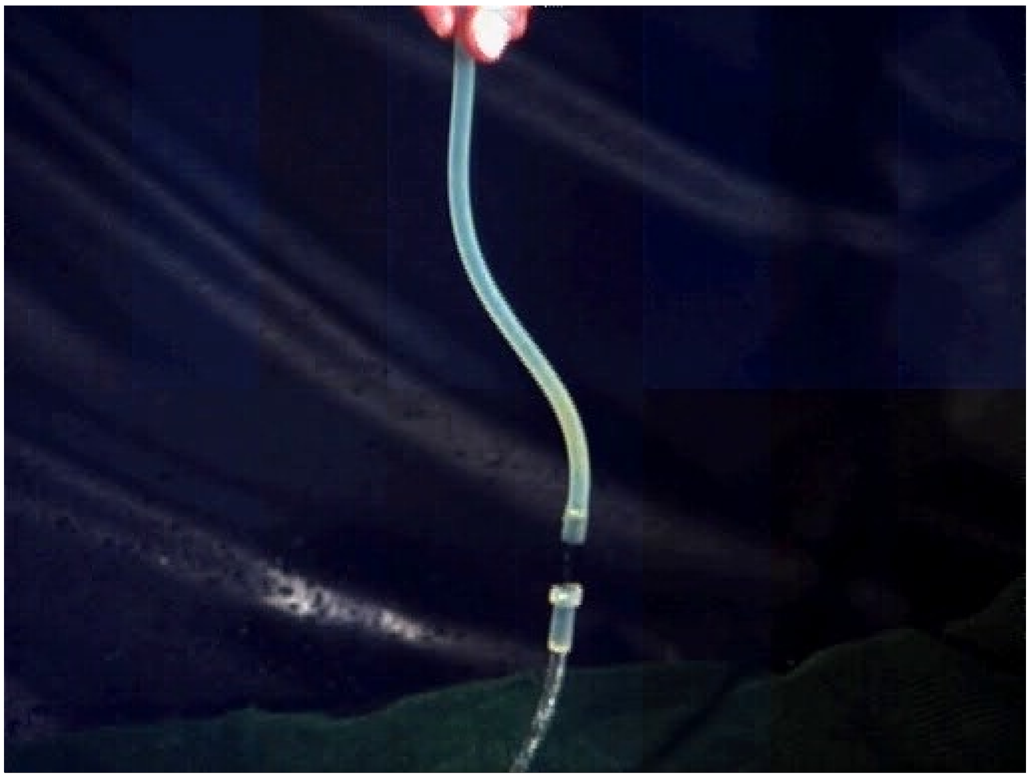
- When phase difference is positive, the work done by the fluid on the structure is positive → amplified oscillations

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

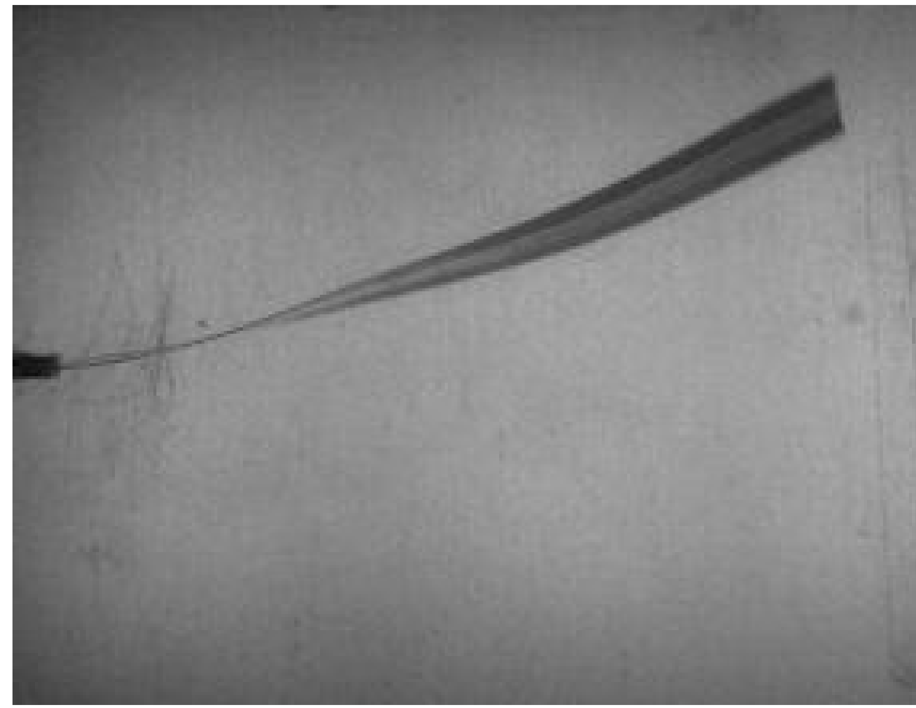
Flutter of flags, plates, fluid-conveying pipes is a coupled modes flutter

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Observation of the instability



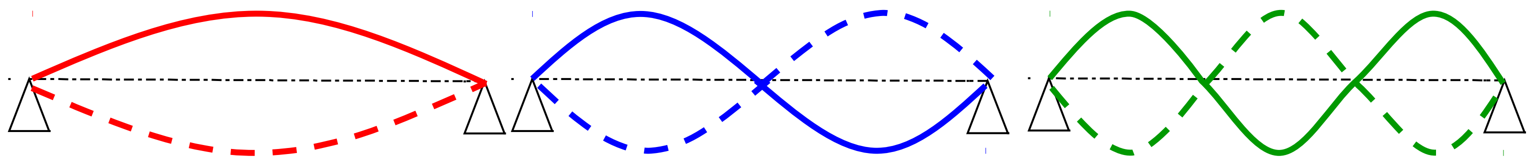
Fluid-conveying pipe



Fluttering flag

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

General methodology (not detailled in the present course)



Kinematic condition (equality of displacements)

$$\underline{U} = \sum_n \underline{\phi}_n(\underline{X})\dot{X}_n(t) \quad \text{on } \partial\Omega$$

STRUCTURE
Coupling by mass, damping and stiffness terms

FLUID

(Linearized equations)

Dynamic condition (equality of efforts)

$$f_n = \int_{\partial\Omega_0} (\underline{\underline{\sigma}}_f \cdot \underline{n}) \cdot \underline{\phi}_n \, ds$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

- Dynamical equation in the modal space with additionnal stiffness, damping, inertia terms, du to the presence of flow :

$$M\ddot{\vec{X}} + C\dot{\vec{X}} + K\vec{X} = 0$$

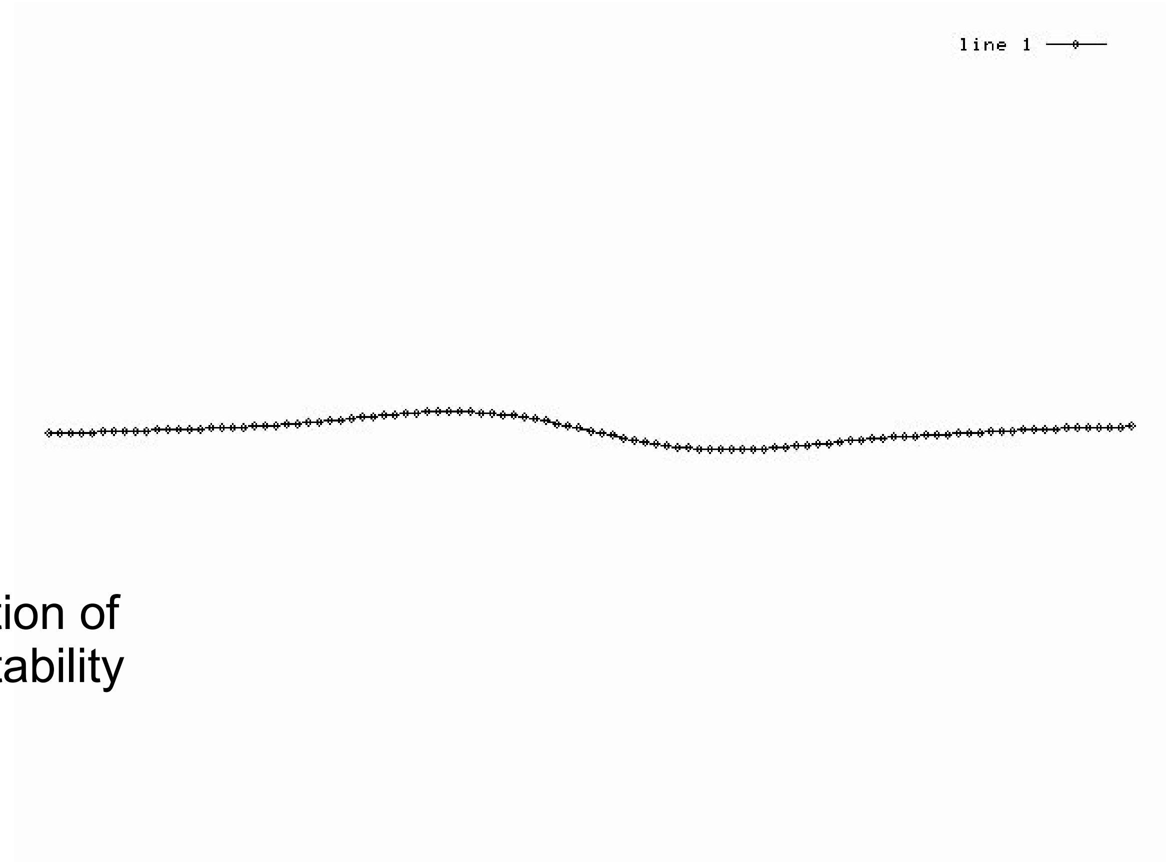
- Solutions though in the form :

$$\vec{X} = \vec{V}\mathrm{e}^{\mathrm{i}\omega t}$$

- → Eigenvalue problem

- Eigenvalues with positive negative imaginary part may be found

- The eigenvector gives the combination of modes that is associated to this instability



Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

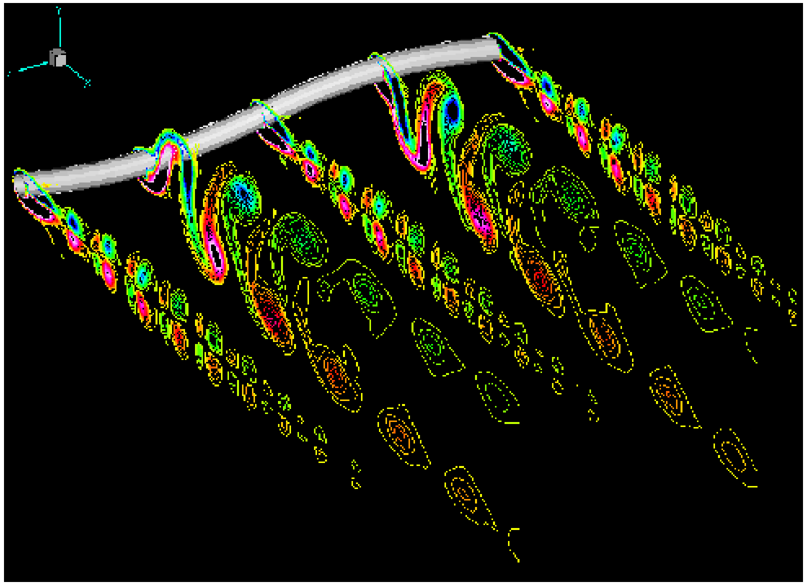
Conclusion on modes coupling

- The presence of fluid, flowing or not, induces a coupling between the eigenmodes of the structure

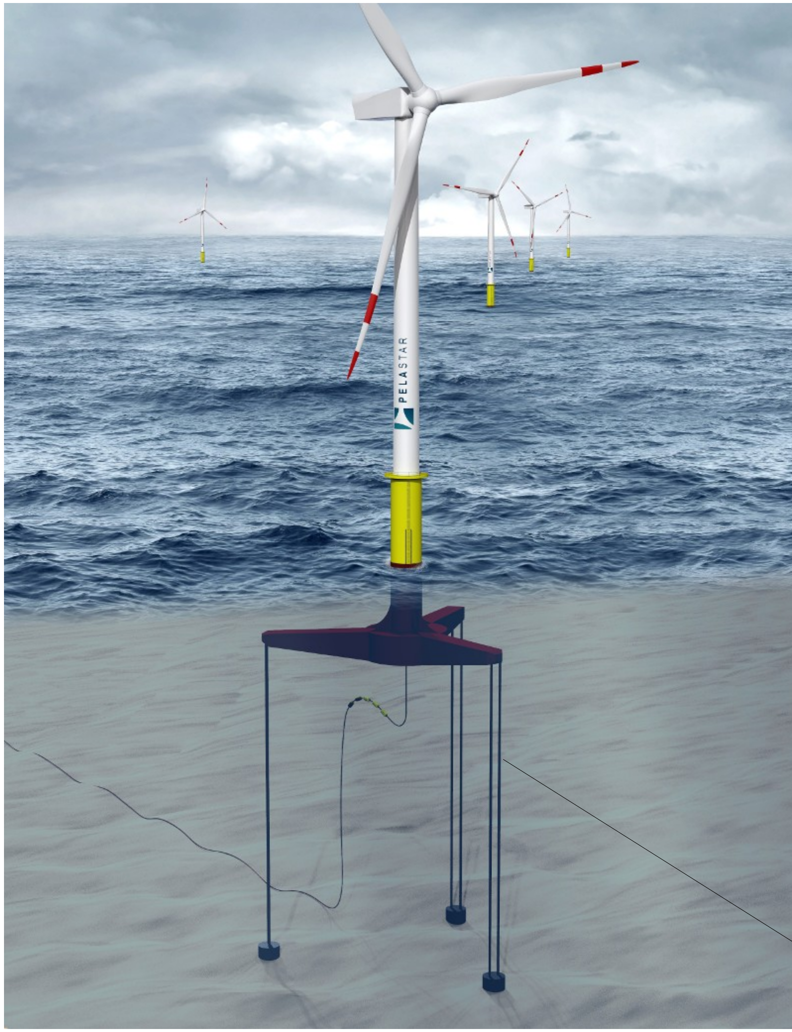
  → Matrices of mass, damping, stiffness are ful matrices

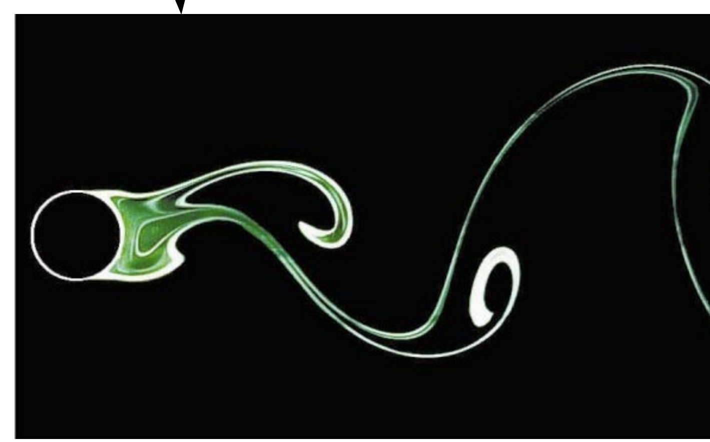
- This coupling can modify the stability properties of the mechanical system

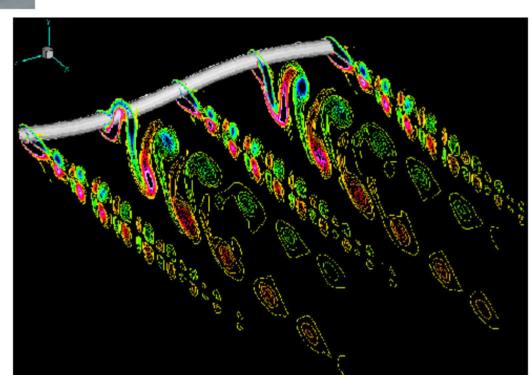
IV - Vortex induced vibrations



Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

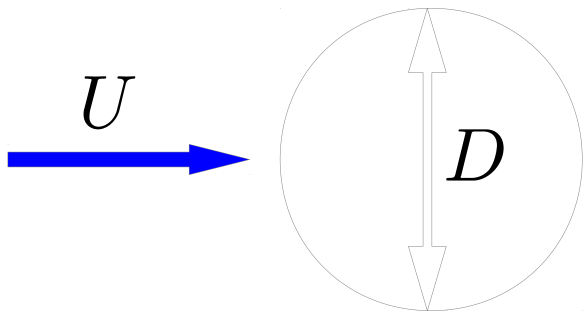






Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Vortex shedding around a circular cylinder



$$R_e = \frac{UD}{\nu}$$

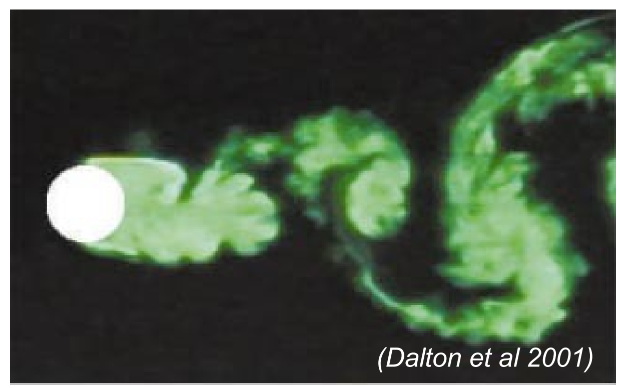
$R_e = 100$


(Dalton et al 2001)

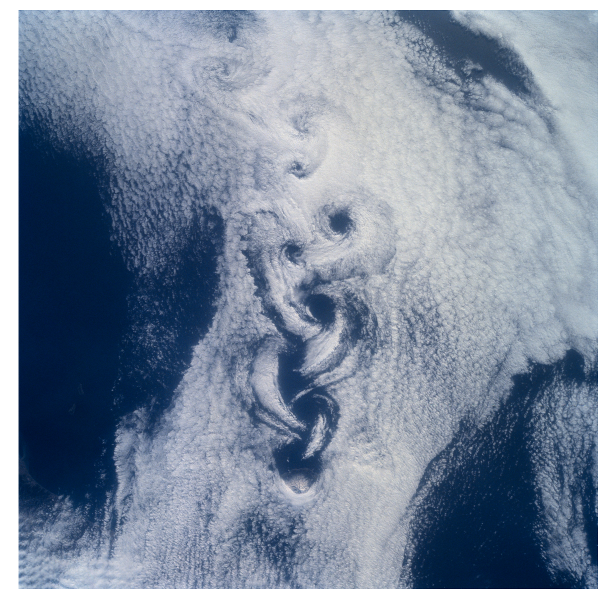
$R_e = 1000$


(Dalton et al 2001)

$R_e = 3000$


(Dalton et al 2001)

$R_e = 6 \times 10^9$

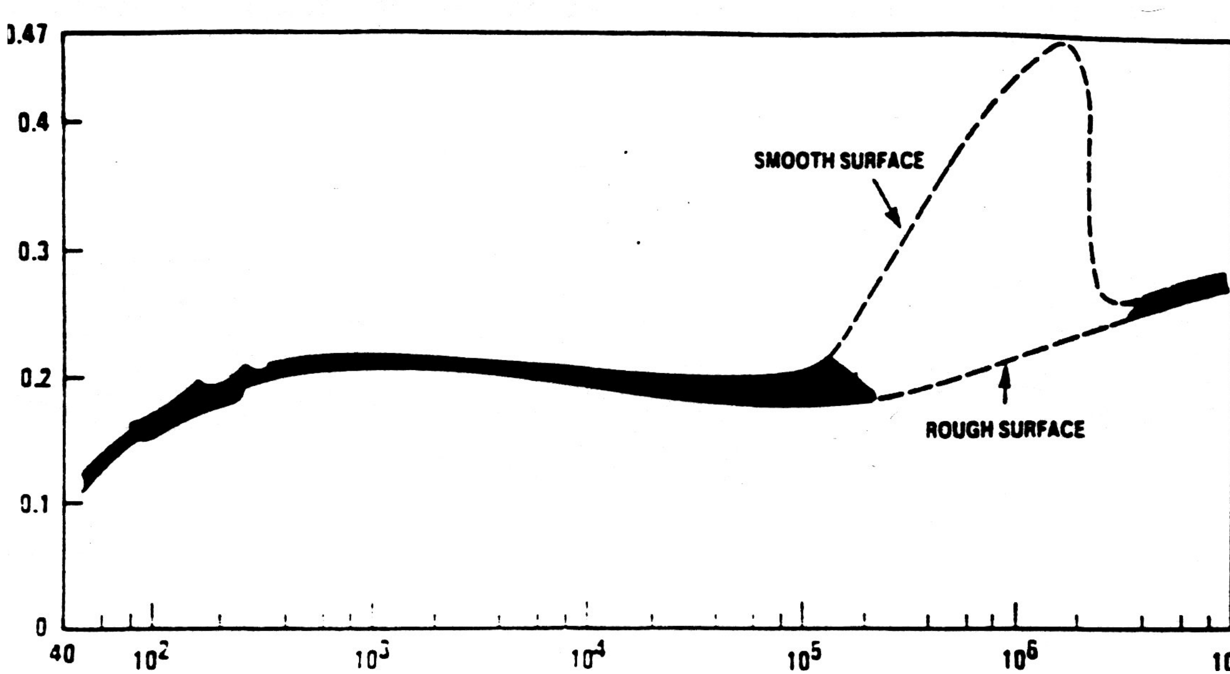

Rishiri island (source wikipedia)


Thiria & Cadot

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Frequency of the vortex street behind a cylinder

Strouhal number $S_t = \frac{FD}{U}$



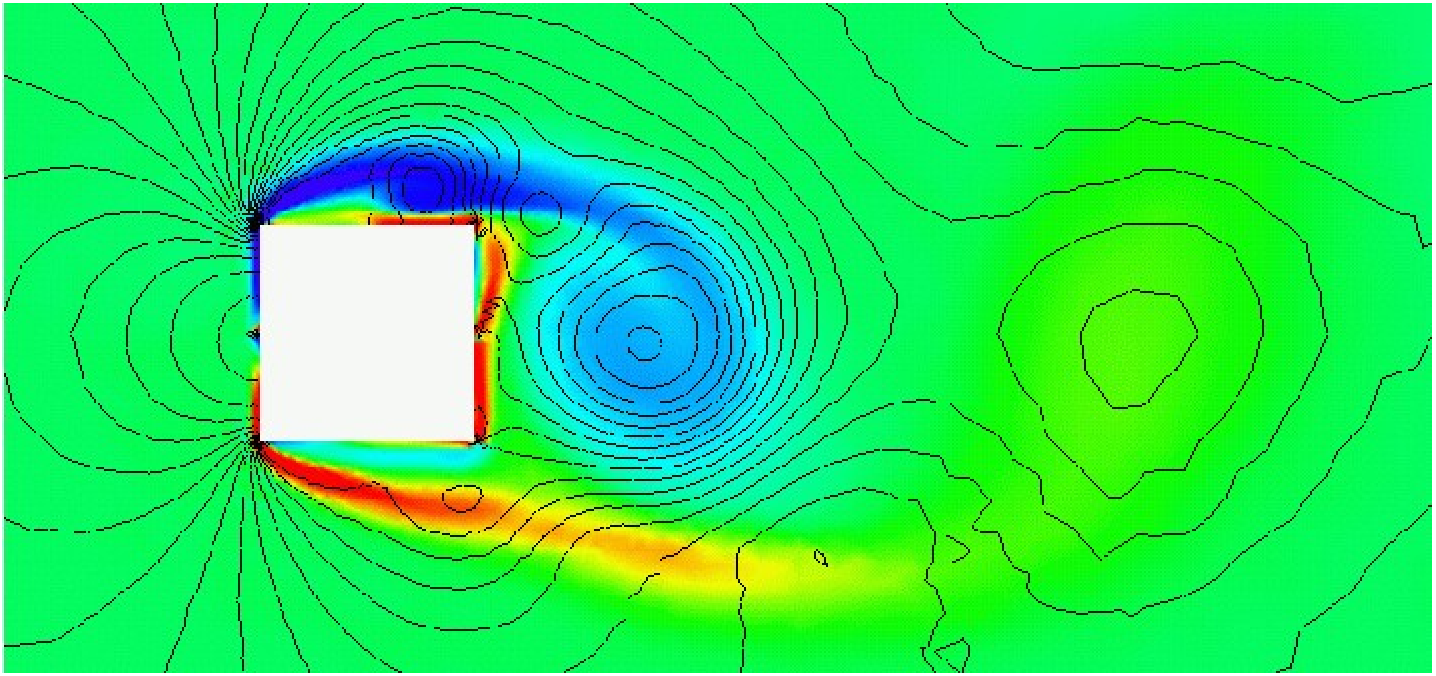
Blevins, 1990

Reynolds number $R_e = \frac{UD}{\nu}$

- Strouhal number almost constant (~0.2, 0.3)
- Frequency of the vortex shedding varies almost linearly with the flow velocity

Other geometries

Square section St ~ 0.16



- Each cross-section has a unique Strouhal number
- Robust, generic, and predictible phenomenon

$$F = \frac{S_t U}{D}$$

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

This vortex shedding acts like a fluctuatig force on the structure

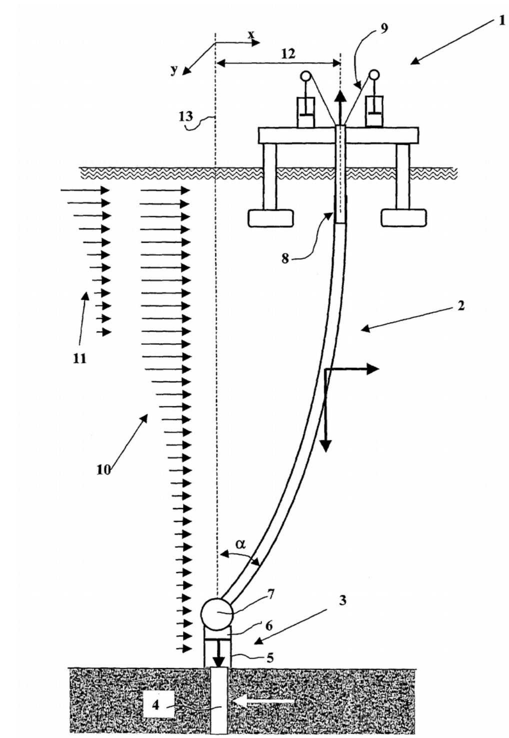
Forcing → Vibrating structure → Response ?

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

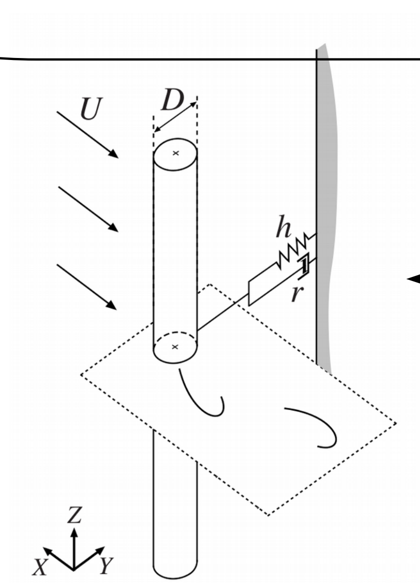
Structural vibrations : what is a scruture and how to model it ?

Bridge

Cables conveying electricity



Risers in the offshore industry

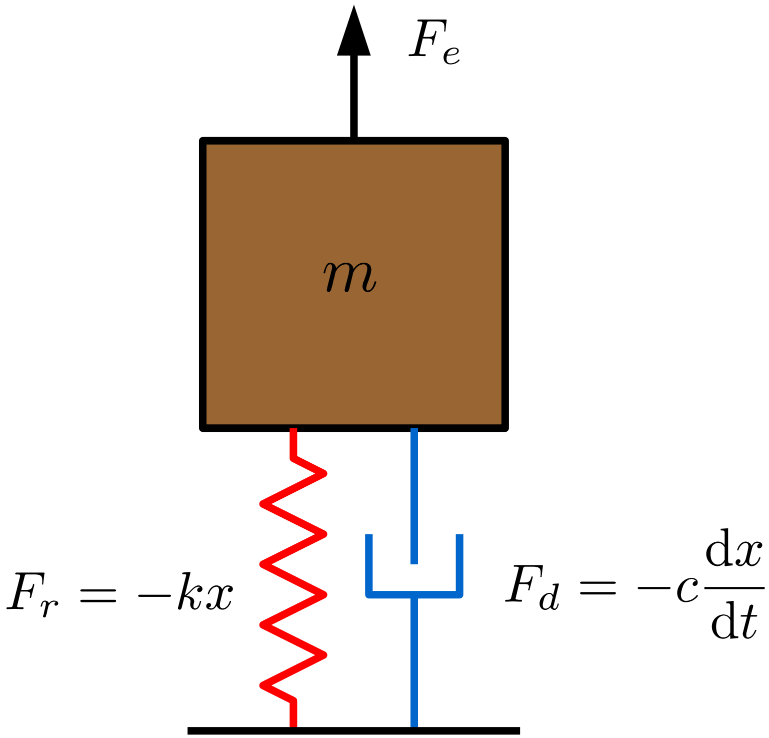


~

Single oscillator coupled to a fluctuating lift due to a wake

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Harmonic oscillator : forced motion



$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_e$$

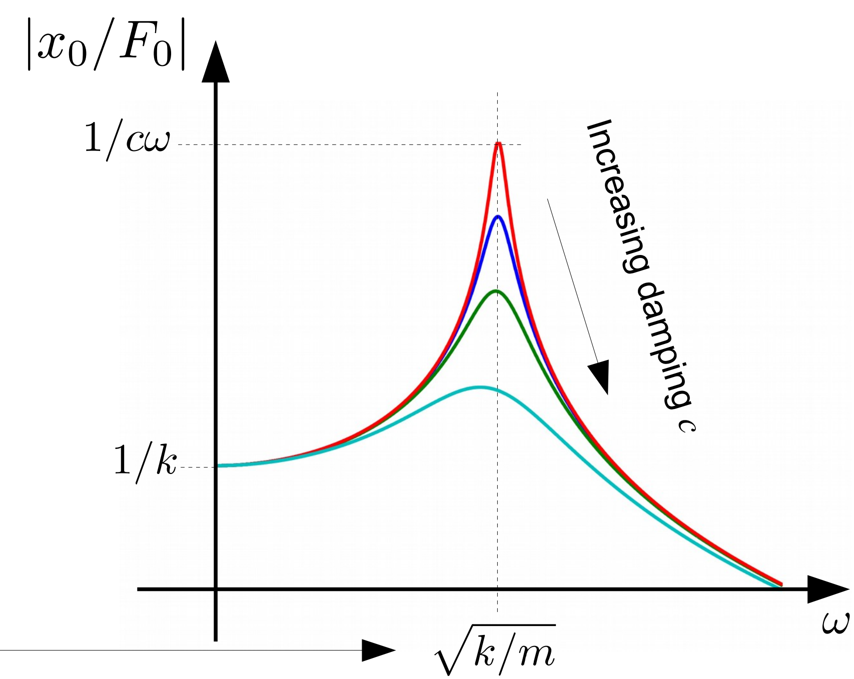
- Harmonic forcing :
  $F_e = \mathrm{Re}(F_0 e^{i\omega t}) \quad F_0 \in \mathbb{C}$

- Hyp. : response at the same frequency :
  $x = \mathrm{Re}(x_0 e^{i\omega t}) \quad x_0 \in \mathbb{C}$

$$(-\omega^2 m + ic\omega + k)x_0 e^{i\omega t} = F_0 e^{i\omega t}$$

$$\frac{x_0}{F_0} = \frac{1}{m(\frac{k}{m} - \omega^2) + ic\omega}$$

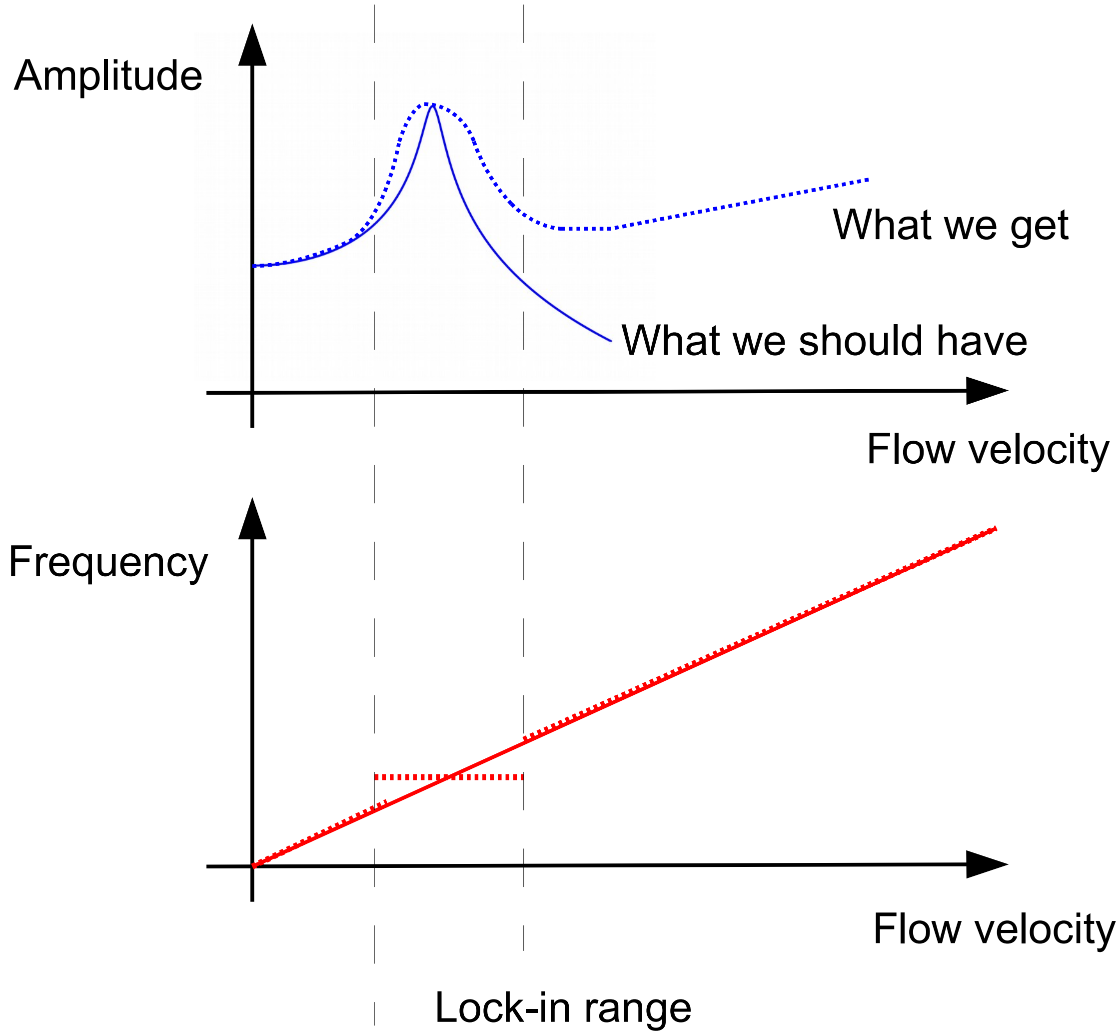
(transfer function)



This is the frequency of free vibrating system

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Lock-in phenomenon

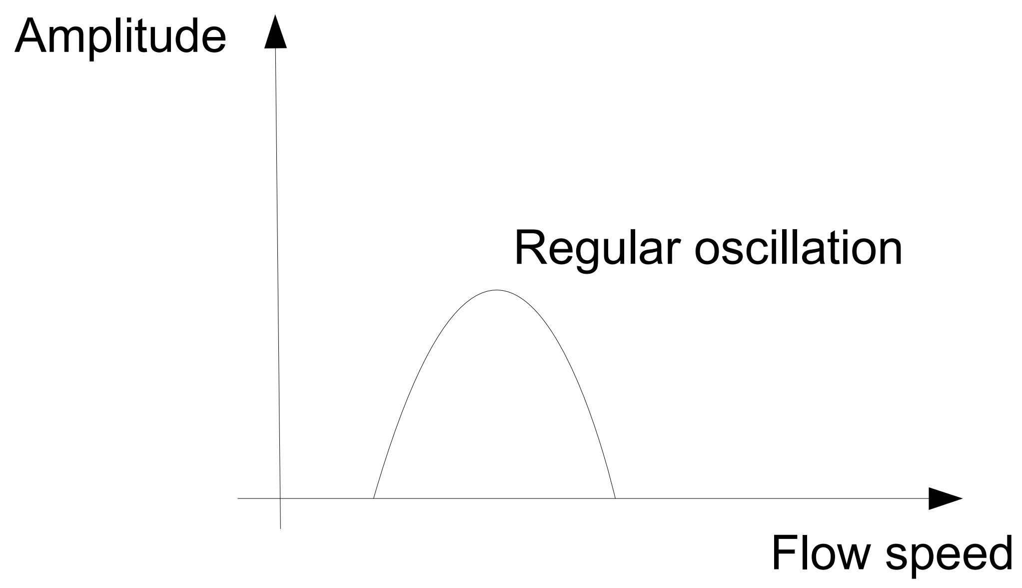


In the lock-in range, fluid-structure coupling is improved, frequency is locked to the eigenfrequency of the oscillator

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Conclusion on vortex-induced vibration

VIV phomenon predicts vibrations in a frequency range around the fundamental frequency of the oscillator



Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr

Conclusion

- Dynamics of structures in presence of still fluid or in presence of flow
- Added mass, stiffness or damping phenomena
- Coupling between structural modes through inertial, stiffness or damping terms
- Possible instabilities due to the coupling with a flow
- Possibility of synchronization with the dynamics of a fluid (VIV)

Olivier Doaré – ENSTA-Paristech – olivier.doare@ensta-paristech.fr