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QUADRATIC 0-1 PROGRAMMING : TIGHTENING LINEAR OR QUADRATIC CONVEX REFORMULATION BY USE OF RELAXATIONS

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Abstract. Many combinatorial optimization problems can be formulated as the minimization of a 0-1 quadratic function subject to linear constraints. In this paper, we are interested in the exact solution of this problem through a two phases general scheme. The first phase consists in reformulating the initial problem either into a compact integer linear program or into an integer quadratic convex program. The second phase simply consists of submitting the reformulated problem to a standard solver. The efficiency of this scheme strongly depends on the quality of the reformulation obtained in phase 1. We show that a good compact linear reformulation can be obtained by solving a continuous linear relaxation of the initial problem. We also show that a good quadratic convex reformulation can be obtained by solving a semi-definite relaxation. In both cases, the obtained reformulation profits from the quality of the underlying relaxation. Hence, the proposed scheme gets around, in a sense, the difficulty to incorporate these costly relaxations in a branch-and-bound algorithm.

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Résumé. Le problème de la minimisation d'une fonction quadratique en variables 0-1 sous contraintes linéaires permet de modéliser de nombreux problèmes d'Optimisation Combinatoire. Nous nous intéressons à sa résolution exacte par un schéma général en deux phases. La première phase permet de reformuler le problème de départ en un programme en nombres entiers, soit linéaire compact, soit quadratique convexe. La deuxième phase consiste simplement à soumettre le problème reformulé à un solveur standard. L'efficacité de ce schéma est étroitement liée à la qualité de la reformulation obtenue à la fin de la phase 1. Nous montrons qu'une bonne reformulation linéaire compacte peut être obtenue par la résolution d'une relaxation linéaire. De même, une bonne reformulation quadratique convexe peut être obtenue par une relaxation semi-définie positive. Dans les deux cas, la reformulation obtenue tire profit de la qualité de la relaxation sur laquelle elle se base. Ainsi, le schéma proposé contourne, d'une certaine façon, la difficulté d'intégrer les relaxations, coûteuses en temps de calcul, dans un algorithme de branch-and-bound.

Introduction

Consider the following linearly-constrained zero-one quadratic program :

$$Q01 \;:\; \text{Min} \left\{ F(x) = \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} q_{ij} x_i x_j : x \in X,\ x \in \{0,1\}^n \right\}$$

where $X = \left\{ x : \sum_{i=1}^{n} a_{ki} x_i = b_k, k = 1, .., m; \sum_{i=1}^{n} a'_{\ell i} x_i \leq b'_\ell, l = 1, .., p;\ x \in \{0,1\}^n \right\}$ is the feasible solution set of $Q01$ and q_i, q_{ij}, a_{ki}, b_k, $a'_{\ell i}$, b'_ℓ are real numbers. Without loss of generality, we can assume that $q_{ij} = q_{ji}$. We denote by $\overline{X}$ the continuous-relaxation solution set of $Q01$. Set $\overline{X}$ is obtained from X by replacing $x \in \{0,1\}$ by $x \in [0,1]$.

Problem $Q01$ is NP-hard [10]. It allows to formulate many Combinatorial Optimization problems such as graph bipartition, quadratic knapsack, and quadratic assignment. It also has several applications. Due to its complexity, many heuristic solution methods have been proposed for it. For example, [2], [12], [18], and [19] apply tabu search for the unconstrained problem and local search heuristics are applied to graph bipartition problems in [16], [17].

Exact solution methods have also been proposed for solving $Q01$. In this paper, we focus on two main approaches : linear reformulation and quadratic convex reformulation. Linear reformulation techniques transform $Q01$ into a mixed integer linear program. The most frequently used linearization was first introduced by Fortet in [8], [9] and is sometimes called the *classical linearization*. It consists in replacing each product $x_i x_j$ by a new variable y_{ij} and adding a set of linear constraints that force y_{ij} to be equal to $x_i x_j$, i.e. $y_{ij} \leq x_i$, $y_{ij} \geq x_i + x_j - 1$, $y_{ij} \geq 0$, and $y_{ij} = y_{ji}$ for all $i \neq j$. A strengthening of the classical linearization by a family of valid inequalities was later proposed by Sherali et al. in [21], [22]. They developed a Reformulation and Linearization Technique (RLT). Using quite different ideas, Glover [11] introduced an alternative linearization strategy that requires a much smaller set of additional variables and constraints than the classical linearization and RLT. This is why it is often called a *compact* linearization. Compact linearization, and variants of it, was further used by several authors [1], [6], [13].

Another class of exact solution methods aims at reformulating the objective function of $Q01$ by a quadratic convex function. Doing this, solving the continuous relaxation of the reformulated problem becomes tractable in polynomial time. For example, Hammer and Rubin [15] devise a simple convexification method based on a smallest eigenvalue computation. Later, Carter [7] then Billionnet and Elloumi [3], Plateau et al. [20], and Billionnet et al. [4] study several families of quadratic convex reformulations and provide theoretical and computational comparisons between these families. All the reformulations proposed in [3], [20], and [4] are based on an exact solution of a quadratic semidefinite program. Among these papers, the reformulation proposed in [4] and called QCR gives the provably best bound by continuous relaxation.

In this paper, we introduce a *positive* compact linearization method and we recall the quadratic convex reformulation method QCR [4]. We will focus on the fact that each of these methods takes profit from a preprocessing phase. This phase aims both at finding a suitable reformulation of the problem and at making the further resolution process as efficient as possible. To achieve this last objective, we use the common criteria that prefers reformulations yielding bounds as tight as possible by continuous relaxation. We will show how we build a positive linear compact reformulation once an RLT relaxation is solved (or computed) and how it captures the bound obtained by this relaxation. In a similar way, QCR is built once an appropriate SDP relaxation is solved and it captures the bound obtained by SDP relaxation.

The rest of the paper is organized as follows: Linear compact reformulations and convex quadratic reformulations are presented in Section 1. In Section 2, we show how to build a positive compact linearization from the RLT solution. In Section 3, we show how to build a QCR reformulation from the solution of an SDP relaxation. Section 4 is a conclusion.

<u>Example:</u>
All along this paper, we will use the same example for illustration purposes. Consider the following 0-1 quadratic programming instance E, which optimal value is -65:

$$
\begin{array}{lll}
E: & \text{Min} & \phi(x) = -9x_1 - 7x_2 + 2x_3 + 23x_4 + 12x_5 - 48x_1x_2 + 4x_1x_3 + 36x_1x_4 \\
 & & -24x_1x_5 - 7x_2x_3 + 36x_2x_4 - 84x_2x_5 + 40x_3x_4 + 4x_3x_5 - 88x_4x_5 \\
 & \text{s.t.} & \\
 & & x_1 - 2x_2 + 5x_3 + 2x_4 - 2x_5 \geq 2 \\
 & & x_1 + x_2 + x_4 + x_5 = 2 \\
 & & x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}
\end{array}
$$

1. Reformulation of quadratic 0-1 programs

In the context of this paper, a reformulation of $Q01$ is any equivalent mathematical program P with integer or mixed integer variables, that is either linear or quadratic convex. These equivalent reformulations are expected to preserve the feasible solution domain and the optimal value. Among all the possible reformulations, we choose two reformulation schemes that either use the same variables and constraints as program $Q01$ or require a reduced number of additional variables and constraints. Let us first recall the linear compact reformulation of Glover.

1.1. The compact linearization of Glover [11]

The linear reformulation method proposed in [11] aims to replace non-linear expressions of $Q01$ by a set of continuous variables. More precisely, take $z_j = x_j \sum\limits_{i=1, i\neq j}^{n} q_{ij} x_i$ and reformulate $Q01$ by:

$$
\begin{array}{lll}
RL_g: & \text{Min} & F(x) = \sum\limits_{i=1}^{n} q_i x_i + \sum\limits_{j=1}^{n} z_j \\
 & \text{s.t.} & \\
 & & \sum\limits_{i=1}^{n} a_{ki} x_i = b_k \;\; k = 1, \ldots, m \\
 & & \sum\limits_{i=1}^{n} a'_{\ell i} x_i \leq b'_\ell \;\; \ell = 1, \ldots, p \\
 & & L_j x_j \leq z_j \leq U_j x_j \;\; j = 1, \ldots, n \\
 & & \sum\limits_{i=1, i\neq j}^{n} q_{ij} x_i - U_j(1 - x_j) \leq z_j \;\; j = 1, \ldots, n \\
 & & z_j \leq \sum\limits_{i=1, i\neq j}^{n} q_{ij} x_i - L_j(1 - x_j) \;\; j = 1, \ldots, n \\
 & & x \in \{0, 1\}^n, z_j \in \mathbb{R}
\end{array}
$$

where L_j and U_j are respectively lower and upper bounds of the linear functions $\sum_{i=1, i\neq j}^{n} q_{ij}x_i$, computed as : $L_j = Min\left\{\sum_{i=1, i\neq j}^{n} q_{ij}x_i : x \in X\right\}$ and $U_j = Max\left\{\sum_{i=1, i\neq j}^{n} q_{ij}x_i : x \in X\right\}$

By optimality considerations, since coefficients of the z_j variables in the objective function are nonnegative, inequalities $z_j \leq U_j x_j$ and $z_j \leq \sum_{i=1, i\neq j}^{n} q_{ij}x_i - L_j(1-x_j)$ can be discarded.

In the following, we present a family of compact linearizations that are inspired from the linearization of Glover. They are built from a first reformulation of the objective function as a constant plus a nonnegative function over $\overline{X}$.

1.2. Positive linear compact reformulation

Suppose one has identified a constant c and functions $L(x)$, $f_i(x)$, and $g_i(x)$ that are nonnegative on the relaxed domain $\overline{X}$, and that satisfy:

$$\forall x \in X,\ F(x) = c + L(x) + \sum_{i=1}^{n} x_i f_i(x) + \sum_{i=1}^{n}(1-x_i)g_i(x) \qquad (1)$$

This first reformulation of $F(x)$ is always possible since a quadratic pseudo-boolean function can be written as a constant plus a quadratic posiform. Function $\sum_{k=1}^{K} C_k T_k$ is a quadratic posiform if all coefficients C_k are nonnegative and every T_k is a literal or the product of two literals. A literal is either a variable x_i or its complement $(1-x_i)$. Writing a quadratic pseudoboolean function as a constant, as large as possible, plus a posiform has addressed much interest in literature and we know it can be done by at least three different methods, see for example [14]. For our example E, the largest constant equals -160 and we can write $\phi(x) = -160+41(1-x_1)+42(1-x_2)+x_1[48(1-x_2)+24(1-x_5)]+71x_2(1-x_5)+x_3(40x_4+4x_5)+x_4[7x_5+85(1-x_5)]+(1-x_1)[4(1-x_3)+36(1-x_4)]+(1-x_2)[7x_3+36(1-x_4)+13x_5]+(1-x_3)(1-x_5)$

From the first reformulation as (1), we can now build our positive compact linearization. For $i = 1, ..., n$, we need an upper bound $\overline{f}_i$ (resp. $\overline{g}_i$) for function $f_i(x)$ (resp. $g_i(x)$) over the feasible solution set X. Let RL_{comp} be the following mixed 0-1 linear program :

$$
\begin{array}{lll}
RL_{comp}: & \text{Min} & F_L(x) = \ c + L(x) + \displaystyle\sum_{i=1}^{n} h_i + \sum_{i=1}^{n} h'_i \\
& \text{s.t.} & \\
& & \displaystyle\sum_{i=1}^{n} a_{ki} x_i = b_k \ k = 1, \ldots, m \\
& & \displaystyle\sum_{i=1}^{n} a'_{\ell i} x_i \leq b'_\ell \ \ell = 1, \ldots, p \\
& & h_i \geq f_i(x) - \overline{f}_i (1 - x_i) \ i = 1, \ldots, n \\
& & h'_i \geq g_i(x) - \overline{g}_i x_i \ i = 1, \ldots, n \\
& & x \in \{0, 1\}^n \\
& & h_i \geq 0 \ h'_i \geq 0 \ i = 1, \ldots, n
\end{array}
$$

The inequalities on h_i and h'_i impose that, for any optimal solution (x, h, h') for RL_{comp}, if $x_i = 0$ then $h_i = 0$ and $h'_i = g_i(x)$; if $x_i = 1$ then $h_i = f_i(x)$ and $h'_i = 0$. Hence, variable h_i (resp. h'_i) is equal to $x_i f_i(x)$ (resp. $(1 - x_i) g_i(x)$). The following property states the equivalence of $Q01$ and RL_{comp}:

Proposition 1.1. *Problems $Q01$ and RL_{comp} are equivalent in the sense that, from any optimal solution of the one, we can build a solution to the other, with the same objective value.*

Program RL_{comp} has the n variables x_i and the $m + p$ constraints of the initial program $Q01$. In addition, it has $2n$ nonnegative variables h_i and h'_i, and $2n$ additional constraints.

Other linear compact reformulations are reported in literature. Most of them are inspired from [11]. Our positive compact linearization can also be considered as a variation of [11] and has the advantage of working on a first reformulation of the objective function as a constant c plus a quadratic function, nonnegative over $\overline{X}$. Hence, we can be sure that the lower bound computed by continuous relaxation of RL_{comp} is at least c. Observe also that this last property holds for any correctly chosen values of the upper bounds $\overline{f}_i$ et $\overline{g}_i$ on functions $f_i(x)$ and $g_i(x)$.

1.3. Quadratic convex reformulation [4]

A quadratic convex reformulation of $Q01$ consists in finding a quadratic convex function $F_c(x)$ that satisfies $F(x) = F_c(x)$ for any feasible solution x in X. Hence, the following problem RQ_{conv} is equivalent to $Q01$:

$$
\begin{aligned}
RQ_{conv}: \quad \text{Min} \quad & F_c(x) = q'_0 + \sum_{i=1}^{n} q'_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} q'_{ij} x_i x_j \\
\text{s.t.} \quad & \\
& \sum_{i=1}^{n} a_{ki} x_i = b_k \;\; k = 1, \ldots, m \\
& \sum_{i=1}^{n} a'_{\ell i} x_i \leq b'_\ell \;\; \ell = 1, \ldots, p \\
& x \in \{0,1\}^n
\end{aligned}
$$

Function F_c is convex if and only if its Hessian matrix $Q' = (q'_{ij})$ is positive semidefinite.

Such reformulation of $Q01$ is always possible. One can for example use equality $x_i^2 = x_i$ satisfied by any binary variable x_i in order to raise up the diagonal coefficients of matrix Q' by a large-enough constant. This diagonal perturbation can then immediately be balanced by the coefficients of the linear terms q'_i. Based on this idea, a simple convex reformulation was already proposed in [15] and consists in a perturbation of the diagonal terms of Q' by its smallest eigenvalue. For our example E, the smallest eigenvalue of the Hessian matrix equals -56.87 and $\phi(x)$ can be reformulated as $\phi_c(x) = \phi(x) + 56.87 \sum_{i=1}^{5} (x_i^2 - x_i)$ and the minimum of ϕ_c over $\overline{X}$ is equal to -119.31.

Problem RQ_{conv} has precisely the same number of variables and constraints as problem $Q01$.

1.4. A comparison criteria for reformulations

The linear and quadratic reformulations presented above have the following as a common property: solving their continuous relaxation is tractable in polynomial time, and better still is quick to compute. Indeed, the reformulated problem has, roughly speaking, the same size as the original problem $Q01$. Hence a natural way to solve the mixed-01 reformulated problems is to submit them to standard solvers whose exact solution procedures are based on branch-and-bound and continuous relaxation. Further, we know that the efficiency of a branch-and-bound algorithm is strongly dependant upon the quality of the bound at the root of the branch-and-bound tree, computed as the optimal value of the continuous relaxation of the reformulated problem. This provides us with a criteria for comparing reformulations. We consider that the quality of a reformulation is given by the tightness of its continuous relaxation. Within a given reformulation scheme, reformulation

$F1$ is better than reformulation $F2$ if the continuous relaxation of $F1$ leads to a tighter bound than the continuous relaxation of $F2$.

2. Building a compact linear reformulation by use of the bound of a linear relaxation

2.1. The RLT-relaxation

The relaxation we use here in known in literature under the name "RLT-level 1" [21], [22]. It consists in adding quadratic valid inequalities to $Q01$ before a classical linearization step. The valid inequalities are obtained by multiplying every equality by x_i and every inequality by x_i and $(1-x_i)$. Then, each product of two variables x_ix_j is replaced by a new real variable y_{ij}. Linearization constraints ensure the validity of the substitution. Finally, a continuous relaxation step provides the following linear program PL_p that is precisely called the RLT-relaxation:

$$PL_p: \quad \text{Min} \quad F(x,y) = \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} q_{ij} y_{ij}$$

s.t.

$$\sum_{i=1}^{n} a_{ki} x_i = b_k \;\; k = 1, \ldots, m$$

$$\sum_{i=1}^{n} a_{ki} y_{ij} = b_k x_j \;\; k = 1, \ldots, m; j = 1, \ldots, n$$

$$y_{ij} = y_{ji} \;\; i = 1, \ldots, n; j = 1, \ldots, n : j \neq i$$

$$y_{ii} = x_i \;\; i = 1, \ldots, n$$

$$\sum_{i=1}^{n} a'_{\ell i} x_i \leq b'_\ell \;\; \ell = 1, \ldots, p \tag{2}$$

$$\sum_{i=1}^{n} a'_{\ell i} y_{ij} \leq b'_\ell x_j \;\; \ell = 1, \ldots, p; j = 1, \ldots, n \tag{3}$$

$$\sum_{i=1}^{n} a'_{\ell i}(x_i - y_{ij}) \leq b'_\ell(1 - x_j) \;\; \ell = 1, \ldots, p; j = 1, \ldots, n \tag{4}$$

$$y_{ij} \leq x_i \;\; i = 1, \ldots, n; j = 1, \ldots, n, j \neq i \tag{5}$$

$$x_i + x_j - y_{ij} \leq 1 \;\; i = 1, \ldots, n; j = i+1, \ldots, n \tag{6}$$

$$x_i \leq 1 \tag{7}$$

$$x_i \geq 0 \;\; y_{ij} \geq 0$$

This relaxations can be viewed as a strengthening trick for the classical linearization recalled in the Introduction. The bound computed by solving problem PL_p is known to be quite efficient but rather slow to compute because of the important size of PL_p. This makes it difficult to incorporate these bounds into a branch-and-bound algorithm.

2.2. Using a dual solution of PL_p in order to build a positive linear compact reformulation

The following proposition shows how a reformulation of the objective function $F(x)$, in the form (1), can be obtained from any dual solution of the RLT-relaxation.

Proposition 2.1. For any dual feasible solution of PL_p, having objective value V, there exists $L(x)$, $f_i(x)$, and $g_i(x)$ such that, $\forall x \in X$

$$F(x) \;=\; V + L(x) + \sum_{i=1}^{n} x_i f_i(x) + \sum_{i=1}^{n} (1-x_i) g_i(x)$$

where $L(x)$, $f_i(x)$, and $g_i(x)$ are linear functions, nonnegative over the relaxed set $\overline{X}$.

Proof. (Use the Property in Appendix A) Let s_i (resp. s_{ij}) be the slack variables of the inequalities in the dual of PL_p associated to x_i (resp. y_{ij}). Given a dual feasible solution of PL_p, having objective value V, we take $\tilde{s}_i$ and $\tilde{s}_{ij}$, values of variables s_i and s_{ij}, and the dual variables values $\tilde{v}^1$ (resp. $\tilde{v}^2$, $\tilde{v}^3$, $\tilde{v}^4$, $\tilde{v}^5$, $\tilde{v}^6$) associated to inequalities (2) (resp. (3), (4), (5), (6), (7)). We get,

For any feasible solution (x, y) of PL_p :
$F(x,y) = V + \sum_{i=1}^{n} \tilde{s}_i x_i + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \tilde{s}_{ij} y_{ij} + \sum_{\ell=1}^{p} \tilde{v}^1_\ell (b'_\ell - \sum_{i=1}^{n} a'_{\ell i} x_i) + \sum_{\ell=1}^{p} \sum_{j=1}^{n} \tilde{v}^2_{\ell j} (b'_\ell x_j - \sum_{i=1}^{n} a'_{\ell i} y_{ij}) + \sum_{\ell=1}^{p} \sum_{j=1}^{n} \tilde{v}^3_{\ell j} (b'_\ell (1-x_j) - \sum_{i=1}^{n} a'_{\ell i}(x_i - y_{ij})) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \tilde{v}^4_{ij} (x_i - y_{ij}) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \tilde{v}^5_{ij} (1 + y_{ij} - x_i - x_j) + \sum_{i=1}^{n} \tilde{v}^6_i (1-x_i)$

For any $x \in X$, there exists a unique y such that (x, y) is a feasible solution to PL_p. This y is defined as $y_{ij} = x_i x_j$, and gives, for any $x \in X$:
$F(x) = V + \sum_{i=1}^{n} \tilde{s}_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{s}_{ij} x_i x_j + \sum_{\ell=1}^{p} \tilde{v}^1_\ell (b'_\ell - \sum_{i=1}^{n} a'_{\ell i} x_i) + \sum_{\ell=1}^{p} \sum_{j=1}^{n} \tilde{v}^2_{\ell j} x_j (b'_\ell - \sum_{i=1}^{n} a'_{\ell i} x_i) + \sum_{\ell=1}^{p} \sum_{j=1}^{n} \tilde{v}^3_{\ell j} (1-x_j)(b'_\ell - \sum_{i=1}^{n} a'_{\ell i} x_i) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \tilde{v}^4_{ij} x_i (1-x_j) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \tilde{v}^5_{ij} (1-x_i)(1-x_j) + \sum_{i=1}^{n} \tilde{v}^6_i (1-x_i)$

Let:

$L(x) = \sum_{i=1}^{n} \tilde{s}_i x_i + \sum_{i=1}^{n} \tilde{v}_i^6 (1 - x_i) + \sum_{\ell=1}^{p} \tilde{v}_\ell^1 (b'_\ell - \sum_{i=1}^{n} a'_{\ell i} x_i)$

$f_i(x) = \sum_{j=1}^{n} \tilde{s}_{ij} x_j + \sum_{j=1, j \neq i}^{n} \tilde{v}_{ij}^4 (1 - x_j) + \sum_{\ell=1}^{p} \tilde{v}_{\ell i}^2 (b'_\ell - \sum_{j=1}^{n} a'_{\ell j} x_j)$

$g_i(x) = \sum_{j=i+1}^{n} \tilde{v}_{ij}^5 (1 - x_j) + \sum_{\ell=1}^{p} \tilde{v}_{\ell i}^3 (b'_\ell - \sum_{j=1}^{n} a'_{\ell j} x_j)$

observing that, for any $x \in \overline{X}$, the above functions are nonnegative, we obtain the reformulation of $F(x)$ as (1), i.e.:

$$\forall x \in X, \; F(x) = V + L(x) + \sum_{i=1}^{n} x_i f_i(x) + \sum_{i=1}^{n} (1 - x_i) g_i(x)$$

□

Finally, we determine the upper bounds : $\overline{f}_i = Max \{f_i(x) : x \in X\}$ and $\overline{g}_i = Max \{g_i(x) : x \in X\}$ in order to get a positive linear compact reformulation associated to the given dual solution of PL_p.

Corollary 2.2. For any dual solution of PL_p having objective value V, we can build a positive linear compact reformulation which continuous relaxation value is at least V. The best value of V is obviously obtained from an optimal solution.

Let us observe that Adams et al. [1] build a different compact linearization, more directly inspired from [11] and whose continuous relaxation is equal to the optimal value of PL_p. An additional advantage of Corollary 2.2 is that it allows to reformulate the objective function F as the optimal value plus a nonnegative function over $\overline{X}$. This reformulation can be used for other exact or approximate solution approaches.

Example :

Let us reformulate the objective function $\phi(x)$ of E once the RLT-relaxation has been solved:

$$\begin{aligned}
\phi(x) = \quad & -67.52 \\
& +x_1 (109.83x_4 + 30.07x_5) \\
& +x_2 (129.90x_4 + 3.52 (1 - x_3)) \\
& +x_3 (1.26 [-2 - (-x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5)]) \\
& +x_4 (6.95 [-2 - (-x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5)]) \\
& +x_5 (52.34 (1 - x_3) + 11.22 [-2 - (-x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5)]) \\
& + (1 - x_2) (0.74 [-2 - (-x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5)])
\end{aligned}$$

We can now define functions $f_i(x)$ and $g_i(x)$,

$$\begin{aligned}
f_1(x) = &\quad 109.83x_4 + 30.07x_5\\
f_2(x) = &\quad 129.90x_4 + 3.52(1 - x_3)\\
f_3(x) = &\quad 1.26(-2 - (-x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5))\\
f_4(x) = &\quad 6.95(-2 - (-x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5))\\
f_5(x) = &\quad 52.34(1 - x_3) + 11.22(-2 - (-x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5))\\
g_2(x) = &\quad 0.74(-2 - (-x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5))
\end{aligned}$$

deduce the upper bounds,

$$\overline{f}_1 = 139.90,\ \overline{f}_2 = 133.42,\ \overline{f}_3 = 7.56,\ \overline{f}_4 = 41.7,\ \overline{f}_5 = 67.32,\ \overline{g}_2 = 4.44.$$

and finally, build the following program RL_{comp_E}.

$$\begin{array}{lll}
RL_{comp_E}: & \text{Min} & \phi(x) = -67.52 + \sum_{i=1}^{5} h_i + h'_2\\
& \text{s.t.} & \\
& & -x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5 \leq -2\\
& & x_1 + x_2 + x_4 + x_5 = 2\\
& & h_i \geq f_i(x) - \overline{f}_i(1 - x_i)\ i = 1, \ldots, 5\\
& & h'_2 \geq g_2(x) - \overline{g}_2 x_2\\
& & x \in \{0,1\}^n\\
& & h_i \geq 0\ i = 1, \ldots, 5\\
& & h'_2 \geq 0
\end{array}$$

The optimal value of the continuous relaxation of RL_{comp_E} is -67.52.

In Appendix B we apply the linear compact reformulation of Glover [11] to the initial problem E. Its continuous relaxation value is -110.78 that is significantly lower than the continuous relaxation value of our positive linear compact reformulation.

3. Building a quadratic convex reformulation by use of the bound of a semidefinite relaxation

3.1. A semidefinite relaxation of $Q01$

Let us consider again problem $Q01$. We multiply every equality constraint $\sum_{i=1}^{n} a_{ki}x_i = b_k$ by x_i and then replace the products $x_i x_j$ by variables X_{ij}, we obtain the following equivalent problem:

$$
\begin{array}{lll}
\text{Min} & F(x) = \sum\limits_{i=1}^{n} q_i x_i & + \sum\limits_{i=1}^{n} \sum\limits_{j=i, j\neq i}^{n} q_{ij} X_{ij} \\
\text{s.t.} & & \\
& \sum\limits_{i=1}^{n} a_{ki} x_i = b_k & k = 1, \ldots, m \\
& \sum\limits_{i=1}^{n} a_{ki} X_{ij} = b_k x_j & k = 1, \ldots, m; j = 1, \ldots, n \\
& \sum\limits_{i=1}^{n} a'_{\ell i} x_i \leq b'_\ell & \ell = 1, \ldots, p \\
& X_{ij} = x_i x_j & i = 1, \ldots, n; j = 1, \ldots, n \\
& x \in \{0,1\}^n &
\end{array}
$$

The semidefinite relaxation consists in replacing the set of constraints $X_{ij} = x_i x_j$ by the linear matrix inequality $X - xx^t \succeq 0$. By the Lemma of Schur, $X - xx^t \succeq 0$ is equivalent to $\begin{pmatrix} 1 & x^t \\ x & X \end{pmatrix} \succeq 0$.

The obtained SDP relaxation is the following:

$$
\begin{array}{lll}
SDP: & \text{Min} & \sum\limits_{i=1}^{n} q_i x_i + \sum\limits_{i=1}^{n} \sum\limits_{j=i, j\neq i}^{n} q_{ij} X_{ij} \\
& \text{s.t.} & \\
& & \sum\limits_{i=1}^{n} a_{ki} x_i = b_k \ \ k = 1, \ldots, m \\
& & \sum\limits_{j=1}^{n} a_{kj} X_{ij} = b_k x_j \ \ k = 1, \ldots, m; j = 1, \ldots, n \qquad (8) \\
& & \sum\limits_{i=1}^{n} a'_{\ell i} x_i \leq b'_\ell \ \ \ell = 1, \ldots, p \\
& & X_{ii} = x_i \ \ i = 1, \ldots, n \qquad (9) \\
& & \begin{pmatrix} 1 & x^t \\ x & X \end{pmatrix} \succeq 0 \\
& & x \in \mathbb{R}^n, \ X \in \mathbb{R}^{n \times n}
\end{array}
$$

3.2. Using an optimal solution to SDP in order to build a quadratic reformulation

The QCR method consists in reformulating problem $Q01$ by adding a combination of quadratic functions that vanish on the feasible solution set X. For any $\alpha \in \mathbb{R}^{m \times n}$ and $u \in \mathbb{R}^n$, let us consider the following quadratic function:

$$F_{\alpha,u}(x) = \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n-1} \sum_{j=i,j\neq i}^{n} q_{ij} x_i x_j + \sum_{k=1}^{m} \left(\sum_{i=1}^{n} \alpha_{ki} x_i \right) \left(\sum_{j=1}^{n} a_{kj} x_j - b_k \right) + \sum_{i=1}^{n} u_i \left(x_i^2 - x_i \right)$$

Function $F_{\alpha,u}$ is a reformulation of F since for all $x \in X$, $F_{\alpha,u}(x)$ is equal to $F(x)$. We focus our interest on reformulations of F by $F_{\alpha,u}$ where $F_{\alpha,u}(x)$ is convex over $\mathbb{R}^n$. Once $F(x)$ is transformed into a convex function, the reformulated problem can be solved by mixed integer convex quadratic programming. Our objective now is to find values for parameters $\alpha \in \mathbb{R}^{m\times n}$ and $u \in \mathbb{R}^n$ such that $F_{\alpha,u}(x)$ is convex on the one hand and the optimal value of the continuous relaxation of the reformulated problem is maximized. It was proved in [4] that solving the above semidefinite relaxation SDP allows to deduce optimal values for parameters α and u. More precisely, the optimal values u_i^* of u_i $(i = 1 \ldots, n)$ are given by the optimal values of the dual variables associated to constraints (9) and the optimal values α_{ik}^* of α_{ik} $(i = 1, \ldots, n; k = 1, \ldots, m)$ are given by the optimal values of the dual variables associated to constraints (8).

The obtained quadratic convex reformulation is then:

$$\begin{array}{lll} RQ_{conv}: & \text{Min} & F_{\alpha^*,u^*}(x) \\ & \text{s.t.} & \\ & & \displaystyle\sum_{i=1}^{n} a_{ki} x_i = b_k \;\; k = 1, \ldots, m \\ & & \displaystyle\sum_{i=1}^{n} a'_{\ell i} x_i \leq b'_\ell \;\; \ell = 1, \ldots, p \\ & & x \in \{0,1\}^n \end{array}$$

It is proved in [4] that v_{SDP}, the optimal value to SDP is equal to the optimal value of the continuous relaxation of RQ_{conv}.

Remark 3.1. In a similar way as in the linear reformulation, we reformulate the objective function $F(x)$ as a constant plus a convex quadratic function that is non-negative for any feasible solution $x \in \overline{X}$. Indeed, let $g(x) = F_{\alpha^*,u^*}(x) - v_{SDP}$, we have $F(x) = v_{SDP} + g(x)$ and $g(x) \geq 0 \; \forall x \in \overline{X}$.

Example:
The SDP relaxation of our example E is:

$$
\begin{array}{lll}
SDP_E: & \text{Min} & -9x_1 - 7x_2 + 2x_3 + 23x_4 + 12x_5 - 48X_{12} + 4X_{13} + 36X_{14} \\
& & -24X_{15} - 7X_{23} + 36X_{24} - 84X_{25} + 40X_{34} + 4X_{35} - 88X_{45} \\
& \text{s.t.} & \\
& & x_1 - 2x_2 + 5x_3 + 2x_4 - 2x_5 \geq 2 \\
& & x_1 + x_2 + x_4 + x_5 = 2 \\
& & X_{12} + X_{14} + X_{15} = x_1 \qquad \leftarrow \alpha_1^* \\
& & X_{12} + X_{24} + X_{25} = x_2 \qquad \leftarrow \alpha_2^* \\
& & X_{13} + X_{23} + X_{34} + X_{35} = 2x_3 \qquad \leftarrow \alpha_3^* \\
& & X_{14} + X_{24} + X_{45} = x_4 \qquad \leftarrow \alpha_4^* \\
& & X_{15} + X_{25} + X_{45} = x_5 \qquad \leftarrow \alpha_5^* \\
& & X_{11} = x_1 \qquad \leftarrow u_1^* \\
& & X_{22} = x_2 \qquad \leftarrow u_2^* \\
& & X_{33} = x_3 \qquad \leftarrow u_3^* \\
& & X_{44} = x_4 \qquad \leftarrow u_4^* \\
& & X_{55} = x_5 \qquad \leftarrow u_5^* \\
& & \begin{pmatrix} 1 & x^t \\ x & X \end{pmatrix} \succeq 0 \\
& & x \in \mathbb{R}^n, \ X \in \mathbb{R}^{n \times n}
\end{array}
$$

The optimal solution value of SDP_E equals -81.32. It is also the optimal solution value of the continuous relaxation of the reformulated problem RQ_{conv_E}. Parameters u^* and α^* that allow to build RQ_{conv_E} are obtained from the optimal solution of SDP_E:

$$
\begin{array}{lll}
RQ_{conv_E}: & \min & \phi(x) + (8066.79x_1 + 8076.64x_2 - 8.79x_3 + 8040.33x_4 + 8088.93x_5) \\
& & (x_1 + x_2 + x_4 + x_5 - 2) + 8.63(x_1^2 - x_1) + 0.17(x_2^2 - x_2) \\
& & +12.44(x_3^2 - x_3) + 66.36(x_4^2 - x_4) - 7.70(x_5^2 - x_5) \\
& \text{s.t.} & \\
& & x_1 - 2x_2 + 5x_3 + 2x_4 - 2x_5 \geq 2 \\
& & x_1 + x_2 + x_4 + x_5 = 2 \\
& & \\
& & x \in \{0,1\}^n
\end{array}
$$

4. Conclusion

Nowadays, efficient mathematical programming solvers are available to solve mixed-integer linear or convex quadratic problems. Consequently, try to use them for the general 0-1 quadratic problem is attractive. For that, a preprocessing phase is necessary in order to reformulate the initial problem into a tractable form. The efficiency of this exact solution approach strongly depends on the quality of the bound given by the continuous relaxation of the reformulation and also on the size of this reformulation. For both classes of solvers - mixed-integer linear or mixed-integer convex quadratic - we have proposed a concise reformulation technique that incorporates, in a sense, the optimal values of tight linear or positive semi-definite relaxations. Experimental results presented in [1], [3], [4] and [5] show the potential of such an approach. Future research topics include obtaining reformulations based on even tigher relaxations.

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Appendix A

Let P be the following linear program:

$$\begin{aligned}
\text{P}: \quad \text{Min} \quad & f(x) = \sum_{i=1}^{n} c_i x_i \\
\text{s.t.} \quad & \\
& \sum_{i=1}^{n} a_{ki} x_i = b_k \; k = 1, \ldots, m \\
& \sum_{i=1}^{n} a'_{\ell i} x_i \leq b'_\ell \; \ell = 1, \ldots, p \\
& x_i \geq 0
\end{aligned}$$

and let D be its dual problem :

$$\begin{aligned}
\text{D}: \quad \text{Max} \quad & g(u,v) = \sum_{k=1}^{m} (-b_k) u_k + \sum_{\ell=1}^{p} (-b'_\ell) v_\ell \\
\text{s.t.} \quad & \\
& \sum_{k=1}^{m} (-a_{ki}) u_k + \sum_{\ell=1}^{p} (-a'_{\ell i}) v_\ell \leq c_i \; i = 1, \ldots, n \qquad (10) \\
& u_k \in R \; ; \; v_\ell \geq 0
\end{aligned}$$

Proposition

Let s_i, $i = 1, \ldots, n$ be the non-negative slack variables associated to constraints (10) and let $(\tilde{u}, \tilde{v}, \tilde{s})$ be a feasible solution for the dual problem D. Then, for any feasible solution x of the primal P, we have:

$f(x) = g(\tilde{u}, \tilde{v}) + \sum_{\ell=1}^{p} \tilde{v}_\ell (b'_\ell - \sum_{i=1}^{n} a'_{\ell i} x_i) + \sum_{i=1}^{n} \tilde{s}_i x_i$

Proof. By definition of the slack variables s_i: $c_i = \sum_{k=1}^{m} (-a_{ki}) \tilde{u}_k + \sum_{\ell=1}^{p} (-a'_{\ell i}) \tilde{v}_\ell + \tilde{s}_i$.

We can now use the last identity in the expression of $f(x)$.

We get, for any vector x:

$f(x) = \sum_{i=1}^{n} (\sum_{k=1}^{m} (-a_{ki}) \tilde{u}_k + \sum_{\ell=1}^{p} (-a'_{\ell i}) \tilde{v}_\ell + \tilde{s}_i) x_i$

$= \sum_{k=1}^{m} \tilde{u}_k (- \sum_{i=1}^{n} a_{ki} x_i) + \sum_{\ell=1}^{p} \tilde{v}_\ell (- \sum_{i=1}^{n} a'_{\ell i} x_i) + \sum_{i=1}^{n} \tilde{s}_i x_i$

$= \sum_{k=1}^{m} (-b_k) \tilde{u}_k + \sum_{\ell=1}^{p} (-b'_\ell) \tilde{v}_\ell + \sum_{k=1}^{m} \tilde{u}_k (b_k - \sum_{i=1}^{n} a_{ki} x_i) + \sum_{\ell=1}^{p} \tilde{v}_\ell (b'_\ell - \sum_{i=1}^{n} a'_{\ell i} x_i) + \sum_{i=1}^{n} \tilde{s}_i x_i$

$= g(\tilde{u}, \tilde{v}) + \sum_{k=1}^{m} \tilde{u}_k (b_k - \sum_{i=1}^{n} a_{ki} x_i) + \sum_{\ell=1}^{p} \tilde{v}_\ell (b'_\ell - \sum_{i=1}^{n} a'_{\ell i} x_i) + \sum_{i=1}^{n} \tilde{s}_i x_i$

now, for any feasible solution x to P, we have $b_k - \sum_{i=1}^{n} a_{ki}x_i = 0$, and then:

$$f(x) = g(\tilde{u}, \tilde{v}) + \sum_{\ell=1}^{p} \tilde{v}_\ell(b'_\ell - \sum_{i=1}^{n} a'_{\ell i}x_i) + \sum_{i=1}^{n} \tilde{s}_i x_i$$

Observe that for any feasible solution x, $b'_k - \sum_{i=1}^{n} a'_{\ell i}x_i \geq 0$, $x_i \geq 0$ and coefficients $\tilde{v}_\ell$ et $\tilde{s}_i$ are ≥ 0. $\square$

Appendix B

Recall our example E presented all along the paper:

$$
\begin{array}{lll}
E: & \text{Min} & \phi(x) = -9x_1 - 7x_2 + 2x_3 + 23x_4 + 12x_5 - 48x_1x_2 + 4x_1x_3 + 36x_1x_4 \\
 & & -24x_1x_5 - 7x_2x_3 + 36x_2x_4 - 84x_2x_5 + 40x_3x_4 + 4x_3x_5 - 88x_4x_5 \\
 & \text{s.t.} & \\
 & & x_1 - 2x_2 + 5x_3 + 2x_4 - 2x_5 \geq 2 \\
 & & x_1 + x_2 + x_4 + x_5 = 2 \\
 & & x_1, x_2, x_3, x_4, x_5 \in \{0,1\}
\end{array}
$$

Let us apply the classical linearization to it:

$$
\begin{array}{ll}
\text{Min} & -9x_1 - 7x_2 + 2x_3 + 23x_4 + 12x_5 - 48y_{12} + 4y_{13} + 36y_{14} \\
 & -24y_{15} - 7y_{23} + 36y_{24} - 84y_{25} + 40y_{34} + 4y_{35} - 88y_{45} \\
\text{s.t.} & \\
 & -x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5 \leq -2 \\
 & x_1 + x_2 + x_4 + x_5 = 2 \\
 & y_{ij} \leq x_i \ i < j; c_{ij} < 0 \\
 & y_{ij} \leq x_j \ i < j; c_{ij} < 0 \\
 & 1 - x_i - x_j + y_{ij} \geq 0 \ i < j; c_{ij} > 0 \\
 & y_{ij} \geq 0 \ i < j; c_{ij} > 0 \\
 & x_1, \ldots, x_5 \in \{0,1\}
\end{array}
$$

The optimal solution value to LR_E equals -115.

If one applies the compact linearization of Glover [11] to E (see Section 1.1), we get the following linear program :

$$
\begin{array}{ll}
\text{Min} & -9x_1 - 7x_2 + 2x_3 + 23x_4 + 12x_5 + z_1 + z_2 + z_3 + z_4 + z_5 \\
\text{s.t.} & \\
 & -x_1 + 2x_2 - 5x_3 - 2x_4 + 2x_5 \leq -2 \\
 & x_1 + x_2 + x_4 + x_5 = 2 \\
 & z_1 \geq -30x_1 \\
 & z_1 \geq -24x_2 + 2x_3 + 18x_4 - 12x_5 - 20(1 - x_1) \\
 & z_2 \geq -69.5x_2 \\
 & z_2 \geq -24x_1 - 3.5x_3 + 18x_4 - 42x_5 - 16.6(1 - x_2) \\
 & z_3 \geq -1.5x_3 \\
 & z_3 \geq 2x_1 - 3.5x_2 + 20x_4 + 2x_5 - 22(1 - x_3) \\
 & z_4 \geq -36x_4 \\
 & z_4 \geq 18x_1 + 18x_2 + 20x_3 - 44x_5 - 56(1 - x_4) \\
 & z_5 \geq -85.2x_5 \\
 & z_5 \geq -12x_1 - 42x_2 + 2x_3 - 44x_4 + 10(1 - x_5) \\
 & x_1, \ldots, x_5 \in \{0,1\}
\end{array}
$$

which optimal value of the continuous relaxation equals -110.78.

Below, we summarize the optimal values of different relaxations associated with example E:

Opt. value	Classical lin.	Compact lin. of Glover	RLT-1	**Our positive compact lin.**	Eigenvalue reform.	**Our quad. conv. reformulation**
-65	-115	-110.78	-67.52	-67.52	-119.31	-81.39