Numerical methods for time harmonic scalar wave equation in locally perturbed periodic media - Part 4

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POEMS (UMR 7231 CNRS-INRIA-ENSTA)

Mainly based on joint works with Julien Coatleven, Patrick Joly and Jing-Rebecca Li
Presentation of the problem

1D Case

Waveguide case

2D case

Simple scattering

Multiple scattering

1D Case

Waveguide case

2D case

Simple scattering

Multiple scattering
The 2D plane problem - Multiple scattering

Time harmonic Scalar wave Problem **with absorption**

\[
(P^i_\varepsilon) \quad \begin{align*}
-\Delta u^i_\varepsilon - \rho(x)(\omega^2 + i\varepsilon \omega) u^i_\varepsilon &= f(x), \quad \text{in } \Omega^i \\
\frac{\partial u^i_\varepsilon}{\partial n} + \Lambda^i_\varepsilon u^i_\varepsilon &= 0 \quad \text{on } \Sigma^i = \partial \Omega^i
\end{align*}
\]

\[
\Omega^i = \Omega^i_1 \cup \Omega^i_2 \quad \Sigma^i = \Sigma^i_1 \cup \Sigma^i_2
\]

2D case - simple scattering

\(\forall \varphi \in H^{1/2}(\Sigma^i), \ u^e_\varepsilon(\varphi) \) is the unique \(H^1\) solution of the exterior problem

\[
(P^e_\varepsilon) \quad \begin{align*}
-\Delta u^e_\varepsilon - \rho_{per}(x)(\omega^2 + i\varepsilon \omega) u^e_\varepsilon &= 0, \quad \text{in } \Omega^e \\
u^e_\varepsilon &= \varphi, \quad \text{on } \Sigma^i \\
\Lambda^e_\varepsilon \varphi &= \left. \frac{\partial u^e_\varepsilon(\varphi)}{\partial n} \right|_{\Sigma^i}
\end{align*}
\]
The 2D plane problem - Multiple scattering

2D case - simple scattering

∀ ϕ ∈ H^{1/2}(Σ^i), u_ε^e(ϕ) is the unique H^1 solution of the exterior problem

\[(P_ε^e) \quad \begin{cases} -\Delta u_ε^e - ρ_{per}(x)(ω^2 + iεω)u_ε^e = 0, & \text{in } Ω_ε^e \\ u_ε^e = ϕ, & \text{on } Σ^i \end{cases}\]

\[Λ_ε \varphi = \frac{∂u_ε^e(ϕ)}{∂n} \bigg|_{Σ^i}\]

∀ ϕ_1 ∈ H^{1/2}(Σ_1^i), u_1^e(ϕ_1) is the unique H^1 solution of the exterior problem posed in Ω_1^e

Λ_1 corresponding DtN operator

Theorem

If \( Σ_1^i \cap Σ_2^i = \emptyset \), ∀ ϕ ∈ H^{1/2}(Σ^i), ∃!(ϕ_1, ϕ_2) ∈ H^{1/2}(Σ_1^i) × H^{1/2}(Σ_2^i)

\[u^e(ϕ) = u_1^e(ϕ_1)\bigg|_{Ω_1^e} + u_2^e(ϕ_2)\bigg|_{Ω_ε^e}\]

∀ ϕ_2 ∈ H^{1/2}(Σ_2^i), u_2^e(ϕ_2) is the unique H^1 solution of the exterior problem posed in Ω_2^e

Λ_2 corresponding DtN operator
2D case - simple scattering

∀φ ∈ H^{1/2}(Σ^i), u^e_ε(φ) is the unique H^1 solution of the exterior problem

\[(\mathcal{P}^e_ε)\]

\[-Δ u^e_ε - \rho_{per}(x)(ω^2 + εω) u^e_ε = 0, \quad \text{in } Ω^e\]

\[u^e_ε = φ, \quad \text{on } Σ^i\]

\[\Lambda_ε φ = \frac{∂ u^e_ε(φ)}{∂ n} |_{Σ^i}\]

Theorem

If Σ_1^i \cap Σ_2^i = \emptyset, ∀φ ∈ H^{1/2}(Σ^i), ∃!(φ_1, φ_2) ∈ H^{1/2}(Σ_1^i) × H^{1/2}(Σ_2^i)

\[u^e(φ) = u^e_1(φ_1)|_{Ω^e} + u^e_2(φ_2)|_{Ω^e}\]

\[φ|_{Σ_1^i} = φ_1 + u^e_2(φ_2)|_{Σ_1^i}\]

\[\Theta_{12} φ_2\]

\[φ|_{Σ_2^i} = u^e_1(φ_1)|_{Σ_2^i} + φ_2\]

\[\Theta_{21} φ_1\]

\[\left(φ|_{Σ_1^i}, φ|_{Σ_2^i}\right) = Θ(φ_1, φ_2)\]

where

\[Θ = \begin{bmatrix} \mathbb{I} & Θ_{12} \\ Θ_{21} & \mathbb{I} \end{bmatrix}\]
The 2D plane problem - Multiple scattering

2D case - simple scattering

\( \forall \varphi \in H^{1/2}(\Sigma^i), u^e_\varepsilon(\varphi) \) is the unique \( H^1 \) solution of the exterior problem

\[
\begin{aligned}
(\mathcal{P}^e_\varepsilon) & \quad - \Delta u^e_\varepsilon - \rho_{\text{per}}(x)(\omega^2 + i\varepsilon\omega) u^e_\varepsilon = 0, \quad \text{in } \Omega^e \\
& \quad u^e_\varepsilon = \varphi, \quad \text{on } \Sigma^i \\
\Lambda_\varepsilon \varphi &= \frac{\partial u^e_\varepsilon(\varphi)}{\partial n} \bigg|_{\Sigma^i}
\end{aligned}
\]

Theorem

If \( \Sigma^i_1 \cap \Sigma^i_2 = \emptyset \), \( \forall \varphi \in H^{1/2}(\Sigma^i), \exists! (\varphi_1, \varphi_2) \in H^{1/2}(\Sigma^i_1) \times H^{1/2}(\Sigma^i_2) \)

\[
u^e(\varphi) = \left. u^e_1(\varphi_1) \right|_{\Omega^e} + \left. u^e_2(\varphi_2) \right|_{\Omega^e}
\]

\[
\begin{aligned}
\Lambda_\varepsilon \varphi \big|_{\Sigma^i_1} &= \Lambda_1 \varphi_1 + \left. \frac{\partial}{\partial n} u^e_2(\varphi_2) \right|_{\Sigma^i_1} \\
\Lambda_\varepsilon \varphi \big|_{\Sigma^i_2} &= \left. \frac{\partial}{\partial n} u^e_1(\varphi_1) \right|_{\Sigma^i_2} + \Lambda_2 \varphi_2 \\
& \quad \underbrace{\Lambda_{12} \varphi_2}_{\Sigma^i_1} \\
& \quad \underbrace{\Lambda_{21} \varphi_1}_{\Sigma^i_2}
\end{aligned}
\]

\[
\left( \Lambda_\varepsilon \varphi \big|_{\Sigma^i_1}, \Lambda_\varepsilon \varphi \big|_{\Sigma^i_2} \right) = \Lambda (\varphi_1, \varphi_2)
\]

where \( \Lambda = \left[ \begin{array}{cc} \Lambda_1 & \Lambda_{12} \\ \Lambda_{21} & \Lambda_2 \end{array} \right] \)
The 2D plane problem - Multiple scattering

2D case - simple scattering

\[ \forall \phi \in H^{1/2}(\Sigma^i), u^e_\varepsilon(\phi) \text{ is the unique } H^1 \text{ solution of the exterior problem} \]

\[ (P^e_\varepsilon) \]

\[ -\triangle u^e_\varepsilon - \rho_{\text{per}}(x)(\omega^2 + i\varepsilon\omega)u^e_\varepsilon = 0, \quad \text{in } \Omega^e \]

\[ u^e_\varepsilon = \phi, \quad \text{on } \Sigma^i \]

\[ \Lambda_\varepsilon \phi = \frac{\partial u^e_\varepsilon(\phi)}{\partial n} \mid_{\Sigma^i} \]

Theorem

\[ \text{If } \Sigma_1^i \cap \Sigma_2^i = \emptyset, \forall \phi \in H^{1/2}(\Sigma^i), \exists!(\phi_1, \phi_2) \in H^{1/2}(\Sigma_1^i) \times H^{1/2}(\Sigma_2^i) \]

\[ u^e(\phi) = u^e_1(\phi_1) \mid_{\Omega^e} + u^e_2(\phi_2) \mid_{\Omega^e} \]

Theorem

\[ \Lambda_\varepsilon = \Lambda \Theta^{-1} \]

where \[ \Theta = \begin{bmatrix} \mathbb{I} & \Theta_{12} \\ \Theta_{21} & \mathbb{I} \end{bmatrix} \]

and \[ \Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_{12} \\ \Lambda_{21} & \Lambda_2 \end{bmatrix} \]

Numerical results
Numerical results
Numerical results
Numerical results
The 2D plane problem - Multiple scattering

$\zeta(x)$

$x$ is

-4 -2 0 2 4
The 2D plane problem - Multiple scattering

Solution globale

6 -4 -2 0 2 4
The 2D plane problem - Multiple scattering

$\Omega^e$
The 2D plane problem - Multiple scattering

Media without the defect

Solution without the defect

Media with the defect

Solution with the defect
The 2D plane problem - Multiple scattering

Media without the defect

Media with the defect

Wave diffracted by the defect
Numerical methods for time harmonic scalar wave equation in locally perturbed periodic media - Part 4

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Numerical methods for time harmonic scalar wave equation in locally perturbed periodic media - Conclusions

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Conclusions for the 1D and waveguide problems

Absorbing media

The theory is complete and the numerical method works well

Non absorbing media

Limiting absorption principle for the definition of the DtN operators

Under some conditions on the periodic media, the problem is Fredholm

The problem is well posed except for a countable set of frequencies?
Conclusions for the 2D problems

Absorbing media

The theory is complete and the numerical method works well

The numerical analysis has to be done

Non absorbing media

A numerical limiting absorption method is done for the other cases

The corresponding theory still raises challenging open questions
Numerical results for time domain problems

\[
f = \begin{cases} 
A_f e^{-\frac{(x-c_x)^2}{w_x^2}} \cos(\omega t) & \text{dans } \Omega_0 \\
0 & \text{dans } \mathbb{R}^2 \setminus \Omega_0 
\end{cases}
\]

\begin{align*}
A_f &= 3 & c_x &= 0.5 & w_x &= 0.2 & \omega &= 5
\end{align*}

\[r = 2, \theta = 1/4, \Delta t = 0.04\]
Numerical results for time domain problems

$r = 3, \theta = 1/4, \Delta t = 0.04$
Numerical results for time domain problems

$r = 4, \theta = 1/4, \Delta t = 0.04$
Numerical results for time domain problems
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Thank you for your attention