Complexity of Nonholonomic Motion Planning

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Abstract

The complexity of motion planning amidst obstacles is a well modeled and understood notion. What is the increase of the complexity when the problem is to plan the trajectories of a nonholonomic robot? We show that this quantity can be seen as a function of paths and of the distance between the paths and the obstacles. We propose various definitions of it, from both topological and metric points of view, and compare their values. For two of them we give estimates which involve some ε-norm on the tangent space to the configuration space. Finally we apply these results to compute the complexity needed to park a car-like robot with trailers.

Key words: complexity, mobile robots, motion planning, non-linear control, nonholonomic robotic.

1 Introduction

How many maneuvers are needed to park a car? In a more general way, what is the algorithmic complexity required to plan the trajectories of a nonholonomic robot amidst obstacles?

A robot is said nonholonomic when its kinematic is represented by a non-linear control system of the form

$$\dot{x} = u_1 X_1(x) + \cdots + u_m X_m(x), \quad x \in CS, \tag{Σ}$$

the dimension n of the configuration space CS being greater than the number m of controls. In particular a path in CS can be not a trajectory of Σ), that is non-feasible for the robot.

Different approaches can be used to plan nonholonomic motions (see (Laumond *et al.*, 1998) for guidelines). However, most of them are based on the same scheme; namely first find a non-feasible path and then 'approximate' it by a nonholonomic motion (the meaning of approximation differs according to algorithms). What interested us is the complexity of the second step, the one due to nonholonomy.

A crucial point for this problem is to measure how far is the nonholonomic robot from an obstacle. This measure is given by the *sub-Riemannian distance* attached to the robot (or to the control system (Σ)). We define first the length of a trajectory γ_u of (Σ) as the integral of $\sqrt{u_1^2(t) + \cdots + u_m^2(t)}$ along the trajectory, where the $u_i(t)$'s are controls associated to γ_u. The sub-Riemannian distance between two points is the infimum of the length of γ_u taken on all the trajectories γ_u linking the two points (see (Bellaïche, 1996) or (Bellaïche *et al.*, 1998, §1) for general presentations).

The complexity due to nonholonomy appears then as a function of some path and of the distance between this path and the obstacles. (Bellaïche *et al.*, 1991; Laumond, 1993) have given first topological definitions of this complexity, but no estimates. We have also suggested a metric definition in a previous work (Bellaïche *et al.*, 1998).

In this paper, we resume these definitions and add a new one, the entropy. We show how to compute a function equivalent to the entropy and, under some conditions, to the metric complexity (Theorem 3). This equivalent is the integral along the path of some ε-norm of its tangent. A comparison between the various definitions of complexity allows then to bound the topological complexities (Theorem 1). Finally we study the complexity required to park a car-like robot pulling trailers.

Notice that the pincipal result, Theorem 3, is not proved in the paper. The proof of this result is long and technical. It is given in (Jean, 1999), which deals with entropy, complexity and Hausdorff dimension of submanifolds in sub-Riemannian geometry. The aim here is to present and use these geometrical results in the frame of nonholonomic robotics.

The outline of the paper is as follows. After introducing the nonholonomic motion planning problem, we give in § 2 several definitions of complexity and compare one with the other.

In § 3 we state our results on entropy and metric complexity. We describe first the structure of the tangent space (singular points, basis) and present the ε-norm on this space. We give a sufficient condition for the metric complexity to be equivalent to the entropy and detail the case of a path without singular points.

Finally § 4 is dedicated to the application of the previous results to the car-like robot with n trailers.

2 Complexity of nonholonomic motion planning

2.1 Nonholonomic motion planning amidst obstacles

We consider a nonholonomic robot. Each configuration of the robot is represented as a point in the configuration space CS. The motions of the robot are described by a nonholonomic control system (Σ), which is controllable. We assume here that CS is a real analytic manifold and that the vector fields $X_1, \dots, X_m$ defining (Σ) are analytic.

In the configuration space CS, the obstacles are represented as closed subsets O. The open set $CS - O$ is called the *free space*. The motion planning problem for the robot is:
given a and b in $CS - O$, find a trajectory of (Σ) contained in the free space, and joining a to b.

From Chow's Theorem (Chow, 1940), this problem has a solution if and only if a and b lie in the same connected component of the free space. Since $CS - O$ is an open set, this is equivalent to the existence of a path in $CS - O$ linking a to b. This argument suggests a general method to solve the problem. This method, called 'Approximation of a collision-free holonomic path' (Laumond *et al.*, 1998), has two steps:

1. find a path $\mathcal{C}$ in the free space linking a to b ($\mathcal{C}$ is the 'collision-free holonomic path');

2. approximate $\mathcal{C}$ by a trajectory of (Σ), close enough to be contained in the free space.

The complexity of the first step (motion planning problem for holonomic systems) is well modeled and understood. It depends on the geometric complexity of the environment, that is on the complexity of the geometric primitives modeling the obstacles and the robot in the real world (Canny, 1988; Schwartz & Sharir, 1983).

We are interested here in the second step: its complexity represents the increase of complexity due to the nonholonomy. This complexity can be modeled as the one of an output trajectory (that is a trajectory solution of the problem). We have then to define the complexity of a trajectory approximating a given path.

Let $\mathcal{C}$ be a path in the free space obtained by solving the first step of the method. For $\rho > 0$, we denote by $\mathrm{Tube}(\mathcal{C}, \rho)$ the tube of radius ρ centred at $\mathcal{C}$, that is

$$\mathrm{Tube}(\mathcal{C}, \rho) = \bigcup_{x \in \mathcal{C}} B_{SR}(x, \rho).$$

where $B_{SR}(x, \rho)$ is the open ball centred at x of radius ρ for the sub-Riemannian distance attached to (Σ).

We denote by ε the biggest radius ρ for which $\mathrm{Tube}(\mathcal{C}, \rho)$ is contained in the free space $CS - O$. We call ε the *size of the free space around* $\mathcal{C}$. We say that a trajectory of (Σ) approximate $\mathcal{C}$ if it is contained in $\mathrm{Tube}(\mathcal{C}, \varepsilon)$ and if it has the same extremities as $\mathcal{C}$.

Let us assume that we have already defined the complexity of a trajectory (it will be done in § 2.2). The complexity of the second step appears then as a function $\sigma(\mathcal{C}, \varepsilon)$ of the chosen path $\mathcal{C}$ and of the size ε of the free space around $\mathcal{C}$. This function is defined as the infimum of the complexity of γ, for all trajectories γ approximating $\mathcal{C}$. We call $\sigma(\mathcal{C}, \varepsilon)$ *the complexity of approximating* $\mathcal{C}$ *within* ε, or simply *the complexity of the path* $\mathcal{C}$.

Remark 1. This definition of complexity is quite restrictive, since it takes account only of trajectories lying in the tube. It is adapted in very obstructed spaces, where the distance to the obstacles is similar and small all along the path. Moreover it is better suited to paths roughly equidistant from the obstacles, like Voronoï diagrams.

Nevertheless it is useful in general to have such a notion of 'worst case' complexity. It gives at least an upper bound for the complexity. It can also be used to compute the complexity of a part of the path close to obstacles.

2.2 Complexity of a path

We have not yet defined the complexity of a trajectory. As proposed in our previous works (Bellaïche *et al.*, 1998; Jean, 1999), we present here several definitions.

We first restrict to 'bang-bang' trajectories. These trajectories correspond to controls $(u_1, \ldots, u_m)$ such that, at every time, one control equals ± 1 and the other are zero. The number of switches in the controls defines a complexity for this kind of trajectories. We set:

$$\sigma_{sw}(\mathcal{C}, \varepsilon) = \inf \left\{ \#\text{ of switches} \;\middle|\; \begin{array}{l} \gamma \text{ bang-bang trajectory of } (\Sigma) \\ \gamma \text{ joins } a \text{ to } b \\ \gamma \subset \text{Tube}(\mathcal{C}, \varepsilon) \end{array} \right\}.$$

The second definition is derived from the topological complexity of a real-valued function (it is the number of changes in the sign of variation of the function). The complexity of a trajectory appears then as the total number of sign changes for all the controls associated to the trajectory γ (for a bang-bang trajectory, this definition coincides with the previous one). This yields the *topological complexity*:

$$\sigma_t(\mathcal{C}, \varepsilon) = \inf \left\{ \begin{array}{r} \#\text{ of sign changes in the} \\ \text{controls assoc. to } \gamma \end{array} \;\middle|\; \begin{array}{l} \gamma \text{ trajectory of } (\Sigma) \\ \gamma \text{ joins } a \text{ to } b \\ \gamma \subset \text{Tube}(\mathcal{C}, \varepsilon) \end{array} \right\}.$$

A third way to define the complexity of a trajectory is to use the length of the trajectory γ. Since ε is the size of the free space around $\mathcal{C}$, we consider that an elementary step is 'to build a trajectory of length ε'. The number of elementary steps in a trajectory is then $\text{length}(\gamma)/\varepsilon$. We call *metric complexity* the complexity obtained in this way and we denote it by

$$\sigma_m(\mathcal{C}, \varepsilon) = \frac{1}{\varepsilon} \inf \left\{ \text{length}(\gamma) \;\middle|\; \begin{array}{l} \gamma \text{ trajectory of } (\Sigma) \\ \gamma \text{ joins } a \text{ to } b \\ \gamma \subset \text{Tube}(\mathcal{C}, \varepsilon) \end{array} \right\}.$$

We can introduce also a fourth definition of complexity. Assume that the sub-Riemannian distance function, and so the balls, are known (it is the case for the car-like robot, see (Souères & Laumond, 1996)). In this case, an efficient way to approximate a path is to use a covering of the path by collision-free balls (Khatib *et al.*, 1997). This algorithm suggests to define the complexity as the minimum number of balls of radius ε needed to cover $\mathcal{C}$. This complexity, denoted $e(\mathcal{C}, \varepsilon)$, is called the *$\varepsilon$-entropy of $\mathcal{C}$*.

Remark 2. In fact the complexity of the algorithm above should be defined as the minimum number of collision-free balls needed to cover $\mathcal{C}$. This definition would be adapted to spaces with few obstacles. An even better definition, proposed by Laumond, would involve visible sets instead of balls (Vendittelli & Laumond, 1997). Unfortunately the study of such a complexity is very intricated. Indeed it requires global estimations of the distance and the shortest paths, and such results are in general not available.

2.3 Relations between various complexities

From now on we are interested by the behaviour of a complexity $\sigma(\mathcal{C}, \varepsilon)$ for ε small (that is, in very obstructed spaces, see Remark 1). For a given path $\mathcal{C}$, the notation $f(\mathcal{C}, \varepsilon) \preceq g(\mathcal{C}, \varepsilon)$ means that there exists $k > 0$ such that, for ε small enough, $f(\mathcal{C}, \varepsilon) \leq kg(\mathcal{C}, \varepsilon)$. We write also $f(\mathcal{C}, \varepsilon) \asymp g(\mathcal{C}, \varepsilon)$ when $f(\mathcal{C}, \varepsilon) \preceq g(\mathcal{C}, \varepsilon)$ and $g(\mathcal{C}, \varepsilon) \preceq f(\mathcal{C}, \varepsilon)$.

Theorem 1. *For any path $\mathcal{C}$,*

$$\sigma_t(\mathcal{C}, \varepsilon) \leq \sigma_{sw}(\mathcal{C}, \varepsilon), \tag{1}$$

$$\sigma_m(\mathcal{C}, \varepsilon) \preceq e(\mathcal{C}, \varepsilon). \tag{2}$$

For a path $\mathcal{C}$ never tangent to the distribution $\langle X_1, \dots, X_m \rangle$,

$$\sigma_m(\mathcal{C}, \varepsilon) \preceq \sigma_t(\mathcal{C}, \varepsilon). \tag{3}$$

Proof. Inequality (1) is straightforward, since for a bang-bang trajectory the sign changes in the controls coincide with the switches.

Let us consider now a covering of a path $\mathcal{C}$ with M balls of radius ε. The centres of two adjacent balls can be linked by a trajectory of length $\leq 2\varepsilon$ included in the union of the two balls. Therefore the extremities of $\mathcal{C}$ can be linked by a trajectory of length $\leq 2M\varepsilon$ included in the union of balls, and so in $\text{Tube}(\mathcal{C}, \varepsilon)$. This proves (2).

Let $\mathcal{C}$ be a path never tangent to the distribution $\langle X_1, \dots, X_m \rangle$. Thus, for any $x \in \mathcal{C}$, the angle between $T_x\mathcal{C}$ and $\langle X_1, \dots, X_m \rangle(x)$ is greater than one specific $\theta > 0$. Consider a piece of trajectory of a X_i contained in $\text{Tube}(\mathcal{C}, \varepsilon)$. Its length cannot exceed $\varepsilon / \sin\theta + O(\varepsilon) \leq 2\varepsilon / \sin\theta$. It implies that, for a bang-bang trajectory γ included in the tube, the number of switches is greater than $\sin\theta \, \text{length}(\gamma)/(2\varepsilon)$. This proves that $\sigma_m(\mathcal{C}, \varepsilon) \preceq \sigma_{sw}(\mathcal{C}, \varepsilon)$.

We can obtain Inequality (3) in the same way. Indeed, if a trajectory of (Σ) contained in $\text{Tube}(\mathcal{C}, \varepsilon)$ has no sign changes in the controls associated, then its length is $\leq K\varepsilon / \sin\theta$ for some $K > 0$. This fact can easily be proved by using the coordinates and the estimation of Lemma 2 below. □

In addition to these relations, we will see below that σ_m and e are equivalent for generic paths. We show also how to estimate these complexities. Thus we have equivalents for σ_m and e, and lower bounds for σ_t and σ_{sw} (resulting from (3) and (1)) when the path is never tangent to the distribution. Unfortunately, at our knowledge there are no general upper bounds for these last two complexities. However some results are known for particular systems (Σ), for instance when the Lie algebra generated by the distribution is nilpotent (Jean & Koseleff, 1997).

3 Metric complexity and entropy

3.1 Paths and singular points

What is a path? It is first a one-dimensional submanifold of CS. It also has two extremities and it is connected. Moreover it is certainly piecewise regular, for instance piecewise analytic.

Let us forget the term 'piecewise' for the moment (we will see below in Remark 4 how to take it into account). The previous properties implies that a path $\mathcal{C}$ is an analytic submanifold of CS diffeomorphic to a closed interval in $\mathbb{R}$. It can be seen as a parameterized arc, whose parameterization $x : [0, T] \to \mathcal{C}$, $t \mapsto x(t)$, is an analytic diffeomorphism.

Although a path is a smooth submanifold, some points seem 'singular' for the nonholonomic structure. Let us precise this notion.

Let $s \geq 1$ be an integer and $x \in CS$ a point. We denote by $L^s(x)$ the linear subspace of T_xCS generated by the values at x of brackets $[[\dots[X_{i_1}, X_{i_2}], \dots], X_{i_k}]$ with no more than s elements (each i_j belongs to $\{1, \dots, m\}$). We call *growth vector at x* the following increasing sequence of dimensions

$$1 \leq \dim L^1(x) \leq \dots \leq \dim L^s(x) \leq \dots \leq \dim L^{r(x)}(x) = n.$$

The smaller $r(x)$ such that $\dim L^{r(x)}(x) = n$ is called *the degree of nonholonomy.*

A point $x \in \mathcal{C}$ is said to be *regular for $\mathcal{C}$* if the growth vector is constant on $\mathcal{C}$ near x. Otherwise x is said to be *singular for $\mathcal{C}$*. Regular points for $\mathcal{C}$ form an open dense set in $\mathcal{C}$ and singular points are isolated in $\mathcal{C}$.

This definition holds not only for paths but for any submanifold of CS. In particular one usually says singular point instead of point singular for CS. It is worth to notice that a point singular for $\mathcal{C}$ is singular but that a point regular for $\mathcal{C}$ can be singular.

3.2 The ε-norm

To determine complexities and entropy we need an estimation of the sub-Riemannian distance. This estimation holds on a compact set $\Omega \subset CS$. We denote by r the maximum of the degree of nonholonomy on Ω (this maximum exists because the X_i's are analytic vector fields).

Let $x \in \Omega$ and $\varepsilon > 0$. We consider the families of vector fields $(Y_1, \dots, Y_n)$ such that each Y_j is a bracket $[[X_{i_1}, X_{i_2}], \dots, X_{i_s}]$ of length $\ell(Y_j) = s \leq r$. On the (finite) set of these families, we have a function

$$\left| \det \left(Y_1(x)\varepsilon^{\ell(Y_1)}, \dots, Y_n(x)\varepsilon^{\ell(Y_n)} \right) \right|.$$

We say that the family $(Y_1, \dots, Y_n)$ is associated with (x, ε) if it achieves the maximum of this function. The value at x of an associated family is a basis of T_xCS.

Remark 3. Fix $x \in \Omega$. For ε smaller than some $\varepsilon_1(x) > 0$ any family associated with (x, ε) is of minimal total length, that is

$$\sum_{i=1}^{n} \ell(Y_i) = \sum_{s \geq 1} s(\dim L^s(x) - \dim L^{s-1}(x)).$$

Such families are often said to be adapted to the flag $L^s(x)$, $j \geq 1$, and are involved in the construction of privileged coordinates (Bellaïche, 1996).

After that we have a sequence $0 < \varepsilon_1(x) < \cdots < \varepsilon_N(x) < 1$ with the following property: the set of families associated with (x, ε) is constant for ε in an interval $(\varepsilon_k, \varepsilon_{k+1})$. Moreover all these families have the same total length $\sum_i \ell(Y_i)$ which is an increasing function of k. It is worth to notice that, on a submanifold N (CS itself or a path) $\varepsilon_1(x)$ is a continuous function of x on the set of regular points for N and tends to 0 as x tends to a singular point for N. But $\varepsilon_1(x)$ is always nonzero, even at singular points.

We can now state the result on sub-Riemannian balls, proved in (Jean, 1999).

Lemma 2. *Let $\Omega \subset CS$ be a compact set (for instance a path). There exist constants k, K and $\varepsilon_0 > 0$ such that, for all $x \in \Omega$ and $\varepsilon \leq \varepsilon_0$, if $(Y_1, \ldots, Y_n)$ is a family associated with (x, ε), then,*

$$B_{\mathrm{Y}}(x, k\varepsilon) \subset B(x, \varepsilon) \subset B_{\mathrm{Y}}(x, K\varepsilon), \tag{4}$$

where $B_{\mathrm{Y}}(x, \varepsilon) = \{x \exp(u_n Y_n) \cdots \exp(u_1 Y_1),\ |u_i| < \varepsilon^{\ell(Y_i)},\ 1 \leq i \leq n\}$.

This lemma extends the classical Ball-Box Theorem (Bellaïche, 1996; Gromov, 1996). Indeed, x being fixed and ε smaller than $\varepsilon_1(x)$ (see Remark 3), the estimate (4) is equivalent to the one of Ball-Box Theorem. In this case the u_i's are the canonical coordinates of the second kind and then are privileged coordinates at x (Bellaïche, 1996; Hermes, 1991). However $\varepsilon_1(x)$ can be infinitely small for x close to a singular point, though (4) holds uniformly on Ω for $\varepsilon \leq \varepsilon_0$, independent of x.

Lemma 2 suggests to introduce the following notation. Let $v \in T_xCS$. We denote by $(v_1^{\mathrm{Y}}, \ldots, v_n^{\mathrm{Y}})$ the coordinates of v in a basis $\mathrm{Y} = (Y_1(x), \ldots, Y_n(x))$ of T_xCS. We define then an *ε-norm* on T_xCS as

$$\|v\|_{x,\varepsilon} = \max\left\{ \frac{|v_i^{\mathrm{Y}}|}{\varepsilon^{\ell(Y_i)}} \mid 1 \leq i \leq n,\ \mathrm{Y} \text{ associated with } (x, \varepsilon) \right\}.$$

When v is the tangent $\dot{x}(t)$ to a path, the dependence with respect to $x(t)$ is implicit and we write the ε-norm as $\|\dot{x}(t)\|_\varepsilon$. It becomes then a function of t, which is piecewise continuous and so integrable on $[0, T]$.

3.3 Estimate of entropy and metric complexity

All the results given in this section are proved in (Jean, 1999).

Theorem 3. *For a path $\mathcal{C} \subset CS$, with a parameterization $x(t)$, $t \in [0, T]$,*

$$e(\mathcal{C}, \varepsilon) \asymp \int_0^T \|\dot{x}(t)\|_\varepsilon dt. \tag{5}$$

Let $t_1 < \cdots < t_k$ be the parameters of the points which are singular for $\mathcal{C}$ but not an extremity (that is $0 < t_1$ and $t_k < T$). The metric complexity of $\mathcal{C}$ satisfies

$$\int_0^{t_1-\varepsilon} \|\dot{x}(t)\|_\varepsilon dt + \int_{t_1+\varepsilon}^{t_2-\varepsilon} \|\dot{x}(t)\|_\varepsilon dt + \cdots + \int_{t_k+\varepsilon}^{T} \|\dot{x}(t)\|_\varepsilon dt \ \preceq\ \sigma_m(\mathcal{C}, \varepsilon) \ \preceq\ \int_0^T \|\dot{x}(t)\|_\varepsilon dt. \tag{6}$$

For a path with singular points, we can not know directly if complexity and entropy are equivalent or not. For instance we give in (Jean, 1999) a path for which they are not equivalent. However the inequalities of Theorem 3 grouped together provides a sufficient condition (we set $t_0 = 0$ and $t_{k+1} = T$):

$$\textit{if} \int_{t_{i-1}}^{t_i} \|\dot{x}(t)\|_\varepsilon dt \asymp \int_{t_{i-1}+\varepsilon}^{t_i-\varepsilon} \|\dot{x}(t)\|_\varepsilon dt \textit{ for } i = 1, \ldots, k+1, \quad \textit{then} \quad \sigma_m(\mathcal{C}, \varepsilon) \asymp e(\mathcal{C}, \varepsilon).$$

This condition is generically satisfied on the space of paths.

Remark 4. Theorem 3 still holds if we define the paths as continuous piecewise analytic submanifolds (see § 3.1). Indeed, in this case the inequalities of the theorem can be applied to each analytic part of the path. Equivalence (5) results then from the additivity of the entropy: $e(\cup \mathcal{C}_i, \varepsilon) \asymp \sum e(\mathcal{C}_i, \varepsilon)$. For the metric complexity, we just have to consider points of non-analyticity as singular points for $\mathcal{C}$ in the inequality (6).

For $x \in CS$ we have

$$T_x CS = L^1(x) \oplus L^2(x)/_{L^1(x)} \oplus \cdots \oplus L^r(x)/_{L^{r-1}(x)}. \tag{7}$$

This direct sum gives the decomposition $\dot{x}(t) = \dot{x}_1(t) + \cdots + \dot{x}_r(t)$ on $T_{x(t)}CS$. For a path $\mathcal{C}$ without singular points, the spaces in the direct sum above have constant dimensions and are continuous along $\mathcal{C}$. Then each $\dot{x}_s(t)$ is a continuous function of t on $[0, T]$. Moreover, for $\varepsilon < \varepsilon_1(x)$ the families associated with (x, ε) are adapted to the decomposition (7) and $\varepsilon_1(x)$ is continuous on $\mathcal{C}$ (see Remark 3). It implies that $\min_{x \in \mathcal{C}} \varepsilon_1(x) > 0$. For $\varepsilon < \min_{x \in \mathcal{C}} \varepsilon_1(x)$, the ε-norm of $\dot{x}(t)$ is then equivalent to $\max_s(\varepsilon^{-s}\|\dot{x}_s(t)\|)$, and the multiplicative constants in the equivalence are uniform on $\mathcal{C}$ (here $\|\cdot\|$ denotes an Euclidean norm on $T_{x(t)}CS$).

Corollary 4. *Let $\mathcal{C}$ be a path containing no singular points for $\mathcal{C}$. We write as above $\dot{x}(t) = \dot{x}_1(t) + \cdots + \dot{x}_r(t)$. Then*

$$\sigma_m(\mathcal{C}, \varepsilon) \asymp e(\mathcal{C}, \varepsilon) \asymp \int_0^T \max\left\{\varepsilon^{-1}\|\dot{x}_1(t)\|, \ldots, \varepsilon^{-r}\|\dot{x}_r(t)\|\right\} dt.$$

In particular, if $\mathcal{C}$ is everywhere tangent to $L^s(x)$ but not to $L^{s-1}(x)$, its entropy and metric complexity are equivalent to ε^{-s}.

4 Application to the car-like robot with trailers

A classical example of nonholonomic robot is the car-like robot pulling trailers (see (Bellaïche *et al.*, 1998, §2) for a survey). The car and each one of the n trailers are represented as two wheels connected by an axle. Each trailer is connected to the preceding one by means of a rigid bar, which is hitched in the middle of the axle in front of it (we consider the car as the trailer 0).

A configuration of the robot is parameterized by $q = (x, y, \theta_0, \dots, \theta_n)$, where (x, y) are the coordinates of the middle of the last trailer's axle and θ_i the orientation angle of the trailer $(n-i)$ (w.r.t. the x-axis, cf. figure 1). The two controls are ω and v, the tangential and the angular velocity of the car. The motion of the system is characterized by the equation:

$$\dot{q} = \omega X_1(q) + v X_2(q)$$

with

$$X_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} \cos\theta_0 \prod_{j=1}^{n} \cos(\theta_j - \theta_{j-1}) \\ \sin\theta_0 \prod_{j=1}^{n} \cos(\theta_j - \theta_{j-1}) \\ \sin(\theta_1 - \theta_0) \prod_{j=2}^{n} \cos(\theta_j - \theta_{j-1}) \\ \vdots \\ \sin(\theta_n - \theta_{n-1}) \\ 0 \end{pmatrix}.$$

A point is singular if at least one trailer, different from the last one, is perpendicular to the previous one (that is $\cos(\theta_i - \theta_{i-1}) = 0$ for some $i \geq 2$). We refer to (Jean, 1996) for a complete description of the system (growth vector, degree of nonholonomy, singular locus, basis of T_qCS).

We want to know the complexity required to park the robot and in particular to reverse into a parking space. This motion is represented by a path which involves a lateral moving of the last trailer. The tangent $T_q\mathcal{C}$ to such a path belongs to $L^{r(q)}(q)$ but not to $L^{r(q)-1}(q)$ for any $q \in \mathcal{C}$.

[Insert figure 1 about here]

a. Path included in the regular locus. We consider a path $\mathcal{C}_1(t)$, $t \in [0, T]$, with the following properties:

(i) the last trailer is moved at constant velocity and the angle between the direction of its moving and its orientation is $\phi(t)$;

(ii) $\cos(\theta_i - \theta_{i-1}) \neq 0 \; \forall i \geq 2$ along $\mathcal{C}_1$ (the path is then included in the regular locus);

(iii) $s(t) = \sin\big((\theta_1 - \theta_0)(t)\big) \neq 0$; this last assumption ensures that a tangential moving of the last trailer belongs to $L^{r-1}(q)$ but not to $L^{r-2}(q)$.

This path has no singular points, so we are going to apply Corollary 4.

At any regular point, the only basis adapted to the decomposition (7) is Y = $(Y_1(q), \ldots, Y_{n+3}(q))$ where the length of the bracket Y_j is $\ell(Y_j) = 1$ if $j = 1, 2$ and $\ell(Y_j) = j - 1$ if $j \geq 3$. On the other hand the tangent to the path is

$$\dot{q}(t) = \sin\phi(t)\ Y_{n+3}\big(q(t)\big) + s(t)\cos\phi(t)\ Y_{n+2}\big(q(t)\big) + \text{terms in } Y_{n+1}, \ldots, Y_1.$$

Since $\ell(Y_{n+3}) = n+2$, the first term is $\dot{q}_{n+2}$, the second $\dot{q}_{n+1}$ and so on (we use the notations of Corollary 4). Recall that the ε-norm of $\dot{q}(t)$ is equivalent to $\max_s(\varepsilon^{-s}\|\dot{q}_s(t)\|)$ and that $s(t) \neq 0$ (property (iii) above). Then for ε small enough, only $\dot{q}_{n+2}$ and $\dot{q}_{n+1}$ can appear in the ε-norm. Applying Corollary 4 we obtain

$$\sigma_m(\mathcal{C}_1, \varepsilon) \asymp e(\mathcal{C}_1, \varepsilon) \asymp \frac{1}{\varepsilon^{n+2}} \int_0^T \max\big(\varepsilon\,|s(t)\cos\phi(t)|, |\sin\phi(t)|\big)\, dt.$$

We keep the maximum in this expression to emphasize the behaviour of σ_m and e when $|\sin\phi(t)|$ tends to 0. Of course if $|\sin\phi(t)|$ is everywhere greater than a positive constant, then the complexity is equivalent to $1/\varepsilon^{n+2}$ (property (iii) is unnecessary in this case). For instance a parallel parking with all the trailers in line is a path satisfying this last condition. Thus to reverse a car into a parking space needs $1/\varepsilon^2$ maneuvers, and $1/\varepsilon^{n+2}$ maneuvers if the car is pulling n trailers.

b. Paths included in the singular locus ($n \geq 2$). We consider now a path $\mathcal{C}_2$ with the properties (i), (ii) and (iii) as above, except that $\cos(\theta_n - \theta_{n-1})$ is identically zero along the path.

The growth vector is constant along $\mathcal{C}_2$, so the path contains only regular points for $\mathcal{C}_2$. The basis adapted to the decomposition (7) is now Z = $(Z_1(q), \ldots, Z_{n+3}(q))$ where the length of the bracket Z_j is $\ell(Z_j) = \ell(Y_j)$ if $j = 1, 2, 3$ and $\ell(Z_j) = 2j - 5$ if $j \geq 4$. The tangential movings belong to $L^{2n-1}(q)$ but not to $L^{2n-2}(q)$ (notice that $L^{2n}(q)$ is equal to $L^{2n-1}(q)$). As before the terms appearing in the ε-norm are

$$\dot{q}_{2n+1} = \sin\phi(t)\ Z_{n+3}\big(q(t)\big) \quad \text{and} \quad \dot{q}_{2n-1} = s(t)\cos\phi(t)\ Z_{n+2}\big(q(t)\big).$$

From Corollary 4 the complexity is:

$$\sigma_m(\mathcal{C}_2, \varepsilon) \asymp e(\mathcal{C}_2, \varepsilon) \asymp \frac{1}{\varepsilon^{2n+1}} \int_0^T \max\big(\varepsilon^2|s(t)\cos\phi(t)|, |\sin\phi(t)|\big)\, dt.$$

The greatest complexity is obtained for a path $\mathcal{C}_3$ satisfying the properties (i) and (iii), but with $\cos(\theta_i - \theta_{i-1})$ identically zero along $\mathcal{C}_3$ for any $i \geq 2$. All points in the path are regular for $\mathcal{C}_3$, but their degree of nonholonomy is the maximal one. It is equal to F_{n+3}, the $(n+3)$th Fibonacci number, defined by the induction formula $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$. The tangential movings correspond to a bracket of length F_{n+2} and we have

$$\sigma_m(\mathcal{C}_3, \varepsilon) \asymp e(\mathcal{C}_3, \varepsilon) \asymp \frac{1}{\varepsilon^{F_{n+3}}} \int_0^T \max\big(\varepsilon^{F_{n+1}}|s(t)\cos\phi(t)|, |\sin\phi(t)|\big)\, dt.$$

As conjectured in (Laumond *et al.*, 1998), the maximal complexity of a path is then equivalent to $1/\varepsilon^{F_{n+3}}$.

c. Path going through the singular locus ($n \geq 2$). For this last example, we choose a path $\mathcal{C}_4$ corresponding to a purely lateral moving of the last trailer ($\phi = \pm\pi/2$). The angle $\theta_n - \theta_{n-1}$ goes linearly from $3\pi/4$ to $\pi/4$. All the other $\cos(\theta_i - \theta_{i-1})$ are nonzero along $\mathcal{C}_4$. The path can be parameterized by $t = \cot(\theta_n - \theta_{n-1})$, $t \in [-1, 1]$. Thus $t = 0$ is the parameter of the only singular point for $\mathcal{C}_4$.

Let us first determine the families associated with $(q(t), \varepsilon)$. When $t = 0$, Z is the only family associated with $(q(0), \varepsilon)$ as soon as $\varepsilon < \varepsilon_1(q(0))$. When $t \neq 0$ the basis with minimal total length is Y. Each Y_j can be written (Jean, 1996)

$$Y_j\big(q(t)\big) = \frac{t^{\ell(Z_j)-\ell(Y_j)}}{(\ell(Z_j) - \ell(Y_j))!} Z_j\big(q(t)\big) + \text{ lin. comb. of } Z_k,\ k < j. \tag{8}$$

We have then

$$\left|\det\left(\mathrm{Y}\big(q(t)\big)\varepsilon^{\ell(\mathrm{Y})}\right)\right| = \frac{1}{\prod_j (\ell(Z_j) - \ell(Y_j))!}\left(\frac{|t|}{\varepsilon}\right)^{\ell(\mathrm{Z})-\ell(\mathrm{Y})}\left|\det\left(\mathrm{Z}\big(q(t)\big)\varepsilon^{\ell(\mathrm{Z})}\right)\right|$$

where $\ell(\mathrm{Y})$ (resp. $\ell(\mathrm{Z})$) denotes the total length $\sum \ell(Y_j)$ (resp. $\sum \ell(Z_j)$). More generally any family W which can be associated with $(q(t), \varepsilon)$ has a total length $\ell(\mathrm{W})$ in $[\ell(\mathrm{Y}), \ell(\mathrm{Z})]$ and satisfies

$$\left|\det\left(\mathrm{W}\big(q(t)\big)\varepsilon^{\ell(\mathrm{W})}\right)\right| = a_{\mathrm{W}}\left(\frac{|t|}{\varepsilon}\right)^{\ell(\mathrm{Z})-\ell(\mathrm{W})}\left|\det\left(\mathrm{Z}\big(q(t)\big)\varepsilon^{\ell(\mathrm{Z})}\right)\right|.$$

Finally we have a sequence $1 = \lambda_1 < \cdots < \lambda_N$ such that:

- if $|t| < \varepsilon$: Z is the only family associated with $(q(t), \varepsilon)$;
- if $|t| > \varepsilon\lambda_N$: Y is the only family associated with $(q(t), \varepsilon)$;
- if $|t| \in [\varepsilon\lambda_i, \varepsilon\lambda_{i+1}]$: the families W associated with $(q(t), \varepsilon)$ have a length $\ell(\mathrm{W})$ contained in $[\ell(\mathrm{Y}), \ell(\mathrm{Z})]$.

We can now compute the ε-norm. We have chosen the path in such a way that its tangent is

$$\dot{q}(t) = Z_{n+3}\big(q(t)\big) + \text{terms in } Z_{n+2}, \ldots, Z_1.$$

Using (8) and the definition of the ε-norm, we obtain:

- if $|t| \leq \varepsilon$, $\|\dot{q}(t)\|_\varepsilon = \varepsilon^{-2n-1}$;
- if $|t| \geq \varepsilon\lambda_N$, $\|\dot{q}(t)\|_\varepsilon = (n-1)!|t|^{-n+1}\varepsilon^{-n-2}$.

For the intermediate values of $|t|$ we can show that the ε-norm is boundfed from below and from above respectively by its value at $\varepsilon\lambda_N$ and at ε. We neglect then these values in the integral of the ε-norm:

$$\int_{-1}^{1} \|\dot{q}(t)\|_\varepsilon dt \asymp \int_0^\varepsilon \varepsilon^{-2n-1} dt + \int_\varepsilon^1 \varepsilon^{-n-2}\frac{dt}{t^{n-1}}. \tag{9}$$

The second integral is greater than the first one. It results from Theorem 3 that metric complexity and entropy are equivalent and

$$\sigma_m(\mathcal{C}_4, \varepsilon) \asymp e(\mathcal{C}_4, \varepsilon) \asymp \frac{1}{\varepsilon^{2n}} \qquad \text{or} \qquad \frac{1}{\varepsilon^4}\log(\frac{1}{\varepsilon}) \qquad \text{if } n = 2.$$

This complexity is always greater than the complexity of a path like $\mathcal{C}_1$ contained in the regular locus. On the other hand it is less than the complexity of a path like $\mathcal{C}_2$, but the difference is smaller since we have, for $n \geq 3$, $\sigma_m(\mathcal{C}_4, \varepsilon) \asymp \varepsilon\sigma_m(\mathcal{C}_2, \varepsilon)$ though $\sigma_m(\mathcal{C}_4, \varepsilon) \asymp \varepsilon^{-n+2}\sigma_m(\mathcal{C}_1, \varepsilon)$.

Remark 5. In the second member of (9), the first integral represents the complexity required to go from $t = -\varepsilon$ to ε, that is to go through the singular locus. On the other hand the second integral is the complexity required to go near the singular locus.

Then going near the singular locus is more complicated than going through it.

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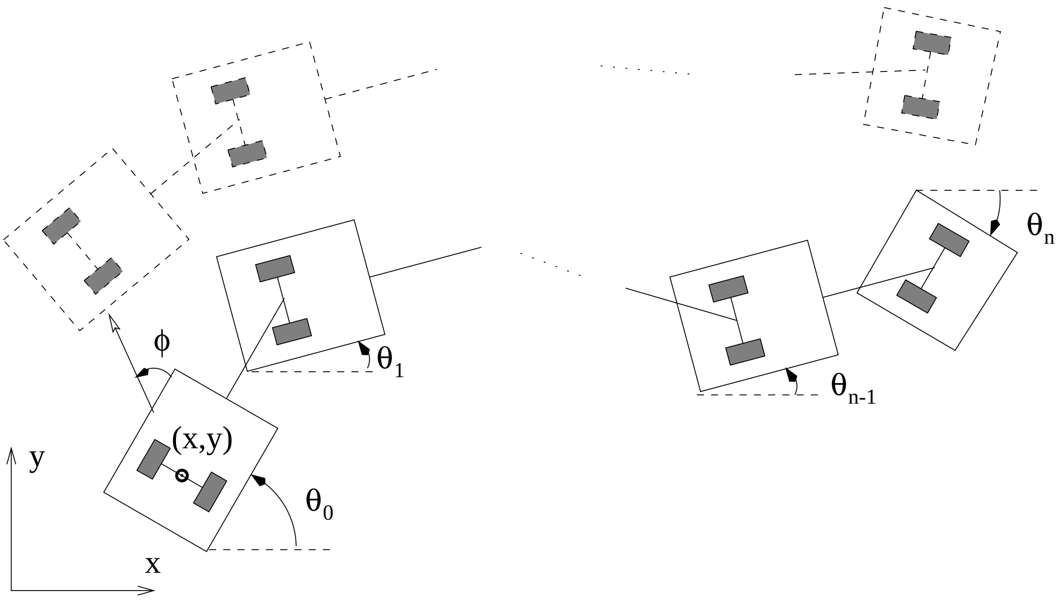

Figure 1: Configuration of the robot during a lateral moving.