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ON THE INVERSE OPTIMAL CONTROL PROBLEMS OF THE HUMAN LOCOMOTION: STABILITY AND ROBUSTNESS OF THE MINIMIZERS

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Abstract. In recent papers models of the human locomotion by means of an optimal control problem have been proposed. In this paradigm, the trajectories are assumed to be solutions of an optimal control problem whose cost has to be determined. The purpose of the present paper is to analyze the class of optimal control problems defined in this way. We prove strong convergence result for their solutions on the one hand for perturbations of the initial and final points (stability), and on the other hand for perturbations of the cost (robustness).

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1. Introduction

An important question in the study of human motor control is to determine which law governs a particular body movement, such as arm pointing motions or goal-oriented locomotion. A nowadays widely accepted paradigm in neurophysiology is that, among all possible movements, the accomplished ones satisfy suitable optimality criteria (see [17] for a review). Once a dynamical model of the movements under consideration is given, one is then led to solve an *inverse optimal control problem*: given recorded experimental data, infer a cost function such that the recorded movements are solutions of the associated optimal control problem.

In the theory of linear-quadratic control, the question of which quadratic cost is minimized in order to control a linear system along certain trajectories was already raised by R. Kalman [13]. Some methods allowed deducing cost functions from optimal behavior in system and control theory (linear matrix inequalities [9]) and in Markov decision processes (inverse reinforcement learning [15]). A new and promising approach has been developed in [7, 12] for the pointing movements of the arm. In that approach, based on Thom transversality theory, the cost structure is deduced from qualitative properties highlighted by the experimental data. However all these methods have been conceived for very specific systems and they are not suitable for general inverse optimal control problems.

This paper focuses on the inverse optimal control problem associated with the goal-oriented human locomotion. In the approach initiated in [2–4], goal-oriented human locomotion is understood as the motion in the 3-D space of both its position (x, y) and its orientation θ of a person walking from an initial point (x_0, y_0, θ_0) to a final point (x_1, y_1, θ_1) (see Figure 1).

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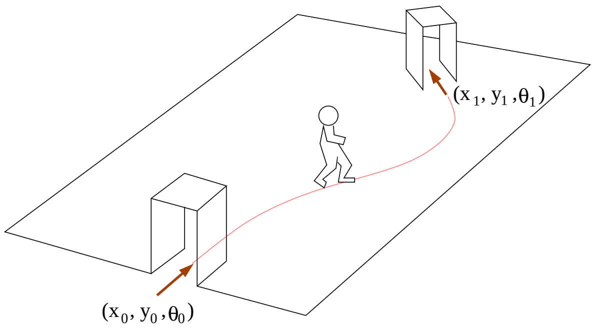

Fig. 1. Goal-oriented human locomotion.

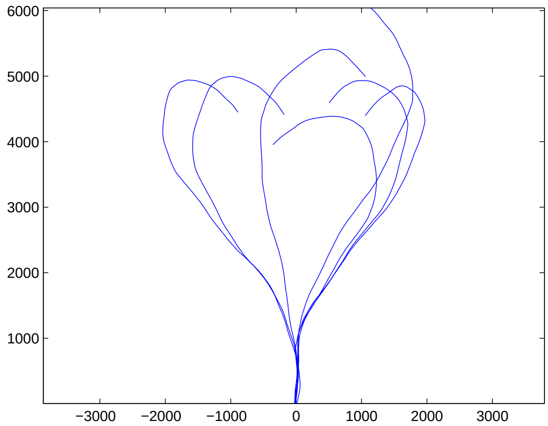

Fig. 2. Examples of trajectories for several final positions, drawn by using the data recorded by Arechavaleta et al. [3] (with the kind permission of the authors).

We take advantage here of the analysis carried out in a previous work [10] (see also [6]) in modeling the human locomotion by means of optimal control problems. We assume that every admissible locomotion trajectory is solution of a control system that describes the dynamics, exploiting the nonholonomic nature of human locomotion [3]. Among these trajectories, the chosen one minimizes a cost that has to be sought in a particular class $\mathcal{L}$ of functionals. Thus, with each cost function in $\mathcal{L}$, one can associate the set of solutions of the corresponding optimal control problem. The inverse optimal control problem consists in determining an inverse of this mapping. From a practical point of view, it amounts to find in $\mathcal{L}$ a cost whose minimizing trajectories fit accurately experimental data (see Figure 2). To this purpose Mombaur et al. [14] proposed a purely numerical approach based on parameter identification that is used to implement human-like motion on humanoid robots. This method furnishes a plausible solution but does not give further insight into the considered inverse optimal control problem.

The purpose of the present paper is to analyze the direct problem, and in particular to understand the behavior of minimizing trajectories in function of the cost. This qualitative analysis represents a step forward a global qualitative and numerical analysis of the inverse optimal control problem. We will show that the solutions of the optimal control problem corresponding to costs in $\mathcal{L}$ depend continuously on the one hand on the initial and final points (stability), and on the other hand on the cost (robustness). The stability of the minimizers, as well as their existence, are properties that arise naturally in the modeling of a biological motion. The fact that they are satisfied here only enhance the relevance of our model. The robustness has more consequences for the associated inverse optimal control problem. Due to the experimental nature of the recorded data, it implies that if a cost is solution of the inverse control problem, then its small enough perturbations are solutions too. Thus the class of possible costs can be reduced to any dense subclass and, in order to further reduce the class of costs in the spirit of [7, 12], we can look for properties of the minimizers that are only approximately satisfied. We intend to exploit this property in forthcoming studies.

The optimal control problems modeling the locomotion are described in Section 2. Two types of models are given: in the first one the control is the angular acceleration $\ddot{\theta}$, in the second one it is the angular velocity $\dot{\theta}$. We also give in this section the equations resulting by the application of the Pontryagin maximum principle.

The existence and stability results are given in Section 3. We first prove the existence of minimizers and the continuity of the value function. Then we show that, for a converging sequence of boundary conditions, the corresponding minimizers and their adjoint vectors uniformly converge.

Section 4 is dedicated to the proof of robustness. We first show that, if a sequence of costs depending on $\ddot{\theta}$ converges to a cost also depending on $\ddot{\theta}$, then the minimizers with given terminal conditions and their adjoint vectors uniformly converge. We then prove the same result when the limit of the sequence of costs does not depend on $\ddot{\theta}$ but only on $\dot{\theta}$.

2. The models and the adjoint equations

2.1. The models of human locomotion. We present in this section the model of the human locomotion developed in [10]. The dynamics are described by control systems of the form

$$\begin{cases} \dot{x} = \cos\theta, \\ \dot{y} = \sin\theta, \\ \theta^{(k)} = u, \end{cases} \tag{1}$$

where $k > 0$ is an integer. System (1) captures the nonholonomic nature of the human locomotion pointed out in [3], which is valid when the subject walks toward a far enough target point. Note also that the paths (x, y) are parameterized by arc-length.

The locomotion trajectories between two arbitrary configurations (x_0, y_0, θ_0) and (x_1, y_1, θ_1) appear as the solutions of an optimal control problem of the following form: minimize the cost

$$J = \int_0^T L(\dot{\theta}, \dots, \theta^{(k-1)}, u)dt$$

among all admissible controls $u(\cdot)$ steering system (1) from $q(0) = (x_0, y_0, \theta_0)$ to $q(T) = (x_1, y_1, \theta_1)$. Notice that the final time T and the boundary values of $\theta^{(m)},\ m = 1, \dots, k-1$, are free.

The inverse optimal control problem consists in finding the proper cost function L. Of course it is necessary to precise the class $\mathcal{L}$ of functionals in which L has to be sought. This class contains different types of functionals depending on the integer k characterizing system (1).

The first task is to understand how the integer k affects the structure of the corresponding set of optimal trajectories.

Keeping in mind that the cost should model a human behavior, and that the angular acceleration $\ddot{\theta}$ is equivalent to a third-order derivative (so-called *jerk*) of the planar path, we make the rather natural assumption that $k \leq 2$ in the sequel (see also [2, 10]). Thus in the following the class $\mathcal{L}$ will be divided into two sub-classes $\mathcal{L}_1$ and $\mathcal{L}_2$ that depend on whether the highest derivative of the angle θ explicitely appearing in the cost is the first or the second one.

We first define the class $\mathcal{L}_2$ as the set of functions $L : \mathbb{R}^2 \to \mathbb{R}$ which satisfy the following three assumptions:

(H1) $L(\cdot,\cdot)$ is non-negative, C^2 in the pair, and satisfies $L(0,0) = 0$. Moreover $L(r,0) \leq L(r,s)$ for every $(r,s) \in \mathbb{R}^2$;

(H2) $L(\cdot,\cdot)$ is strictly convex with respect to the second variable, with $\frac{\partial^2 L}{\partial s^2}(r,s) > 0$ for every (r,s);

(H3) there exist two constants $C, R > 0$ and $p > 1$ such that

$$L(r,s) \geq C|s|^p \quad \forall\, r \in \mathbb{R},\ \forall |s| > R.$$

Moreover, there exist two L^∞_{loc} functions $\gamma, \delta : \mathbb{R} \to \mathbb{R},\ \gamma > 0$ such that

$$\left| \frac{\partial L}{\partial r}(r,s) \right| \leq \gamma(r)|s|^p + \delta(r). \tag{2}$$

The integral cost associated with a function $L \in \mathcal{L}_2$ is

$$J_L(T, \kappa_0, u) = \int_0^T 1 + L(\kappa(t), u(t))\, dt,$$

and has to be evaluated along the trajectory $Q(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot), \kappa(\cdot))$ of the control system

$$\begin{cases} \dot{x} = \cos\theta \\ \dot{y} = \sin\theta \\ \dot{\theta} = \kappa \\ \dot{\kappa} = u \end{cases}, \tag{3}$$

with initial condition $(0,0,0,\kappa_0)$ and corresponding to the control function $u(\cdot)$. The corresponding optimal control problem is

$\mathbf{P}_2(L)$ Minimize $J_L(T, \kappa_0, u)$ among all controls $u \in L^p([0,T])$ and all values $\kappa(0) = \kappa_0$ such that the corresponding trajectories $Q(\cdot)$ satisfy (3) with boundary conditions $q_0 = (x(0), y(0), \theta(0)) = (0,0,0)$ and $q_1 = (x(T), y(T), \theta(T)) = (x_1, y_1, \theta_1)$. The final time T and the initial and final values of the curvature, that is $\kappa(0) = \kappa_0$ and $\kappa(T)$, are free.

Note that the model is valid only when the target (x_1, y_1) is far enough from the initial position $(0, 0)$, so we will always assume $(x_1, y_1) \neq 0$.

Remark 1. *In [10], for the need of the analysis, the cost functions L are assumed to depend separately on κ and u. We remove this unnecessary assumption here. Hypothesis* **(H3)** *then writes in a slightly different – and more general – form.*

The second class of costs we consider is the one in which $\ddot{\theta}$ and higher derivatives of θ do not appear. We define this class $\mathcal{L}_1$ as the set of functions $\ell : \mathbb{R} \to \mathbb{R}$ which satisfy the following three assumptions:

(H1′) $\ell(\cdot)$ is a non-negative C^2 function satisfying $\ell(0) = 0$;
(H2′) $\ell(\cdot)$ is strictly convex, with $\ell''(r) > 0$ for every r;
(H3′) there exist two constants $C, R > 0$ and $p > 1$ such that

$$\ell(r) \geq C|r|^p \quad \forall |r| > R.$$

The integral cost associated with a function $\ell \in \mathcal{L}_1$ is

$$J_\ell(T, \kappa) = \int_0^T 1 + \ell(\kappa(t))\, dt,$$

and has to be evaluated along the trajectories $q(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot))$ of the control system

$$\begin{cases} \dot{x} = \cos\theta \\ \dot{y} = \sin\theta \\ \dot{\theta} = \kappa \end{cases}, \tag{4}$$

where κ is the control. The corresponding optimal control problem is

$\mathbf{P}_1(\ell)$ Minimize $J_\ell(T, \kappa)$ among all controls $\kappa(\cdot) \in L^p([0, T])$ such that the corresponding trajectories $q(\cdot)$ satisfy (4) with free final time T and boundary conditions $q_0 = (x(0), y(0), \theta(0)) = (0, 0, 0)$ and $q_1 = (x(T), y(T), \theta(T)) = (x_1, y_1, \theta_1)$.

Remark 2. *With a little abuse of notations, when speaking about solutions of* $\mathbf{P}_2(L)$ *and* $\mathbf{P}_1(\ell)$ *we mean the corresponding optimal trajectories.*

Finally the class $\mathcal{L}$ is defined as the union $\mathcal{L}_1 \cup \mathcal{L}_2$. Thus with every function L or ℓ in $\mathcal{L}$ is associated the optimal synthesis of $\mathbf{P}_2(L)$ or $\mathbf{P}_1(\ell)$, that is the set of solutions of this problem for all $q_1 \in \mathbb{R}^2 \times S^1$. The inverse optimal control problem of the goal-oriented locomotion consists in finding an inverse to this mapping.

The main purpose of this paper, as a first step toward the analysis of this inverse problem, is to show for that mapping a continuity property that we call robustness (Section 4). We will also prove in Section 3 the existence of the solutions $\mathbf{P}_2(L)$ and $\mathbf{P}_1(\ell)$, and their continuity with respect to q_1. Both parts make use of the equations of the Pontryagin maximum principle that we state now.

2.2. Application of the Pontryagin Maximum Principle. The Pontryagin maximum principle (PMP) establishes some first-order necessary conditions to be satisfied by every optimal trajectory. However, the PMP in its classical form [1, 16] requires that the optimal control is bounded in the L^∞ topology, which is an information that we do not possess at this stage. Nevertheless, refined versions of the PMP deal also with unbounded controls. In particular it is easy to check that our problems meet all the hypotheses required in [5, Theorem 2.3], and thus we can state the following.

Proposition 1. *Every solution of* $\mathbf{P}_2(L)$ *(resp.* $\mathbf{P}_1(\ell)$*) satisfies the PMP.*

Remark 3. *The PMP does not actually guarantee the existence of optimal trajectories. In Section 3 we prove the existence of solutions for every final point* q_1*, both for problem* $\mathbf{P}_1(\ell)$ *and* $\mathbf{P}_2(L)$*.*

In order to apply the PMP to the problem $\mathbf{P}_2(L)$, we define the following Hamiltonian function:

$$H(Q, P, \nu, u) = p_x \cos\theta + p_y \sin\theta + p_\theta \kappa + p_\kappa u + \nu(1 + L(\kappa, u)),$$

where $P = (p_x, p_y, p_\theta, p_\kappa) \in \mathbb{R}^4$ and $\nu \in \mathbb{R}$.

We recall that the PMP states that if $Q(\cdot)$ is an optimal trajectory corresponding to a control $u(\cdot)$ defined on an interval $[0, T]$, then there exist an absolutely continuous function $P(\cdot) : [0, T] \to \mathbb{R}^4$ and $\nu \leq 0$ such that the pair $(P(\cdot), \nu)$ is non-trivial and $(Q(\cdot), P(\cdot))$ satisfies the Hamiltonian system

$$\begin{cases} \dot{Q}(t) = \frac{\partial H}{\partial P}(Q(t), P(t), \nu, u(t)) \\ \dot{P}(t) = -\frac{\partial H}{\partial Q}(Q(t), P(t), \nu, u(t)) \end{cases}. \tag{5}$$

Moreover, the PMP also states that the n-tuple $(Q(\cdot), P(\cdot), \nu, u(t))$ satisfies the maximality condition

$$H(Q(t), P(t), \nu, u(t)) = \max_{v \in \mathbb{R}} H(Q(t), P(t), \nu, v)$$

for $t \in [0, T]$. This condition establishes a relation between the optimal control $u(\cdot)$, $\kappa(\cdot)$ and $p_\kappa(\cdot)$:

$$p_\kappa(t) = -\nu \frac{\partial L}{\partial u}(\kappa(t), u(t)) \tag{6}$$

for $t \in [0, T]$.

As the final time is free, the Hamiltonian is identically zero along the optimal trajectory, i.e. $H(Q(t), P(t), \nu, u(t)) \equiv 0$. Since the values $\kappa(0)$ and $\kappa(T)$ are free, we get also the following transversality conditions:

$$p_\kappa(0) = p_\kappa(T) = 0.$$

The equation for Q in (5) is nothing but (3), while the equation for the *adjoint vector* P, also called the *adjoint equation*, can be explicitly rewritten as

$$\begin{cases} \dot{p}_x = 0 \\ \dot{p}_y = 0 \\ \dot{p}_\theta = p_x \sin\theta - p_y \cos\theta \\ \dot{p}_\kappa = -p_\theta - \nu \frac{\partial L}{\partial \kappa} \end{cases} \tag{7}$$

If $\nu \neq 0$, we can always normalize it by putting $\nu = -1$. In this case, we say that the pair $(Q(\cdot), P(\cdot))$ is a normal extremal, while if $\nu = 0$ the pair $(Q(\cdot), P(\cdot))$ is said to be an abnormal extremal. Thanks to Remark 4 below, in the following we always set $\nu = -1$.

Remark 4. *The problem $P_2(L)$ does not admit abnormal extremals. Indeed, from the maximality condition applied to the Hamiltonian $H(Q, P, 0, u) = p_x \cos\theta + p_y \sin\theta + p_\theta \kappa + p_\kappa u$ we obtain that $p_\kappa \equiv 0$, and then, from $\dot{p}_\kappa = -p_\theta$, that $p_\theta \equiv 0$. Since $\dot{p}_\theta = p_x \sin\theta - p_y \cos\theta \equiv 0$ and $H(Q, P, 0, u) = p_x \cos\theta + p_y \sin\theta \equiv 0$, we get that the whole adjoint vector vanishes, contradicting the PMP. Moreover, we deduce that with any solution $Q(\cdot)$ of $\mathbf{P}_2(L)$ it is associated one and only one normal extremal trajectory $(Q(\cdot), P(\cdot))$. Indeed, if $(Q(\cdot), P_1(\cdot))$ and $(Q(\cdot), P_2(\cdot))$ were two normal extremals (with $\nu = -1$), then it is easy to check that $(Q(\cdot), P_1(\cdot) - P_2(\cdot))$ would be an abnormal extremal.*

Notice that the absence of abnormal extremals, together with (6)*, implies that all the pairs $(Q(\cdot), P(\cdot))$ satisfying the PMP are $\mathcal{C}^1$.*

Concerning the problem $\mathbf{P}_1(\ell)$, its associated Hamiltonian is

$$H(q, p, \nu, \kappa) = p_x \cos\theta + p_y \sin\theta + p_\theta \kappa + \nu(1 + \ell(\kappa)),$$

where $p = (p_x, p_y, p_\theta) \in \mathbb{R}^3$ and $\nu \leq 0$, and the Hamiltonian equation for the pair $(q(\cdot), p(\cdot))$ is

$$\begin{cases} \dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t), \nu, \kappa(t)) \\ \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t), \nu, \kappa(t)) \end{cases}. \tag{8}$$

The n-tuple $(q(\cdot), p(\cdot), \nu, \kappa(\cdot))$ satisfies the maximality condition

$$H(q(t), p(t), \nu, \kappa(t)) = \max_{v \in \mathbb{R}} H(q(t), p(t), \nu, v)$$

for $t \in [0, T]$, and, as the final time is free, the Hamiltonian is zero along the optimal trajectory: $H(q(t), p(t), \nu, \kappa(t)) \equiv 0$. The maximality condition links p_θ with κ:

$$p_\theta(t) = -\nu \ell'(\kappa(t)) \tag{9}$$

for $t \in [0, T]$. We notice that in this case there are no transversality conditions.

The adjoint equation is

$$\begin{cases} \dot{p}_x = 0 \\ \dot{p}_y = 0 \\ \dot{p}_\theta = p_x \sin\theta - p_y \cos\theta \end{cases}$$

We remark that also the problem $\mathbf{P}_1(\ell)$ does not admit abnormal extremals, so that we set, without loss of generality, $\nu = -1$.

3. Existence and stability of the optimal solutions

Given a function L or ℓ in $\mathcal{L}$, with every terminal point $q_1 \in \mathbb{R}^2 \times S^1$ one associates the corresponding solutions of $\mathbf{P}_2(L)$ or $\mathbf{P}_1(\ell)$. The aim of this section is to study this mapping; we first prove that it is well-defined (existence of solutions of $\mathbf{P}_2(L)$ or $\mathbf{P}_1(\ell)$), and then that it is continuous with respect to q_1 (stability of the solutions). Namely, we will show that when a sequence of final points converges to a point q_1, any sequence of corresponding optimal trajectories converges uniformly to the set of optimal trajectories having q_1 as final point.

3.1. Existence of optimal solutions. First of all, we notice that the set of trajectories of (3) (resp. (4)) satisfying the boundary conditions of problem $\mathbf{P}_2(L)$ (resp. $\mathbf{P}_1(\ell)$) is non-empty. This is a straightforward consequence of classical controllability results (see for instance [1]). Concerning problem $\mathbf{P}_2(L)$, the main result of this section, stated below, guarantees that, among these trajectories, there always exists a minimizer.

Theorem 1. *For every choice of $q_1 = (x_1, y_1, \theta_1) \in \mathbb{R}^2 \times S^1$ such that $(x_1, y_1) \neq (0,0)$ there exist $\bar{T} > 0$, a control function $\bar{u} \in L^p([0, \bar{T}])$ and an initial value $\bar{\kappa}_0$ such that the corresponding trajectory $\bar{Q}(\cdot)$ solves problem $\mathbf{P}_2(L)$.*

In order to prove the previous theorem we will make use of a standard result in calculus of variation, and a technical lemma, stated below.

Theorem 2 ([11, Theorem 1.3])**.** *Let $n, N \in \mathbb{N}$, $p \geq 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}$ be a non-negative continuous function and*

$$J(\varphi(\cdot)) = \int_\Omega f(x, \varphi(x), \nabla\varphi(x))dx.$$

If the function $s \mapsto f(x, r, s)$ is convex, then J is (sequentially) weakly lower semicontinuous in $W^{1,p}(\Omega)$.

Lemma 1. *Let $u^n : [0, T^n] \to \mathbb{R}$, $\kappa_0^n \in \mathbb{R}$ satisfy $J_L(T^n, \kappa_0^n, u^n) \leq M$ for every n, for some $M > 0$. Consider the solution $Q^n(\cdot) = (x^n(\cdot), y^n(\cdot), \theta^n(\cdot), \kappa^n(\cdot))$ of the control system (3) associated with the control $u^n(\cdot)$ and with initial condition $Q^n(0) = (0, 0, 0, \kappa_0^n)$, and assume that $\lim_n q^n(T^n) = \lim_n (x^n(T^n), y^n(T^n), \theta^n(T^n)) = q_1$, for some $q_1 = (x_1, y_1, \theta_1) \in \mathbb{R}^2 \times S^1$, with $(x_1, y_1) \neq (0,0)$.*

Then there exist a time $\bar{T} > 0$ and a function $\bar{u} : [0, \bar{T}] \to \mathbb{R}$ such that, up to subsequences, $\bar{T} = \lim_n T^n$ and, possibly putting $u^n = 0$ on $[T^n, \bar{T}]$ if $T^n \leq \bar{T}$, $(u^n(\cdot))^n$ weakly converges to $\bar{u}(\cdot)$ in $L^p([0, \bar{T}])$. Moreover, the functions $Q^n(\cdot)$ converge uniformly to a solution $\bar{Q}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{\theta}(\cdot), \bar{\kappa}(\cdot))$ of (3) associated with $\bar{u}(\cdot)$. In particular, $(\bar{x}(\bar{T}), \bar{y}(\bar{T}), \bar{\theta}(\bar{T}))) = q_1$.

Finally, the following relation holds

$$J_L(\bar{T}, \bar{\kappa}(0), \bar{u}) \leq \liminf_n J_L(\bar{T}, \kappa_0^n, u^n) \leq \liminf_n J_L(T^n, \kappa_0^n, u^n). \tag{10}$$

Proof. By **(H3)** and the special structure of the cost, it is easy to see that T^n and $\|u^n\|_{L^p}$ are uniformly bounded and therefore, up to subsequences, we have that $(T^n)^n$ converges to $\bar{T}$. Whenever $T^n < \bar{T}$, we extend u^n to the whole $[0, \bar{T}]$ taking $u^n(t) = 0$ for $t \in [T^n, \bar{T}]$. Without loss of generality, we can assume that $(u^n(\cdot))^n$ weakly converges to some $\bar{u}(\cdot)$ in $L^p([0, \bar{T}])$. Since

$$|\kappa^n(t) - \kappa^n(t')| \leq \int_{t'}^{t} |u^n(s)|\, ds \leq |t - t'|^{1/q} \|u^n\|_{L^p}, \tag{11}$$

where as usual $1/q + 1/p = 1$, we see that, for every n, $\kappa^n(\cdot)$ is Hölder continuous and the family $(\kappa^n(\cdot))^n$ is uniformly equicontinuous.

We claim that the sequence $(\kappa^n(\cdot))^n$ is uniformly bounded. By contradiction, assume that $\|\kappa^n(\cdot)\|_\infty$ goes to infinity. Since T^n and $\|u^n\|_{L^p}$ are bounded and by (11), for n large enough $\kappa^n(\cdot)$ has constant sign on $[0, T^n]$. We fix n sufficiently large and, without loss of generality, we assume that $\kappa^n(\cdot)$ is positive. Let α be defined by the relation $(x^n(T^n), y^n(T^n)) =$

$|(x^n(T^n), y^n(T^n))|(\cos\alpha, \sin\alpha)$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$. Let $I_h^+, h = 1, \ldots, N_+$ and $I_h^-, h = 1, \ldots, N_-$ be the subintervals of $[0, T^n]$ where $\cos(\theta^n(t) - \alpha)$ is respectively non-negative or non-positive. Then $|N_- - N_+| \le 1$ and, since for every h (except possibly $h = 1, N_-, N_+$) it holds $\pi = \int_{I_h^\pm} \kappa^n(s)ds \le \mu(I_h^\pm)\|\kappa^n\|_\infty$, we get $N_-, N_+ \le \frac{T^n\|\kappa^n\|_\infty}{2\pi} + \frac{3}{2}$ (here and in the following we denote by $\mu(J)$ the measure of a measurable set $J \subset \mathbb{R}$).

Let us observe that from (11) we get that $\kappa^n(t) \ge \frac{\|\kappa^n\|_\infty}{1+\varepsilon}$, where $\varepsilon = \frac{(T^n)^{1/q}\|u^n\|_p}{\|\kappa^n\|_\infty - (T^n)^{1/q}\|u^n\|_p} > 0$ if n is large enough. Then for every $h = 1, \ldots, N_+$ we have that $\int_{I_h^+} \cos(\theta^n(t) - \alpha)dt \le \frac{1+\varepsilon}{\|\kappa^n\|_\infty} \int_{I_h^+} \cos(\theta^n(t) - \alpha)\, \kappa^n(t)dt \le 2\frac{1+\varepsilon}{\|\kappa^n\|_\infty}$.

Moreover for every $h = 2, \ldots, N_- - 1$ we have that $\int_{I_h^-} \cos(\theta^n(t) - \alpha)dt \le \frac{1}{\|\kappa^n\|_\infty} \int_{I_h^-} \cos(\theta^n(t) - \alpha)\, \kappa^n(t)dt = -\frac{2}{\|\kappa^n\|_\infty}$. Then

$$
\begin{aligned}
|(x^n(T^n), y^n(T^n))| &= \langle (x^n(T^n), y^n(T^n)), (\cos\alpha, \sin\alpha)\rangle \\
&= \int_0^{T^n} \langle (\cos\theta^n(t), \sin\theta^n(t)), (\cos\alpha, \sin\alpha)\rangle dt \\
&= \int_0^{T^n} \cos(\theta^n(t) - \alpha)dt \le 2N_+ \frac{1+\varepsilon}{\|\kappa^n\|_\infty} - \frac{2N_- - 4}{\|\kappa^n\|_\infty} \le \frac{\varepsilon T^n}{\pi} + \frac{6+3\varepsilon}{\|\kappa^n\|_\infty}.
\end{aligned}
$$

This leads to a contradiction, since ε tends to zero as n goes to infinity, while $|(x^n(T^n), y^n(T^n))|$ is bounded from below and T^n is bounded.

Thus $(\kappa^n(\cdot))^n$ is uniformly bounded and therefore, by Ascoli-Arzelà Theorem, the functions $\kappa^n(\cdot)$ converge uniformly, up to extracting a subsequence, to a function $\bar{\kappa} : [0, \bar{T}] \to \mathbb{R}$. This implies the uniform convergence of $Q^n(\cdot)$ to $\bar{Q}(\cdot)$ on $[0, \bar{T}]$.

Finally, in view of (**H2**), equation (10) is a direct consequence of Theorem 2. □

Proof of Theorem 1. Fix $q_1 = (x_1, y_1, \theta_1)$ and consider a minimizing sequence $\kappa^n \in \mathbb{R}$, $u^n : [0, T^n] \to \mathbb{R}$ for problem $\mathbf{P}_2(L)$, that is

$$\lim_n J_L(T^n, \kappa_0^n, u^n) = \inf J_L(T, \kappa_0, u),$$

where κ_0^n, u^n and κ_0, u are such that the trajectories $Q^n(\cdot) = (x^n(\cdot), y^n(\cdot), \theta^n(\cdot), \kappa^n(\cdot)), Q(\cdot) = (x(\cdot), y(\cdot), \theta(\cdot), \kappa(\cdot))$ of (3) with initial condition $Q^n(0) = (0, 0, 0, \kappa_0^n)$ and $Q(0) = (0, 0, 0, \kappa_0)$, respectively, satisfy $(x^n(T^n), y^n(T^n), \theta^n(T^n)) = (x(T), y(T), \theta(T)) = q_1$. It is easy to prove that the sequence $(u^n(\cdot))^n$ satisfies the hypotheses of Lemma 1, and therefore it weakly converges to some $\bar{u}(\cdot)$ in $L^p([0, \bar{T}])$ whose corresponding trajectory $\bar{Q}(\cdot)$ solves problem $\mathbf{P}_2(L)$. □

The counterpart of Theorem 1 for problem $\mathbf{P}_1(\ell)$ is the following.

Theorem 3. *For every choice of $q_1 \in \mathbb{R}^2 \times S^1$ there exist $\bar{T} > 0$ and a control function $\bar{\kappa}(\cdot) \in L^{p'}([0, \bar{T}])$ such that the corresponding trajectory $\bar{q}(\cdot)$ solves problem $\mathbf{P}_1(\ell)$.*

The previous result follows directly from the following adaptation of Lemma 1, which, in turn, is an easy consequence of Ascoli-Arzelà Theorem and of Theorem 2 applied with $\varphi = \theta$.

Lemma 2. *Let $\kappa^n : [0, T^n] \to \mathbb{R}$ satisfy $J_\ell(T^n, \kappa^n) \le M$ for every n and some $M > 0$. Let $q^n(\cdot)$ be a solution of the control system (4) associated with the control $\kappa^n(\cdot)$ with $q^n(0) = q_0 = (0,0,0)$, and assume that $\lim_n q^n(T^n) = q_1$, for some $q_1 \in \mathbb{R}^2 \times S^1$. Then there exist a time $\bar{T} > 0$ and a function $\bar{\kappa} : [0, \bar{T}] \to \mathbb{R}$ such that up to subsequences, $\bar{T} = \lim_n T^n$ and, possibly putting $\kappa^n = 0$ on $[T^n, \bar{T}]$ if $T^n \le \bar{T}$, $\kappa^n(\cdot)$ weakly converges to $\bar{\kappa}(\cdot)$ in $L^p([0, \bar{T}])$. Moreover, the functions $q^n(\cdot)$ converge uniformly to the corresponding solution $\bar{q}(\cdot)$ of (4) associated with $\bar{\kappa}(\cdot)$, and*

$$J_\ell(\bar{T}, \bar{\kappa}) \le \liminf_n J_\ell(\bar{T}, \kappa^n) \le \liminf_n J_\ell(T^n, \kappa^n).$$

3.2. Stability of the optimal solutions. We start this section by stating an accessory result about the regularity of the value function associated with problems $\mathbf{P}_2(L)$ and $\mathbf{P}_1(\ell)$. For every $q_1 \in \mathbb{R}^2 \times S^1$ we define the value function $J_L^*(q_1)$ (resp. $J_\ell^*(q_1)$) as the minimum of the functional J_L (resp. J_ℓ) among all trajectories of (3) (resp. (4)) steering $q_0 = (0,0,0)$ to q_1.

Proposition 2. *For $\natural = L, \ell$, the value function $J_\natural^*(\cdot)$ is continuous at every $q_1 = (x_1, y_1, \theta_1)$ such that $(x_1, y_1) \ne (0,0)$.*

Proof. Take $\natural = L$; the proof for $\natural = \ell$ is analogous. Let us consider a time-reparametrization of the control system (3)

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = v\kappa \\ \dot{\kappa} = vu \end{cases}, \tag{12}$$

where $v(\cdot) > 0$ is essentially bounded. Equation (12) is considered as a control system with control function $(u(\cdot), v(\cdot))$. Let $Q_0 = (0,0,0,\kappa_0)$ for some $\kappa_0 \in \mathbb{R}$ and, for $T > 0$, let $E_{Q_0,T}(\cdot) : L^\infty([0,T], \mathbb{R}^2) \to \mathbb{R}^2 \times S^1 \times \mathbb{R}$ denote the *end-point mapping*, that is $E_{Q_0,T}(u,v) = \widehat{Q}(T)$, where $\widehat{Q}(\cdot)$ is the solution of (12), associated with the control (u,v) and satisfying $\widehat{Q}(0) = Q_0$. It is easy to see that the control system (12) does not admit singular controls, that is controls (u,v) such that the (Fréchet) differential at (u,v) of the end-point mapping is not surjective (see [8]). Indeed, any singular control satisfies the Weak maximum principle [8, Theorem 6] with associated Hamiltonian $H = v(p_x\cos\theta + p_y\sin\theta + p_\theta\kappa + p_\kappa u)$, and the maximality condition implies that $p_x\cos\theta + p_y\sin\theta + p_\theta\kappa + p_\kappa u = 0$; there are no extremals $(Q(\cdot), P(\cdot))$ satisfying the latter condition, since this would entail the existence of abnormal extremals for problem $\mathbf{P}_2(L)$, which is excluded by Remark 4.

Let $\bar{u} : [0, \bar{T}] \to \mathbb{R}$ be an optimal control for $\mathbf{P}_2(L)$ with final condition q_1, and let $\bar{Q}(\cdot)$ denote the associated trajectory. Then the absence of singular controls for (12) implies that the mapping $E_{\bar{Q}(0),\bar{T}}(\cdot)$ is open at $(\bar{u}, 1)$.

It is easy to see that the functional $\widehat{J}(\bar{\kappa}_0, u, v) = \int_0^{\bar{T}} v(t)(1 + L(\kappa(t), u(t)))\,dt$, evaluated on the solution $\widehat{Q}(\cdot)$ of (12) with $\widehat{Q}(0) = \bar{Q}(0) = (0,0,0,\bar{\kappa}_0)$ associated with (u,v), is continuous with respect to the $L^\infty([0,\bar{T}], \mathbb{R}^2)$ topology for (u,v), and moreover we have that $\widehat{J}(\bar{\kappa}_0, u, v)$ coincides with the value of the functional J_L evaluated on a time reparametrization of $\widehat{Q}(\cdot)$ depending on v. Then for every $\varepsilon > 0$ there exists an open neighborhood $\mathcal{U}$ of $(\bar{u}, 1)$ such that $\widehat{J}(\bar{\kappa}_0, u, v) \le J_L^*(q_1) + \varepsilon$ for every $(u,v) \in \mathcal{U}$, while by definition of J_L^* we have

$J_L^*(E_{\bar{Q}(0),\bar{T}}(u,v)) \leq \widehat{J}(\bar{\kappa}_0,u,v)$. Since the end-point mapping is open and ε is arbitrary, we get that $\limsup_{q\to q_1} J_L^*(q) \leq J_L^*(q_1)$, that is J_L^* is upper semicontinuous.

It remains to prove that J_L^* is lower semicontinuous. Let $(q_1^n)^n$ be a sequence of points converging to q_1, and, for every n, $u^n : [0,T^n] \to \mathbb{R}$ be an optimal control steering the system from q_0 to q_1^n, with initial curvature κ_0^n. By the upper semicontinuity of J_L^* we get the uniform boundedness of $J_L(T^n,\kappa_0^n,u^n)$, thus we can apply Lemma 1 and obtain $\liminf_n J_L^*(q_1^n) = \liminf_n J_L(T^n,\kappa_0^n,u^n) \geq J_L(\bar{T},\bar{\kappa}_0,\bar{u})$, where $\bar{u} : [0,\bar{T}] \to \mathbb{R}$ steers the system from q_0 to q_1, with initial curvature $\bar{\kappa}_0$. By definition of J_L^* we have that $J_L(\bar{T},\bar{\kappa}_0,\bar{u}) \geq J_L^*(q_1)$, and this completes the proof. □

The previous result shows that the cost (value function) is continuous with respect to the final point. We will prove now a continuity property of the optimal trajectories and the associate adjoint vectors with respect to the final point.

Fix a point $q_1 = (x_1,y_1,\theta_1) \in \mathbb{R}^2 \times S^1$. Given a cost $L \in \mathcal{L}_2$, we define $\mathcal{E}_2(L,q_1)$ as the set of curves $(Q(\cdot),P(\cdot))$ such that $Q(\cdot)$ is a solution of $\mathbf{P}_2(L)$ with final condition q_1 and $P(\cdot)$ the associated adjoint vector. We endow the set of such continuous curves $(Q(\cdot),P(\cdot))$ with the distance d_∞ associated with the norm $\|\cdot\|_\infty$ of the uniform convergence.

Similarly, given a cost $\ell \in \mathcal{L}_1$ we define $\mathcal{E}_1(\ell,q_1)$ as the set of curves $(q(\cdot),p(\cdot))$ such that $q(\cdot)$ is a solution of $\mathbf{P}_1(\ell)$ with final condition q_1 and $p(\cdot)$ the associated adjoint vector. As above, we endow the set of such continuous curves $(q(\cdot),p(\cdot))$ with the distance associated with the norm of the uniform convergence that we denote again d_∞.

Theorem 4. *Let $q_1 = (x_1,y_1,\theta_1)$ be a point in $\mathbb{R}^2 \times S^1$ and $(q_1^n)^n$ a sequence of points converging to q_1.*

- *Fix $L \in \mathcal{L}_2$ and assume $(x_1,y_1) \neq (0,0)$. Then, for any sequence $(Q^n(\cdot),P^n(\cdot))^n$ such that, for every n, $(Q^n(\cdot),P^n(\cdot)) \in \mathcal{E}_2(L,q_1^n)$, we have*

$$\lim_{n\to\infty} d_\infty\left((Q^n(\cdot),P^n(\cdot)),\mathcal{E}_2(L,q_1)\right) = 0.$$

- *Fix $\ell \in \mathcal{L}_1$ and assume $(x_1,y_1) \neq (0,0)$. Then, for any sequence $(q^n(\cdot),p^n(\cdot))^n$ such that, for every n, $(q^n(\cdot),p^n(\cdot)) \in \mathcal{E}_1(\ell,q_1^n)$, we have*

$$\lim_{n\to\infty} d_\infty\left((q^n(\cdot),p^n(\cdot)),\mathcal{E}_1(\ell,q_1)\right) = 0.$$

The first item of this theorem is a direct consequence of the lemma below and the second one results from a straighforward adaptation of this lemma.

Lemma 3. *Fix $L \in \mathcal{L}_2$, $q_1 = (x_1,y_1,\theta_1) \in \mathbb{R}^2 \times S^1$ such that $(x_1,y_1) \neq 0$, and let $(q_1^n)^n$ be a sequence of points converging to q_1. Then, for any sequence $(Q^n(\cdot),P^n(\cdot))^n$ with $(Q^n(\cdot),P^n(\cdot)) \in \mathcal{E}_2(L,q_1^n)$, there exists a subsequence $(Q^{n_k}(\cdot),P^{n_k}(\cdot))^k$ which uniformly converges to an extremal $(\bar{Q}(\cdot),\bar{P}(\cdot))$ in $\mathcal{E}_2(L,q_1)$.*

Proof. For every integer n, let u^n be the optimal control associated with $(Q^n(\cdot),P^n(\cdot))$. By the continuity of the value function, we have that $J_L^*(q_1^n)$ is uniformly bounded. Thus the sequence $(u^n)^n$ satisfies the hypotheses of Lemma 1, which implies the uniform convergence of $Q^n(\cdot)$, up to subsequences, to a trajectory $\bar{Q}(\cdot) = (\bar{x}(\cdot),\bar{y}(\cdot),\bar{\theta}(\cdot),\bar{\kappa}(\cdot))$ associated with $\bar{u} : [0,\bar{T}] \to \mathbb{R}$. By (10) and the continuity of the value function, we have that $\bar{Q}(\cdot)$ is a solution of $\mathbf{P}_2(L)$ with final point q_1.

Let us now prove the uniform convergence of the adjoint vector. For every integer n we set $Q^n(\cdot) = (x^n(\cdot), y^n(\cdot), \theta^n(\cdot), \kappa^n(\cdot))$ and $P^n(\cdot) = (p_x^n(\cdot), p_y^n(\cdot), p_\theta^n(\cdot), p_\kappa^n(\cdot))$. We then proceed in two step: first of all, we prove that the sequence of the initial conditions $P^n(0)$ is bounded (and therefore it converges, up to subsequences); then, since for every n the extremal $(Q^n(\cdot), P^n(\cdot))$ satisfies the Hamiltonian system (5), we get the convergence of the whole adjoint vector.

By the adjoint equation we get that p_x^n, p_y^n are constant and we write $(p_x^n, p_y^n) = \rho^n(\cos\alpha^n, \sin\alpha^n)$, $\rho^n > 0$. Let us first show that ρ^n is uniformly bounded. Assume by contradiction that this is not true. Up to restricting to an appropriate subsequence we can assume that $(Q^n(\cdot))^n$ uniformly converges to $\bar{Q}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{\theta}(\cdot), \bar{\kappa}(\cdot))$, $(\alpha^n)^n$ converges to some α^{**} and $(\rho^n)^n$ goes to infinity.

Consider the case in which $\bar{\theta}(\cdot)$ is not constant. Then $|\sin(\bar{\theta}(t) - \alpha^{**})| > C$ for t belonging to some (closed) interval $I = [a, a+\tau] \subset [0, \bar{T}]$ and for some $C > 0$, which implies that $|\sin(\theta^n(t) - \alpha^n)| \geq C$ on I, for n suitably large. Then by equation (7) we get $|\dot{p}_\theta^n| \geq C\rho^n$ on I and therefore there exist $\eta > 0$ and a family of intervals $I^n \subset I$ with $\mu(I^n) \geq \eta$ such that $\lim_n \inf_{I^n} |p_\theta^n(t)| = \infty$. Indeed, for every $t \in [a, a+\tau/3]$, every $t' \in [a + 2\tau/3, a+\tau]$ and for every n we have that $|p_\theta^n(t) - p_\theta^n(t')| \geq C\rho^n\tau/3$, therefore it must be $|p_\theta^n(t)| \geq C\rho^n\tau/6$ on $[a, a+\tau/3]$ or $[a + 2\tau/3, a+\tau]$.

For every n, consider $p_\kappa^n(t)$ for $t \in I^n$. Since $\dot{p}_\kappa^n = -p_\theta^n + \frac{\partial L}{\partial \kappa}$, from (2) we have that there exists a family of subintervals $I^{n\prime} \subset I^n$ with $\mu(I^{n\prime}) \geq c > 0$ such that $\lim_n \inf_{I^{n\prime}} |p_\kappa^n| = \infty$. Since $\kappa^n(\cdot)$ is uniformly bounded and by (6) we get that $|u^n|$ is unbounded in $I^{n\prime}$, which contradicts the uniform boundedness of u^n in the L^p norm. Then ρ^n is uniformly bounded.

Let us now consider the case in which $\bar{\theta}(\cdot)$ is constant, that is $\bar{\theta}(\cdot) \equiv \bar{\kappa}(\cdot) \equiv \bar{u}(\cdot) \equiv 0$. First of all, we notice that $\alpha^{**} = 0$; indeed, if not, then we can proceed as above and get a contradiction. The Hamiltonian at $t = 0$ writes $H = \rho^n \cos\alpha^n + p_\theta^n(0)\kappa^n(0) - 1 - L(\kappa^n(0), 0) = 0$. If $(\rho^n)^n$ goes to infinity, then also $|p_\theta^n(0)|$ does, as well as the ratio $|p_\theta^n(0)|/\rho^n$. This comes from the fact that the Hamiltonian is null and $\kappa^n(0)$ tends to zero. Since $|\dot{p}_\theta^n|/\rho^n \leq 1$, we get as above that $|p_\theta^n(\cdot)|$ goes to infinity on $[0, T^n]$, which leads to a contradiction. We can then conclude that ρ^n is uniformly bounded.

In particular, we obtain that $|\dot{p}_\theta^n|$ is uniformly bounded with respect to n, and therefore that $|p_\theta^n(0)|$ is uniformly bounded: if not, in fact, $|p_\theta^n(\cdot)|$ would go to infinity on the whole interval $[0, T^n]$, and then also $|p_\kappa^n(\cdot)|$ would go to infinity (on some appropriate subinterval), and we would get a contradiction.

Then $|P^n(0)|$ is uniformly bounded and therefore, up to subsequences, we get that $(P^n(0))^n$ converges to some P^*.

Let us now consider the system

$$\begin{cases} \dot{Q}^n = \frac{\partial H}{\partial P}(Q^n, P^n, -1, \Upsilon(\kappa^n, p_\kappa^n)) \\ \dot{P}^n = -\frac{\partial H}{\partial Q}(Q^n, P^n, -1, \Upsilon(\kappa^n, p_\kappa^n)) \end{cases} \tag{13}$$

where $\Upsilon(\kappa, p_\kappa)$ is the control expressed in terms of κ and p_κ by means of the implicit function theorem applied to (6), namely $\Upsilon(\kappa, p_\kappa) = \left(\frac{\partial L}{\partial u}(\kappa, \cdot)\right)^{-1}(p_\kappa)$.

Since the initial conditions $(Q^n(0), P^n(0))$ of (13) converge to $(\bar{Q}(0), P^*)$, we have the convergence of the solutions $(Q^n(\cdot), P^n(\cdot))$ to some solution $(\bar{Q}(\cdot), P^*(\cdot))$ of (13). By uniqueness of the adjoint vector (see Remark 4) we get that $P^*(\cdot) = \bar{P}(\cdot)$. □

4. Robustness

In this section we study the continuity of the mapping which associates with a cost in $\mathcal{L}$ the set of solutions of the corresponding optimal problem $\mathbf{P}_2(L)$ or $\mathbf{P}_1(\ell)$ with fixed terminal point. We will show that when a sequence of costs converge to a cost L or ℓ in $\mathcal{L}$, any sequence of optimal trajectories associated with these costs converges uniformly to the set of optimal trajectories of respectively $\mathbf{P}_2(L)$ or $\mathbf{P}_1(\ell)$. This can be understood as a property of robustness of these optimal control problems. The class of costs $\mathcal{L}$ being defined as the disjoint union of two sub-classes $\mathcal{L}_1$ and $\mathcal{L}_2$, we have to distinguish several cases: convergence of a sequence in $\mathcal{L}_2$ to a cost in $\mathcal{L}_2$ as well, of a sequence in $\mathcal{L}_2$ to a cost in $\mathcal{L}_1$, and of a sequence in $\mathcal{L}_1$ to a cost in $\mathcal{L}_1$. In each case, we have to specify the notion of convergence we use.

Robustness of $\mathbf{P}_2(L)$. Consider functions L_0 and L_ε, $\varepsilon \in (0, \bar{\varepsilon}]$, belonging to $\mathcal{L}_2$, and assume that L_0 satisfies **(H3)** with $p = \hat{p}$. Let L_ε converge to L_0 in the following sense:

(H4) there exists a constant $C > 0$ such that for every (r, s) one has

$$|L_\varepsilon(r,s) - L_0(r,s)| + \left|\frac{\partial L_\varepsilon}{\partial r}(r,s) - \frac{\partial L_0}{\partial r}(r,s)\right| \leq C\varepsilon(|s|^{\hat{p}} + 1).$$

(H5) $\frac{\partial^2 L_\varepsilon}{\partial s^2}(r,s)$ converges uniformly to $\frac{\partial^2 L_0}{\partial s^2}(r,s)$ on compact subsets of $\mathbb{R}^2$.

We notice that hypothesis **(H4)** in particular implies that for every $\varepsilon > 0$ small enough L_ε satisfies condition **(H3)** with $p = \hat{p}$.

As in Section 3.2, we use $\mathcal{E}_2(L, q_1)$ to denote the set of curves $(Q(\cdot), P(\cdot))$ such that $Q(\cdot)$ is a solution of $\mathbf{P}_2(L)$ with final condition q_1 and $P(\cdot)$ the associated adjoint vector and we endow the set of continuous curves $(Q(\cdot), P(\cdot))$ with the distance d_∞ associated with the norm of the uniform convergence.

Theorem 5. *Fix $q_1 \in \mathbb{R}^2 \times S^1$ and let L_ε, $\varepsilon \geq 0$, be a family of costs in $\mathcal{L}_2$ satisfying hypotheses* **(H4)–(H5)***, with L_0 satisfying* **(H3)** *for $p = \hat{p}$. Then, for any family $(Q^\varepsilon(\cdot), P^\varepsilon(\cdot)) \in \mathcal{E}_2(L_\varepsilon, q_1)$, $\varepsilon \geq 0$, we have*

$$\lim_{\varepsilon \to 0} d_\infty \left((Q^\varepsilon(\cdot), P^\varepsilon(\cdot)), \mathcal{E}_2(L_0, q_1) \right) = 0.$$

This theorem is a direct consequence of the following lemma.

Lemma 4. *Under the hypotheses of Theorem 5, if $(\varepsilon_n)^n$ is a sequence converging to 0, then, for any sequence $(Q^{\varepsilon_n}(\cdot), P^{\varepsilon_n}(\cdot))^n$ such that, for every n, $(Q^{\varepsilon_n}(\cdot), P^{\varepsilon_n}(\cdot)) \in \mathcal{E}_2(L_{\varepsilon_n}, q_1)$, there exists a subsequence which uniformly converges to an extremal $(Q^*(\cdot), P^*(\cdot))$ in $\mathcal{E}_2(L_0, q_1)$.*

Proof. We choose first an extremal $(Q^0(\cdot), P^0(\cdot)) \in \mathcal{E}_2(L_0, q_1)$. For $\varepsilon = 0$ and $\varepsilon \in (\varepsilon_n)^n$, we call $u^\varepsilon : [0, T^\varepsilon] \to \mathbb{R}$ the optimal control of the problem $\mathbf{P}_2(L_\varepsilon)$ associated with the extremal $(Q^\varepsilon(\cdot), P^\varepsilon(\cdot))$ and we denote $Q^\varepsilon(\cdot) = (x^\varepsilon(\cdot), y^\varepsilon(\cdot), \theta^\varepsilon(\cdot), \kappa^\varepsilon(\cdot))$ and $P^\varepsilon(\cdot) = (p_x^\varepsilon, p_y^\varepsilon, p_\theta^\varepsilon(\cdot), p_\kappa^\varepsilon(\cdot))$.

Let us introduce the functional $J_2^\varepsilon(T, \kappa_0, u) = \int_0^T (1 + L_\varepsilon(\kappa(s), u(s)))\, ds$, for every $\varepsilon \in [0, \bar{\varepsilon}]$, which is evaluated along trajectories of (3) starting from $(0, 0, 0, \kappa_0)$. By definition, for every integer n we have $J_2^{\varepsilon_n}(T^{\varepsilon_n},\kappa_0^{\varepsilon_n},u^{\varepsilon_n}) \le J_2^{\varepsilon_n}(T^0,\kappa_0^0,u^0)$, and moreover

$$|J_2^{\varepsilon_n}(T^0,\kappa_0^0,u^0) - J_2^0(T^0,\kappa_0^0,u^0)| \le \int_0^{T^0} |L_{\varepsilon_n}(\kappa^0(s),u^0(s)) - L_0(\kappa^0(s),u^0(s))|\, ds \le C\varepsilon_n(\|u^0\|_{L^{\hat p}}^{\hat p} + T^0), \quad (14)$$

so that $J_2^{\varepsilon_n}(T^{\varepsilon_n},\kappa_0^{\varepsilon_n},u^{\varepsilon_n}) \le J_2^0(T^0,\kappa_0^0,u^0) + C\varepsilon_n(\|u^0\|_{L^{\hat p}}^{\hat p} + T^0)$. By **(H3)**, **(H4)** and the special structure of the cost we thus have that both T^{ε_n} and $\|u^{\varepsilon_n}\|_{L^{\hat p}}$ are uniformly bounded. In particular there exists a subsequence, that we denote again $(\varepsilon_k)^k$, converging to zero such that $(T^{\varepsilon_k})^k$ converges to some $T^* > 0$. Whenever $T^{\varepsilon_k} < T^*$, we extend the control $u^{\varepsilon_k}(\cdot)$ on the interval $[0,T^*]$, putting it equal to zero on $(T^{\varepsilon_k},T^*]$. Since $(u^{\varepsilon_k})^k$ is a bounded sequence in $L^{\hat p}([0,T^*])$, up to subsequences it weakly converges to some $u^*(\cdot) \in L^{\hat p}([0,T^*])$.

We can then repeat the argument of Lemma 1 to prove that $\kappa_0^{\varepsilon_k}$ is bounded, so that it converges to some κ_0^*, up to subsequences. We call $Q^*(\cdot) = (x^*(\cdot),y^*(\cdot),\theta^*(\cdot),\kappa^*(\cdot))$ the solution of (3) associated with $u^*(\cdot)$ with initial condition $Q^*(0) = (0,0,0,\kappa_0^*)$. Then, the weak convergence of $(u^{\varepsilon_k}(\cdot))^k$ to $u^*(\cdot)$ implies that $(\kappa^{\varepsilon_k}(\cdot))^k$ converges to $\kappa^*(\cdot)$ pointwise on $[0,T^*]$ and that $\kappa^{\varepsilon_k}(\cdot)$ is (uniformly) Hölder continuous for every $\varepsilon_k > 0$. Then by Ascoli-Arzelà Theorem, we obtain that $(\kappa^{\varepsilon_k}(\cdot))^k$ converges uniformly to $\kappa^*(\cdot)$.

Therefore, we obtain that the trajectories $Q^{\varepsilon_k}(\cdot)$ uniformly converge to $Q^*(\cdot)$. In particular, $(x^*(T^*),y^*(T^*),\theta^*(T^*)) = q_1$.

Since by definition u^0 is a minimum for $\mathbf{P}_2(L_0)$, it holds $J_2^0(T^0,\kappa_0^0,u^0) \le J_2^0(T^*,\kappa_0^*,u^*)$. From Theorem 2 we get that J_2^0 is weakly lower semicontinuous, and therefore

$$\liminf_{k\to\infty} J_2^0(T^{\varepsilon_k},\kappa^{\varepsilon_k},u^{\varepsilon_k}) \ge \liminf_{k\to\infty} J_2^0(T^*,\kappa_0^{\varepsilon_k},u^{\varepsilon_k}) \ge J_2^0(T^*,\kappa_0^*,u^*),$$

where we used the fact that $u^{\varepsilon_k}(\cdot)$ is set equal to zero on $[T^{\varepsilon_k},T^*]$ if $T^{\varepsilon_k} < T^*$. Moreover, by hypothesis **(H4)**, we have that for every $\varepsilon \in (\varepsilon_n)^n$,

$$J_2^0(T^\varepsilon,\kappa_0^\varepsilon,u^\varepsilon) \le J_2^\varepsilon(T^\varepsilon,\kappa_0^\varepsilon,u^\varepsilon) + C\varepsilon(T^\varepsilon + \|u^\varepsilon\|_{L^{\hat p}}^{\hat p}) \le J_2^\varepsilon(T^0,\kappa_0^0,u^0) + C\varepsilon(T^\varepsilon + \|u^\varepsilon\|_{L^{\hat p}}^{\hat p}).$$

Passing to the lim inf and applying (14) we obtain

$$J_2^0(T^*,\kappa_0^*,u^*) \le \liminf_{k\to\infty}\left(J_2^{\varepsilon_k}(T^0,\kappa_0^0,u^0) + C\varepsilon_k(T^{\varepsilon_k} + \|u^{\varepsilon_k}\|_{L^{\hat p}}^{\hat p})\right) = J_2^0(T^0,\kappa_0^0,u^0),$$

that is, u^* is a minimum.

Let us now prove the convergence of the adjoint vectors. As above, we first show that $P^{\varepsilon_k}(0)$ is bounded, and then we prove that the right-hand side of the adjoint equation (5) converges uniformly on compact sets: then also its solution converges uniformly.

As usual, we write $(p_x^\varepsilon,p_y^\varepsilon) = \rho^\varepsilon(\cos\alpha^\varepsilon,\sin\alpha^\varepsilon)$, $\rho^\varepsilon > 0$. Let us prove that ρ^{ε_k} is uniformly bounded. Assume by contradiction that this is not true. Up to restricting to an appropriate subsequence we can assume that $(\alpha^{\varepsilon_k})^k$ converges to α^* while ρ^{ε_k} goes to infinity. By previous results $\theta^{\varepsilon_k}(\cdot)$ converges to $\theta^*(\cdot)$; as in Lemma 3, we can assume that $\theta^*(\cdot)$ may be constant or not.

In the first case, if θ^* is constant then $\kappa^* \equiv u^* \equiv 0$, and therefore $Q^*(\cdot)$ is the straight line connecting $q_0 = (0,0,0)$ to $q_1 = (T^*,0,0)$. Moreover, since the Hamiltonian is identically zero along extremal trajectories and from the transversality conditions we get that $P^*(\cdot) \equiv (1,0,0,0)$; since we know that the only solution for $\mathbf{P}_2(L_\varepsilon)$ with final point $q_1 = (T^*,0,0)$ is the straight line for every $\varepsilon > 0$, we obtain that $T^{\varepsilon_k} = T^*$ and $P^{\varepsilon_k}(t) \equiv (1, 0, 0, 0)$, for every k.

Consider the second case, that is $\theta^*(\cdot)$ is not constant. Applying the same argument of Lemma 3, we can assume that there exist $\eta > 0$ and a family of intervals I^{ε_k} with $\mu(I^{\varepsilon_k}) \geq \eta$ such that $\lim_k \inf_{I^{\varepsilon_k}} |p_\theta^{\varepsilon_k}(t)| = \infty$. Since $\dot{p}_\kappa^{\varepsilon_k} = -p_\theta^{\varepsilon_k} + \frac{\partial L_{\varepsilon_k}}{\partial \kappa}$ and by (2) and hypothesis **(H4)**, we obtain that there exist a $c > 0$ and a family of subintervals $I^{\varepsilon_k\prime} \subset I^{\varepsilon_k}$ with $\mu(I^{\varepsilon_k\prime}) \geq c > 0$ such that $\lim_k \inf_{I^{\varepsilon_k\prime}} |p_\kappa^{\varepsilon_k}| = \infty$. This implies that also $|u^{\varepsilon_k}|$ is unbounded in $I^{\varepsilon_k\prime}$, which contradicts the uniform boundedness of $\|u^{\varepsilon_k}\|_{L^{\hat{p}}}$. Then we obtain that ρ^{ε_k} is uniformly bounded.

In particular, this implies that $\dot{p}_\theta^{\varepsilon_k}$ is uniformly bounded with respect to ε_k. As in Lemma 3, we can conclude that also $p_\theta^{\varepsilon_k}(0)$ is uniformly bounded and then we get the convergence of the initial conditions $P^{\varepsilon_k}(0)$ to some $(p_x^{**}, p_y^{**}, p_\theta^{**}(0), 0)$, up to subsequences.

Let us now consider the system, for $\varepsilon \in (\varepsilon_n)^n$,

$$\begin{cases} \dot{p}_\theta^\varepsilon = p_x^\varepsilon \cos\theta^\varepsilon - p_y^\varepsilon \sin\theta^\varepsilon \\ \dot{p}_\kappa^\varepsilon = -p_\theta^\varepsilon + \frac{\partial L_\varepsilon}{\partial \kappa}(\kappa^\varepsilon, \Upsilon_\varepsilon(\kappa^\varepsilon, p_\kappa^\varepsilon)) \\ \dot{\theta}^\varepsilon = \kappa^\varepsilon \\ \dot{\kappa}^\varepsilon = \Upsilon_\varepsilon(\kappa^\varepsilon, p_\kappa^\varepsilon) \end{cases}, \tag{15}$$

where $\Upsilon_\varepsilon(\kappa, p_\kappa)$ is the control expressed in function of κ and p_κ by means of the implicit function theorem and the maximality condition (6), namely $\Upsilon_\varepsilon(\kappa, p_\kappa) = \left(\frac{\partial L_\varepsilon}{\partial u}(\kappa, \cdot)\right)^{-1}(p_\kappa)$

Let us prove that $\Upsilon_{\varepsilon_k}(\cdot,\cdot)$ converges to $\Upsilon_0(\cdot,\cdot)$ uniformly on the compact subsets of $\mathbb{R}^2$. Fix r and consider $\upsilon_{\varepsilon_k}^r(s) = \Upsilon_{\varepsilon_k}(r, s)$. The function $\upsilon_{\varepsilon_k}^r$ is the solution of the Cauchy problem

$$\begin{cases} \upsilon_{\varepsilon_k}^{r\prime} = 1 \Big/ \frac{\partial^2 L_{\varepsilon_k}}{\partial u^2}(r, \upsilon_{\varepsilon_k}^r) \\ \upsilon_{\varepsilon_k}^r(0) = 0 \end{cases}.$$

Thanks to hypothesis **(H5)**, the right-hand side of the equation converges uniformly on compact sets, so that also the solution of the Cauchy problem converges. Then $\Upsilon_{\varepsilon_k}(\cdot,\cdot)$ converges to $\Upsilon_0(\cdot,\cdot)$ uniformly on the compact subsets of $\mathbb{R}^2$.

The latter, together with hypothesis **(H4)** and the uniform convergence of the pair $(p_x^{\varepsilon_k}, p_y^{\varepsilon_k})$, implies that the right-hand side of (15) converges uniformly on compact subsets as ε_k goes to zero. Since also the initial conditions of (15) converge, we obtain that the quadruple $(p_\theta^{\varepsilon_k}(\cdot), p_\kappa^{\varepsilon_k}(\cdot), \theta^{\varepsilon_k}(\cdot), \kappa^{\varepsilon_k}(\cdot))$ converges uniformly on $[0, \bar{T}]$ to the solution $(p_\theta^{**}(\cdot), p_\kappa^{**}(\cdot), \theta^*(\cdot), \kappa^*(\cdot))$ of the system

$$\begin{cases} \dot{p}_\theta^{**} = p_x^{**} \cos\theta^* - p_y^{**} \sin\theta^* \\ \dot{p}_\kappa^{**} = -p_\theta^{**} + \frac{\partial L_0}{\partial \kappa}(\kappa^*, \Upsilon_0(\kappa^*, p_\kappa^{**})) \\ \dot{\theta}^* = \kappa^* \\ \dot{\kappa}^* = \Upsilon_0(\kappa^*, p_\kappa^{**}) \end{cases} \tag{16}$$

with initial condition $(p_\theta^{**}, 0, 0, \kappa^*)$. Call $P^{**}(\cdot) = (p_x^{**}, p_y^{**}, p_\theta^{**}(\cdot), p_\kappa^{**}(\cdot))$; since the pair $(Q^*(\cdot), P^{**}(\cdot))$ satisfies the PMP, by the uniqueness of the adjoint vector we have that $P^{**}(\cdot) = P^*(\cdot)$. □

Robustness of $\mathbf{P}_1(\ell)$ w.r.t. $\mathbf{P}_2(L)$. Consider a cost ℓ_0 in $\mathcal{L}_1$, and a family of costs L_ε in $\mathcal{L}_2$, $\varepsilon \in (0, \bar{\varepsilon}]$ converging to ℓ_0 in the following sense:

(H4′) there exist three constants $C, C_1, C_2 > 0$ such that for every (r, s) one has

$$C_1\varepsilon(|s|^{\hat{p}} - 1) \leq L_\varepsilon(r, s) - \ell_0(r) \leq C_2\varepsilon(|s|^{\hat{p}} + 1) \tag{17}$$

$$\left|\frac{\partial L_\varepsilon}{\partial r}(r, s) - \ell_0'(r)\right| \leq C\varepsilon(|s|^{\hat{p}} + 1). \tag{18}$$

Remark 5. *It is not our purpose here to find the minimal assumptions under which robustness results can be obtained. If needed, the hypotheses above could certainly be weakened. We notice moreover that* **(H4′)** *implies that, for $\varepsilon \in (0, \bar{\varepsilon}]$, L_ε satisfies* **(H3)** *with $p = \hat{p}$.*

In the following we consider again the sets $\mathcal{E}_2(L, q_1)$ and $\mathcal{E}_1(\ell, q_1)$ defined in Section 3.2, and we also use $\widetilde{\mathcal{E}}_1(\ell, q_1)$ to denote the set of curves $(Q(\cdot), P(\cdot))$ such that $Q(\cdot) = (q(\cdot), \kappa(\cdot))$ and $P(\cdot) = (p(\cdot), 0)$, where $(q(\cdot), p(\cdot)) \in \mathcal{E}_1(\ell, q_1)$ and $\kappa(\cdot)$ is the associated optimal control. We endow these sets with the distance d_∞ associated with the norm of the uniform convergence.

Theorem 6. *Fix $q_1 \in \mathbb{R}^2 \times S^1$. Let ℓ_0 be a cost in $\mathcal{L}_1$, and let L_ε, $\varepsilon \in (0, \bar{\varepsilon}]$ be a family of functions in $\mathcal{L}_2$. Then, for any family $(Q^\varepsilon(\cdot), P^\varepsilon(\cdot)) \in \mathcal{E}_2(L_\varepsilon, q_1)$, $\varepsilon \geq 0$, we have*

$$\lim_{\varepsilon \to 0} d_\infty\left((Q^\varepsilon(\cdot), P^\varepsilon(\cdot)), \widetilde{\mathcal{E}}_1(\ell_0, q_1)\right) = 0.$$

This theorem is a direct consequence of the following lemma.

Lemma 5. *Under the hypotheses of Theorem 6, if $(\varepsilon_n)^n$ is a sequence converging to 0, then, for any sequence $(Q^{\varepsilon_n}(\cdot), P^{\varepsilon_n}(\cdot))^n$ such that, for every n, $(Q^{\varepsilon_n}(\cdot), P^{\varepsilon_n}(\cdot)) \in \mathcal{E}_2(L_{\varepsilon_n}, q_1)$, there exists a subsequence which uniformly converges to a curve $((q^*(\cdot), \kappa^*(\cdot)), (p^*(\cdot), 0))$ in $\widetilde{\mathcal{E}}_1(\ell_0, q_1)$.*

Proof. For every $\varepsilon \in (0, \bar{\varepsilon}]$ we denote by $J_2^\varepsilon(T, \kappa_0, u)$ the functional $\int_0^T 1 + L_\varepsilon(\kappa(s), u(s))\, ds$ associated with a solution of (3) starting from $(0, 0, 0, \kappa_0)$ and with control function $u(\cdot)$, and we set $J_1^0(T, \kappa) = \int_0^T 1 + \ell_0(\kappa(s))\, ds$.

For every $\varepsilon \in (\varepsilon_n)^n$, we call $u^\varepsilon : [0, T^\varepsilon] \to \mathbb{R}$ the optimal control of the problem $\mathbf{P}_2(L_\varepsilon)$ associated with the extremal $(Q^\varepsilon(\cdot), P^\varepsilon(\cdot))$ and we denote $Q^\varepsilon(\cdot) = (x^\varepsilon(\cdot), y^\varepsilon(\cdot), \theta^\varepsilon(\cdot), \kappa^\varepsilon(\cdot))$ and $P^\varepsilon(\cdot) = (p_x^\varepsilon, p_y^\varepsilon, p_\theta^\varepsilon(\cdot), p_\kappa^\varepsilon(\cdot))$.

Let $\kappa^0(\cdot) : [0, T^0] \to \mathbb{R}$ be an optimal control of the problem $\mathbf{P}_1(\ell_0)$ with final point q_1. From the PMP (in particular, see Remark 4) we know that $\kappa^0(\cdot) \in \mathcal{C}^1([0, T^0])$, so that we can define $u^0(\cdot) = \dot{\kappa}^0(\cdot)$, which is a continuous function on $[0, T^0]$. Hypothesis **(H4′)** implies that, for every integer n,

$$\begin{aligned} C_1\varepsilon_n(\|u^{\varepsilon_n}\|_{L^{\hat{p}}}^{\hat{p}} - T^{\varepsilon_n}) &\leq J_2^{\varepsilon_n}(T^{\varepsilon_n}, \kappa^{\varepsilon_n}(0), u^{\varepsilon_n}) - J_1^0(T^{\varepsilon_n}, \kappa^{\varepsilon_n}) \leq J_2^{\varepsilon_n}(T^{\varepsilon_n}, \kappa_0^{\varepsilon_n}, u^{\varepsilon_n}) - J_1^0(T^0, \kappa^0) \leq \\ &\leq J_2^{\varepsilon_n}(T^0, \kappa^0(0), u^0) - J_1^0(T^0, \kappa^0) \leq \int_0^{T^0} L_{\varepsilon_n}(\kappa^0(s), u^0(s)) - \ell_0(\kappa^0(s))\, ds \\ &\leq C_2\varepsilon_n(\|u^0\|_{L^{\hat{p}}}^{\hat{p}} + T^0). \end{aligned}$$

Since moreover $J_2^{\varepsilon_n}(T^{\varepsilon_n}, \kappa^{\varepsilon_n}(0), u^{\varepsilon_n}) \geq T^{\varepsilon_n}$, we therefore obtain that both T^{ε_n} and $\|u^{\varepsilon_n}\|_{L^{\hat{p}}}$ are uniformly bounded.

As in the proof of Lemma 4, up to taking a subsequence, T^{ε_k} converges to some $T^* > 0$. As above, we extend all the controls u^{ε_k} such that $T^{\varepsilon_k} < T^*$ putting them equal to zero on $(T^{\varepsilon_k}, T^*]$. Then we have that $(u^{\varepsilon_k}(\cdot))^k$ weakly converges to some $u^*(\cdot) \in L^{\hat{p}}([0, T^*])$, up to subsequences.

Again reasoning as in Lemma 1, we obtain that the trajectories $Q^{\varepsilon_k}(\cdot)$ uniformly converge to $Q^*(\cdot)$, where $Q^*(\cdot) = (x^*(\cdot), y^*(\cdot), \theta^*(\cdot), \kappa^*(\cdot))$ is the solution of (3) associated with $u^*(\cdot)$ with initial condition $Q^*(0) = (0, 0, 0, \kappa_0^*)$, and κ_0^* is the limit of $\kappa^\varepsilon(0)$, up to subsequences. In particular, $q^*(\cdot) = (x^*(\cdot), y^*(\cdot), \theta^*(\cdot))$ is the solution of (4) associated with the control $\kappa^*(\cdot)$ with initial condition q_0 and final condition q_1. By definition, $J_1^0(T^0, \kappa^0) \leq J_1^0(T^*, \kappa^*)$.

By hypothesis **(H2′)** and Theorem 2 we obtain that J_1^0 is weakly lower semicontinuous in L^p, and since $\kappa^{\varepsilon_k}(\cdot)$ converges uniformly to $\kappa^*(\cdot)$ we have

$$J_1^0(T^*, \kappa^*) \leq \liminf_{k\to\infty} J_1^0(T^*, \kappa^{\varepsilon_k}) = \liminf_{k\to\infty} J_1^0(T^{\varepsilon_k}, \kappa^{\varepsilon_k}),$$

Moreover, by hypothesis **(H4′)**, we have that for every $\varepsilon \in (\varepsilon_n)^n$,

$$J_1^0(T^\varepsilon, \kappa^\varepsilon) \leq J_2^\varepsilon(T^\varepsilon, \kappa^\varepsilon(0), u^\varepsilon) \leq J_2^\varepsilon(T^0, \kappa^0(0), u^0).$$

We thus obtain

$$J_1^0(T^*, \kappa^*) \leq \liminf_{k\to\infty} J_2^{\varepsilon_k}(T^0, \kappa^0(0), u^0) = J_1^0(T^0, \kappa^0),$$

that is $q^*(\cdot)$ is a solution of the problem $\mathbf{P}_1(\ell_0)$ with final condition q_1. Moreover, as proved above, the trajectories $(x^{\varepsilon_k}(\cdot), y^{\varepsilon_k}(\cdot), \theta^{\varepsilon_k}(\cdot))$ uniformly converge to $(x^*(\cdot), y^*(\cdot), \theta^*(\cdot))$.

Let us now investigate the convergence of the adjoint vectors. Since the first two components of the adjoint vector are constant with respect to time, we write $(p_x^{\varepsilon_k}, p_y^{\varepsilon_k}) = \rho^{\varepsilon_k}(\cos\alpha^{\varepsilon_k}, \sin\alpha^{\varepsilon_k})$, $\rho^{\varepsilon_k} > 0$.

Repeating the same argument of Lemma 3 and 4, we have that $(\rho^{\varepsilon_k})^k$ converges to some ρ^{**} and $(\alpha^{\varepsilon_k})^k$ converges to α^{**}, up to subsequences. Indeed, as above, if ρ^{ε_k} was not uniformly bounded, then it is easy to see that $\|u^{\varepsilon_k}\|_{L^{\hat{p}}}$ would not be uniformly bounded.

The same argument proves that also $p_\theta^{\varepsilon_k}(0)$ is uniformly bounded and then we get the convergence of the initial conditions $(P^{\varepsilon_k}(0))^k$ to some $(p_x^{**}, p_y^{**}, p_\theta^{**}(0), 0)$, up to subsequences.

Let us now consider the system

$$\begin{cases} \dot{p}_\theta^{\varepsilon_k} = p_x^{\varepsilon_k}\cos\theta^{\varepsilon_k} - p_y^{\varepsilon_k}\sin\theta^{\varepsilon_k} \\ \dot{\theta}^{\varepsilon_k} = \kappa^{\varepsilon_k} \end{cases} . \tag{19}$$

The right-hand side of (19) converges uniformly on $[0, T^*]$, then we obtain that the pair $(p_\theta^{\varepsilon_k}(\cdot), \theta^{\varepsilon_k}(\cdot))$ converges uniformly on $[0, T^*]$ to the solution $(p_\theta^{**}(\cdot), \theta^*(\cdot))$ of the system

$$\begin{cases} \dot{p}_\theta^{**} = p_x^{**}\cos\theta^* - p_y^{**}\sin\theta^* \\ \dot{\theta}^* = \kappa^* \end{cases} \tag{20}$$

with initial condition $(p_\theta^{**}(0), 0)$. Moreover we have that $p_\kappa^{\varepsilon_k}(\cdot)$ converges to zero uniformly. This comes immediately from the equation $\dot{p}_\kappa^{\varepsilon_k}(\cdot) = -p_\theta^{\varepsilon_k}(\cdot) + \frac{\partial L_{\varepsilon_k}}{\partial\kappa}$, the convergence of the trajectories, and equations (18) and (9).

Call $p^{**}(\cdot) = (p_x^{**}, p_y^{**}, p_\theta^{**}(\cdot))$; since the pair $(q^*(\cdot), p^{**}(\cdot))$ satisfies the PMP, by the uniqueness of the adjoint vector we have that $p^{**}(\cdot) = p^*(\cdot)$. □

Robustness of $\mathbf{P}_1(\ell)$ w.r.t. $\mathbf{P}_1(\ell)$. Consider a cost ℓ_0 in $\mathcal{L}_1$ satisfying **(H3′)** with $p = \hat{p}$, and a family of costs ℓ_ε in $\mathcal{L}_1$, $\varepsilon \in (0, \bar{\varepsilon}]$, converging to ℓ_0 in the following sense:

(H4″) there exists a constant $C > 0$ such that for every r one has

$$|\ell_\varepsilon(r) - \ell_0(r)| \leq C\varepsilon(|r|^{\hat{p}} + 1).$$

(H5″) $\ell''_\varepsilon(r)$ converges uniformly to $\ell''_0(r)$ on compact subsets of $\mathbb{R}$.

Here again we consider the set of curves $\mathcal{E}_1(\ell, q_1)$ defined in Section 3.2, endowed with the distance d_∞ associated with the norm of the uniform convergence.

Theorem 7. *Fix $q_1 \in \mathbb{R}^2 \times S^1$ and let ℓ_ε be a family of costs in $\mathcal{L}_1$ satisfying hypotheses* **(H4″)–(H5″)**. *Then, for any family $(q^\varepsilon(\cdot), p^\varepsilon(\cdot)) \in \mathcal{E}_1(\ell_\varepsilon, q_1)$, $\varepsilon \geq 0$, we have*

$$\lim_{\varepsilon \to 0} d_\infty\left((q^\varepsilon(\cdot), p^\varepsilon(\cdot)), \mathcal{E}_1(\ell_0, q_1)\right) = 0.$$

The proof of this result is an easy adaptation of the one of Theorem 5.

5. Conclusion

In this paper we analyze continuity properties of a class of optimal control problems that is suitable to describe the human locomotion. These properties reflect the physiological nature of the problem, in the sense that human behavior is homogeneous and stable with respect to different kinds of perturbations. Since costs not satisfying such continuity properties are not suitable to describe the locomotion problem, this analysis is a necessary step to possibly reduce the class of candidate cost functions that can modelize the system.

The first property analyzed is what we call stability, namely, for an optimal control problem associated with a fixed cost, the continuity of the solutions with respect to small perturbations of the final point. This very natural property has been highlighted in the experiments performed by Laumond et al. [2–4].

For an optimal control problem associated with a final point, we call robustness the continuity of the solutions with respect to small perturbations of the cost. This property is also natural and is important to specify a candidate cost, in order to perform numerical simulations of the model. Indeed, if the class of cost functions is robust, then the solution of the problem does not depend critically on the choice of a particular cost function, but it is possible to reduce the problem to the study of a simpler cost function. This is in particular very effective if we can reduce the number of parameters of the problem by choosing a cost function that does not depend on the angular acceleration and its derivatives. In this paper we performed a first step in this direction, showing a continuity property between the problems associated with the class of costs that do depend on the angular acceleration and the problems associated with the class of cost that do only depend on the angular velocity.

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