Singular trajectories of driftless and control-affine systems

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***Abstract*— We establish generic properties for singular trajectories, first for driftless, and then for control-affine systems, extending results of [17], [16]. We show that, generically – for the Whitney topology – nontrivial singular trajectories are of minimal order and of corank one. As a consequence, if the number of vector fields of the system is greater than or equal to 3, then there exists generically no singular minimizing trajectory.**

I. INTRODUCTION

Let M be a smooth (i.e. C^∞) manifold of dimension n, $x_0 \in M$ and T a positive real number. Consider the control system (Σ) defined on M by

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where the mapping f, defined on $M \times U$, is smooth, and U is an open subset of $\mathbb{R}^m$, $m \geq 1$. A control $u \in L^\infty([0,T], U)$ is said to be *admissible* if the trajectory $x(\cdot, x_0, u)$ of (Σ) solution of (1), associated to the control u, and such that $x(0, x_0, u) = x_0$, is well defined on $[0,T]$. Let $\mathcal{U}$ denote the set of admissible controls; it is an open subset of $L^\infty([0,T], U)$. Define on $\mathcal{U}$ the *end-point mapping* by

$$E_{x_0,T}(u) := x(T, x_0, u).$$

With the assumptions made previously, $E_{x_0,T}$ is a smooth map.

Definition 1.1: A control $u \in \mathcal{U}$ is said to be *singular* on $[0,T]$ if u is a critical point of the end-point mapping $E_{x_0,T}$, i.e. its differential at u, $DE_{x_0,T}(u)$, is not surjective. A trajectory $x(t, x_0, u)$ is said to be *singular* on $[0,T]$ if u is singular and of corank one if the codimension in T_xM of the range of $E_{x_0,T}(u)$ is equal to one.

Let $x \in M$. Consider the following optimal control problem: among all the trajectories of (Σ) steering x_0 to x, determine a trajectory minimizing the *cost*

$$C_T(u) = \int_0^T f^0(x, u)dt,$$

where $f^0 : M \times U \to \mathbb{R}$ is smooth. Then the *value function* S_T at the point x is defined as the infimum over the costs of the trajectories of (Σ) steering x_0 to x in time T. The Pontryagin Maximum Principle (see [26]) provides the following necessary condition for optimality. If the trajectory $x(\cdot)$ associated to $u \in \mathcal{U}$ is optimal on $[0,T]$, then there exists a nonzero pair $(\lambda(\cdot), \lambda^0)$, where λ^0 is a nonpositive real number and $\lambda(\cdot)$ is an absolutely continuous covector function on $[0,T]$ called the *adjoint vector*, such that $\lambda(t) \in T^*_{x(t)}M$ and the following equations are satisfied for almost all $t \in [0,T]$:

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H}{\partial \lambda}(x(t), \lambda(t), \lambda^0, u(t)), \\ \dot{\lambda}(t) &= -\frac{\partial H}{\partial x}(x(t), \lambda(t), \lambda^0, u(t)), \\ &\frac{\partial H}{\partial u}(x(t), \lambda(t), \lambda^0, u(t)) = 0, \end{aligned} \tag{2}$$

where

$$H(x, \lambda, \lambda^0, u) := \langle \lambda, f(x,u) \rangle + \lambda^0 f^0(x,u)$$

is the *hamiltonian* of the system.

An *extremal* is a 4-tuple $(x(\cdot), \lambda(\cdot), \lambda^0, u(\cdot))$ solution of the system of equations (2). The extremal is said to be *normal* if $\lambda^0 \neq 0$ and *abnormal* if $\lambda^0 = 0$.

In particular a trajectory is singular if and only if it is the projection of an abnormal extremal. A singular trajectory is said to be *strictly abnormal* if it is not the projection of a normal extremal.

Note that a singular trajectory is of corank one if and only if it admits a unique abnormal extremal lift. It is strictly abnormal and of corank one if and only if it admits a unique extremal lift which is abnormal.

Singular trajectories play a major role in optimal control theory. They appear as singularities in the set of solutions of a control system; as a result, they are not dependent on the specific minimization problem. In particular, the consideration of abnormal extremals with null hamiltonian is crucial. The issue of such singular trajectories was already well-known in the classical theory of calculus of variations (see for instance [10]) and proved to be a major focus, during the forties, when the whole issue eventually developed into optimal control theory. Their role in the nonlinear control theory is reviewed in [11] and [29]. For a long time, there had been a suspicion that such minimizing singular trajectories actually existed: Carathéodory and Hilbert were already familiar with the rigidity phenomenon (see [31]), while Bismut provides clear evidence of their existence in [9]. Attempts have been made, however, to ignore singular trajectories, on the (false) grounds that they are never optimal. In [23], Montgomery offers both an example of a minimizing strictly abnormal extremal in sub-Riemannian geometry and a list of false demonstrations (by several authors) allegedly showing that an abnormal extremal cannot be optimal. These findings gave

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impetus to wide-ranging research with view to identifying the role of abnormal extremals in sub-Riemannian geometry.

The optimality status of singular trajectories was chiefly investigated by [13], [30] in relation to control-affine systems, by [2], [22], [30] regarding driftless systems and by [4], [27] more generally, as these singularities are addressed in a generic context. This research leads to results showing the rigidity (see also [15]) of singular trajectories, which means that they are locally isolated from trajectories having the same boundary conditions; thus they are locally *optimal*.

Besides, the existence of minimizing singular trajectories is closely related to the regularity of the value function, see [29]. First, in terms of sub-Riemannian geometry, in [5], [6], the authors are showing that, in the absence of a nontrivial minimizing singular trajectory, the sub-Riemannian distance $d_{SR}(0,.)$ to zero is subanalytic in a pointed neighborhood of zero and that, consequently, the spheres with small positive radius are subanalytic. In [7], the authors are showing that this situation is valid for a dense set of distributions (for the Whitney topology) of rank superior or equal to three. In terms of control-affine systems, it is proved in [28] that the absence of a minimizing singular trajectory implies the subanalyticity of the value function.

In this paper, we investigate generic properties for singular trajectories, both for driftless and for control-affine systems. We first adapt techniques and ideas of [17] to driftless systems, and then, extend them to control-affine systems. The results we obtain generalize those of [22] and [14], which are dealing respectively with driftless systems with two vector fields and single-input control-affine systems; we also improve some results of [7] and finally we list several consequences of these properties.

II. SINGULAR TRAJECTORIES FOR DRIFTLESS CONTROL SYSTEMS

A. Definitions

Let M be a smooth, n-dimensional manifold, and T be a positive real number. Consider the driftless control system

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{3}$$

where $(f_1, \ldots, f_m)$ is an m-tuple of smooth vector fields on M, and the set of admissible controls $u = (u_1, \ldots, u_m)$ is an open subset of $L^\infty([0,T], U)$.

Note that the set of trajectories of (3) is not in general a manifold: its singularities correspond exactly to singular trajectories.

Following the Pontryagin Maximum Principle [26], every singular trajectory $x(\cdot)$ is the projection of an abnormal extremal. Let $\lambda(\cdot)$ be an adjoint vector associated to $x(\cdot)$.

For every $t \in [0,T]$ and $i, j \in \{1, \ldots, m\}$, we define

$$h_i(t) := \langle \lambda(t), f_i(x(t)) \rangle,$$

$$h_{ij}(t) := \langle \lambda(t), [f_i, f_j](x(t)) \rangle,$$

where $[\cdot,\cdot]$ stands for the Lie bracket between vector fields. Hence, along abnormal extremals, the following relations hold:

$$h_i \equiv 0, \quad i = 1, \ldots, m. \tag{4}$$

By differentiating (4), one gets for $i = 1, \ldots, m$,

$$\sum_{j=1}^{m} h_{ij}(t) u_j(t) = 0, \quad \text{for almost all } t \in [0,T]. \tag{5}$$

Definition 2.1: Along an abnormal extremal $(x(\cdot), \lambda(\cdot), 0, u(\cdot))$, the *Goh matrix* at time $t \in [0,T]$ is the $m \times m$ skew-symmetric matrix given by

$$G(t) := \big(h_{ij}(t)\big)_{1 \le i,j \le m}. \tag{6}$$

It is clear that the rank $r(t)$ of $G(t)$ is even. If moreover m is even, the determinant of $G(t)$ is the square of a polynomial $P(t)$ in the $h_{ij}(t)$ with degree $m/2$, called the *Pfaffian*. Along the abnormal extremal, there holds $P(t) = 0$, and, after differentiation, one gets

$$\sum_{i=1}^{m} u_j(t) \{P, h_j\}(t) = 0. \tag{7}$$

Define the $(m+1) \times m$ matrix $\widetilde{G}(t)$ as $G(t)$ augmented with the row $(\{P, h_j\}(t))_{1 \le j \le m}$.

As a consequence of (5), one gets that, along an abnormal extremal, at almost all $t \in [0,T]$, the corresponding singular control $u = (u_1, \ldots, u_m)$ is in the kernel of the Goh matrix, i.e.

$$G(t) u(t) = 0.$$

If m is even, using (7) there holds moreover

$$\widetilde{G}(t) u(t) = 0.$$

Thus, if m is odd and $r(t) = m - 1$ (resp. if m is even and $\tilde{r}(t) = m - 1$), one can deduce from that relation an expression for $u(t)$, up to the sign. This fact motivates the following definition.

Definition 2.2: With the notations above, if m is odd (resp. even), a singular trajectory is said to be *of minimal order* if it admits an abnormal extremal lift along which the set of times $t \in [0,T]$ where $r(t) = m - 1$ (resp. $\tilde{r}(t) = m - 1$) is of full Lebesgue measure in $[0,T]$.

Remark 1: This set is moreover open. Note that this definition is stronger than the corresponding one of [14], in which the set is assumed to be dense only.

On the opposite, for arbitrary m, a singular trajectory is said to be a *Goh trajectory* if it admits an abnormal extremal lift along which the Goh matrix is identically equal to zero.

B. Main result

For singular trajectories of driftless systems, we have the following result, which follows readily from [17].

Theorem 2.3: Let m be a positive integer such that $2 \le m < n$ and let $\mathcal{F}_m$ be the set of m-tuples of independent vector fields on M endowed with the C^∞ Whitney topology. There exists an open set O_m dense in $\mathcal{F}_m$ so that, for every m-tuple $(f_1, \dots, f_m)$ in O_m, every nontrivial singular trajectory of (3) is of minimal order and of corank one.

In addition, for every integer k, the set O_m can be chosen so that its complement has codimension greater than k. Let O_m^∞ be the intersection over all k of the latter subsets; then O_m^∞ shares the same properties as the set O_m with the following differences: O_m^∞ may fail to be open, but its complement has infinite codimension.

Corollary 2.4: With the notations of Theorem 2.3, if $m \geq 3$ then there exists an open set O_m dense in $\mathcal{F}_m$ so that, for every m-tuple $(f_1, \dots, f_m)$ in O_m, the system (3) has no nontrivial Goh singular trajectory.

Remark 2: If m is odd, there exists an open dense subset of M such that through every point of this subset passes a nontrivial singular trajectory (see also [24]).

III. SINGULAR TRAJECTORIES FOR CONTROL-AFFINE SYSTEMS

A. Definitions

Let M be a smooth, n-dimensional manifold and let T be a positive real number. Consider the control-affine system given by

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \qquad (8)$$

where $(f_0, \dots, f_m)$ is an $(m+1)$-tuple of smooth vector fields on M and the set of admissible controls $u = (u_1, \dots, u_m)$ is an open subset of $L^\infty([0,T], U)$.

Recall that a singular trajectory $x(\cdot)$ is the projection of an abnormal extremal $(x(\cdot), \lambda(\cdot))$. Similarly to the previous section, we define, for $t \in [0,T]$ and $i, j \in \{0, \dots, m\}$,

$$h_i(t) := \langle \lambda(t), f_i(x(t)) \rangle,$$
$$h_{ij}(t) := \langle \lambda(t), [f_i, f_j](x(t)) \rangle.$$

Along an abnormal extremal, we have for all $t \in [0,T]$,

$$h_0(t) = \text{constant}, \quad h_i(t) = 0, \quad i = 1, \dots, m. \qquad (9)$$

Differentiating (9), one gets for $i \in \{0, \dots, m\}$,

$$h_{i0}(t) + \sum_{j=1}^{m} h_{ij}(t) u_j(t) = 0. \qquad (10)$$

Similarly to Definition 2.1, we set the following.

Definition 3.1: Along an abnormal extremal $(x(\cdot), \lambda(\cdot), u(\cdot))$ of the system (8), the *Goh matrix* $G(t)$ (resp. the *augmented Goh matrix* $\overline{G}(t)$) at time $t \in [0,T]$ is the $m \times m$ skew-symmetric matrix given by

$$G(t) := \big(h_{ij}(t)\big)_{1 \leq i,j \leq m} \qquad (11)$$

(resp. $\overline{G}(t) := \big(h_{ij}(t)\big)_{0 \leq i,j \leq m}$).
If moreover m is odd, the determinant of $\overline{G}(t)$ is the square of a polynomial $\overline{P}(t)$ in the $h_{ij}(t)$ with degree $(m+1)/2$, called the *Pfaffian*. Along the extremal, $\overline{P}(t) = 0$, and, after differentiation, one gets

$$\{\overline{P}, h_0\}(t) + \sum_{i=1}^{m} u_j(t) \{\overline{P}, h_j\}(t) = 0. \qquad (12)$$

Define the $(m+2) \times (m+1)$ matrix $\widetilde{G}(t)$ as $\overline{G}(t)$ augmented with the row $(\{\overline{P}, h_j\}(t))_{0 \leq j \leq m}$.

If m is even and the Goh matrix $G(t)$ at time t is invertible (resp. if m is odd and $\widetilde{G}(t)$ is of rank m), then, as done in the driftless case, we can deduce from Equations (10) and (12) the singular control $u(t)$. Let us then set the following definition.

Definition 3.2: If m is even (resp. odd), a singular trajectory is said to be *of minimal order* if it admits an abnormal extremal lift along which the set of times $t \in [0,T]$ where rank $G(t) = m$ (resp. rank $\widetilde{G}(t) = m$) is of full Lebesgue measure in $[0,T]$.

On the opposite, for arbitrary m, a singular trajectory is said to be a *Goh trajectory* if it admits an abnormal extremal lift along which the Goh matrix is identically equal to 0.

B. Main result

Theorem 3.3: Let m be a positive integer with $1 \leq m < n$ and $\mathcal{F}_{m+1}$ be the set of $(m+1)$-tuples of linearly independent smooth vector fields on M, endowed with the C^∞ Whitney topology. There exists an open set O_{m+1} dense in $\mathcal{F}_{m+1}$ so that, for all $(m+1)$-tuple $(f_0, \dots, f_m)$ of O_{m+1}, every singular trajectory of the associated control-affine system

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)),$$

is of minimal order and of corank one. In addition, the complementary of O_{m+1} in $\mathcal{F}_{m+1}$ is of infinite codimension.

Corollary 3.4: With the notations of Theorem 3.3 and if $m \geq 2$, there exists an open set O_{m+1} dense in $\mathcal{F}_{m+1}$ so that every control-affine system defined with an $(m+1)$-tuple of O_{m+1} does not admit Goh singular trajectories.

We next deduce another corollary but before doing so, we need the following definition.

Definition 3.5: Let $(f_0, \dots, f_m)$ be an $(m+1)$-tuple of smooth vector fields on M and its associated control-affine system be defined by (8). A trajectory $x(\cdot)$ of (8) associated to a control $u(\cdot)$ is said to be *rigid* on $[0,T]$ if there exists $\varepsilon > 0$ such that, for every $t \in [T-\varepsilon, T+\varepsilon]$ and for every admissible control $v \in L^\infty([0,t], U)$, we have

$$E_{x_0,t}(v) \neq E_{x_0,T}(u).$$

In other words, the point $x(T)$ is reachable for times t close to T only with the control u. (For results regarding rigid curves, see for instance [3], [15].)

We have the following result.

Corollary 3.6: With the notations of Theorem 3.3 and if $m \geq 2$, there exists an open set O_{m+1} dense in $\mathcal{F}_{m+1}$ so that every control-affine system, defined with an $(m+1)$-tuple of O_{m+1}, does not admit rigid trajectories.

IV. CONSEQUENCES IN OPTIMAL CONTROL

We keep here the notations of the previous sections. Let (Σ) be a control system, which is either driftless, of the type (3), or control-affine, of the type (8). Consider the optimal control problem associated to (Σ), corresponding to the minimization of the quadratic cost given by

$$C_T(u) = \int_0^T \Big(u(t)^T U u(t) + g(x(t))\Big) dt, \qquad (13)$$

where U is a $(m \times m)$ real positive definite matrix,

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix},$$

m is a positive integer, and g is a smooth function on M.

Let $x_0 \in M$ and $T > 0$ be fixed. Recall that the *value function* associated to this optimal control problem is defined by

$$S_{x_0,T}(x) := \inf\{C_T(u) \mid E_{x_0,T}(u) = x\} \qquad (14)$$

The regularity of the associated value function was studied in [5], [7] for driftless systems, and in [28] for control-affine systems. Its subanalyticity is intimately related to the existence of nontrivial minimizing trajectories starting from x_0.

A. Driftless control systems

The next result, adapted from [12], states the genericity of the strictly abnormal property.

Proposition 4.1: There exists an open dense subset O_m of $\mathcal{F}_m$ such that every nontrivial singular trajectory of a driftless system defined by a m-tuple $(f_1, \dots, f_m)$ of O_m is strictly abnormal.

As a byproduct of the above proposition and Corollary 2.4, we get the next result.

Corollary 4.2: Let $m \geq 3$ be an integer. There exists an open dense set O_m of $\mathcal{F}_m$ such that every driftless system defined with a m-tuple of O_m does not admit nontrivial minimizing singular trajectories.

This result implies the subanalyticity of the value function in the analytic case (for a general definition of subanalyticity, see e.g. [20]).

Corollary 4.3: In the context of Corollary 4.2, if in addition the function g and the vector fields of the m-tuple in O_m are analytic, then the associated value function S_T is continuous and subanalytic on its domain of definition.

Remark 3: The previous results may be interpreted in the context of sub-Riemannian geometry, for $U = Id$ and $g = 0$ (see [17]). In particular, the above value function is related to the *sub-Riemannian distance* (and thus is always continuous).

Remark 4: If there exists a nontrivial minimizing singular trajectory, then the value function may fail to be subanalytic (see for instance the Martinet case in [1]).

B. Control-affine systems

The next three results correspond respectively to Proposition 4.1, Corollary 4.2, and Corollary 4.3, in the control-affine case.

Proposition 4.4: There exists an open dense subset O_{m+1} of $\mathcal{F}_{m+1}$ such that every nontrivial singular trajectory of a control-affine system defined by a $(m+1)$-tuple $(f_0, \dots, f_m)$ of O_{m+1} is strictly abnormal.

Corollary 3.4 together with Proposition 4.4 yield the next corollary.

Corollary 4.5: Let $m \geq 2$ be an integer. There exists an open set O_{m+1} dense in $\mathcal{F}_{m+1}$ so that every control-affine system defined with a $(m+1)$-tuple of O_{m+1} does not admit minimizing singular trajectories.

Corollary 4.6: In the context of Corollary 4.5, if in addition the function g and the vector fields of the $(m+1)$-tuple in O_{m+1} are analytic, then the associated value function S_T is continuous and subanalytic on its domain of definition.

Remark 5: If there exists a nontrivial minimizing trajectory, the value function may fail to be subanalytic, even continuous. For example, consider the control-affine system in $\mathbb{R}^2$

$$\begin{aligned} \dot{x}(t) &= 1 + y(t)^2, \\ \dot{y}(t) &= u(t), \end{aligned} \qquad (15)$$

and the cost

$$C_T(u) = \int_0^T u(t)^2 dt. \qquad (16)$$

The trajectory $(x(t) = t, y(t) = 0)$, associated to the control $u = 0$, is a nontrivial minimizing singular trajectory, and the value function $S_{(0,0),T}$ is not continuous at $(0, 0)$ (see [28] for details).

V. CONCLUSION

In this paper, we have shown that a large class of systems (generic in a strong sense) enjoys important properties regarding their singular trajectories. Namely, the latter are of *minimal order* and of *corank one*, and excluded from optimality of many quadratic optimal control problems. These properties should have further consequences for motion planning, stabilization, and in Hamilton-Jacobi-Bellman theory.

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