Paths in Sub-Riemannian Geometry

Frédéric Jean

ENSTA, Laboratoire de Mathématiques Appliquées, 32, bd Victor, 75739 Paris cedex 15, France.

Abstract. In sub-Riemannian geometry only horizontal paths – i.e. tangent to the distribution – can have finite length. The aim of this talk is to study non-horizontal paths, in particular to measure them and give their metric dimension. For that we introduce two metric invariants, the entropy and the complexity, and corresponding measures of the paths depending on a small parameter ε.

We give estimates for the entropy and the complexity, and a condition for these quantities to be equivalent. The estimates depend on a ε-norm on the tangent space, which tends to the sub-Riemannian metric as ε goes to zero. The results are based on an estimation of sub-Riemannian balls depending uniformly of their radius.

1 Length of a Path

Let M be a real analytic manifold and $X_1, \dots, X_m$ analytic vector fields on M. We define a *sub-Riemannian metric g on M* by setting, for each $q \in M$ and $v \in T_qM$,

$$g_q(v) = \inf \left\{ u_1^2 + \cdots + u_m^2 \;\middle|\; u_1 X_1(q) + \cdots + u_m X_m(q) = v \right\} .$$

The *length* of an absolutely continuous path $c(t)$ $(0 \leq t \leq \tau)$ is defined as

$$\text{length}(c) = \int_0^\tau \sqrt{g_{c(t)}(\dot{c}(t))} dt .$$

The *sub-Riemannian distance* is $d(p,q) = \inf \text{length}(c)$, where the infimum is taken on all the absolutely continuous paths joining p to q.

An *horizontal path* is an absolutely continuous path tangent almost everywhere to the distribution $\langle X_1, \dots, X_m \rangle$. Only horizontal paths can have finite length. We are interested in the following questions: how to measure non-horizontal paths? how to compare the length of two such paths? More generally, which values can take metric invariants like Hausdorff dimension, entropy?

For an equiregular sub-Riemannian manifold, the Hausdorff dimension of submanifolds is known (Gromov [2]). However there are neither estimates of the entropy nor results in the general case.

In this talk, we will look at things from the point of view of approximating a path by finite sets. According to the chosen kind of finite set, a net or a chain, we introduce two metric invariants of a path, the entropy and the complexity, and the corresponding measures of the path.

We give computable estimates of these quantities (Theorems 2 and 3) and show in which case they are equivalent (Corollary 1). The result on the entropy allows in particular to compute Hausdorff and entropy dimensions. The estimates appear as integrals of some ε-norm of the tangent to the path. This ε-norm comes from a description of the shape of the sub-Riemannian balls depending uniformly of their radius (Theorem 1).

2 Nets and Chains

From now on we restrict the definition of path. What we call a path here is an analytic parameterized curve $c : [0,1] \to M$.

Let c be a path in M. For $\varepsilon > 0$, the set $Z \subset M$ is called an *ε-net* for c if for any q in c there is $z \in Z$ with $d(q,z) \leq \varepsilon$. We define the *metric entropy* $e(c,\varepsilon)$ as the minimal number of the elements in ε-nets for c. It is the minimal number of closed balls of radius ε needed to cover c.

The notion of entropy has been introduced by Kolmogorov [4]. Notice that it is usually defined as the logarithm of $e(c,\varepsilon)$.

The asymptotic behavior of $e(c,\varepsilon)$ as ε tends to 0 reflects the geometry of c in M. This behavior is characterized essentially by the *entropy dimension*

$$\dim_e c = \overline{\lim_{\varepsilon \to 0}} \frac{\log e(c,\varepsilon)}{\log(\frac{1}{\varepsilon})} \ .$$

In other words $\dim_e c$ is the infimum of β for which $e(c,\varepsilon) \leq (1/\varepsilon)^\beta$ for ε small enough.

A maybe more usual characterization of the geometry of a space uses the Hausdorff dimension and measure. The entropy is however easier to evaluate. Moreover in our case the Hausdorff dimension can be deduced from the entropy dimension thanks to the following properties: $\dim_{\mathrm{H}} c \leq \dim_e c$ and, if $e(c,\varepsilon) \sim \varepsilon^{-\beta_0}$, $\dim_{\mathrm{H}} c = \dim_e c = \beta_0$.

For $\varepsilon > 0$, we call *ε-chain* for c a sequence of points $v_0 = c(0), v_1, \ldots, v_k = c(1)$ in c where $d(v_i, v_{i+1}) \leq \varepsilon$ for $i = 0, \ldots, k-1$. We define the *complexity* $\sigma(c,\varepsilon)$ as the minimum number of points in an ε-chain for c.

Finally, as suggested by Gromov [2, p. 278], we propose two definitions of an ε-length of a path:

$$\mathrm{length}^e_\varepsilon(c) = \varepsilon \times e(c,\varepsilon) \qquad \text{and} \qquad \mathrm{length}^\sigma_\varepsilon(c) = \varepsilon \times \sigma(c,\varepsilon) \ .$$

We want to compare these definitions. For that we use the notations $\succeq$, $\preceq$ to denote the corresponding inequalities $\geq$, $\leq$ up to multiplicative constants (uniform with respect to ε, for ε small enough). We say that f and g are equivalent, and we write $f \asymp g$, when $f \succeq g$ and $f \preceq g$.

Remark first that we can construct a 4ε-chain from an ε-net. This shows a first inequality

$$\sigma(c,\varepsilon) \preceq e(c,\varepsilon) \ .$$

To obtain a reverse inequality, an additional metric property of the path is needed. Let $\mathrm{Tube}(c,\varepsilon)$ be the tube of radius ε centered at c. We say that a point q in the interior of c is *a cusp of* c if, for any constant $k \geq 1$, $\mathrm{Tube}(c,\varepsilon)$ without the ball $B(q,k\varepsilon)$ is connected for ε small enough. We use the term cusp by analogy with curve's singularity. Notice however that here the path is smooth and that the property is a metric one.

Now, if c has no cusp, an ε-chain is a $k\varepsilon$-net for some k. This shows the following property.

Lemma 1. *For a path c without cusp, $\sigma(c,\varepsilon) \asymp e(c,\varepsilon)$.*

One can show that the cusps are isolated in c. We will see in Sect. 4 that we have to compare these points with another set of isolated points. Let us define it.

For $s \geq 1$ and $q \in M$, we denote by $L^s(q)$ the subspace of T_qM spanned by values at q of the brackets of length $\leq s$ of vector fields $X_1, \ldots, X_m$. We say that $q \in c$ is a *regular point for* c if the integers $\dim L^s(q)$ $(s = 1, 2, \ldots)$ remain constant on c near q. Otherwise we say that q is a *singular point for* c. Singular points for c are isolated in c.

Notice that a cusp can be regular or singular for c and, conversely, that a singular point for c can be a cusp or not.

3 The ε-norm

Let $\Omega \subset M$ be a compact set (for instance a path). We denote by r the maximum of the degree of nonholonomy on Ω (recall that the degree of nonholonomy at q is the smaller s such that $\dim L^s(q) = n$).

Let $q \in \Omega$ and $\varepsilon > 0$. We consider the families of vector fields $(Y_1, \ldots, Y_n)$ such that each Y_j is a bracket $[[X_{i_1}, X_{i_2}], \ldots, X_{i_s}]$ of length $\ell(Y_j) = s \leq r$. On the set of these families, we have a function

$$\det\Big(Y_1(q)\varepsilon^{\ell(Y_1)}, \ldots, Y_n(q)\varepsilon^{\ell(Y_n)}\Big) \ .$$

We say that the family $\mathrm{Y} = (Y_1, \ldots, Y_n)$ is associated with (q,ε) if it achieves the maximum of this function. In particular the value at q of a family associated with (q,ε) forms a basis of T_qM.

With this definition we can describe the shape of the sub-Riemannian balls.

Theorem 1. *There exist constants k, K and $\varepsilon_0 > 0$ such that, for all $q \in \Omega$ and $\varepsilon \leq \varepsilon_0$, if $\mathrm{Y} = (Y_1, \ldots, Y_n)$ is a family associated with (q,ε), then,*

$$B_{\mathrm{Y}}(q,k\varepsilon) \ \subset \ B(q,\varepsilon) \ \subset \ B_{\mathrm{Y}}(q,K\varepsilon) \ , \tag{1}$$

where $B_{\mathrm{Y}}(q,\varepsilon) = \{q \exp(x_nY_n) \cdots \exp(x_1Y_1),\ |x_i| < \varepsilon^{\ell(Y_i)},\ 1 \leq i \leq n\}$.

This theorem is proved in [3]. The proof is inspired by works of Bellaïche [1] and Nagel, Stein and Wainger [5].

Remark 1. This theorem extends the classical Ball-Box Theorem (Bellaïche [1], Gromov [2]). Indeed, q being fixed, for ε smaller than some $\varepsilon_1(q)$, the estimate (1) is equivalent to the one of Ball-Box Theorem. However $\varepsilon_1(q)$ can be infinitely small for q close to a singular point, though (1) holds for $\varepsilon \leq \varepsilon_0$, independent of q.

The theorem suggests to introduce the following notation. Let $u \in T_qM$ and $\mathrm{Y} = (Y_1, \dots, Y_n)$ be a family of vector fields such that $(Y_1(q), \dots, Y_n(q))$ is a basis of T_qM. We denote by $(u_1^{\mathrm{Y}}, \dots, u_n^{\mathrm{Y}})$ the coordinates of u in this basis. We define then *the ε-norm on T_qM* as

$$\|u\|_{q,\varepsilon} = \max_{\substack{\mathrm{Y} \text{ associated} \\ \text{with } (q,\varepsilon)}} \left(\left[\frac{u_1^{\mathrm{Y}}}{\varepsilon^{\ell(Y_1)}} \right]^2 + \cdots + \left[\frac{u_n^{\mathrm{Y}}}{\varepsilon^{\ell(Y_n)}} \right]^2 \right)^{1/2} .$$

Notice that, for a fixed $\varepsilon > 0$, the ε-norm induces a Riemannian metric. This metric, multiplied by ε, tends to the sub-Riemannian metric as ε goes to zero.

When u is the tangent $\dot{c}(t)$ to a path, the dependence with respect to $c(t)$ is implicit and we write the ε-norm as $\|\dot{c}(t)\|_\varepsilon$. It becomes then a function of t, which is piecewise continuous and so integrable on $[0, 1]$. Notice that in this case we choose $\Omega = c$.

4 Estimate of Entropy and Complexity

The theorems presented here are proved in [3].

Theorem 2. *For any path $c \subset M$,*

$$e(c, \varepsilon) \asymp \int_0^1 \|\dot{c}(t)\|_\varepsilon dt \, . \tag{2}$$

This estimate of the entropy allows to compute the Hausdorff dimension. For instance for a path c containing no singular points for c, $\dim_{\mathrm{H}} c$ is equal to the smallest integer β such that $\dot{c}(t) \in L^\beta(c(t))$ for all $t \in [0, 1]$.

More generally, $\dim_{\mathrm{H}} c$ belongs to the interval $[\beta_{\mathrm{reg}}, \beta_{\mathrm{sing}}[$, where

- β_{reg} is the smallest integer β such that $\dot{c}(t) \in L^\beta(c(t))$ for all point $c(t)$ regular for c;
- β_{sing} is the smallest integer β such that $\dot{c}(t) \in L^\beta(c(t))$ for all $t \in [0, 1]$.

The Hausdorff dimension can take not only the integer values belonging to the interval, but also rational ones (see [3] for examples).

Theorem 3. *Let $t_1 < \cdots < t_k$ be the parameters of the points which are both a cusp and singular for c ($0 < t_1$ and $t_k < 1$). The complexity of c satisfies*

$$\int_0^1 \|\dot{c}(t)\|_\varepsilon dt - \sum_{i=1}^{k} \int_{t_i-\varepsilon}^{t_i+\varepsilon} \|\dot{c}(t)\|_\varepsilon dt \ \preceq\ \sigma(c,\varepsilon) \ \preceq\ \int_0^1 \|\dot{c}(t)\|_\varepsilon dt\ . \tag{3}$$

It results from these two theorems that, for a path without points both singular and cusps, complexity and entropy are equivalent. It is not always the case: we give in [3] an example where they are not equivalent. The inequality (3) provides however a sufficient condition on the integral of the ε-norm for complexity and entropy to be equivalent (we set $t_0 = 0$ and $t_{k+1} = 1$):

$$\text{if} \quad \int_{t_{i-1}}^{t_i} \|\dot{c}(t)\|_\varepsilon dt \asymp \int_{t_{i-1}+\varepsilon}^{t_i-\varepsilon} \|\dot{c}(t)\|_\varepsilon dt \quad \text{for } i = 1,\dots,k+1\ , \tag{4}$$
$$\text{then} \quad \sigma(c,\varepsilon) \asymp e(c,\varepsilon)\ .$$

More generally, a necessary and sufficient condition can be derived from the definition of a cusp. Let $q \in c$ and $\varepsilon > 0$. We denote by $\varrho = \varrho(q,\varepsilon)$ the biggest r such that Tube(c,ε) without the open ball $B(q,r)$ is connected. By definition, q is a cusp if $\varrho(q,\varepsilon) \succeq \varepsilon$.

We set $c^\varrho = c \cap B(q,\varrho)$. The difference between the complexity and the entropy is that $\sigma(c,\varepsilon)$ is equivalent to $\sigma(c \setminus c^\varrho,\varepsilon)$ when $e(c,\varepsilon)$ is equivalent to $e(c \setminus c^\varrho,\varepsilon) + e(c^\varrho,\varepsilon)$. Notice that $c \setminus c^\varrho$ is a union of disjoined paths, so we defined its complexity (resp. entropy) as the sum of the complexities (resp. entropy) of each one of these paths.

It results from Lemma 1 that the complexity and the entropy of a path c are equivalent if and only if, for any cusp $q \in c$, $e(c^\varrho,\varepsilon) \preceq e(c \setminus c^\varrho,\varepsilon)$. A consequence of Theorem 3 is that this condition is always satisfied at a cusp regular for c.

Corollary 1. *We have $\sigma(c,\varepsilon) \asymp e(c,\varepsilon)$ if and only if, for any cusp singular for c, $e(c^\varrho,\varepsilon) \preceq e(c \setminus c^\varrho,\varepsilon)$.*

Remark 2. It is in general difficult to evaluate ϱ and to decide if a point is a cusp. On the other hand, the integral of $\|\dot{c}(t)\|_\varepsilon$ – and then the entropy – is computable as soon as $c(t)$ is known. Thus we use Condition (4) in the following way. Let us compute the two integrals of Condition (4) at each point $c(t_i)$ singular for c and compare them:

- if the integrals are equivalent at each point singular for c, then $\sigma(c,\varepsilon) \asymp e(c,\varepsilon)$;
- if the integrals at $c(t_i)$ are not equivalent, then $c(t_i)$ is a cusp.

For the ε-lengths, we see that the two definitions are not equivalent: the length of a ε-chain can be smaller than length$^e_\varepsilon$. Theorem 2 suggests a third definition, equivalent to length$^e_\varepsilon$:

$$\text{length}_\varepsilon(c) = \int_0^1 \varepsilon \|\dot{c}(t)\|_\varepsilon dt\ .$$

When ε goes to zero, the limit of $\text{length}_\varepsilon(c)$ is the length of c which is infinite if c is not horizontal. We can then compare the lengths of two non-horizontal paths by computing the limit of the ratio of their ε-lengths.

References

1. Bellaïche A. (1996) The Tangent Space in Sub-Riemannian Geometry. In: Bellaïche A., Risler J.-J. (Eds), Sub-Riemannian Geometry. Progress in Mathematics, Birkhäuser.
2. Gromov M. (1996) Carnot-Carathéodory Spaces Seen from Within. In: Bellaïche A., Risler J.-J. (Eds), Sub-Riemannian Geometry. Progress in Mathematics, Birkhäuser.
3. Jean F. (1999) Entropy and Complexity of a Path in Sub-Riemannian Geometry. Technical Report, ENSTA, Paris.
4. Kolmogorov A. N. (1956) On Certain Asymptotics Characteristics of some Completely Bounded Metric Spaces. Soviet Math. Dokl. **108**, 385–388.
5. Nagel A., Stein E. M., Wainger S. (1985) Metrics Defined by Vector Fields. Acta Math. **155**, 103–147.