Introduction à la méthode de sensibilité topologique pour l'identification de défauts. Aspects théoriques et numériques

Marc Bonnet[1]

[1]UMA (Dept. of Appl. Math.), POems, UMR 7231 CNRS-INRIA-ENSTA
ENSTA, PARIS, France
mbonnet@ensta.fr

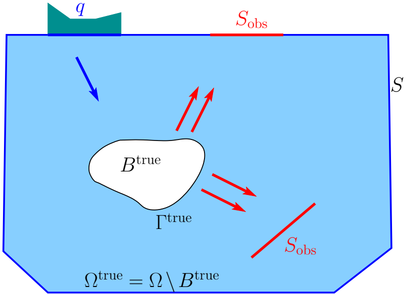
Formation **Problèmes inverses - Contrôle des EDP**, Amiens, 5-6 septembre 2011

Plan

To-do list

- champ total dans lippmann-schwinger
- estimations au moyen de δ^a
- separer inner and outer expansions
- definir $\bar{v}$ mais pas $\bar{u}$
- define polarization tensor from far-field representation

Forward and inverse problems



$$\mathcal{J}_{\mathsf{LS}}(\Omega_\Gamma) = \frac{1}{2}\int_{S_{\text{obs}}} \left| u_\Gamma - u_{\text{obs}}(\boldsymbol{\xi}) \right|^2 \, \mathrm{d}\Omega_\xi$$

$$\mathcal{J}(\Omega_\Gamma) = \int_{S_{\text{obs}}} \varphi(u_\Gamma(\boldsymbol{\xi}), \boldsymbol{\xi}) \, \mathrm{d}\Omega_\xi$$

where $u_{\text{obs}}(\boldsymbol{\xi}) = u_{\text{true}}(\boldsymbol{\xi})|_{S_{\text{obs}}} + (\text{noise})$

- **Insulated or conductive inhomogeneities (heat transfer, electrostatics)**
- **Sound-hard or penetrable objects (linear acoustics)**
- **Cavities, inclusions or cracks (elasticity)**
- Electromagnetic inverse scattering

...

Background

- Extensive literature on (acoustic, electromagnetic, elastic) inverse scattering e.g. books by Colton & Kress (93), Ramm (92), Cakoni & Colton (06), Kirsch (96), Kirsch and Grinberg (08)
- Iterative solution techniques (optimization-based, Newton): facilitated by sensitivity formulae, e.g.:

$$\delta\mathcal{J} = \int_\Gamma b(\mathbf{u}, \hat{\mathbf{u}}; \mathcal{C}, \rho) v_n \, \mathrm{d}S \qquad \text{(shape sensitivity)}$$

  with $\hat{\mathbf{u}}$: **adjoint solution** such that $\boldsymbol{\sigma}[\hat{\mathbf{u}}].\mathbf{n} = \partial_u \varphi$

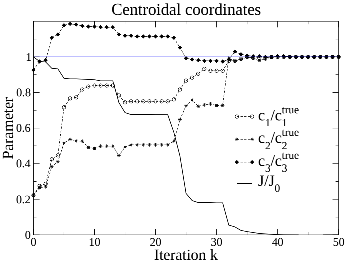
  v_n: normal perturbation of unknown surface Γ

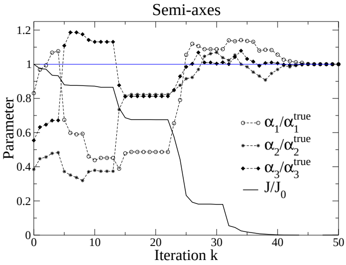
  Extension to combined shape-material sensitivity: $b(\mathbf{u}, \hat{\mathbf{u}}; \mathcal{C}, \rho, \Delta\mathcal{C}, \Delta\rho)$

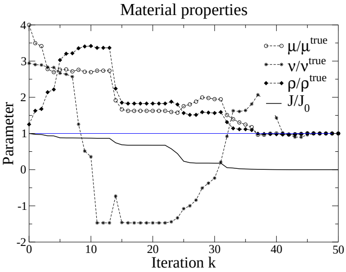
- Combined optimization and level-set approaches (Dorn and Lesselier 06, *topical review*)

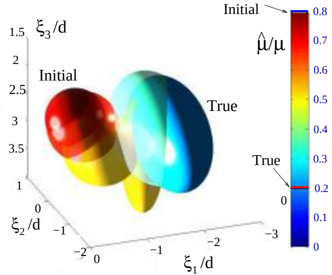
MB *Eng. Anal. with Bound. Elem.* (1995)
S. Nintcheu Fata, B. Guzina, MB, *Comput. Mech.* (2003)
MB and B. Guzina, *J. Comput. Phys.* (2008)

Optimization-based identification: example (inclusion, elastodynamics)







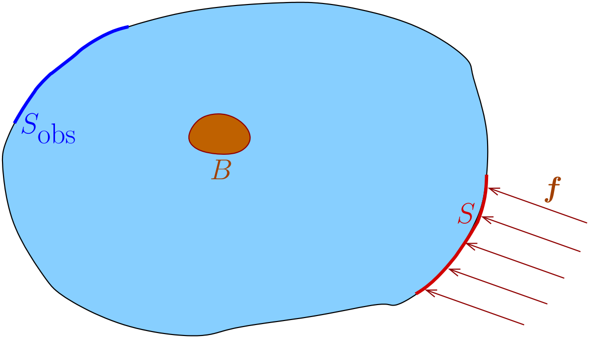


M. Bonnet, B. B. Guzina, *J. Comput. Phys (2009)*

Background

- **Iterative solution techniques:**
  Computationally expensive: ≥ 1 forward solution(s) per iteration;
  Success strongly depends on choice of initial guess.
- **Non-iterative, fast preliminary 'probing' (or 'sampling') methods** (Potthast 06, IP topical review):
  (a) Linear sampling (Colton, Kirsch 96)
  Extension to near-field elastodynamic data (Guzina, Nintcheu Fata 04);
  (b) **Topological sensitivity**
  Mathematical studies (Sokolowski-Zochowski 97, Guillaume-Sid Idris 04)
  Asymptotic methods (Maz'ya-Nazarov-Plamenevskij 91, Il'in 92, Vogelius-Volkov 00, Ammari-Kang 04...)
  Initially proposed for topological optimization (Eschenauer et al. 1994);
  Inverse problems (Guzina, MB 04, Feijoo 04, Masmoudi 05, MB 06...)

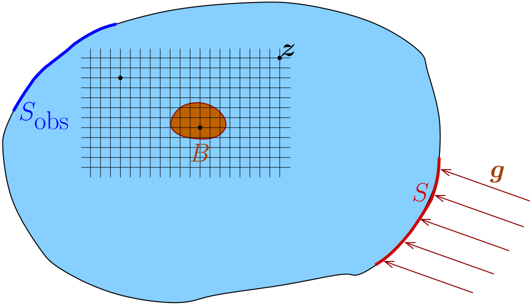
Linear sampling



- Scattering operator:

$$\mathbf{u}_{\rm obs}(\mathbf{x}) = [\mathcal{R}(B)\mathbf{f}](\mathbf{x}) \qquad (\mathbf{x} \in S_{\rm obs})$$

Linear sampling



- Linear sampling equation (with $\mathcal{G}$: Green's tensor):

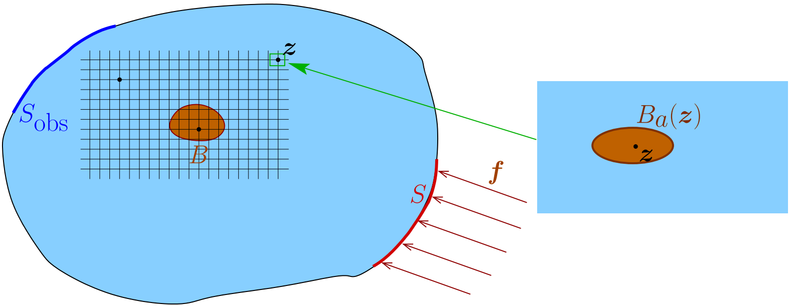
$$\exists ?\mathbf{g}(\boldsymbol{\xi}), \quad \|\mathcal{G}(\mathbf{z};\mathbf{x}).\mathbf{e} - [\mathcal{R}(B)\mathbf{g}_z^\varepsilon](\mathbf{x})\| \leq \varepsilon \ \ (\mathbf{x} \in S_{\rm obs})$$

$$\mathbf{z} \in B \implies \lim_{\mathbf{z}\to\partial B} \|\mathbf{g}_z^\varepsilon\|_{L^2(S)} = \infty$$

$$\mathbf{z} \notin B \implies \lim_{\varepsilon\to 0} \|\mathbf{g}_z^\varepsilon\|_{L^2(S)} = \infty$$

- Defect indicator function: $\mathbf{z} \mapsto 1/\|\mathbf{g}_z^\varepsilon\|_{L^2(S)}$

Topological sensitivity



- Small-obstacle expansion of cost function:

$$J(B_a(\mathbf{z})) = J(\emptyset) + \delta(a)\mathcal{T}(\mathbf{z}) + o(\delta(a))$$

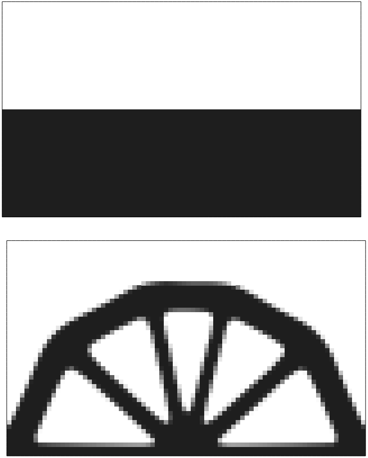
where $\delta(a)$ quantifies the perturbation behavior about $a = 0$;
$\mathcal{T}(\mathbf{z})$ is the **topological derivative** of J;
$\mathbf{z}$ acts as a sampling point.

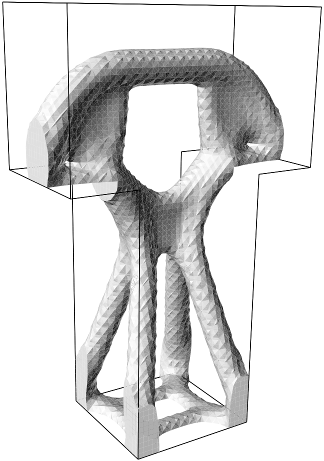
- **Heuristic idea: find regions in Ω where field $\mathcal{T}$ is** most negative.
- Proposed defect indicator function:

$$\boxed{\mathbf{z} \mapsto \mathcal{T}(\mathbf{z})}$$

Concept of topological sensitivity

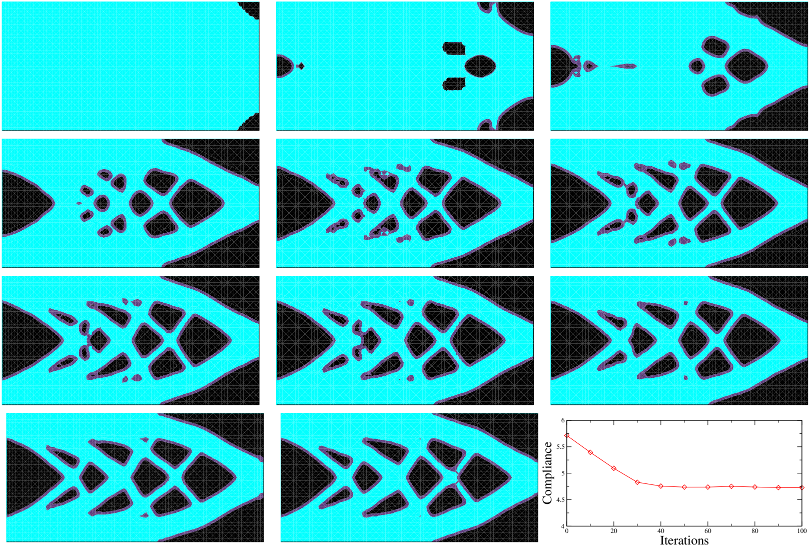
- **Sensitivity analysis tool** initially proposed for topological optimization (Eschenauer *et al* 94; Sokolowski, Zochowski 99; Garreau *et al* 01; Allaire *et al* 05. . .)





Allaire, de Gournay, Jouve, Toader (2005)

Example (Topological optimisation)



Thèse X-EADS "Optimisation géometrique et topologique du drapage des composites" (2010–2013, G. Delgado, dir. G. Allaire) — Algorithm combining topological and shape derivatives

Background problem (BP)

- Media endowed with scalar (thermal, electrostatic...) conductivity κ
  Generalization to anisotropic (tensorial) conductivity not addressed, but relatively straightforward
- Primary field variable u is scalar (temperature, electrostatic potential...)

Background problem (κ: background conductivity): defines the physical state in the absence of inclusion

$$\operatorname{div}(\kappa \boldsymbol{\nabla} u) + f = 0 \text{ in } \Omega, \qquad \kappa \partial_n u = g \text{ on } S_{\mathrm{N}}, \qquad u = u^D \text{ on } S_{\mathrm{D}}$$

i.e., in weak form:

$$\text{Find } u \in W(u^D), \qquad \langle u, w \rangle_\Omega^\kappa = F(w), \;\; \forall w \in W_0$$

with $W(u^D) = \{ w \in H^1(\Omega),\; w = u^D \text{ on } S_{\mathrm{D}} \}$. $W_0 = W(0)$ and

$$\langle u, w \rangle_\Omega^\kappa := \int_\Omega \kappa \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} w \,\mathrm{d}V, \qquad F(w) := \int_\Omega f w \,\mathrm{d}V + \int_{S_{\mathrm{N}}} g w \,\mathrm{d}S$$

Generic transmission problem (TP)

- Generic (possibly multiple) inclusion $B \subset \Omega$ and conductivity κ_B:

$$\kappa_B = [1 - \chi(B)]\kappa + \chi(B)\kappa^\star = \kappa + \chi(B)\Delta\kappa$$

- Transmission problem:

$$\text{div}\,(\kappa_B \boldsymbol{\nabla} u_B) + f = 0 \text{ in } \Omega, \qquad \kappa\partial_n u = g \text{ on } S_\text{N}, \qquad u = u^D \text{ on } S_\text{D}$$

  i.e., in weak form:

$$\text{Find } u_B \in W(u^D), \qquad \langle u_B, w\rangle_\Omega^\kappa + \langle u_B, w\rangle_B^{\Delta\kappa} = F(w), \;\; \forall w \in W_0$$

- Both formulations implicitly enforce the transmission conditions

$$u_B\,|_+ = u_B\,|_- \quad \text{and} \quad \kappa\partial_n u_B\,|_+ = \kappa^\star\partial_n u_B\,|_- \qquad \text{on } \partial B$$

- TP in terms of solution perturbation $v_B := u_B - u$:

$$\boxed{\text{Find } v_B \in W_0, \qquad \langle v_B, w\rangle_\Omega^\kappa + \langle v_B, w\rangle_B^{\Delta\kappa} = -\langle u, w\rangle_B^{\Delta\kappa}, \;\; \forall w \in W_0}$$

Free-space transmission problem (FSTP)

- Auxiliary TP for an inclusion $\mathcal{B}$ embedded in an infinite medium $\Omega = \mathbb{R}^3$ subjected to a given remote uniform gradient $\mathbf{E} \in \mathbb{R}^3$ through the background solution $u(\boldsymbol{\xi}) = \mathbf{E}\cdot\boldsymbol{\xi} =: \psi[\mathbf{E}](\boldsymbol{\xi})$, i.e.:

$$\begin{cases} \operatorname{div}\left(\kappa_{\mathcal{B}} \boldsymbol{\nabla} u_{\mathcal{B}}[\mathbf{E}]\right) = 0 \text{ in } \mathbb{R}^3, \\ u_{\mathcal{B}}[\mathbf{E}](\boldsymbol{\xi}) - \psi[\mathbf{E}](\boldsymbol{\xi}) = O(\|\boldsymbol{\xi}\|^{-2}) \text{ as } \|\boldsymbol{\xi}\| \to \infty \end{cases}$$

  with $\kappa_{\mathcal{B}} := \kappa + \chi(\mathcal{B})\Delta\kappa$.

- Weak formulation for the solution perturbation $v_{\mathcal{B}} := u_{\mathcal{B}} - \psi[\mathbf{E}]$:

$$\text{Find } v_{\mathcal{B}} \in H^1(\mathbb{R}^3), \qquad \langle v_{\mathcal{B}}, w\rangle^{\kappa}_{\mathbb{R}^3} = -\langle \psi[\mathbf{E}], w\rangle^{\Delta\kappa}_{\mathcal{B}}, \quad \forall w \in H^1(\mathbb{R}^3)$$

Cost functional

Consider generic cost functionals of the form

$$J(\kappa_B) = \mathbb{J}(u_B) \qquad \text{with} \;\; \mathbb{J}(w) := \int_\Omega \varphi_{\text{V}}(\cdot, w)\,\text{d}V + \int_{\partial\Omega} \varphi_{\text{S}}(\cdot, w)\,\text{d}S$$

with twice-differentiable densities $w \mapsto \varphi_{\text{V}}(\cdot, w)$ and $w \mapsto \varphi_{\text{S}}(\cdot, w)$

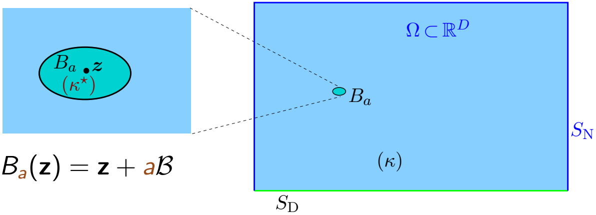
- Example 1: weighted least-squares output residuals (identification based on overdetermined data g, u^{obs} on S_{N})

$$\mathbb{J}(w) = \frac{1}{2}\int_{S_{\text{N}}} Q|w - u^{\text{obs}}|^2\,\text{d}S$$

- Example 2: potential energy (expressed as boundary integrals)

$$\mathbb{J}(w) = \frac{1}{2}\int_{S_{\text{D}}} (\kappa\partial_n w)u^D\,\text{d}S - \frac{1}{2}\int_{S_{\text{N}}} gw\,\text{d}S$$

Concept of topological sensitivity



$B_a(\mathbf{z}) = \mathbf{z} + a\mathcal{B}$

Definition (Topological derivative)

Assume that $J(u^a)$ can be expanded in the form

$$J(\kappa_a) = J(\kappa) + \delta(a)\mathcal{T}(\mathbf{z}) + o(\delta(a))$$

where $\delta(a)$ is such that $\delta(a) = o(1)$ as $a \to 0$ and characterizes the small-inclusion behavior of $J(\kappa_B)$.
Then, the coefficient $\mathcal{T}(\mathbf{z})$ is called the topological derivative of J at $\mathbf{z} \in \Omega$.

- $\mathcal{T}(\mathbf{z})$ also depends on shape $\mathcal{B}$ and conductivity contrast $\beta := \kappa^\star/\kappa$;
- Terminology for $\mathcal{T}$ varies, with “gradient” or “sensitivity” sometimes used instead of “derivative”.

Small-inclusion asymptotic behavior of u^a

Established using a domain integral equation (DIE) formulation of support B_a

- The Green's function $\mathcal{G}$ is defined by

$$\left.\begin{aligned} \text{div}\big(\kappa\boldsymbol{\nabla}\mathcal{G}(\cdot,\mathbf{x})\big)+\delta(\cdot-\mathbf{x})=0 &\quad \text{in } \Omega \\ \mathcal{G}(\cdot,\mathbf{x})=0 &\quad \text{on } S_{\text{D}} \\ \kappa\partial_n\mathcal{G}(\cdot,\mathbf{x})=0 &\quad \text{on } S_{\text{N}} \end{aligned}\right\} \qquad (\mathbf{x}\in\Omega)$$

and verifies

$$\langle\mathcal{G}(\cdot,\mathbf{x}),w\rangle_{\Omega}^{\kappa}=w(\mathbf{x}) \qquad \mathbf{x}\in\Omega,\ \forall w\in W_0\cap C_0(\Omega)$$

- Split G into singular and nonsingular parts:

$$\mathcal{G}(\boldsymbol{\xi},\mathbf{x})=G(\boldsymbol{\xi}-\mathbf{x})+\mathcal{G}_{\text{c}}(\boldsymbol{\xi},\mathbf{x})$$

where $G(\mathbf{r})=1/(4\pi\|\mathbf{r}\|)$ is the (singular) full-space fundamental solution and the complementary Green's function $\mathcal{G}_{\text{c}}$ is bounded at $\boldsymbol{\xi}=\mathbf{x}$ (and in fact C^∞ for $\boldsymbol{\xi},\mathbf{x}\in\Omega$ by virtue of solving a BVP with regular boundary data).

- Note that G and $\boldsymbol{\nabla}G$ are homogeneous of degree -1 and -2, respectively:

$$G(\lambda\mathbf{r})=|\lambda|^{-1}G(\mathbf{r}), \qquad \boldsymbol{\nabla}G(\lambda\mathbf{r})=|\lambda|^{-2}\text{sgn}(\lambda)\boldsymbol{\nabla}G(\mathbf{r})$$

$$(\mathbf{r}\in\mathbb{R}^3\setminus\{\mathbf{0}\},\lambda\in\mathbb{R}\setminus\{0\})$$

Small-inclusion asymptotic behavior of u^a

- **Domain integral equation (DIE) for the TP:**
  combine weak formulations for the TP (with $w = \mathcal{G}(\cdot,\mathbf{x})$) and the Green's function (with $w = v^a$):

$$\begin{cases} \langle v^a, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\kappa}_{\Omega} + \langle v^a, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\kappa}_{B_a} = -\langle u, w\rangle^{\Delta\kappa}_{B_a} \\ \langle \mathcal{G}(\cdot,\mathbf{x}), v^a\rangle^{\kappa}_{\Omega} = v^a(\mathbf{x}) \end{cases} \qquad (\mathbf{x}\in\Omega)$$

  to obtain

$$\mathcal{L}_a[v^a](\mathbf{x}) = -\langle u, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\kappa}_{B_a} \qquad \begin{cases} \mathbf{x}\in B_a & \text{(integral equation for } v^a) \\ \mathbf{x}\in\Omega\setminus\bar{B}_a & \text{(integral representation for } v^a) \end{cases}$$

  with the linear operator $\mathcal{L}_a$ defined by $\mathcal{L}_a[w](\mathbf{x}) := w(\mathbf{x}) + \langle w, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\kappa}_{B_a}$

  i.e. (in expanded form):

$$\mathcal{L}_a[w](\mathbf{x}) = w(\mathbf{x}) + \int_{B_a} \Delta\kappa \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla}\mathcal{G}(\cdot,\mathbf{x}) \,\mathrm{d}V$$

- **DIE for the FSTP:** (similarly)

$$\mathcal{L}_{\mathcal{B}}[v_{\mathcal{B}}](\mathbf{x}) = -\langle \psi[\mathbf{E}], \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\kappa}_{\mathcal{B}} \quad \text{with} \quad \mathcal{L}_{\mathcal{B}}[w](\mathbf{x}) := w(\mathbf{x}) + \langle w, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\kappa}_{\mathcal{B}}$$

Small-inclusion asymptotic behavior of u^a

- Perturbation due to small inclusion governed by domain integral of support B_a

$$\mathcal{L}_a[v^a](\mathbf{x}) = -\langle u, \mathcal{G}(\cdot,\mathbf{x})\rangle_{B_a}^{\Delta\kappa}$$

  Facilitates investigation of limiting behavior as $a \to 0$
- Introduce normalized coordinates

$$\text{(a)}\ (\boldsymbol{\xi},\mathbf{x}) = \mathbf{z} + a(\bar{\boldsymbol{\xi}},\bar{\mathbf{x}}), \qquad \text{(b)}\ \ \text{d}V_\xi = a^3\,\text{d}\bar{V}_{\bar{\xi}} \qquad\qquad (\boldsymbol{\xi}\in B_a,\ \bar{\boldsymbol{\xi}}\in\mathcal{B})$$

Asymptotic behavior of v^a as $a \to 0$: Inside B_a, one has (inner expansion)

$$v^a(\mathbf{x}) = a v_{\mathcal{B}}[\boldsymbol{\nabla} u(\mathbf{z})](\bar{\mathbf{x}}) + o(a) \qquad \boldsymbol{\xi}\in B_a,\ \bar{\boldsymbol{\xi}}\in\mathcal{B}$$

Moreover, at any fixed location $\mathbf{x}\neq\mathbf{z}$ (outer expansion), one has

$$v^a(\mathbf{x}) = -a^3\boldsymbol{\nabla}_1\mathcal{G}(\mathbf{z},\mathbf{x})\cdot\mathbf{A}\cdot\boldsymbol{\nabla} u(\mathbf{z}) + o(a^3) \qquad \mathbf{x}\neq\mathbf{z}$$

where $\mathbf{A}=\mathbf{A}(\kappa,\kappa^\star,\mathcal{B})$ is the (second-order) *polarization tensor*, defined by

$$\mathbf{A}\cdot\mathbf{E} = \int_{\mathcal{B}} \Delta\kappa\boldsymbol{\nabla} u_{\mathcal{B}}[\mathbf{E}]\,\text{d}V = \int_{\mathcal{B}} \Delta\kappa(\mathbf{E}+\boldsymbol{\nabla} v_{\mathcal{B}}[\mathbf{E}])\,\text{d}V \qquad \forall\mathbf{E}\in\mathbb{R}^3$$

($u_{\mathcal{B}}[\mathbf{E}] = \psi[\mathbf{E}] + v_{\mathcal{B}}[\mathbf{E}]$: solution to the FSTP)

Small-inclusion asymptotic behavior of u^a (proof sketch 1/2)

Proof based on seeking and exploiting the limiting form of the DIE, and using scaled coordinates.

(i) Asymptotic behavior of the Green's function:

$$\boldsymbol{\nabla}_1\mathcal{G}(\boldsymbol{\xi},\mathbf{x}) = a^{-2}\boldsymbol{\nabla} G(\bar{\boldsymbol{\xi}}-\bar{\mathbf{x}}) + \boldsymbol{\nabla}_1\mathcal{G}_{\mathrm{c}}(\mathbf{z},\mathbf{z}) + o(1)$$

(ii) Defining $\bar{v}^a(\bar{\boldsymbol{\xi}}) := v^a(\mathbf{z}+a\bar{\boldsymbol{\xi}})$, $\bar{u}(\bar{\boldsymbol{\xi}}) := u(\mathbf{z}+a\bar{\boldsymbol{\xi}})$, one finds

$$\boldsymbol{\nabla} v^a(\boldsymbol{\xi}) = a^{-1}\boldsymbol{\nabla}\bar{v}^a(\bar{\boldsymbol{\xi}}), \qquad \boldsymbol{\nabla}\bar{u}(\bar{\boldsymbol{\xi}}) = \boldsymbol{\nabla} u(\mathbf{z}) + o(1)$$

(iii) Substitute above expansions and rescale coordinates in DIE:

$$\mathcal{L}_a[v^a](\mathbf{x}) = \mathcal{L}_{\mathcal{B}}[\bar{v}^a](\bar{\mathbf{x}}) + aO(\|\boldsymbol{\nabla}\bar{v}^a\|_{2,\mathcal{B}})$$

$$\langle u,\mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\kappa}_{B_a} = a\langle\psi[\boldsymbol{\nabla} u(\mathbf{z})], G(\cdot,\bar{\mathbf{x}})\rangle^{\Delta\kappa}_{\mathcal{B}} + o(a)$$

(iv) Make ansatz $\bar{v}^a(\bar{\boldsymbol{\xi}}) = aV(\bar{\boldsymbol{\xi}}) + o(a)$ and isolate lowest-order (in a) contributions in DIE, which correspond to the FSTP:

$$\mathcal{L}_{\mathcal{B}}[V](\bar{\boldsymbol{\xi}}) = -\langle\psi[\boldsymbol{\nabla} u(\mathbf{z})], G(\cdot,\bar{\mathbf{x}})\rangle^{\Delta\kappa}_{\mathcal{B}}$$

Hence:

$$\boxed{V(\bar{\boldsymbol{\xi}}) = v_{\mathcal{B}}[\boldsymbol{\nabla} u(\mathbf{z})](\bar{\boldsymbol{\xi}}) \qquad \bar{\boldsymbol{\xi}}\in\mathcal{B}}$$

Small-inclusion asymptotic behavior of u^a (proof sketch 2/2)

(v) Now, consider $\mathbf{x} \neq \mathbf{z}$ and a small enough. In that case, one has

$$\boldsymbol{\nabla}_1 \mathcal{G}(\boldsymbol{\xi}, \mathbf{x}) = \boldsymbol{\nabla}_1 \mathcal{G}(\mathbf{z}, \mathbf{x}) + o(1), \qquad \boldsymbol{\xi} \in B_a, \mathbf{x} \notin B_a.$$

(vi) Substitute $\bar{v}^a(\bar{\boldsymbol{\xi}}) = aV(\bar{\boldsymbol{\xi}}) + o(a)$ and the above into the integral representation outside B_a, to obtain

$$\begin{aligned} v^a(\mathbf{x}) &= -a^3 \boldsymbol{\nabla}_1 \mathcal{G}(\mathbf{z}, \mathbf{x}) \cdot \int_{\mathcal{B}} \Delta\kappa \boldsymbol{\nabla} v_{\mathcal{B}}[\boldsymbol{\nabla} u(\mathbf{z})](\bar{\boldsymbol{\xi}}) \, \mathrm{d}\bar{V}_{\bar{\xi}} + o(a^3) \\ &= -a^3 \boldsymbol{\nabla}_1 \mathcal{G}(\mathbf{z}, \mathbf{x}) \cdot \mathbf{A} \cdot \boldsymbol{\nabla} u(\mathbf{z}) + o(a^3) \end{aligned}$$

Topological derivative of cost functionals

(i) Expand cost functional w.r.t. v^a:

$$J(\kappa_a) = J(\kappa) + \mathbb{J}'(u; v^a) + o(\|v^a\|_{2,\Omega} + \|v^a\|_{2,\partial\Omega})$$

with $\quad \mathbb{J}'(u; w) := \int_\Omega \nabla_2\varphi_\mathrm{V}(\cdot, u) w \,\mathrm{d}V + \int_{S_\mathrm{N}} \nabla_2\varphi_\mathrm{S}(\cdot, u) w \,\mathrm{d}S$

(ii) Interpret $\mathbb{J}'(u; v^a)$ as component of a weak formulation with test function v^a and define the adjoint solution $\hat{u}$ by

$$\text{Find } \hat{u} \in W_0, \qquad \langle \hat{u}, w\rangle^\kappa_\Omega = \mathbb{J}'(u; w), \quad \forall w \in W_0$$

(iii) Reciprocity between transmission and adjoint problems:

$$\langle v^a, \hat{u}\rangle^\kappa_\Omega + \langle v^a, \hat{u}\rangle^{\Delta\kappa}_{B_a} = -\langle u, \hat{u}\rangle^{\Delta\kappa}_{B_a}, \qquad \langle \hat{u}, v^a\rangle^\kappa_\Omega = \mathbb{J}'(u; v^a)$$

to obtain $\quad \mathbb{J}'(u; v^a) = -\langle \hat{u}, u\rangle^{\Delta\kappa}_{B_a} - \langle \hat{u}, v^a\rangle^{\Delta\kappa}_{B_a} = -\langle \hat{u}, u^a\rangle^{\Delta\kappa}_{B_a}$

(iv) Exploit known limiting behavior of v^a in B_a (inner expansion), yielding

$$\mathbb{J}'(u; v^a) = -\langle \hat{u}, u^a\rangle^{\Delta\kappa}_{B_a} = -a^3 \nabla\hat{u}(\mathbf{z})\cdot\mathbf{A}\cdot\nabla u(\mathbf{z}) + o(a^3)$$

Topological expansion of J: The topological expansion of J is given by

$$J(\kappa_a) = J(\kappa) + a^3\mathcal{T}(\mathbf{z}) + o(a^3), \qquad \mathcal{T}(\mathbf{z}) = -\nabla\hat{u}(\mathbf{z})\cdot\mathbf{A}\cdot\nabla u(\mathbf{z})$$

Remarks on topological derivative and polarization tensor

(i) $\mathbf{A}$ is symmetric and has the same sign as $\Delta\kappa$ (Cedyo-Fengya, Volkov, Vogelius, 1998);

(ii) By superposition:

$$V(\bar{\mathbf{x}}) = v_{\mathcal{B}}[\boldsymbol{\nabla} u(\mathbf{z})](\bar{\mathbf{x}}) = \sum_{i=1}^{3} \big(\mathbf{e}_i \cdot \boldsymbol{\nabla} u(\mathbf{z})\big) v_{\mathcal{B}}[\mathbf{e}_i](\bar{\mathbf{x}})$$

One thus needs only compute the $v_{\mathcal{B}}[\mathbf{e}_i]$, which are independent on $\mathbf{z}$.

(iii) **Spherical inclusion** (i.e. $\mathcal{B}$ is unit sphere): FSTP solution is given by

$$v_{\mathcal{B}}[\mathbf{E}](\bar{\mathbf{x}}) = \frac{3}{2+\beta}\mathbf{E}\cdot\bar{\mathbf{x}} \quad \text{with } \beta := \kappa^\star/\kappa \qquad (\bar{\mathbf{x}} \in \mathcal{B}).$$

Hence:

$$\boxed{\mathbf{A}(\kappa, \kappa^\star, \mathcal{B}) = 4\pi\kappa\frac{(\beta-1)}{\beta+2}\mathbf{I}};$$

(iv) Analytical FSTP solutions available for other shapes (e.g. ellipsoids); otherwise, solve numerically (once and for all) the normalized DIE with $\mathbf{E} = \mathbf{e}_i$ $(i = 1, 2, 3)$;

(v) Special case $\beta = 0$: **insulated** (i.e. impenetrable) inclusion.

Remarks on topological derivative and polarization tensor

(vi) Similar analysis available for (penetrable or insulated) inclusions in 2D media, yielding cost function asymptotics of the form

$$J(\kappa_a) = J(\kappa) + a^2\mathcal{T}(\mathbf{z}) + o(a^2)$$

(vii) If homogeneous Dirichlet conditions are instead considered on ∂B_a, one finds cost function asymptotics of the form (e.g. Guillaume, Sid Idris, 2002)

$$\begin{aligned} J(\kappa_a) &= J(\kappa) + a\mathcal{T}(\mathbf{z}) + o(a) && \text{(three-dimensional case)} \\ &= J(\kappa) - \frac{1}{\ln a}\mathcal{T}(\mathbf{z}) + o\Big(\frac{-1}{\ln a}\Big) && \text{(two-dimensional case)} \end{aligned}$$

(viii) Cost function example 1 (output least squares): adjoint loads are (as usual) measurement residuals, since

$$\mathbb{J}(u_B) = \frac{1}{2}\int_{S_N} Q|u_B - u^{\text{obs}}|^2 \, dS \implies \boxed{\mathbb{J}'(u;w) = \int_{S_N} Q(w - u^{\text{obs}})w \, dS}$$

(ix) Cost function example 2 (potential energy, with $u^D = 0$):

$$\mathbb{J}(u_B) = -\frac{1}{2}\int_{S_N} g u_B \, dS \implies \mathbb{J}'(u;w) = -\frac{1}{2}\int_{S_N} g w \, dS$$

$$\implies \boxed{\hat{u} = -u/2} \implies \boxed{\mathcal{T}(\mathbf{z}) = (1/2)\boldsymbol{\nabla} u \cdot \mathbf{A} \cdot \boldsymbol{\nabla} u}$$

Use sign properties of **A** for interpretation

Background solution

Background solution ($\mathcal{C}$: background elasticity tensor):

$$\operatorname{div}(\mathcal{C}:\varepsilon[\mathbf{u}]) + \mathbf{f} = \mathbf{0} \text{ in } \Omega, \quad \mathbf{t}[\mathbf{u}] = \mathbf{g} \text{ on } S_{\mathrm{N}}, \quad \mathbf{u} = \mathbf{u}^{D} \text{ on } S_{\mathrm{D}}$$

where

$$\varepsilon[\mathbf{u}] = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{T}) \text{ (linearized strain tensor)},$$
$$\mathbf{t}[\mathbf{u}] = (\mathcal{C}:\varepsilon[\mathbf{u}])\cdot\mathbf{n} \text{ (traction vector)}$$

In weak form:

$$\text{Find } \mathbf{u} \in W(\mathbf{u}^{D}), \qquad \langle \mathbf{u}, \mathbf{w} \rangle_{\Omega}^{\mathcal{C}} = F(\mathbf{w}), \quad \forall \mathbf{w} \in W_0$$

with $W(\mathbf{u}^{D}) := \big\{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^3), \, \mathbf{v} = \mathbf{u}^{D} \text{ on } S_{\mathrm{D}} \big\}$. $W_0 = W(\mathbf{0})$ and

$$\langle \mathbf{u}, \mathbf{w} \rangle_{D}^{\mathcal{C}} := \int_D \varepsilon[\mathbf{u}]:\mathcal{C}:\varepsilon[\mathbf{w}] \, \mathrm{d}V = \int_D \nabla\mathbf{u}:\mathcal{C}:\nabla\mathbf{w} \, \mathrm{d}V,$$
$$F(\mathbf{w}) = \int_\Omega \mathbf{f}\cdot\mathbf{w} \, \mathrm{d}V + \int_{S_{\mathrm{N}}} \mathbf{g}\cdot\mathbf{w} \, \mathrm{d}S$$

Generic transmission problem (TP)

- **Generic TP:** (possibly multiple) inclusion $B \subset \Omega$ and elasticity tensor $\mathcal{C}_B = [1-\chi(B)]\mathcal{C} + \chi(B)\mathcal{C}^\star = \mathcal{C} + \chi(B)\Delta\mathcal{C}$

$$\text{div}\,(\mathcal{C}_B : \boldsymbol{\varepsilon}[\mathbf{u}_B]) + \mathbf{f} = \mathbf{0} \text{ in } \Omega, \quad \mathbf{t}[\mathbf{u}_B] = \mathbf{g} \text{ on } S_\text{N}, \quad \mathbf{u}_B = \mathbf{u}^D \text{ on } S_\text{D}$$

i.e., in weak form:

$$\text{Find } \mathbf{u}_a \in W(\mathbf{u}^D), \qquad \langle \mathbf{u}_a, \mathbf{w}\rangle_\Omega^{\mathcal{C}} + \langle \mathbf{u}_a, \mathbf{w}\rangle_{B_a}^{\Delta\mathcal{C}} = F(\mathbf{w}), \quad \forall \mathbf{w} \in W_0$$

Both formulations implicitly enforce the perfect-bonding conditions

$$\mathbf{u}_a|_+ = \mathbf{u}_a|_- \quad \text{and} \quad \mathbf{t}\big[\mathbf{u}_a|_+\big] = \mathbf{t}^\star\big[\mathbf{u}_a|_-\big] \qquad \text{on } \partial B$$

TP in terms of solution perturbation $\mathbf{v}_B := \mathbf{u}_B - \mathbf{u}$:

$$\text{Find } \mathbf{v}_B \in W_0, \quad \langle \mathbf{v}_B, \mathbf{w}\rangle_\Omega^{\mathcal{C}} + \langle \mathbf{v}_B, \mathbf{w}\rangle_B^{\Delta\mathcal{C}} = -\langle \mathbf{u}, \mathbf{w}\rangle_B^{\Delta\mathcal{C}}, \quad \forall \mathbf{w} \in W_0$$

Free-space transmission problem (FSTP)

- Auxiliary TP for an inclusion $\mathcal{B}$ embedded in an infinite medium $\Omega = \mathbb{R}^3$ subjected to a given remote uniform strain $\mathbf{E} \in \mathbb{R}^{3,3}$ through the background solution $\mathbf{u}(\boldsymbol{\xi}) = \mathbf{E}\cdot\boldsymbol{\xi} =: \psi[\mathbf{E}](\boldsymbol{\xi})$, i.e.:
$$\begin{cases} \text{div}\,(\mathcal{C}_{\mathcal{B}} : \boldsymbol{\nabla}\mathbf{u}_{\mathcal{B}}[\mathbf{E}]) = \mathbf{0} & \text{in } \mathbb{R}^3, \\ \mathbf{u}_{\mathcal{B}}[\mathbf{E}](\boldsymbol{\xi}) - \psi[\mathbf{E}](\boldsymbol{\xi}) = O(\|\boldsymbol{\xi}\|^{-2}) & \text{as } \|\boldsymbol{\xi}\| \to \infty \end{cases}$$
with $\mathcal{C}_{\mathcal{B}} := \mathcal{C} + \chi(\mathcal{B})\Delta\mathcal{C}$.
- Weak formulation for the solution perturbation $\mathbf{v}_{\mathcal{B}} := \mathbf{u}_{\mathcal{B}} - \psi[\mathbf{E}]$:
Find $\mathbf{v}_{\mathcal{B}} \in H^1(\mathbb{R}^3;\mathbb{R}^3)$, $\quad \langle \mathbf{v}_{\mathcal{B}}[\mathbf{E}], \mathbf{w}\rangle^{\mathcal{C}}_{\mathbb{R}^3} = -\langle \psi[\mathbf{E}], \mathbf{w}\rangle^{\Delta\mathcal{C}}_{\mathcal{B}}, \quad \forall \mathbf{w} \in H^1(\mathbb{R}^3;\mathbb{R}^3)$

Cost functional

Cost functional:

$$J(\mathcal{C}_B) = \mathbb{J}(\mathbf{u}_B) \qquad \text{with} \;\; \mathbb{J}(\mathbf{w}) := \int_\Omega \varphi_\mathsf{v}(\cdot, \mathbf{w}) \, \mathrm{d}V + \int_{\partial\Omega} \varphi_\mathsf{s}(\cdot, \mathbf{w}) \, \mathrm{d}S$$

with twice-differentiable densities $\mathbf{w} \mapsto \varphi_\mathsf{v}(\cdot, \mathbf{w})$ and $\mathbf{w} \mapsto \varphi_\mathsf{s}(\cdot, \mathbf{w})$

Topological derivative of J: sought by considering the limiting behavior of $j(a) := J(\mathcal{C}_a) = \mathbb{J}(\mathbf{u}_a)$.

Small-inclusion asymptotic behavior of u^a

- The elastostatic Green's tensor is defined by

$$\left.\begin{aligned} \operatorname{div}\big(\mathcal{C}:\varepsilon[\mathcal{G}(\cdot,\mathbf{x})]\big) + \delta(\cdot - \mathbf{x})\mathbf{I} = \mathbf{0} \quad & \text{in } \Omega \\ \mathcal{G}(\cdot,\mathbf{x}) = \mathbf{0} \quad & \text{on } S_{\mathrm{D}} \\ \mathbf{t}[\mathcal{G}(\cdot,\mathbf{x})] = \mathbf{0} \quad & \text{on } S_{\mathrm{N}} \end{aligned}\right\} \qquad (\mathbf{x}\in\Omega)$$

  and verifies

$$\langle \mathcal{G}(\cdot,\mathbf{x}), \mathbf{w}\rangle_{\Omega}^{\mathcal{C}} = \mathbf{w}(\mathbf{x}) \qquad \mathbf{x}\in\Omega,\ \forall \mathbf{w}\in W_0\cap C_0(\Omega)$$

- Split G into singular and nonsingular parts:

$$\mathcal{G}(\boldsymbol{\xi},\mathbf{x}) = \mathbf{G}(\boldsymbol{\xi}-\mathbf{x}) + \mathcal{G}_{\mathrm{c}}(\boldsymbol{\xi},\mathbf{x})$$

  where $\mathbf{G}(\mathbf{r})$ is the (singular) full-space (Kelvin) fundamental solution and the complementary Green's tensor $\mathcal{G}_{\mathrm{c}}$ is bounded at $\boldsymbol{\xi}=\mathbf{x}$ (and in fact C^{∞} for $\boldsymbol{\xi},\mathbf{x}\in\Omega$ by virtue of solving a BVP with regular boundary data).

Small-inclusion asymptotic behavior of u^a

- Applying 3-D Fourier transform, one has in fact

$$\mathbf{G}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \exp(\mathrm{i}\boldsymbol{\eta}\cdot\mathbf{r})\mathbf{N}(\boldsymbol{\eta})\,\mathrm{d}V(\boldsymbol{\eta}) \qquad (\mathbf{r} \in \mathbb{R}^3 \setminus \{\mathbf{0}\})$$

  with $\mathbf{N}(\boldsymbol{\eta}) = \mathbf{K}^{-1}(\boldsymbol{\eta})$ in terms of the *acoustic tensor* $\mathbf{K}(\boldsymbol{\eta})$, defined by $K_{ik}(\boldsymbol{\eta}) = \mathcal{C}_{ijk\ell}\eta_j\eta_\ell$
- This implies that $\boldsymbol{\nabla} G$ is homogeneous of degree -2:

$$\boxed{\boldsymbol{\nabla}\mathbf{G}(\lambda\mathbf{r}) = |\lambda|^{-3}\lambda\boldsymbol{\nabla}\mathbf{G}(\mathbf{r}) = |\lambda|^{-2}\mathrm{sgn}(\lambda)\boldsymbol{\nabla}\mathbf{G}(\mathbf{r})}$$

$$(\mathbf{r} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}, \lambda \in \mathbb{R} \setminus \{0\})$$

Small-inclusion asymptotic behavior of u^a

- **Domain integral equation (DIE) for the TP:**
combine weak formulations for the TP (with $w = \mathcal{G}(\cdot,\mathbf{x})$) and the Green's function (with $w = v^a$):

$$\begin{cases} \langle \mathbf{v}_B, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\mathcal{C}}_{\Omega} + \langle \mathbf{v}_B, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\mathcal{C}}_{B_a} = -\langle \mathbf{u}, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\mathcal{C}}_{B_a} \\ \langle \mathcal{G}(\cdot,\mathbf{x}), \mathbf{v}_a\rangle^{\mathcal{C}}_{\Omega} = \mathbf{v}_a(\mathbf{x}) \end{cases} \qquad (\mathbf{x}\in\Omega)$$

to obtain

$$\mathcal{L}_a[\mathbf{v}_a](\mathbf{x}) = -\langle \mathbf{u}, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\mathcal{C}}_{B_a} \quad \begin{cases} \mathbf{x}\in B_a \text{ (integral equation for } v^a) \\ \mathbf{x}\in\Omega\setminus\bar{B}_a \text{ (integral representation for } v^a) \end{cases}$$

with the linear operator $\mathcal{L}_a$ defined by $\mathcal{L}_a[\mathbf{w}](\mathbf{x}) = \mathbf{w}(\mathbf{x}) + \langle \mathbf{w}, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\mathcal{C}}_{B_a}$

i.e. (in expanded form):

$$\mathcal{L}_a[\mathbf{w}](\mathbf{x}) = \mathbf{w}(\mathbf{x}) + \int_{B_a} \boldsymbol{\nabla}\mathbf{w} : \Delta\mathcal{C} : \boldsymbol{\nabla}\mathcal{G}(\cdot,\mathbf{x}) \,\mathrm{d}V$$

- **DIE for the FSTP:** (similarly)

$$\mathcal{L}_{\mathcal{B}}[\mathbf{v}_{\mathcal{B}}](\mathbf{x}) = -\langle \psi[\mathbf{E}], \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\mathcal{C}}_{\mathcal{B}} \quad \text{with} \quad \mathcal{L}_{\mathcal{B}}[\mathbf{w}](\mathbf{x}) := \mathbf{w}(\mathbf{x}) + \langle \mathbf{w}, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\mathcal{C}}_{\mathcal{B}}$$

Small-inclusion asymptotic behavior of u^a

Asymptotic behavior of $\mathbf{v}_a$ as $a \to 0$: Inside B_a, one has (inner expansion)

$$\mathbf{v}_a(\mathbf{x}) = a\mathbf{v}_{\mathcal{B}}[\nabla \mathbf{u}(\mathbf{z})](\bar{\mathbf{x}}) + o(a) \qquad \mathbf{x} \in B_a,\ \bar{\mathbf{x}} \in \mathcal{B}$$

Moreover, at any fixed location $\mathbf{x} \neq \mathbf{z}$ (outer expansion), one has

$$\mathbf{v}_a(\mathbf{x}) = -a^3 \nabla_1 \mathcal{G}(\mathbf{z}, \mathbf{x}) : \mathcal{A} : \nabla \mathbf{u}(\mathbf{z}) + o(a^3) \qquad (\mathbf{x} \neq \mathbf{z})$$

where $\mathcal{A} = \mathcal{A}(\mathcal{C}, \mathcal{C}^\star, \mathcal{B})$ is the (fourth-order) *elastic moment tensor*, defined by

$$\mathcal{A} : \mathbf{E} = \int_{\mathcal{B}} \Delta\mathcal{C} : \nabla \mathbf{u}_{\mathcal{B}}[\mathbf{E}] \, \mathrm{d}V = \int_{\mathcal{B}} \Delta\mathcal{C} : (\mathbf{E} + \nabla \mathbf{v}_{\mathcal{B}}[\mathbf{E}]) \, \mathrm{d}V \qquad \forall \mathbf{E} \in \mathbb{R}^{3,3}$$

($\mathbf{u}_{\mathcal{B}}[\mathbf{E}] = \psi[\mathbf{E}] + \mathbf{v}_{\mathcal{B}}[\mathbf{E}]$: solution to the FSTP)

Small-inclusion asymptotic behavior of $\mathbf{u}_a$ (proof sketch 1/2)

Proof based on seeking and exploiting the limiting form of the DIE, and using scaled coordinates.

(i) Asymptotic behavior of the Green's tensor:

$$\boldsymbol{\nabla}_1\mathcal{G}(\boldsymbol{\xi},\mathbf{x}) = a^{-2}\boldsymbol{\nabla}\mathbf{G}(\bar{\boldsymbol{\xi}}-\bar{\mathbf{x}}) + \boldsymbol{\nabla}_1\mathcal{G}_{\text{c}}(\mathbf{z},\mathbf{z}) + o(1)$$

(ii) Defining $\bar{\mathbf{v}}_a(\bar{\boldsymbol{\xi}}) := \mathbf{v}_a(\mathbf{z}+a\bar{\boldsymbol{\xi}})$, $\bar{\mathbf{u}}(\bar{\boldsymbol{\xi}}) := \mathbf{u}(\mathbf{z}+a\bar{\boldsymbol{\xi}})$, one finds

$$\boldsymbol{\nabla}\bar{\mathbf{v}}_a(\bar{\boldsymbol{\xi}}) = a^{-1}\boldsymbol{\nabla}\mathbf{v}_a(\boldsymbol{\xi}), \qquad \boldsymbol{\nabla}\bar{\mathbf{u}}(\bar{\boldsymbol{\xi}}) = \boldsymbol{\nabla}\mathbf{u}(\mathbf{z}) + o(1)$$

(iii) Substitute above expansions and rescale coordinates in DIE:

$$\mathcal{L}_a[\mathbf{v}_a](\mathbf{x}) = \mathcal{L}_{\mathcal{B}}[\bar{\mathbf{v}}_a](\bar{\mathbf{x}}) + aO(\|\boldsymbol{\nabla}\bar{\mathbf{v}}_a\|_{2,\mathcal{B}})$$

$$\langle \mathbf{u}, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\mathcal{C}}_{B_a} = a\langle \psi[\boldsymbol{\nabla}\mathbf{u}(\mathbf{z})], \mathbf{G}(\cdot,\bar{\mathbf{x}})\rangle^{\Delta\mathcal{C}}_{\mathcal{B}} + o(a)$$

(iv) Make ansatz $\bar{\mathbf{v}}_a(\bar{\boldsymbol{\xi}}) = a\mathbf{V}(\bar{\boldsymbol{\xi}}) + o(a)$ and isolate lowest-order (in a) contributions in DIE, which correspond to the FSTP:

$$\mathcal{L}_{\mathcal{B}}[\mathbf{V}](\bar{\boldsymbol{\xi}}) = -\langle \psi[\boldsymbol{\nabla}\mathbf{u}(\mathbf{z})], \mathbf{G}(\cdot,\bar{\mathbf{x}})\rangle^{\Delta\mathcal{C}}_{\mathcal{B}}$$

Hence:

$$\boxed{\mathbf{V}(\bar{\boldsymbol{\xi}}) = \mathbf{v}_{\mathcal{B}}[\boldsymbol{\nabla}\mathbf{u}(\mathbf{z})](\bar{\boldsymbol{\xi}}) \qquad \bar{\boldsymbol{\xi}}\in\mathcal{B}}$$

Small-inclusion asymptotic behavior of $\mathbf{u}_a$ (proof sketch 2/2)

(v) Now, consider $\mathbf{x} \neq \mathbf{z}$ and a small enough. In that case, one has

$$\boldsymbol{\nabla}_1 \mathcal{G}(\boldsymbol{\xi}, \mathbf{x}) = \boldsymbol{\nabla}_1 \mathcal{G}(\mathbf{z}, \mathbf{x}) + o(1), \qquad \boldsymbol{\xi} \in B_a, \mathbf{x} \notin B_a.$$

(vi) Substitute $\bar{\mathbf{v}}_a(\bar{\boldsymbol{\xi}}) = a\mathbf{V}(\bar{\boldsymbol{\xi}}) + o(a)$ and the above into the integral representation outside B_a, to obtain

$$\begin{aligned}\mathbf{v}_a(\mathbf{x}) &= -a^3 \boldsymbol{\nabla}_1 \mathcal{G}(\mathbf{z}, \mathbf{x}) : \int_{\mathcal{B}} \Delta \mathcal{C} : \boldsymbol{\nabla} \mathbf{u}_{\mathcal{B}}[\boldsymbol{\nabla} \mathbf{u}(\mathbf{z})](\bar{\boldsymbol{\xi}}) \, \mathrm{d}\bar{V}_{\bar{\xi}} + o(a^3) \\ &= -a^3 \boldsymbol{\nabla}_1 \mathcal{G}(\mathbf{z}, \mathbf{x}) : \boldsymbol{\mathcal{A}} : \boldsymbol{\nabla} \mathbf{u}(\mathbf{z}) + o(a^3)\end{aligned}$$

Topological derivative of cost functionals

(i) Expand cost functional w.r.t. $\mathbf{v}_a$:

$$J(\mathcal{C}_a) = J(\mathcal{C}) + \mathbb{J}'(\mathbf{u};\mathbf{v}_a) + o(\|\mathbf{v}_a\|_{2,\Omega} + \|\mathbf{v}_a\|_{2,\partial\Omega})$$

with $\mathbb{J}'(\mathbf{u};\mathbf{w}) := \int_\Omega \boldsymbol{\nabla}_2\varphi_{\text{V}}(\cdot,\mathbf{u})\cdot\mathbf{w}\,\text{d}V + \int_{S_\text{N}} \boldsymbol{\nabla}_2\varphi_{\text{S}}(\cdot,\mathbf{u})\cdot\mathbf{w}\,\text{d}S$

(ii) Interpret $\mathbb{J}'(\mathbf{u};\mathbf{v}_a)$ as component of a weak formulation with test function $\mathbf{v}_a$ and define the adjoint solution $\hat{\mathbf{u}}$ by

$$\text{Find } \hat{\mathbf{u}} \in W_0, \qquad \langle \hat{\mathbf{u}},\mathbf{w}\rangle_\Omega^{\mathcal{C}} = \mathbb{J}'(\mathbf{u};\mathbf{w}), \quad \forall \mathbf{w} \in W_0$$

(iii) Reciprocity between transmission and adjoint problems:

$$\langle \mathbf{v}_a,\hat{\mathbf{u}}\rangle_\Omega^{\mathcal{C}} + \langle \mathbf{v}_a,\hat{\mathbf{u}}\rangle_{B_a}^{\Delta\mathcal{C}} = -\langle \mathbf{u},\hat{\mathbf{u}}\rangle_{B_a}^{\Delta\mathcal{C}}\,, \qquad \langle \hat{\mathbf{u}},\mathbf{v}_a\rangle_\Omega^{\mathcal{C}} = \mathbb{J}'(\mathbf{u};\mathbf{v}_a)$$

to obtain

$$\mathbb{J}'(\mathbf{u};\mathbf{v}_a) = -\langle \hat{\mathbf{u}},\mathbf{u}\rangle_{B_a}^{\Delta\mathcal{C}} - \langle \hat{\mathbf{u}},\mathbf{v}_a\rangle_{B_a}^{\Delta\mathcal{C}} = -\langle \hat{\mathbf{u}},\mathbf{u}_a\rangle_{B_a}^{\Delta\mathcal{C}}$$

(iv) Exploit known limiting behavior of v^a in B_a (inner expansion), yielding

$$\mathbb{J}'(\mathbf{u};\mathbf{v}_a) = -\langle \hat{\mathbf{u}},\mathbf{u}_a\rangle_{B_a}^{\Delta\mathcal{C}} = -a^3\boldsymbol{\nabla}\hat{\mathbf{u}}(\mathbf{z})\!:\!\boldsymbol{\mathcal{A}}\!:\!\boldsymbol{\nabla}\mathbf{u}(\mathbf{z}) + o(a^3)$$

Topological expansion of J: The topological expansion of J is given by

$$J(\mathcal{C}_a) = J(\mathcal{C}) + a^3\mathcal{T}(\mathbf{z}) + o(a^3), \qquad \mathcal{T}(\mathbf{z}) = -\boldsymbol{\nabla}\mathbf{u}(\mathbf{z})\!:\!\boldsymbol{\mathcal{A}}\!:\!\boldsymbol{\nabla}\hat{\mathbf{u}}(\mathbf{z})$$

Elastic moment tensor (anisotropic elastic inclusion)

1. The EMT $\boldsymbol{\mathcal{A}}$ has major and minor symmetries: for any $(\mathbf{E},\mathbf{E}')\in\mathbb{R}^{3,3}\times\mathbb{R}^{3,3}$, one has

$$\mathbf{E}':\boldsymbol{\mathcal{A}}:\mathbf{E}=\mathbf{E}:\boldsymbol{\mathcal{A}}:\mathbf{E}' \qquad \text{(major symmetry)}$$

$$\mathbf{E}':\boldsymbol{\mathcal{A}}:\mathbf{E}=\mathbf{E}':\boldsymbol{\mathcal{A}}:\mathbf{E}^{\mathsf{T}}=\mathbf{E}'^{\mathsf{T}}:\boldsymbol{\mathcal{A}}:\mathbf{E} \qquad \text{(minor symmetry)}$$

2. By superposition, one has

$$\boxed{\mathbf{V}_{\mathcal{B}}[\boldsymbol{\nabla}\mathbf{u}(\mathbf{z})]=\sum_{1\le i\le j\le 3}\big(\mathbf{E}^{ij}:\boldsymbol{\nabla}\mathbf{u}(\mathbf{z})\big)\mathbf{V}_{\mathcal{B}}[\mathbf{E}^{ij}]}$$

with $\quad\mathbf{E}^{ij}=\mathbf{e}_i\otimes\mathbf{e}_i\ \ (i=j),\quad \mathbf{E}^{ij}=2^{-1/2}(\mathbf{e}_i\otimes\mathbf{e}_j+\mathbf{e}_j\otimes\mathbf{e}_i)\ \ (i\neq j)$

One thus needs only compute the $v_{\mathcal{B}}[\mathbf{E}_{ij}]$ (6 auxiliary solutions independent on $\mathbf{z}$) once and for all.

3. $\mathcal{C}^{\star}=\mathbf{0}$, i.e. $\Delta\mathcal{C}=-\mathcal{C}$, corresponds to a **traction-free cavity**.
Thus, small-cavity asymptotics is special case of small-inclusion asymptotics.

Beretta, Bonnetier, Francini, Mazzucato (2011); B, Delgado (in preparation, 2011)
Isotropic case: Ammari, Kang, Nakamura, Tanuma (2002); Guzina, Chikichev (2006)

Elastic moment tensor (anisotropic elastic inclusion)

4. Let $\mathcal{B}$ be an ellipsoid (of principal semiaxes a_1, a_2, a_3). The Eshelby tensor $\boldsymbol{\mathcal{S}}(a_1, a_2, a_3)$ of $\mathcal{B}$ is given by (Mura, 1987):

$$\mathcal{S}_{ijmn}(a_1, a_2, a_3) = \frac{1}{8\pi}\mathcal{C}_{pqmn} \int_{-1}^{1} \text{d}u \int_{0}^{2\pi} \big\{ G_{ipjq}(\boldsymbol{\eta}(u,\theta)) + G_{jpiq}(\boldsymbol{\eta}(u,\theta)) \big\} \, \text{d}\theta$$

$$G_{ijk\ell}(\boldsymbol{\eta}) := \eta_k \eta_\ell N_{ij}(\boldsymbol{\eta}), \quad \boldsymbol{\eta}(u,\theta) := \sqrt{1-u^2}[a_1^{-1}\cos\theta\, \mathbf{e}_1 + a_2^{-1}\sin\theta \mathbf{e}_2] + a_3^{-1} u\, \mathbf{e}_3$$

Then, $\boldsymbol{\mathcal{A}}$ has the explicit expression (valid for arbitrary anisotropic $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}^\star$)

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{C}}^\star, a_1, a_2, a_3) = \boldsymbol{\mathcal{C}} : \big[\boldsymbol{\mathcal{C}} + \Delta\boldsymbol{\mathcal{C}} : \boldsymbol{\mathcal{S}}(a_1, a_2, a_3) \big]^{-1} : \Delta\boldsymbol{\mathcal{C}}$$

$\boldsymbol{\mathcal{S}}(a_1, a_2, a_3)$ known in closed form for **isotropic** $\boldsymbol{\mathcal{C}}$ (elliptic integrals involved in the case of ellipsoids).

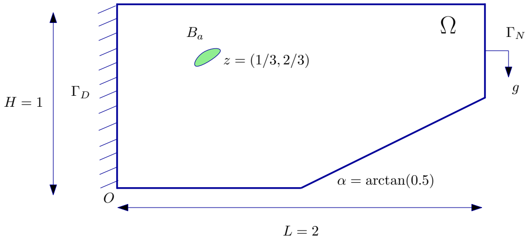
5. For ellipsoidal cavities, one has

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{C}}, a_1, a_2, a_3) = \big[\boldsymbol{\mathcal{S}}(a_1, a_2, a_3) - \boldsymbol{\mathcal{I}} \big]^{-1} : \boldsymbol{\mathcal{C}}$$

6. Closed-form expression for spherical cavity (i.e. when $\mathcal{B}$ is unit sphere):

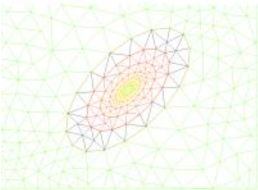
$$\boldsymbol{\mathcal{A}} = -2\mu \Big[\frac{3(1+\nu)}{2(1-2\nu)^2} \boldsymbol{\mathcal{J}} + \frac{15(1-\nu)}{7-5\nu} \boldsymbol{\mathcal{K}} \Big] \qquad (3\boldsymbol{\mathcal{J}} = \mathbf{I} \otimes \mathbf{I},\ \boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{J}})$$

Numerical example



$$J(B_a) = -\frac{1}{2}\int_{S_N} \mathbf{g}\cdot\mathbf{u}_a \,\mathrm{d}S$$

(compliance)

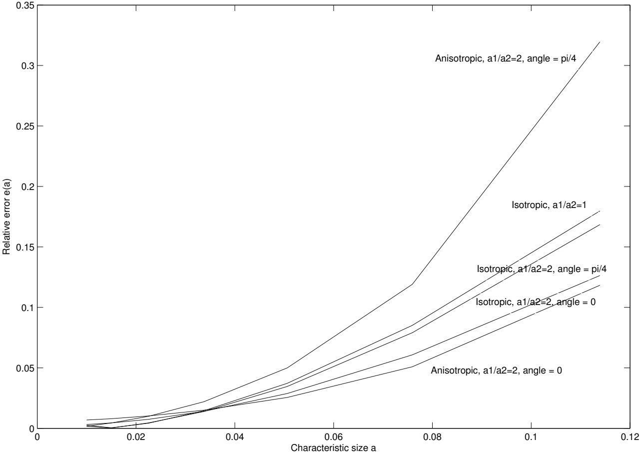


Isotropic tests: $\mathcal{C} = \begin{pmatrix} 1.34 & 0.57 & 0. \\ 0.57 & 1.34 & 0. \\ 0. & 0. & 0.38 \end{pmatrix}, \quad \mathcal{C}^\star = 10^{-9}\mathcal{C}$

Anisotropic tests: $\mathcal{C} = \begin{pmatrix} 1. & 0.5 & 0. \\ 0.5 & 2. & 0. \\ 0. & 0. & 0.04 \end{pmatrix}, \quad \mathcal{C}^\star = \begin{pmatrix} 3. & 0.4 & 0. \\ 0.4 & 1.5 & 0. \\ 0. & 0. & 0.03 \end{pmatrix}.$

B, Delgado (in preparation, 2011)

Numerical example



$$e(a) = \frac{|J(\mathcal{C}_a) - J(\mathcal{C}) - a^2\mathcal{T}(\mathbf{z})|}{|a^2\mathcal{T}(\mathbf{z})|}$$

B, Delgado (in preparation, 2011)

Energy-based cost functional

- 'Neumann' and 'Dirichlet' background fields:

$$\begin{array}{lll} \text{Find } \mathbf{u}^{\mathrm{N}} \in W^{\mathrm{N}}(\mathbf{u}^D), & \langle \mathbf{u}^{\mathrm{N}}, \mathbf{w} \rangle_{\Omega}^{\mathcal{C}} = F_{\mathrm{no}}(\mathbf{w}) + F_{\mathrm{o}}(\mathbf{w}), & \forall \mathbf{w} \in W_0^{\mathrm{N}}. \\ \text{Find } \mathbf{u}^{\mathrm{D}} \in W^{\mathrm{D}}(\mathbf{u}^D), & \langle \mathbf{u}^{\mathrm{D}}, \mathbf{w} \rangle_{\Omega}^{\mathcal{C}} = F_{\mathrm{no}}(\mathbf{w}), & \forall \mathbf{w} \in W_0^{\mathrm{D}}. \end{array}$$

where $W^{\mathrm{N}}(\mathbf{u}^D) = W(\mathbf{u}^D)$, $W_0^{\mathrm{N}} = W_{\mathbf{0}}$ and

$$W^{\mathrm{D}}(\mathbf{u}^D) := \big\{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^3),\ \mathbf{v} = \mathbf{u}^D \text{ on } S_{\mathrm{D}},\ \mathbf{v} = \mathbf{u}_{\mathrm{obs}} \text{ on } S_{\mathrm{o}} \big\}, \qquad W_0^{\mathrm{D}} := W^{\mathrm{D}}(\mathbf{0})$$

- 'Neumann' and 'Dirichlet' fields for small trial inclusion:

$$\begin{array}{lll} \text{Find } \mathbf{v}_a^{\mathrm{N}} \in W_0^{\mathrm{N}}, & \langle \mathbf{v}_a^{\mathrm{N}}, \mathbf{w} \rangle_{\Omega}^{\mathcal{C}} + \langle \mathbf{v}_a^{\mathrm{N}}, \mathbf{w} \rangle_{B_a}^{\Delta\mathcal{C}} = -\langle \mathbf{u}^{\mathrm{N}}, \mathbf{w} \rangle_{B_a}^{\Delta\mathcal{C}}, & \forall \mathbf{w} \in W_0^{\mathrm{N}}. \\ \text{Find } \mathbf{v}_a^{\mathrm{D}} \in W_0^{\mathrm{D}}, & \langle \mathbf{v}_a^{\mathrm{D}}, \mathbf{w} \rangle_{\Omega}^{\mathcal{C}} + \langle \mathbf{v}_a^{\mathrm{D}}, \mathbf{w} \rangle_{B_a}^{\Delta\mathcal{C}} = -\langle \mathbf{u}^{\mathrm{D}}, \mathbf{w} \rangle_{B_a}^{\Delta\mathcal{C}}, & \forall \mathbf{w} \in W_0^{\mathrm{D}}. \end{array}$$

- Energy discrepancy functional:

$$E(\mathcal{C}_a) = \mathbb{E}(\mathbf{u}_a^{\mathrm{N}}, \mathbf{u}_a^{\mathrm{D}}, \mathcal{C}_a) = \frac{1}{2} \langle \mathbf{u}_a^{\mathrm{N}} - \mathbf{u}_a^{\mathrm{D}}, \mathbf{u}_a^{\mathrm{N}} - \mathbf{u}_a^{\mathrm{D}} \rangle_{\Omega}^{\mathcal{C}_a}$$

Energy-based cost functional

Topological derivative of E: it is given by

$$\mathcal{T}_E(\mathbf{z}) = \frac{1}{2}\varepsilon[\mathbf{u}^{\mathrm{N}}](\mathbf{z}):\boldsymbol{\mathcal{A}}:\varepsilon[\mathbf{u}^{\mathrm{N}}](\mathbf{z}) - \frac{1}{2}\varepsilon[\mathbf{u}^{\mathrm{D}}](\mathbf{z}):\boldsymbol{\mathcal{A}}:\varepsilon[\mathbf{u}^{\mathrm{D}}](\mathbf{z})$$

- Energy-based cost functionals are very useful for identification of (possibly heterogeneous) material parameters, see e.g. Kohn, Vogelius (1987), Kohn, McKenney (1989), Constantinescu (1995), Chavent, Kunisch, Roberts (1996).
- Specific treatment needed to find $\mathcal{T}_E$, because (i) $\varphi_{\mathrm{V}}(\cdot, \boldsymbol{\nabla}\mathbf{w}, \boldsymbol{\mathcal{C}}_B)$ rather than $\varphi_{\mathrm{V}}(\cdot, \mathbf{w})$, and (ii) $\mathbf{v}_a = O(a)$ but $\mathbf{v}_a = O(1)$ inside B_a (so care needed in doing expansions about $a=0$).

B, Delgado (in preparation, 2011)

Acoustic transmission problem

Scattering of an acoustic wave by a penetrable inclusion B $(\rho^\star, c^\star)$ embedded in an acoustic medium Ω (ρ, c).

$$\begin{cases} \beta := \rho/\rho^\star & \text{mass density ratio} \\ \gamma := c/c^\star & \text{velocity ratio} \\ \eta := (\rho c^2)/(\rho c^2)^\star & \text{bulk modulus ratio} \end{cases}$$

Let $m := 1 + (\beta - 1)\chi(B)$ and $n := 1 + (\eta - 1)\chi(B)$.

- Field equation (assuming homogeneous inclusion and background) on acoustic pressure p:

$$\text{div}\,(m\boldsymbol{\nabla} u_B) + nk^2 u_B + f = 0 \qquad (\text{in } \Omega)$$

- Transmission conditions implied (continuity of pressure and normal velocity across ∂B)

$$u_B\,|_+ = u_B\,|_- \quad \text{and} \quad \rho^{-1}\partial_n u_B\,|_+ = (\rho^\star)^{-1}\partial_n u_B\,|_- \qquad \text{on } \partial B$$

Martin, P.A. (2003), Acoustic scattering by inhomogeneous obstacles. SIAM J. Appl. Math. **64**:297-308

Weak formulation of TP

Notations: Let $W(u^D) := \{ w \in H^1(\Omega),\ w = u^D \text{ on } S_D \}$. $W_0 = W(0)$ and

$$\langle u, w \rangle_D^m := \int_D m \nabla u \cdot \nabla w \, dV \qquad (u, w)_D^n := \int_D n u w \, dV$$

$$\langle u, w \rangle_D := \int_D \nabla u \cdot \nabla w \, dV \qquad (u, w)_D := \int_D u w \, dV$$

$$F(w) = \int_\Omega f w \, dV + \int_{S_N} g w \, dS$$

- **Background solution** (weak formulation):

$$\boxed{\text{Find } u \in W(u^D) \quad \langle u, w \rangle_\Omega - k^2 (u, w)_\Omega = F(w), \quad \forall w \in W_0}$$

  For unbounded-media idealizations ($\Omega = \mathbb{R}^3$), u is an arbitrary solution to $(\Delta + k^2) u = 0$ (e.g. a plane wave), in particular not constrained by radiation conditions.

- **TP** (weak formulation), in terms of the solution perturbation (scattered field) $v_B := u_B - u$:

$$\boxed{\text{Find } v_B \in W_0, \quad \langle v_B, w \rangle_\Omega^{m_B} - k^2 (v_B, w)_\Omega^{n_B} = -\langle u, w \rangle_B^{\Delta m} + k^2 (u, w)_B^{\Delta n}, \quad \forall w \in W_0}$$

Cost functional

$$J(m_B, n_B) = \mathbb{J}(u_B) \qquad \text{with } \mathbb{J}(w) := \int_\Omega \varphi_\mathsf{v}(\cdot, w)\, \mathrm{d}V + \int_{\partial\Omega} \varphi_\mathsf{s}(\cdot, w)\, \mathrm{d}S$$

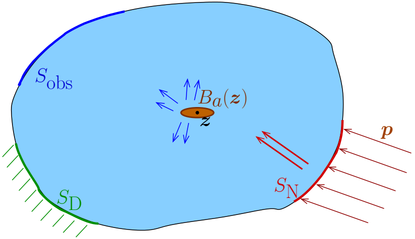
with twice-differentiable real-valued densities $w \mapsto \varphi_\mathsf{v}(\cdot, w)$ and $w \mapsto \varphi_\mathsf{s}(\cdot, w)$.
Example (output least-squares for defect identification):

$$\varphi_\mathsf{v}(\cdot, w) = 0, \quad \varphi_\mathsf{s}(\cdot, w) = \frac{1}{2}(\overline{w - u^\mathsf{obs}})Q(\cdot)(w - u^\mathsf{obs})$$

Topological derivative of J: sought by considering the limiting behavior of

$$j(a) := J(m_a, n_a) = \mathbb{J}(u^a)$$

with $m_a := 1 + (\beta - 1)\chi(B_a)$ and $n_a := 1 + (\eta - 1)\chi(B_a)$



Small-inclusion asymptotic behavior of u^a

- Green's function: defined, for $\mathbf{x} \in \Omega$, by

$$(\Delta + k^2)\mathcal{G}(\cdot, \mathbf{x}; k) + \delta(\cdot - \mathbf{x}) = 0 \quad (\text{in } \Omega),$$
$$\mathcal{G}(\cdot, \mathbf{x}; k) = 0 \quad (\text{on } S_\text{D}), \qquad \partial_n \mathcal{G}(\cdot, \mathbf{x}; k) = 0 \quad (\text{on } S_\text{N})$$

- Split G into singular and nonsingular parts:

$$\boxed{\mathcal{G}(\boldsymbol{\xi}, \mathbf{x}; k) = G(\boldsymbol{\xi} - \mathbf{x}; k) + \mathcal{G}_\text{c}(\boldsymbol{\xi}, \mathbf{x}; k)}$$

where $G(\mathbf{r}; k)$ is the (singular) Helmholtz full-space fundamental solution:

$$G(\mathbf{r}; k) = \frac{e^{\text{i}kr}}{4\pi r} \qquad (r = \|\mathbf{r}\|)$$

- Moreover, one has

$$\boxed{G(\mathbf{r}; k) = G(\mathbf{r}) + G_\text{c}(\mathbf{r}; k)}$$

where $G(\mathbf{r}) = 1/(4\pi r)$ is the Laplace fundamental solution and $G_\text{c}(\mathbf{r}; k)$ is nonsingular for $\mathbf{r} = \mathbf{0}$

Small-inclusion asymptotic behavior of u^a

Domain integral equation (DIE) for the TP:

Combining weak formulations for the TP (with $w = \mathcal{G}(\cdot, \mathbf{x}; k)$) and the Green's function (with $w = v^a$) yields a DIE of Lippmann-Schwinger type:

$$\mathcal{L}_a^k[v^a](\mathbf{x}) = -\langle u, \mathcal{G}(\cdot, \mathbf{x}; k)\rangle_{B_a}^{\Delta m} + k^2 (u, \mathcal{G}(\cdot, \mathbf{x}; k))_{B_a}^{\Delta n}$$

$$\begin{cases} \mathbf{x} \in B_a & \text{(integral equation for } v^a) \\ \mathbf{x} \in \Omega \setminus \bar{B}_a & \text{(integral representation for } v^a) \end{cases}$$

with the linear operator $\mathcal{L}_a^k$ defined by

$$\mathcal{L}_a^k[w](\mathbf{x}) = w(\mathbf{x}) + \langle w, \mathcal{G}(\cdot, \mathbf{x}; k)\rangle_{B_a}^{\Delta m} - k^2 (w, \mathcal{G}(\cdot, \mathbf{x}; k))_{B_a}^{\Delta n}$$

i.e. (in expanded form):

$$\mathcal{L}_a[w](\mathbf{x}) = w(\mathbf{x}) + \int_{B_a} \big[(\beta - 1) \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} \mathcal{G}(\cdot, \mathbf{x}; k) - (\eta - 1) k^2 w \mathcal{G}(\cdot, \mathbf{x}; k) \big] \, \mathrm{d}V$$

Small-inclusion asymptotic behavior of u^a

$$\boxed{\mathcal{L}_a^k[v^a](\mathbf{x}) = -\langle u, \mathcal{G}(\cdot,\mathbf{x})\rangle_{B_a}^{\Delta m} + k^2 (u, \mathcal{G}(\cdot,\mathbf{x}))_{B_a}^{\Delta n}} \qquad \text{(LS)}$$

- Coordinate scaling:

$$\boldsymbol{\xi} = \mathbf{z} + a\bar{\boldsymbol{\xi}}, \quad \mathbf{x} = \mathbf{z} + a\bar{\mathbf{x}}, \quad \mathrm{d}V_\xi = a^3\,\mathrm{d}\bar{V}_{\bar{\xi}}, \quad \boldsymbol{\nabla}_\xi = a^{-1}\boldsymbol{\nabla}_{\bar{\xi}}$$

- Scaling properties of $\mathcal{G}(\boldsymbol{\xi},\mathbf{x};k)$ (by homogeneity properties of $G(\boldsymbol{\xi}-\mathbf{x})$):

$$\mathcal{G}(\boldsymbol{\xi},\mathbf{x};k) = a^{-1}G(\bar{\boldsymbol{\xi}}-\bar{\mathbf{x}}) + O(1)$$
$$\boldsymbol{\nabla}_1\mathcal{G}(\boldsymbol{\xi},\mathbf{x};k) = a^{-2}\boldsymbol{\nabla}G(\bar{\boldsymbol{\xi}}-\bar{\mathbf{x}}) + O(1)$$

- Leading contribution in (LS) using ansatz $\boxed{v^a(\boldsymbol{\xi}) = aV(\bar{\boldsymbol{\xi}}) + o(a) \quad \text{in } B_a}$:

$$\mathcal{L}_a^k[v^a](\mathbf{x}) = a\mathcal{L}_{\mathcal{B}}[V](\bar{\mathbf{x}}) + o(a^1)$$
$$\langle u, \mathcal{G}(\cdot,\mathbf{x})\rangle_{B_a}^{\Delta m} - k^2 (u, \mathcal{G}(\cdot,\mathbf{x}))_{B_a}^{\Delta n} = a\langle \psi[\boldsymbol{\nabla}u(\mathbf{z})], G(\cdot,\bar{\mathbf{x}})\rangle_{\mathcal{B}}^{\Delta m} + o(a^1)$$

Hence, retaining only the $O(a)$ terms, V solves again the **static FSTP for the normalized inclusion**:

$$\boxed{\mathcal{L}_{\mathcal{B}}[V](\bar{\mathbf{x}}) = -\langle \psi[\boldsymbol{\nabla}u(\mathbf{z})], G(\cdot,\bar{\mathbf{x}})\rangle_{\mathcal{B}}^{\Delta m} \quad (\bar{\mathbf{x}}\in\mathcal{B})}$$

Small-inclusion asymptotic behavior of u^a

Inside B_a, one has (inner expansion)

$$v^a(\mathbf{x}) = a v_{\mathcal{B}}[\nabla u(\mathbf{z})](\bar{\mathbf{x}}) + o(a) \qquad \mathbf{x} \in B_a,\ \bar{\mathbf{x}} \in \mathcal{B}$$

where $v_{\mathcal{B}}[\mathbf{E}]$ is again the solution of the **static** FSTP.

Moreover, at any fixed location $\mathbf{x} \neq \mathbf{z}$ (outer expansion), one has

$$v^a(\mathbf{x}) = -a^3 \big[\nabla_1 \mathcal{G}(\cdot, \mathbf{x}; k) \cdot \mathbf{A} \cdot \nabla u - k^2(\eta - 1)|\mathcal{B}| \mathcal{G}(\cdot, \mathbf{x}; k) u \big](\mathbf{z}) + o(a^3) \quad (\mathbf{x} \neq \mathbf{z})$$

where $\mathbf{A} = \mathbf{A}(\beta, \mathcal{B})$ is again the polarization tensor associated with the normalized **static** FSTP.

Topological derivative of cost functionals

(i) Expand cost functional w.r.t. v^a:

$$J(m_a, n_a) = J(1,1) + \mathbb{J}'(u; v^a) + o(\|v^a\|_{2,\Omega} + \|v^a\|_{2,\partial\Omega})$$

(ii) Interpret $\mathbb{J}'(u; v^a)$ as component of a weak formulation with test function v^a and define the adjoint solution $\hat{u}$ by

$$\boxed{\text{Find } \hat{u} \in W_0, \qquad \langle \hat{u}, w\rangle_\Omega - k^2(\hat{u}, w)_\Omega = \mathbb{J}'(u; w), \quad \forall w \in W_0}$$

(iii) Reciprocity between transmission and adjoint problems:

$$\langle v^a, \hat{u}\rangle_\Omega^{m_a} - k^2(v^a, \hat{u})_\Omega^{n_a} = -\langle u, \hat{u}\rangle_{B_a}^{\Delta m} + k^2(u, \hat{u})_{B_a}^{\Delta n}$$

$$\langle \hat{u}, v^a\rangle_\Omega - k^2(\hat{u}, v^a)_\Omega = \mathbb{J}'(u; v^a)$$

to obtain
$$\boxed{\mathbb{J}'(u; v^a) = -\langle \hat{u}, u^a\rangle_{B_a}^{\Delta m} + k^2(\hat{u}, u^a)_{B_a}^{\Delta n}}$$

(iv) Exploit known limiting behavior of v^a in B_a (inner expansion), yielding

$$\mathbb{J}'(u; v^a) = -a^3\big[\boldsymbol{\nabla}\hat{u}(\mathbf{z})\cdot\mathbf{A}\cdot\boldsymbol{\nabla}u(\mathbf{z}) - k^2(\eta-1)|\mathcal{B}|\hat{u}(\mathbf{z})u(\mathbf{z})\big] + o(a^3)$$

Topological derivative of J:

$$\boxed{\mathcal{T}(\mathbf{z}) = -\boldsymbol{\nabla}\hat{u}(\mathbf{z})\cdot\mathbf{A}\cdot\boldsymbol{\nabla}u(\mathbf{z}) + k^2(\eta-1)|\mathcal{B}|\hat{u}(\mathbf{z})u(\mathbf{z})}$$

Comments

Since $\mathbf{A}(\beta,\mathcal{B})$ is associated with the normalized **static** FSTP, the following hold:

- $\mathbf{A}$ is symmetric;
- By superposition:

$$V(\bar{\mathbf{x}}) = v_B[\nabla u(\mathbf{z})](\bar{\mathbf{x}}) = \sum_{i=1}^{3} \big(\mathbf{e}_i \cdot \nabla u(\mathbf{z})\big)\, v_B[\mathbf{e}_i](\bar{\mathbf{x}})$$

  One thus needs only compute the $v_B[\mathbf{e}_i]$, which are independent on $\mathbf{z}$.
- **Spherical vanishing inclusion** (i.e. $\mathcal{B}$ is unit sphere): The FSTP solution is given by

$$v_B[\mathbf{E}](\bar{\mathbf{x}}) = \frac{3}{2+\beta}\mathbf{E}\cdot\bar{\mathbf{x}} \quad \text{with } \beta := \kappa^\star/\kappa \qquad (\bar{\mathbf{x}} \in \mathcal{B}).$$

  Hence:

$$\boxed{\mathbf{A}(\beta,\mathcal{B}) = 4\pi\frac{(1-\beta)}{2+\beta}\mathbf{I}};$$

- FSTP solutions analytically available for other shapes (e.g. ellipsoids); otherwise, solve numerically (and once and for all) the normalized DIE with $\mathbf{E} = \mathbf{e}_i$ $(i = 1,2,3)$;
- The special case $\beta = 0$, i.e. $\rho^\star = \infty$, corresponds to the **impenetrable** inclusion.

Background solution

- Background solution ($\mathcal{C}, \rho$: background elasticity tensor and mass density):

$$\text{div}\,(\mathcal{C}\!:\!\varepsilon[\mathbf{u}]) + \rho\omega^2\mathbf{u} + \mathbf{f} = \mathbf{0} \text{ in } \Omega, \quad \mathbf{t}[\mathbf{u}] = \mathbf{g} \text{ on } S_{\text{N}}, \quad \mathbf{u} = \mathbf{u}^D \text{ on } S_{\text{D}}$$

  In weak form:

$$\text{Find } \mathbf{u} \in W(\mathbf{u}^D), \qquad \langle \mathbf{u}, \mathbf{w} \rangle_\Omega^{\mathcal{C}} - \omega^2 (\mathbf{u}, \mathbf{w})_\Omega^{\rho} = F(\mathbf{w}), \quad \forall \mathbf{w} \in W_0$$

  with $W(\mathbf{u}^D) := \big\{ \mathbf{v} \in H^1(\mathbb{R}^3; \mathbb{C}^3),\ \mathbf{v} = \mathbf{u}^D \text{ on } S_{\text{D}} \big\}$. $W_0 = W(\mathbf{0})$ and

$$\langle \mathbf{u}, \mathbf{w} \rangle_D^{\mathcal{C}} := \int_D \varepsilon[\mathbf{u}]\!:\!\mathcal{C}\!:\!\varepsilon[\mathbf{w}] \,\text{d}V = \int_D \boldsymbol{\nabla}\mathbf{u}\!:\!\mathcal{C}\!:\!\boldsymbol{\nabla}\mathbf{w} \,\text{d}V,$$

$$(\mathbf{u}, \mathbf{w})_\Omega^{\rho} := \int_D \rho\, \mathbf{u}\cdot\mathbf{w} \,\text{d}V$$

$$F(\mathbf{w}) = \int_\Omega \mathbf{f}\cdot\mathbf{w} \,\text{d}V + \int_{S_{\text{N}}} \mathbf{g}\cdot\mathbf{w} \,\text{d}S$$

- For unbounded-media idealizations, $\mathbf{u}$ is an arbitrary solution to the Navier PDE (e.g. a plane wave), which in particular is not constrained by radiation conditions.

Transmission problem (TP)

(possibly multiple) inclusion $B \subset \Omega$ with

$$\begin{aligned} \mathcal{C}_B &= [1-\chi(B)]\mathcal{C} + \chi(B)\mathcal{C}^\star &&= \mathcal{C} + \chi(B)\Delta\mathcal{C} && \text{(elasticity tensor)} \\ \rho_B &= [1-\chi(B)]\rho + \chi(B)\rho^\star &&= \rho + \chi(B)\Delta\rho && \text{(mass density)} \end{aligned}$$

TP:

$$\boxed{\operatorname{div}(\mathcal{C}_B : \varepsilon[\mathbf{u}_B]) + \rho_B\omega^2\mathbf{u} + \mathbf{f} = \mathbf{0} \text{ in } \Omega, \quad \mathbf{t}[\mathbf{u}_B] = \mathbf{g} \text{ on } S_\text{N}, \quad \mathbf{u}_B = \mathbf{u}^D \text{ on } S_\text{D}}$$

i.e., in weak form and in terms of the solution perturbation $\mathbf{v}_B := \mathbf{u}_B - \mathbf{u}$:

$$\boxed{\text{Find } \mathbf{v}_B \in W_0, \quad \langle \mathbf{v}_B, \mathbf{w} \rangle_\Omega^{\mathcal{C}_B} - \omega^2 (\mathbf{v}_B, \mathbf{w})_\Omega^{\rho_B} = -\langle \mathbf{u}, \mathbf{w} \rangle_B^{\Delta\mathcal{C}} + \omega^2 (\mathbf{u}, \mathbf{w})_B^{\Delta\rho}, \quad \forall \mathbf{w} \in W_0}$$

This weak form is also applicable to scattering in unbounded media.

Cost functional

$$J(\mathcal{C}_B, \rho_B) = \mathbb{J}(\mathbf{u}_B) \qquad \text{with } \mathbb{J}(\mathbf{w}) := \int_\Omega \varphi_\mathsf{V}(\cdot, \mathbf{w})\,\mathrm{d}V + \int_{\partial\Omega} \varphi_\mathsf{S}(\cdot, \mathbf{w})\,\mathrm{d}S$$

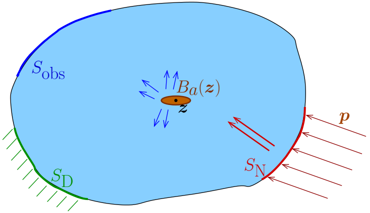
with twice-differentiable real-valued densities $\mathbf{w} \mapsto \varphi_\mathsf{V}(\cdot, \mathbf{w})$ and $\mathbf{w} \mapsto \varphi_\mathsf{S}(\cdot, \mathbf{w})$.
Example (for defect identification, with positive definite weighting matrix $\mathbf{Q}$):

$$\varphi_\mathsf{V}(\cdot, \mathbf{w}) = 0, \quad \varphi_\mathsf{S}(\cdot, \mathbf{w}) = \frac{1}{2}(\overline{\mathbf{w} - \mathbf{u}^{\text{obs}}})^\mathsf{T} \mathbf{Q}(\cdot)(\mathbf{w} - \mathbf{u}^{\text{obs}})$$

Topological derivative of J: sought by considering the limiting behavior of

$$j(a) := J(\mathcal{C}_a, \rho_a) = \mathbb{J}(\mathbf{u}_a)$$

with $\mathcal{C}_a := \mathcal{C} + \chi(B_a)\Delta\mathcal{C}$ and $\rho_a = \rho + \chi(B_a)\Delta\rho$



Small-inclusion asymptotic behavior of u^a

- The elastodynamic Green's tensor is defined, for $\mathbf{x} \in \Omega$ by

$$\text{div}\,\big(\mathcal{C} : \varepsilon[\mathcal{G}(\cdot,\mathbf{x};\omega)]\big) + \rho\omega^2\mathcal{G}(\cdot,\mathbf{x};\omega) + \delta(\cdot - \mathbf{x})\mathbf{I} = \mathbf{0} \quad (\text{in } \Omega),$$
$$\mathcal{G}(\cdot,\mathbf{x};\omega) = \mathbf{0} \quad (\text{on } S_{\text{D}}), \qquad \mathbf{t}[\mathcal{G}(\cdot,\mathbf{x};\omega)] = \mathbf{0} \quad (\text{on } S_{\text{N}})$$

- Split $\mathbf{G}$ into singular and nonsingular parts:

$$\boxed{\mathcal{G}(\boldsymbol{\xi},\mathbf{x};\omega) = \mathbf{G}(\boldsymbol{\xi}-\mathbf{x};\omega) + \mathcal{G}_{\text{c}}(\boldsymbol{\xi},\mathbf{x};\omega)}$$

  where $\mathbf{G}(\mathbf{r};\omega)$ is the (singular) full-space elastodynamic fundamental solution and the complementary Green's tensor $\mathcal{G}_{\text{c}}$ is bounded at $\boldsymbol{\xi} = \mathbf{x}$ (and in fact C^∞ for $\boldsymbol{\xi},\mathbf{x} \in \Omega$).

- Moreover, the singular part of $\mathbf{G}$ corresponds to that of the elastostatic Green's tensor:

$$\boxed{\mathbf{G}(\mathbf{r};\omega) = \mathbf{G}(\mathbf{r}) + \mathbf{G}_{\text{c}}(\mathbf{r};\omega)}$$

  where $\mathbf{G}(\mathbf{r})$ is the Kelvin elastostatic fundamental solution.

Small-inclusion asymptotic behavior of u^a

Domain integral equation (DIE) for the TP:
Combining weak formulations for the TP (with $\mathbf{w} = \mathcal{G}(\cdot, \mathbf{x}; \omega)$) and the Green's tensor (with $\mathbf{w} = \mathbf{v}_a$), as in elastostatics, yields a DIE of Lippmann-Schwinger type:

$$\mathcal{L}_a^\omega[\mathbf{v}_a](\mathbf{x}) = -\langle \mathbf{u}, \mathcal{G}(\cdot, \mathbf{x}) \rangle_{B_a}^{\Delta\mathcal{C}} + \omega^2 (\mathbf{u}, \mathcal{G}(\cdot, \mathbf{x}))_{B_a}^{\Delta\rho}$$

$$\begin{cases} \mathbf{x} \in B_a \text{ (integral equation for } v^a) \\ \mathbf{x} \in \Omega \setminus \bar{B}_a \text{ (integral representation for } v^a) \end{cases}$$

with the linear operator $\mathcal{L}_a^\omega$ defined by

$$\mathcal{L}_a^\omega[\mathbf{w}](\mathbf{x}) = \mathbf{w}(\mathbf{x}) + \langle \mathbf{w}, \mathcal{G}(\cdot, \mathbf{x}; \omega) \rangle_{B_a}^{\Delta\mathcal{C}} - \omega^2 (\mathbf{w}, \mathcal{G}(\cdot, \mathbf{x}; \omega))_{B_a}^{\Delta\rho}$$

i.e. (in expanded form):

$$\mathcal{L}_a[\mathbf{w}](\mathbf{x}) = \mathbf{w}(\mathbf{x}) + \int_{B_a} \big[\boldsymbol{\nabla}\mathbf{w} : \Delta\mathcal{C} : \boldsymbol{\nabla}\mathcal{G}(\cdot, \mathbf{x}; \omega) - \Delta\rho\omega^2 \mathbf{w} \cdot \mathcal{G}(\cdot, \mathbf{x}; \omega) \big] \, \mathrm{d}V$$

Small-inclusion asymptotic behavior of u^a

$$\boxed{\mathcal{L}_a[\mathbf{v}_a](\mathbf{x}) = -\langle \mathbf{u}, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\mathcal{C}}_{B_a} + \omega^2(\mathbf{u}, \mathcal{G}(\cdot,\mathbf{x}))^{\Delta\rho}_{B_a}} \qquad \text{(LS)}$$

- Coordinate scaling:

$$\boldsymbol{\xi} = \mathbf{z} + a\bar{\boldsymbol{\xi}}, \quad \mathbf{x} = \mathbf{z} + a\bar{\mathbf{x}}, \quad \text{d}V_\xi = a^3\,\text{d}\bar{V}_{\bar{\xi}}, \quad \boldsymbol{\nabla}_\xi = a^{-1}\boldsymbol{\nabla}_{\bar{\xi}}$$

- Scaling properties of $\mathcal{G}(\boldsymbol{\xi},\mathbf{x};\omega)$ (by homogeneity properties of $\mathcal{G}(\boldsymbol{\xi}-\mathbf{x})$):

$$\mathcal{G}(\boldsymbol{\xi},\mathbf{x};\omega) = a^{-1}\mathbf{G}(\bar{\boldsymbol{\xi}}-\bar{\mathbf{x}}) + O(1)$$

$$\boldsymbol{\nabla}_1\mathcal{G}(\boldsymbol{\xi},\mathbf{x};\omega) = a^{-2}\boldsymbol{\nabla}\mathbf{G}(\bar{\boldsymbol{\xi}}-\bar{\mathbf{x}}) + O(1)$$

- Leading contribution in (LS) using ansatz $\boxed{\mathbf{v}_a(\boldsymbol{\xi}) = a\mathbf{V}(\bar{\boldsymbol{\xi}}) + o(a) \quad \text{in } B_a}$:

$$\mathcal{L}_a[\mathbf{v}_a](\mathbf{x}) = a\mathcal{L}_\mathcal{B}[\mathbf{V}](\bar{\mathbf{x}}) + o(a)$$

$$\langle \mathbf{u}, \mathcal{G}(\cdot,\mathbf{x})\rangle^{\Delta\mathcal{C}}_{B_a} - \omega^2(\mathbf{u}, \mathcal{G}(\cdot,\mathbf{x}))^{\Delta\rho}_{B_a} = a\langle \psi[\boldsymbol{\nabla}\mathbf{u}(\mathbf{z})], \mathbf{G}(\cdot,\bar{\mathbf{x}})\rangle^{\Delta\mathcal{C}}_{\mathcal{B}} + o(a)$$

Hence, retaining only the $O(a)$ terms, $\mathbf{V}$ solves again the **elastostatic FSTP for the normalized inclusion**:

$$\boxed{\mathcal{L}_\mathcal{B}[\mathbf{V}](\bar{\mathbf{x}}) = -\langle \psi[\boldsymbol{\nabla}\mathbf{u}(\mathbf{z})], \mathbf{G}(\cdot,\bar{\mathbf{x}})\rangle^{\Delta\mathcal{C}}_{\mathcal{B}} \quad (\bar{\mathbf{x}}\in\mathcal{B})}$$

Small-inclusion asymptotic behavior of u^a

Inside B_a, one has (inner expansion)

$$\mathbf{v}_a(\mathbf{x}) = a\mathbf{v}_{\mathcal{B}}[\nabla\mathbf{u}(\mathbf{z})](\bar{\mathbf{x}}) + o(a) \qquad \mathbf{x}\in B_a,\ \bar{\mathbf{x}}\in\mathcal{B}$$

where $\mathbf{v}_{\mathcal{B}}[\mathbf{E}]$ is again the solution of the **elastostatic** FSTP.

Moreover, at any fixed location $\mathbf{x}\neq\mathbf{z}$ (outer expansion), one has

$$\mathbf{v}_a(\mathbf{x}) = -a^3\big[\nabla_1\mathcal{G}(\cdot,\mathbf{x};\omega):\boldsymbol{\mathcal{A}}:\nabla\mathbf{u} - \omega^2\Delta\rho|\mathcal{B}|\mathcal{G}(\cdot,\mathbf{x};\omega)\cdot\mathbf{u}\big](\mathbf{z}) + o(a^3) \quad (\mathbf{x}\neq\mathbf{z})$$

where $\boldsymbol{\mathcal{A}}$ is again the elastic moment tensor associated with the normalized **elastostatic** FSTP.

Elastic moment tensor (anisotropic elastic inclusion)

1. The EMT $\boldsymbol{\mathcal{A}}$, being associated with the normalized **elastostatic** FSTP, is endowed with already-mentioned properties:
   - major and minor symmetries,
   - link to the Eshelby tensor,
   - evaluation through six auxiliary solutions to normalized elastostatic FSTPs,
   - known closed-form solutions for special cases
2. $\boldsymbol{\mathcal{C}}^{\star} = \mathbf{0}$ and $\rho^{\star} = 0$, i.e. $\Delta\boldsymbol{\mathcal{C}} = -\boldsymbol{\mathcal{C}}$ and $\Delta\rho = -\rho$, correspond to a **traction-free cavity**.
   Thus, small-cavity asymptotics is again a special case of small-inclusion asymptotics.

Guzina, B (2004); Guzina, Chikichev (2006)

Topological derivative of cost functionals

(i) Expand cost functional w.r.t. $\mathbf{v}_a$:

$$J(\mathcal{C}_a) = J(\mathcal{C}) + \mathbb{J}'(\mathbf{u};\mathbf{v}_a) + o(\|\mathbf{v}_a\|_{2,\Omega} + \|\mathbf{v}_a\|_{2,\partial\Omega})$$

(ii) Interpret $\mathbb{J}'(\mathbf{u};\mathbf{v}_a)$ as component of a weak formulation with test function $\mathbf{v}_a$ and define the adjoint solution $\hat{\mathbf{u}}$ by

$$\boxed{\text{Find } \hat{\mathbf{u}} \in W_0, \qquad \langle \hat{\mathbf{u}}, \mathbf{w}\rangle^{\mathcal{C}}_{\Omega} - \omega^2(\hat{\mathbf{u}}, \mathbf{w})^{\rho}_{\Omega} = \mathbb{J}'(\mathbf{u};\mathbf{w}), \quad \forall \mathbf{w} \in W_0}$$

(iii) Reciprocity between transmission and adjoint problems:

$$\langle \mathbf{v}_a, \hat{\mathbf{u}}\rangle^{\mathcal{C}_a}_{\Omega} - \omega^2(\mathbf{v}_a, \hat{\mathbf{u}})^{\rho_a}_{\Omega} = -\langle \mathbf{u}, \hat{\mathbf{u}}\rangle^{\Delta\mathcal{C}}_{B_a} + \omega^2(\mathbf{u}, \hat{\mathbf{u}})^{\Delta\rho}_{B_a}$$
$$\langle \hat{\mathbf{u}}, \mathbf{v}_a\rangle^{\mathcal{C}}_{\Omega} - \omega^2(\hat{\mathbf{u}}, \mathbf{v}_a)^{\rho}_{\Omega} = \mathbb{J}'(\mathbf{u};\mathbf{v}_a)$$

to obtain
$$\boxed{\mathbb{J}'(\mathbf{u};\mathbf{v}_a) = -\langle \hat{\mathbf{u}}, \mathbf{u}_a\rangle^{\Delta\mathcal{C}}_{B_a} + \omega^2(\hat{\mathbf{u}}, \mathbf{u}_a)^{\Delta\rho}_{B_a}}$$

(iv) Exploit known limiting behavior of v^a in B_a (inner expansion), yielding

$$\mathbb{J}'(\mathbf{u};\mathbf{v}_a) = a^3\big[\boldsymbol{\nabla}\hat{\mathbf{u}}(\mathbf{z}):\boldsymbol{\mathcal{A}}:\boldsymbol{\nabla}\mathbf{u}(\mathbf{z}) + \omega^2\Delta\rho|\mathcal{B}|\hat{\mathbf{u}}(\mathbf{z})\cdot\mathbf{u}(\mathbf{z})\big] + o(a^3)$$

Topological derivative of J:

$$\boxed{\mathcal{T}(\mathbf{z}) = -\big[\boldsymbol{\nabla}\hat{\mathbf{u}}(\mathbf{z}):\boldsymbol{\mathcal{A}}:\boldsymbol{\nabla}\mathbf{u}(\mathbf{z}) - \omega^2\Delta\rho|\mathcal{B}|\hat{\mathbf{u}}(\mathbf{z})\cdot\mathbf{u}(\mathbf{z})\big]}$$

Usefulness of topological sensitivity for flaw detection

$$J(\mathcal{C}_a, \rho_a) = J(\mathcal{C}, \rho) + a^3 \mathcal{T}(\mathbf{z}) + o(a^3)$$
$$\mathcal{T}(\mathbf{z}) = -\boldsymbol{\nabla}\hat{\mathbf{u}}(\mathbf{z}) : \boldsymbol{\mathcal{A}} : \boldsymbol{\nabla}\mathbf{u}(\mathbf{z}) + \omega^2 \Delta\rho |\mathcal{B}| \hat{\mathbf{u}}(\mathbf{z}) \cdot \mathbf{u}(\mathbf{z})$$

Computation of topological sensitivity field:

- Free and adjoint solutions $\mathbf{u}, \hat{\mathbf{u}}$ defined on **same** (reference) configuration
  Evaluation of $\hat{\mathbf{u}}$ and $\mathcal{T}$ at moderate extra cost
- Computation of $\mathcal{T}(\mathbf{z})$ straightforward with **standard** methods (FEM, BEM...)

Usefulness for inverse problems

- **Mathematically:** $\mathcal{T}$ meaningful only in the small-obstacle limit ($a \to 0$).
- **Heuristically**: seek regions of Ω where $\mathcal{T}$ is most negative
- Qualitative estimation (location, size, number) of buried objects;
- Initialization of full-fledged (and more CPU-intensive) inversion algorithms.

Computation of field $\mathcal{T}(\mathbf{x_s})$

$$\mathcal{T}(\mathbf{z};\Delta\mathcal{C},\Delta\rho) = \text{Re}\Big\{-\boldsymbol{\nabla}\hat{\mathbf{u}}(\mathbf{z}):\boldsymbol{\mathcal{A}}:\boldsymbol{\nabla}\mathbf{u}(\mathbf{z}) + \omega^2\Delta\rho|\mathcal{B}|\hat{\mathbf{u}}(\mathbf{z})\cdot\mathbf{u}(\mathbf{z})\Big\}$$

Configurations for which the Green's tensor is known (e.g. full-space, half-space):

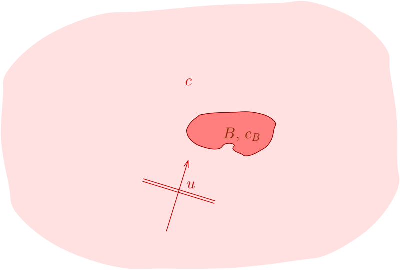
- Explicit formulae for $\mathcal{T}$ in terms of free and adjoint solutions:

$$\mathbf{u}(\mathbf{x}) = \int_{S_\text{N}} \mathcal{G}(\cdot,\mathbf{x}).\mathbf{g}\,\text{d}S, \qquad \hat{\mathbf{u}}(\mathbf{x}) = \int_{S_\text{N}} \mathcal{G}(\cdot,\mathbf{x}).\boldsymbol{\nabla}_2\varphi_\text{s}(\boldsymbol{\xi},\mathbf{u})\,\text{d}S$$

Configurations for which the Green's tensor is not known

- Numerical computation of **u** and **û** using standard techniques;
- Numerical evaluation of field $\mathcal{T}$
  - **(i)** Integral equations → FMM:
    (i) Fast solution on the boundary
    (ii) Fast evaluation of integral representations at large numbers of grid points
  - **(ii)** FEM
  - **(iii)** Other

Analytic solution for TS under Born approximation (acoustics)



Born approximation for weak scatterer ($|1-\eta| \ll 1$, $\eta = c_B^2/c^2$):

$$v_B(\mathbf{x}) = -k^2 \int_B (1-\eta)\mathcal{G}(\boldsymbol{\xi}-\mathbf{x})u(\boldsymbol{\xi})\,\mathrm{d}V_\xi + o(1-\eta)$$

Interrogating incident plane wave (with $\hat{\mathbf{d}} := \mathbf{d}/\|\mathbf{d}\|$):

$$u(\boldsymbol{\xi}) = e^{\mathrm{i}k\hat{\mathbf{d}}\cdot\boldsymbol{\xi}}$$

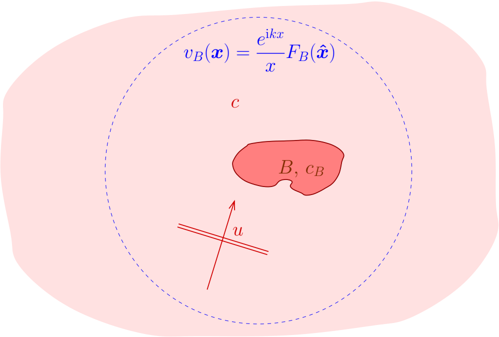
Far-field approximation of v_B:

$$v_B(\mathbf{x}) = \frac{e^{\mathrm{i}kx}}{x}F_B(\hat{\mathbf{x}}), \qquad F_B(\hat{\mathbf{x}}) = -\frac{k^2}{4\pi}\int_B (1-\eta)e^{\mathrm{i}k\boldsymbol{\xi}\cdot(\hat{\mathbf{d}}-\hat{\mathbf{x}})}\,\mathrm{d}V_\xi$$

Analytic solution for TS under Born approximation (acoustics)

Least-squares cost function

$$J(B) = \frac{1}{2} \int_{\hat{S}} |F_B(\hat{\mathbf{x}}) - F_{\text{true}}(\hat{\mathbf{x}})|^2 \, \mathrm{d}S(\hat{\mathbf{x}})$$



Analytic solution for TS under Born approximation (acoustics)

Least-squares cost function

$$\mathcal{J}(B) = \frac{1}{2}\int_{\hat{S}} |F_B(\hat{\mathbf{x}}) - F_{\text{true}}(\hat{\mathbf{x}})|^2 \, \mathrm{d}S(\hat{\mathbf{x}})$$

Small-inclusion behavior of scattered acoustic field (for *trial* contrast $\eta^\star$):

$$F_a(\hat{\mathbf{x}}) = a^3|\mathcal{B}|\,\mathcal{F}(\hat{\mathbf{x}};\mathbf{z},\mathcal{B},\eta^\star) + o(a^3),$$

$$\mathcal{F}(\hat{\mathbf{x}};\mathbf{z},\mathcal{B},\eta^\star) = -k^2(1-\eta^\star)e^{\mathrm{i}k\mathbf{z}\cdot(\hat{\mathbf{d}}-\hat{\mathbf{x}})}$$

Expansion of cost function:

$$\boxed{J(a) := \mathcal{J}(B_a) = J(0) + a^3\,\mathcal{T}(\mathbf{z};\mathcal{B},\eta^\star) + o(a^3)}$$

with (after analytical evaluation of integral over $\hat{S}$)

$$\boxed{\mathcal{T}(\mathbf{z};\mathcal{B},\eta^\star) = -\frac{k^4}{16\pi}|\mathcal{B}|\text{Re}\Big\{\int_{B_{\text{true}}}(1-\eta)(1-\eta^\star)j_0(k\|\boldsymbol{\xi}-\mathbf{z}\|)e^{\mathrm{i}k(\boldsymbol{\xi}-\mathbf{z})\cdot\hat{\mathbf{d}}}\,\mathrm{d}V_\xi\Big\}}$$

Analytical solution for TS under Born approximation (acoustics)

Cumulative topological sensitivity over all incidence directions:

$$J_{\text{cumul}}(B) = \int_{\hat{S}} J(B; \hat{\mathbf{d}})\, \mathrm{d}S(\hat{\mathbf{d}}),$$

$$\mathcal{T}_{\text{cumul}}(\mathbf{z}; \eta^\star) = -|\mathcal{B}| \frac{k^4}{16} \int_{B_{\text{true}}} (1-\eta)(1-\eta^\star) j_0^2(k\|\boldsymbol{\xi} - \mathbf{z}\|)\, \mathrm{d}V_\xi$$

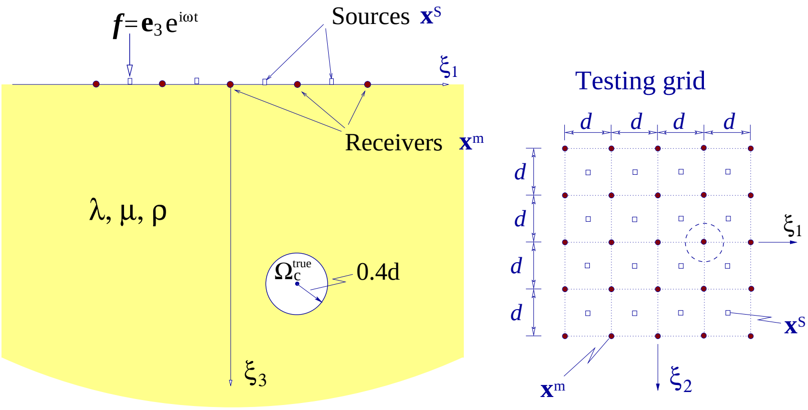
Far-field behavior and sign of $\mathbf{z} \mapsto \mathcal{T}(\mathbf{z}; \eta^\star)$:

$$\left|\mathcal{T}_{\text{cumul}}(\mathbf{z}; \eta^\star)\right| \approx \left[\text{dist}(\mathbf{z}, B_{\text{true}})\right]^2$$
$$\text{sign}\mathcal{T}_{\text{cumul}}(\mathbf{z}; \eta^\star) = -\text{sign}(1-\eta)(1-\eta^\star)$$

- $\mathcal{T}_{\text{cumul}}$ has the "expected" (i.e. consistent with initial heuristic) sign if the trial and actual flaw are of same qualitative nature (here: both are either "slower" or "faster" than the background);
- $|\mathcal{T}_{\text{cumul}}|$ decays with distance $\text{dist}(\mathbf{z}, B_{\text{true}})$;
- Similar results found for weak mass density contrast $|\beta - 1| \ll 1$;

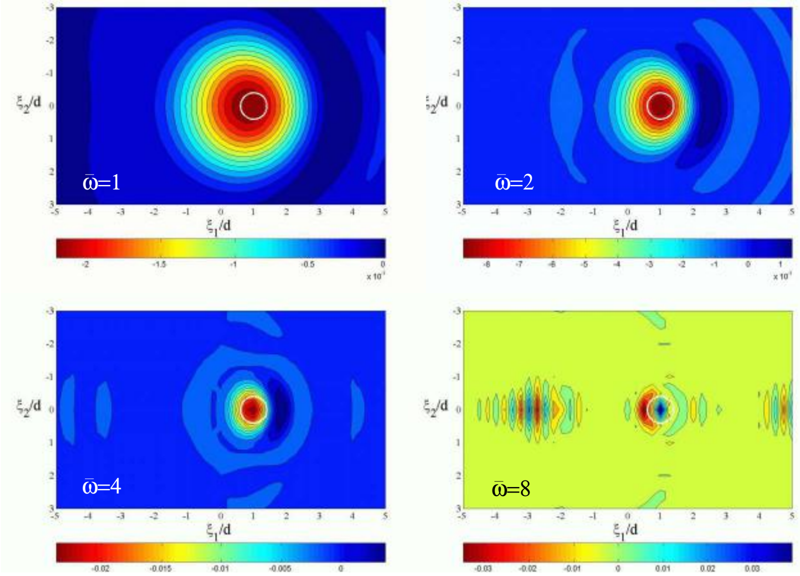
Derivation idea suggested by H. Haddar; see also Ammari, Garnier, Jugnon, Kang 2011

Example: qualitative imaging of spherical void (3-D elastodynamics)



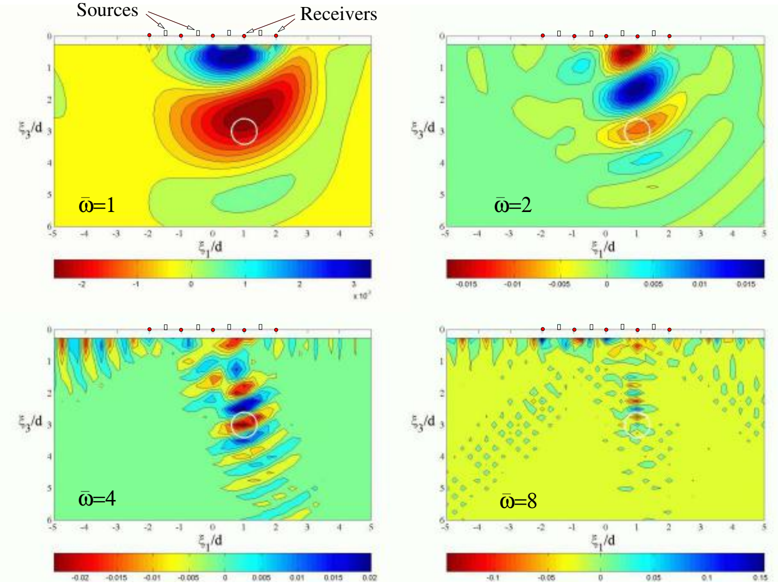
Guzina, B, *Quart. J. Mech. Appl. Math. (2004)*

Example: qualitative imaging of spherical void (3-D elastodynamics)



$$\bar{\omega} = \omega d \sqrt{\rho/\mu}$$

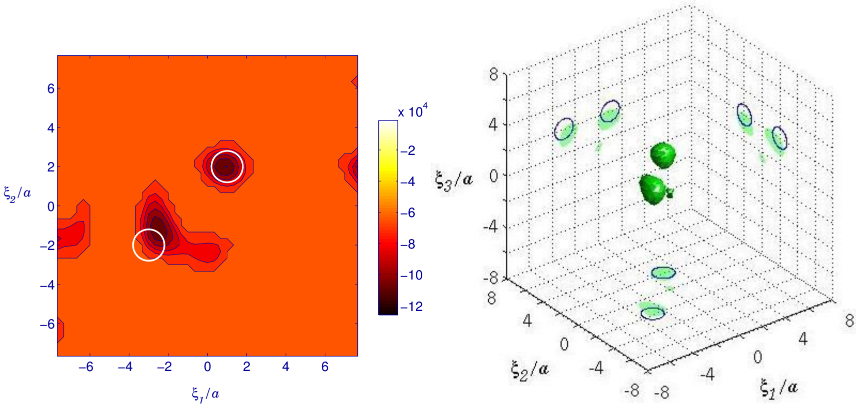
Example: qualitative imaging of spherical void (3-D elastodynamics)



$\bar{\omega} = \omega d\sqrt{\rho/\mu}$

Computation of $\mathcal{T}_3$ using FM-BEM: example

Identification of a dual hard scatterer



Nemitz, B, *Eng. Anal. Bound. Elem* (2008)

Computation of $\mathcal{T}_3$ using FM-BEM: example

FMM: numerical parameters

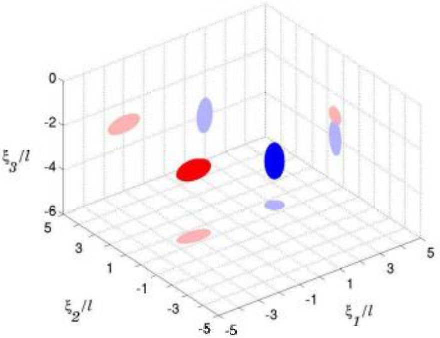
Element and DOF count (FM-BEM):

Cube size	Cube		Obstacle		Total		$\mathcal{G}$
	Elements	DOFs	Elements	DOFs	Elements	DOFs	
$16a$	76800	38402	336	170	77136	38572	100^3
$32a$	307200	153602	336	170	307536	153772	150^3

CPU timing (single-CPU PC) and GMRES iteration count

Cube size	u_{true} on $S \cup \Gamma_{\text{true}}$	u on S	$\hat{u}$ on S	$\mathcal{T}$ on $\mathcal{G}$
$16a$	$1444s$ ($N_{\text{iter}} = 435$)	969s ($N_{\text{iter}} = 282$)	1163s ($N_{\text{iter}} = 342$)	852s
$32a$	$6461s$ ($N_{\text{iter}} = 439$)	5615s ($N_{\text{iter}} = 388$)	6818s ($N_{\text{iter}} = 476$)	1860s

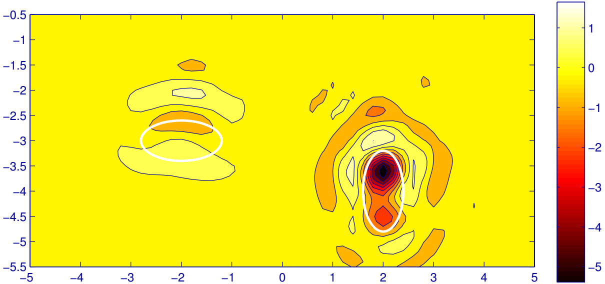
Example (acoustics): two inclusions (one hard, one penetrable) in half-space



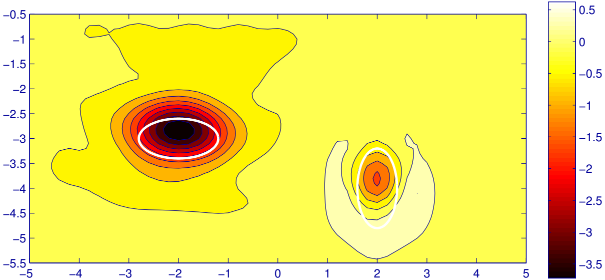
Constants for soft inclusion: $(\beta, \gamma) = (2, 0.5)$

Guzina, B, *Inverse Problems* (2006)

Example (acoustics): two inclusions (one hard, one penetrable) in half-space

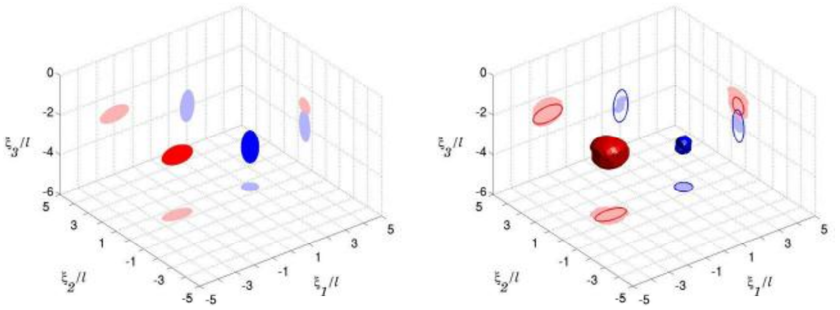


($\mathcal{T}$ for rigid inclusion)



($\mathcal{T}$ for soft inclusion)

Example (acoustics): two inclusions (one hard, one penetrable) in half-space



$$\mathcal{T}_{\text{opt}}(\mathbf{z}) = 0.35 \min_{\mathbf{z}} \mathcal{T}_{\text{opt}}(\mathbf{z})$$

where $\mathcal{T}_{\text{opt}}(\mathbf{z}) = \min_{\beta} \mathcal{T}(\mathbf{z}; \beta, \eta)$ with $\gamma = 0.5$

Acoustic transmission problem (TP)

Scattering of an acoustic wave by a penetrable inclusion B $(\rho^\star, c^\star)$ embedded in an acoustic medium Ω (ρ, c).

$$\begin{cases} \beta := \rho/\rho^\star & \text{mass density ratio} \\ \gamma := c/c^\star & \text{velocity ratio} \\ \eta := (\rho c^2)/(\rho c^2)^\star & \text{bulk modulus ratio} \end{cases}$$

Let $m := 1 + (\beta - 1)\chi(B)$ and $n := 1 + (\eta - 1)\chi(B)$.

- Field equation on acoustic pressure $u(\mathbf{x}, t)$:

$$\boxed{\operatorname{div}(m\boldsymbol{\nabla} u_B) - \frac{n}{c^2}\ddot{u}_B = 0 \quad (\text{in } \Omega)}$$

- Initial conditions: $\boxed{u_B(\cdot, 0) = \dot{u}_B(\cdot, 0) = 0 \quad (\text{in } \Omega)}$

- Transmission conditions implied (continuity of pressure and normal velocity across ∂B)

$$\boxed{u_B\,|_+ = u_B\,|_- \quad \text{and} \quad \rho^{-1}\partial_n u_B\,|_+ = (\rho^\star)^{-1}\partial_n u_B\,|_- \qquad \text{on } \partial B}$$

Acoustic TP: domain integral equation (DIE) formulation

- Time-impulsive Green's function: defined, for $(\mathbf{x}, t) \in \Omega \times \mathbb{R}^+$, by

$$\begin{aligned}(\Delta - c^{-2}\partial_{tt})\mathcal{G}(\cdot, \mathbf{x}, t) + \delta(\cdot - \mathbf{x})\delta(t) = 0 \quad (\text{in } \Omega \times \mathbb{R}^+),\\ \mathcal{G}(\cdot, \mathbf{x}; t) = 0 \quad (\text{on } S_{\mathrm{D}}), \qquad \partial_n \mathcal{G}(\cdot, \mathbf{x}; t) = 0 \quad (\text{on } S_{\mathrm{N}} \times \mathbb{R}^+)\\ \mathcal{G}(\cdot, \mathbf{x}; 0) = \dot{G}(\cdot, \mathbf{x}; 0) = 0 \quad (\text{in } \Omega).\end{aligned}$$

- Convolution of G with the wave equation:

$$\int_0^t \int_\Omega \Big[\mathrm{div}\,(m \boldsymbol{\nabla} u_B) - \frac{n}{c^2} \ddot{u}_B \Big] (\boldsymbol{\xi}, t') \mathcal{G}(\boldsymbol{\xi}, \mathbf{x}, t - t') \, \mathrm{d}V_\xi \, \mathrm{d}t' = 0$$

- Reformulating the above by twice integrating by parts in space and invoking the definition of G, one obtains the DIE formulation on $v_B := u_B - u$:

$$\boxed{\mathcal{L}_B[v_B](\mathbf{x}, t) = -\langle\langle u, \mathcal{G}(\cdot, \mathbf{x}, \cdot) \rangle\rangle_B^{\Delta m}(t) - c^{-2}((\ddot{u}, \mathcal{G}(\cdot, \mathbf{x}, \cdot)))_B^{\Delta n}(t)}$$

with $\mathcal{L}_B[w](\mathbf{x}, t) := w(\mathbf{x}, t) + \langle\langle w, \mathcal{G}(\cdot, \mathbf{x}, \cdot) \rangle\rangle_B^{\Delta m}(t) + c^{-2}((\ddot{w}, \mathcal{G}(\cdot, \mathbf{x}, \cdot)))_B^{\Delta n}(t)$

$$\langle\langle u, w \rangle\rangle_D^m(t) := \int_0^t \langle u(\cdot, t'), w(\cdot, t - t') \rangle_D^m \, \mathrm{d}t'$$

$$((u, w))_D^n(t) := \int_0^t (u(\cdot, t'), w(\cdot, t - t'))_D^n \, \mathrm{d}t'$$

Small-inclusion asymptotic behavior of u^a

- Coordinate scaling (again):

$$\boldsymbol{\xi} = \mathbf{z} + a\bar{\boldsymbol{\xi}}, \quad \mathbf{x} = \mathbf{z} + a\bar{\mathbf{x}}, \quad \mathrm{d}V_{\xi} = a^3 \,\mathrm{d}\bar{V}_{\bar{\xi}}, \quad \boldsymbol{\nabla}_{\xi} = a^{-1}\boldsymbol{\nabla}_{\bar{\xi}}$$

- Leading contribution in DIE using ansatz $\boxed{v^a(\boldsymbol{\xi}, t) = aV(\bar{\boldsymbol{\xi}}, t) + o(a) \quad \text{in } B_a}$:

$$\mathcal{L}_B[v^a](\mathbf{x}, t) = a\mathcal{L}_{\mathcal{B}}[V](\bar{\mathbf{x}}, t) + o(a)$$

$$\langle\langle u, \mathcal{G}(\cdot, \mathbf{x}, \cdot)\rangle\rangle^{\Delta m}_{B_a}(t) + c^{-2}((\ddot{u}, \mathcal{G}(\cdot, \mathbf{x}, \cdot)))^{\Delta n}_{B_a}(t) = a\langle\psi[\boldsymbol{\nabla} u(\mathbf{z}, t)], G(\cdot - \bar{\mathbf{x}})\rangle^{\Delta m}_{\mathcal{B}} + o(a)$$

Hence, retaining only the $O(a)$ terms:

Similarly to the frequency-domain case, V solves the **quasistatic FSTP for the normalized inclusion**:

$$\boxed{\mathcal{L}_{\mathcal{B}}[V](\bar{\mathbf{x}}, t) = -\langle\psi[\boldsymbol{\nabla} u(\mathbf{z}, t)], G(\cdot, \bar{\mathbf{x}})\rangle^{\Delta m}_{\mathcal{B}} \quad (\bar{\mathbf{x}} \in \mathcal{B},\ t \geq 0)}$$

where $G(\mathbf{r}) = 1/(4\pi\|\mathbf{r}\|)$ is the Laplace full-space fundamental solution.

Small-inclusion asymptotic behavior of u^a

Inside B_a, one has (inner expansion)

$$v^a(\boldsymbol{\xi},t) = a v_{\mathcal{B}}[\boldsymbol{\nabla} u(\mathbf{z},t)](\bar{\boldsymbol{\xi}}) + o(a) \qquad \boldsymbol{\xi}\in B_a,\ \bar{\boldsymbol{\xi}}\in\mathcal{B}$$

where $v_{\mathcal{B}}[\mathbf{E}]$ is again the solution of the **static** FSTP.

Moreover, at any fixed location $\mathbf{x}\neq\mathbf{z}$ (outer expansion), one has

$$v^a(\mathbf{x},t) = -a^3\big[\boldsymbol{\nabla}_1\mathcal{G}(\cdot,\mathbf{x};t)\star[(\mathbf{A}\cdot\boldsymbol{\nabla} u(\cdot,t)] + (\eta-1)c^{-2}\mathcal{G}(\cdot,\mathbf{x};t)\star\ddot{u}(\cdot,t)\big](\mathbf{z}) + o(a^3)$$

where $\mathbf{A}=\mathbf{A}(\mathcal{B},\beta)$ is again the polarization tensor associated with the normalized **static** FSTP.

Small-inclusion asymptotic behavior of u^a (proof sketch)

Considering as an example the term

$$\langle\langle u, \mathcal{G}(\cdot, \mathbf{x}, \cdot)\rangle\rangle_{B_a}^{\Delta m}(t)$$

- Split G into singular and nonsingular parts:

$$\boxed{\mathcal{G}(\boldsymbol{\xi}, \mathbf{x}; t) = G(\boldsymbol{\xi} - \mathbf{x}; t) + \mathcal{G}_{\mathrm{c}}(\boldsymbol{\xi}, \mathbf{x}; t)}$$

  with $G(\mathbf{r}; t)$: (singular) full-space time-impulsive fundamental solution

$$G(\mathbf{r}; t) = (4\pi r)^{-1}\delta(t - r/c) \qquad (r = \|\mathbf{r}\|)$$

- Hence

$$\begin{aligned} \boldsymbol{\nabla} G(\mathbf{r}, t) \star \boldsymbol{\nabla} u(\boldsymbol{\xi}, t) &= -(4\pi r^2)^{-1}\big[\boldsymbol{\nabla} u(\boldsymbol{\xi}, t - r/c) + rc^{-1}\big[\boldsymbol{\nabla} \dot{u}(\boldsymbol{\xi}, t - r/c)\big] \\ &= -a^{-2}\big[(4\pi \bar{r}^2)^{-1}\boldsymbol{\nabla} u(\boldsymbol{\xi}, t) + o(a)\big] \end{aligned}$$

  (assuming sufficient regularity of $t \mapsto u(\boldsymbol{\xi}, t)$, e.g. Lipschitz continuity), while

$$\boldsymbol{\nabla}\mathcal{G}_{\mathrm{c}}(\mathbf{r}, t) \star \boldsymbol{\nabla} u(\boldsymbol{\xi}, t) = O(a^0)$$

- One thus has

$$\langle\langle u, \mathcal{G}(\cdot, \mathbf{x}, \cdot)\rangle\rangle_{B_a}^{\Delta m}(t) = a\langle u(\cdot, t), G(\cdot - \mathbf{x})\rangle_{B_a}^{\Delta m} + o(a)$$

One proceeds similarly for the other terms

Cost functional

$$J(m_B, n_B) = \mathbb{J}(u_B) \quad \text{with} \quad \mathbb{J}(w) := \int_0^T \Big\{ \int_\Omega \varphi_{\mathrm{V}}(\cdot,\cdot,w)\,\mathrm{d}V + \int_{\partial\Omega} \varphi_{\mathrm{S}}(\cdot,\cdot,w)\,\mathrm{d}S \Big\}\,\mathrm{d}t$$

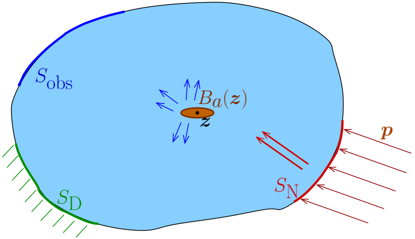
Example (weighted output least-squares for defect identification, with $Q(\mathbf{x},t) > 0$):

$$\varphi_{\mathrm{V}}(\mathbf{x},t,w) = 0, \quad \varphi_{\mathrm{S}}(\mathbf{x},t,w) = \frac{1}{2}(\bar{w} - u^{\mathcal{D}\mathrm{obs}}(\mathbf{x},t))Q(\mathbf{x},t)(w - u^{\mathrm{obs}}(\mathbf{x},t))$$

Topological derivative of J: sought by considering the limiting behavior of

$$j(a) := J(m_a, n_a) = \mathbb{J}(u^a)$$

with $m_a := 1 + (\beta - 1)\chi(B_a)$ and $n_a := 1 + (\eta - 1)\chi(B_a)$



Topological derivative of cost functionals

(i) Expand cost functional w.r.t. v^a:

$$\boxed{J(m_a, n_a) = J(1,1) + \mathbb{J}'(u; v^a) + o(\|v^a\|_{2,\Omega} + \|v^a\|_{2,\partial\Omega})}$$

$$\mathbb{J}'(u; v^a) = \int_0^T \mathbb{J}'(t, u; v^a)\,\mathrm{d}t$$

$$\mathbb{J}'(t, u; v^a) := \int_\Omega \partial_3\varphi_{\mathrm{v}}(\cdot, t, u) v^a(\cdot, t)\,\mathrm{d}V + \int_{\partial\Omega} \partial_3\varphi_{\mathrm{s}}(\cdot, t, u) v^a(\cdot, t)\,\mathrm{d}S$$

(ii) Interpret $\mathbb{J}'(u; v^a)$ as component of a weak formulation with test function v^a and define the adjoint solution $\hat{u}$, for $0 \leq t' \leq T$, by

$$\boxed{\text{Find } \hat{u}, \quad \langle \hat{u}(\cdot, t'), w\rangle_\Omega + c^{-2}(\ddot{\hat{u}}(\cdot, t'), w)_\Omega = \mathbb{J}'(T - t', u; w), \quad \forall w \in W_0}$$

(iii) Weak formulation of acoustic TP, for $0 \leq t' \leq T$:

$$\text{Find } v^a, \quad \langle v^a(\cdot, t'), w\rangle_\Omega^m + c^{-2}(\ddot{v}^a(\cdot, t'), w)_\Omega^n$$
$$= -\langle u(\cdot, t'), w\rangle_{B_a}^{\Delta m} - c^{-2}(\ddot{u}(\cdot, t'), w)_{B_a}^{\Delta n}, \quad \forall w \in W_0$$

Topological derivative of cost functionals

(iv) Set $w = v^a(\cdot, T-t')$ in (ii) and $w = \hat{u}(\cdot, T-t')$ in (iii), integrate both over $0 \le t' \le T$, and subtract the resulting weak formulations in convolutional form.

Noting that $((\ddot{v}^a, \hat{u}))_\Omega(T) = ((\ddot{\hat{u}}, v^a))_\Omega(T)$, one obtains

$$\boxed{\mathbb{J}'(u; v^a) = -\langle\langle \hat{u}, u^a \rangle\rangle^{\Delta m}_{B_a}(T) - c^{-2}((\hat{u}, u^a))^{\Delta n}_{B_a}(T)}$$

Topological derivative of J:

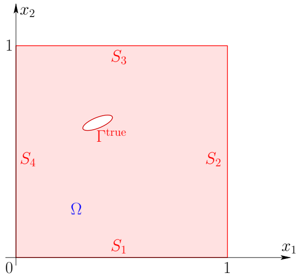
$$\boxed{\mathcal{T}(\mathbf{z}) = -\big[\boldsymbol{\nabla}\hat{u}(\mathbf{z};t) \star [(\mathbf{A}\cdot\boldsymbol{\nabla}u(\mathbf{z},t)] + (\eta-1)c^{-2}|\mathcal{B}|\hat{u}(\mathbf{z},t) \star \ddot{u}(\mathbf{z},t)\big]}$$

- $\mathbf{A} = \mathbf{A}(\mathcal{B}, \beta)$ is again the polarization tensor associated with the normalized **static** FSTP.
- Similar expression available for elastodynamics, using again the elastostatic EMT $\boldsymbol{\mathcal{A}}$:

$$\boxed{\mathcal{T}(\mathbf{z}) = -\big[\boldsymbol{\nabla}\hat{\mathbf{u}}(\mathbf{z};t) \star [(\boldsymbol{\mathcal{A}} : \boldsymbol{\nabla}u(\mathbf{z},t)] + |\Delta\rho|\mathcal{B}|\hat{\mathbf{u}}(\mathbf{z},t) \star \ddot{\mathbf{u}}(\mathbf{z},t)\big]}$$

FEM-based computation of $\mathcal{T}$: example (2D time-domain wave eqn.)

Identification of impenetrable scatterer(s) in 2-D acoustic medium.



- Normalized scalar wave equation

$$\Delta u_\Gamma^{(k)} - \ddot{u}_\Gamma^{(k)} = 0$$

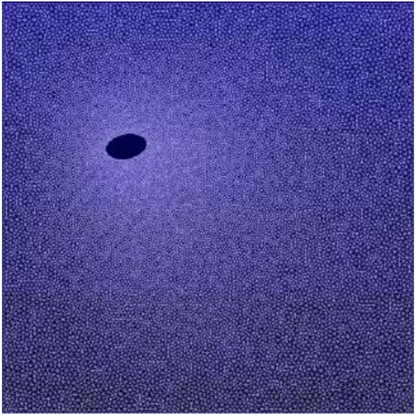
$$\frac{\partial u_\Gamma}{\partial n} = \begin{cases} 1 & (\text{on } S_k) \\ 0 & (\text{on } S_j,\ j \neq k) \end{cases} \qquad \frac{\partial u_\Gamma}{\partial n} = 0 \quad (\text{on } \Gamma)$$

- Matlab (very simple) implementation, T3 elements.
- Newmark inconditionally stable time-marching scheme $(\beta = 1/4, \gamma = 1/2)$

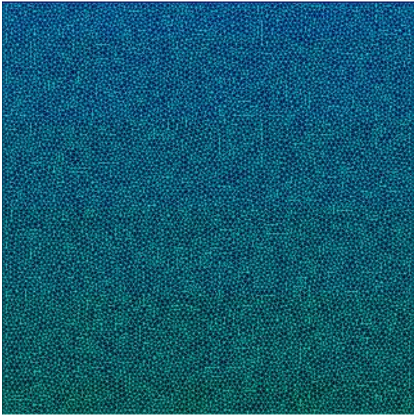
Cost function (simulated measurements over duration T):

$$J^{(k)}(\Omega) = \frac{1}{2} \int_0^T \int_{S_1+S_2+S_3+S_4} |u_\Gamma^{(k)} - u_{\text{true}}^{(k)}|^2 \, \mathrm{d}s \, \mathrm{d}t$$

FEM-based computation of $\mathcal{T}$: identification of a single scatterer

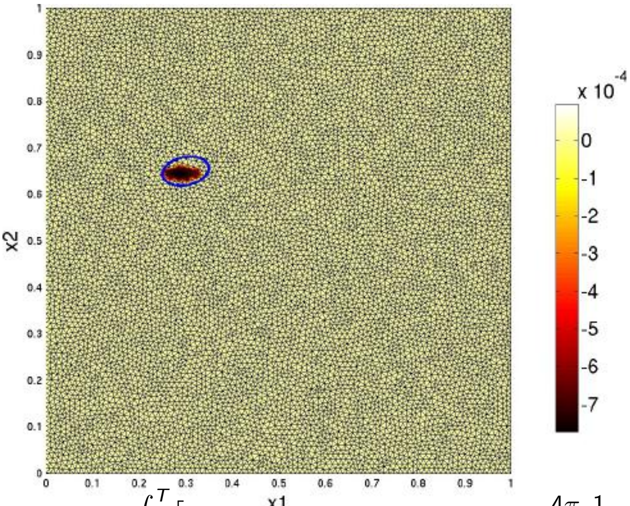


Computation of synthetic data $u^{(k)}$



Computation of $u^{(k)}$ et $\hat{u}^{(k)}$ (9841 DOFs)

FEM-based computation of $\mathcal{T}$: identification of a single scatterer



Thresholded TS field:

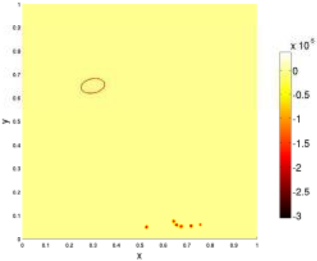
$$\hat{\mathcal{T}}_3 = \mathcal{T}_3 \quad (\mathcal{T}_3 \leq \alpha \mathcal{T}_3^{\text{Min}})$$
$$= 0 \quad (\mathcal{T}_3 > \alpha \mathcal{T}_3^{\text{Min}})$$

$$k = 1, \ T = 2$$
$$\alpha = .75$$

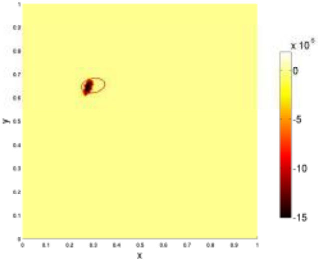
$$\mathcal{T}(\mathbf{z}) = \int_0^T \Big[2\pi \boldsymbol{\nabla} u^{(k)}(\mathbf{z}, t).\boldsymbol{\nabla} \hat{u}^{(k)}(\mathbf{z}, T-t) + \frac{4\pi}{3}\frac{1}{c^2} u^{(k)}(\mathbf{z}, t) \hat{u}^{(k)}(\mathbf{z}, T-t) \Big] \, \mathrm{d}t$$

FEM-based computation of $\mathcal{T}$: identification of a single scatterer

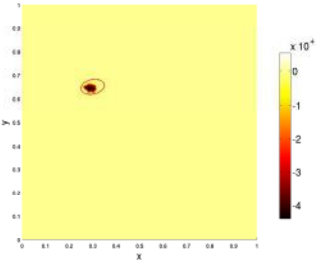
$k = 1, \; T = 1$



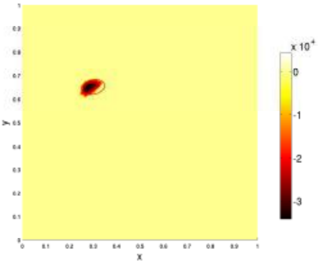
$k = 4, \; T = 1$



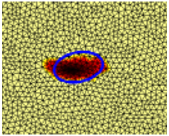
$k = 1, \; T = 2$



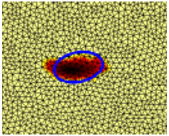
$k = 1, 2, 3, 4, \; T = 1$



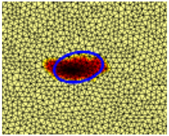
FEM-based computation of $\mathcal{T}$: identification of a single scatterer



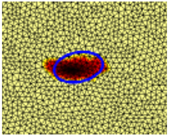
(a) $\alpha = 0.1$



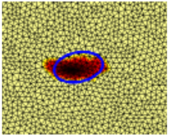
(b) $\alpha = 0.2$



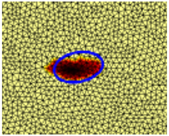
(c) $\alpha = 0.3$



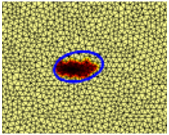
(d) $\alpha = 0.4$



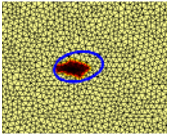
(e) $\alpha = 0.5$



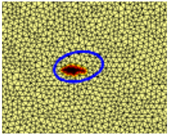
(f) $\alpha = 0.6$



(g) $\alpha = 0.7$



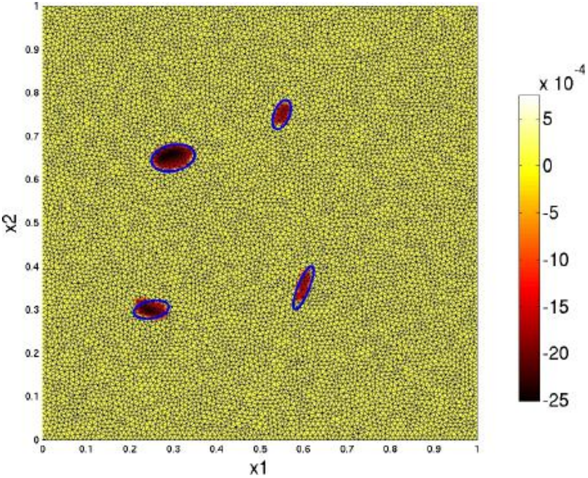
(h) $\alpha = 0.8$



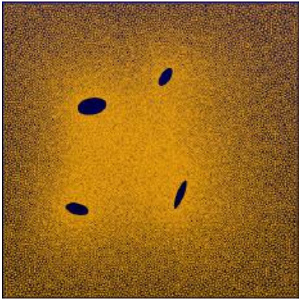
(i) $\alpha = 0.9$

Influence of cut-off parameter α.

FEM-based computation of $\mathcal{T}$: simultaneous identification of a multiple scatterer



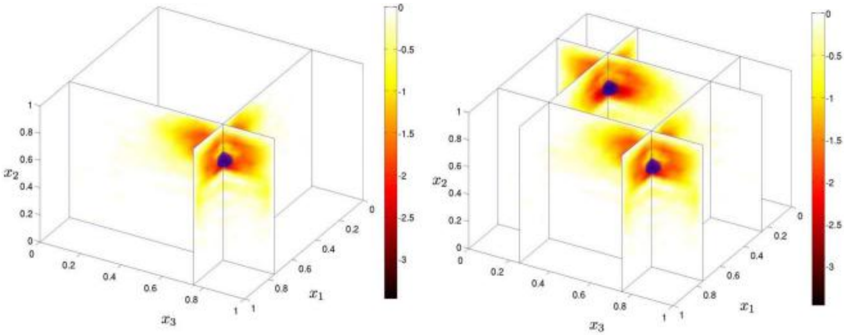
$k = 1, 2, 3, 4, \;\; T = 2, \;\; \alpha = 0.5$



Computation of synthetic data $u^{(k)}$

Bellis, B, *Int. J. Solids Struct.* (2010)

FEM-based evaluation of $\mathcal{T}$: example (3D time-domain elasticity)

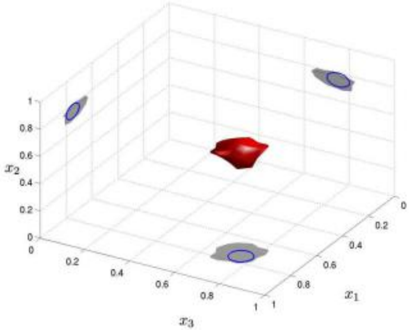


(j) $R = 0.05$, $\mathbf{x}^{\text{true}} = (0.75,\ 0.75,\ 0.75)$

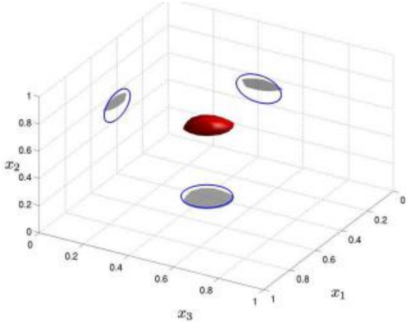
(k) $R_1 = R_2 = 0.05$,
$\mathbf{x}_1^{\text{true}} = (0.25,\ 0.25,\ 0.75)$,
$\mathbf{x}_2^{\text{true}} = (0.75,\ 0.75,\ 0.75)$

C. Bellis, MB, *Int. J. Solids Struct.* (2010)

FEM-based evaluation of $\mathcal{T}$: example (3D time-domain elasticity)

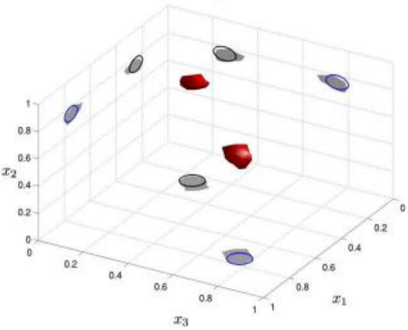


(l) $R = 0.05$, $\alpha = 0.6$

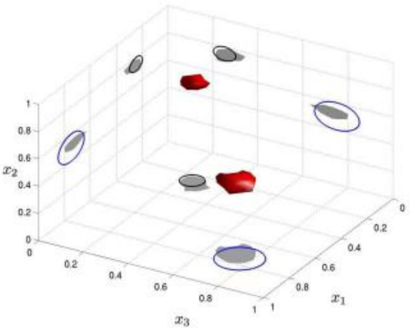


(m) $R = 0.1$, $\alpha = 0.6$

FEM-based evaluation of $\mathcal{T}$: example (3D time-domain elasticity)

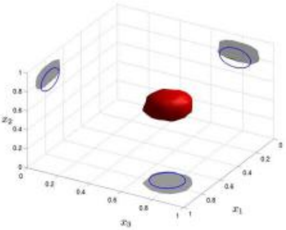


(n) $R_1 = R_2 = 0.05$, $\alpha = 0.6$

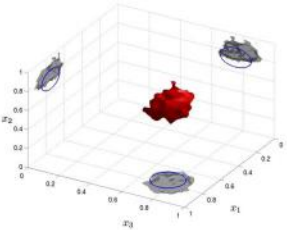


(o) $R_1 = 0.05 \quad R_2 = 0.1$, $\alpha = 0.7$

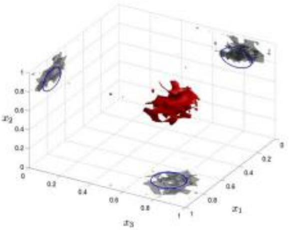
FEM-based evaluation of $\mathcal{T}$: example (3D time-domain elasticity)



(p) $\sigma = 0.1$, $\alpha = 0.75$



(q) $\sigma = 0.5$, $\alpha = 0.75$

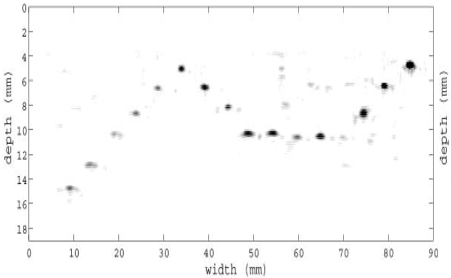


(r) $\sigma = 1$, $\alpha = 0.6$

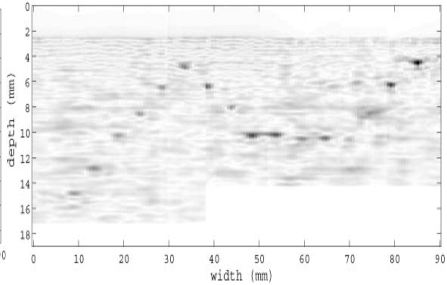
Influence of data noise on $\mathcal{T}$: $B_{\text{eq}}(\alpha)$ for various levels of noise ($\mathbf{x}^{\text{true}} = (0.75, 0.75, 0.75)$).

C. Bellis, MB, *Int. J. Solids Struct.* (2010)

Experimental study 1



TDTE (Dominguez, Gibiat)



conventional B-scan

Dominguez, Gibiat, *Ultrasonics* (2009)

Experimental study 2

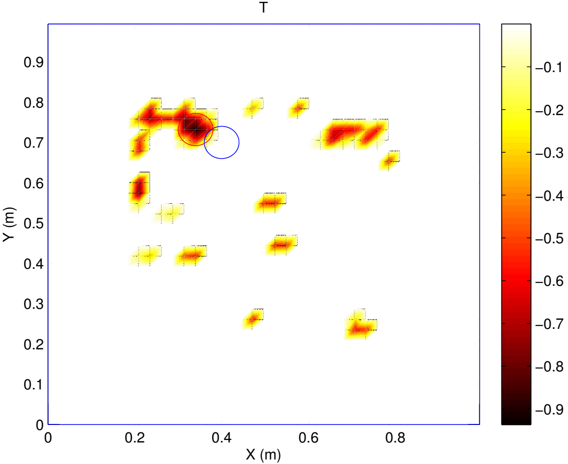
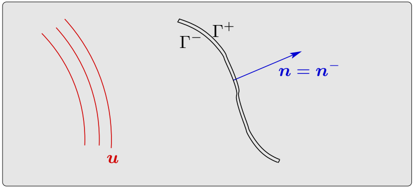


Figure 15: Result of the real experiment. The real defect is represented by a blue circle while the red circle stands for the estimated cavity.

A. Tixier, Ecole Polytechnique research internship project, supervisor B. Guzina (2011)

Scattering of elastic waves by cracks



- Idealized crack: limiting configuration of a traction-free cavity of vanishing thickness;
- Superposition: defining the **scattered displacement** $\mathbf{v}_\Gamma$ by $\mathbf{u}_\Gamma = \mathbf{u} + \mathbf{v}_\Gamma$ (with $\mathbf{u}$: background, or incident, displacement), one has

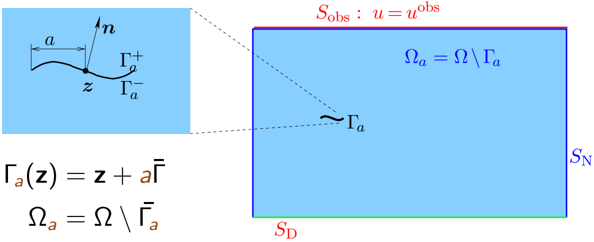
$$\boxed{\mathbf{t}[\mathbf{u}_\Gamma] = \mathbf{0} \quad \Longrightarrow \quad \mathbf{t}[\mathbf{v}_\Gamma] = -\mathbf{t}[\mathbf{u}] \qquad \text{on } \Gamma^\pm}$$

  with $\mathbf{t}[\mathbf{w}] := \boldsymbol{\sigma}[\mathbf{w}]\cdot\mathbf{n} = (\mathcal{C}:\varepsilon[\mathbf{w}])\cdot\mathbf{n}$ (surface traction vector);
- $\mathbf{v}_\Gamma$ is **discontinuous** ($[\![\mathbf{v}_\Gamma]\!] \neq \mathbf{0}$ across Γ;
- $\boldsymbol{\sigma}[\mathbf{u}_\Gamma](\mathbf{x})$, $\varepsilon[\mathbf{u}_\Gamma](\mathbf{x})$ and $\boldsymbol{\nabla}\mathbf{u}_\Gamma(\mathbf{x})$ all have a $O\big([\text{dist}(\mathbf{x},\partial\Gamma)]^{1/2}\big)$ crack-edge singularity

Concept of topological sensitivity for crack nucleation

- Cost function of format $$J(\Omega_\Gamma) = \mathbb{J}(\mathbf{u}_\Gamma) = \int_{S_{\text{obs}}} \varphi(\mathbf{u}_\Gamma)\, \text{d}S$$

- Nucleation of crack of small characteristic length a:



$$\Gamma_a(\mathbf{z}) = \mathbf{z} + a\bar{\Gamma}$$
$$\Omega_a = \Omega \setminus \bar{\Gamma}_a$$

- Seek expansion of cost function according to

$$j(a) := J(\Omega_a) = j(0) + \delta(a)\mathcal{T}(\mathbf{z}; \bar{\Gamma}, \mathbf{n}) + o(\delta(a))$$

$\mathcal{T}(\mathbf{z}; \bar{\Gamma}, \mathbf{n})$: (distributed) **topological sensitivity** associated with $\mathcal{J}$;

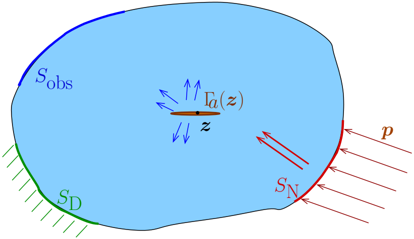
Scattering by small crack

Forward problem for small crack $\Gamma_a(\mathbf{z})$, in terms of scattered field:

$$\begin{gathered} \mathcal{P}^\omega[\mathbf{v}_a] = \mathbf{0} \text{ (in } \Omega_a), \\ \mathbf{t}[\mathbf{u}_a] = \mathbf{0} \text{ (on } S_\mathrm{N}), \quad \mathbf{u}_a = \mathbf{0} \text{ (on } S_\mathrm{D}), \\ \mathbf{t}[\mathbf{v}_a] = -\mathbf{t}[\mathbf{u}] \text{ (on } \Gamma_a) \end{gathered}$$

$$\Omega_a := \Omega \setminus \bar{\Gamma}_a$$

$$\mathcal{P}^\omega[\mathbf{w}] := \operatorname{div}(\mathcal{C} : \boldsymbol{\nabla}\mathbf{w}) + \omega^2 \mathbf{w}$$



Scattering by small crack: integral representation and equation

Integral representation (with $\mathbf{G}$ again the elastodynamic Green's tensor, such that $\mathcal{G}(\cdot,\mathbf{x};\omega)=\mathbf{0}$ on S_D and $\mathbf{t}[\mathcal{G}(\cdot,\mathbf{x};\omega)]$ on S_N):

$$\mathbf{v}_a(\mathbf{x}) = \int_{\Gamma_a} \mathbf{t}[\mathcal{G}(\cdot,\mathbf{x};\omega)]\cdot[\![\mathbf{v}_a]\!]\,\mathrm{d}S \qquad (\mathbf{x}\in\Omega_a)$$

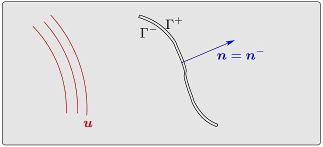
Traction integral equation (in regularized collocation form)

$$\mathcal{L}^\omega_a\big[\,[\![\mathbf{v}_a]\!]\,\big](\mathbf{x}) = -\mathbf{t}[\mathbf{u}](\mathbf{x}) \qquad (\mathbf{x}\in\Gamma_a)$$

with the integral operator $\mathcal{L}^\omega_a$ defined by

$$\mathcal{L}^\omega_a[\mathbf{w}](\mathbf{x}) = \mathbf{n}(\mathbf{x})\cdot\mathcal{C}:\Big\{ \int\!\!\!\!\!\!-_{\Gamma_a(z)} \mathcal{D}_\Gamma\mathbf{w}\cdot(\mathcal{C}:\boldsymbol{\nabla}_1\mathcal{G}(\cdot,\mathbf{x};\omega))\,\mathrm{d}S + \rho\omega^2\int_{\Gamma_a}\mathcal{G}(\cdot,\mathbf{x};\omega)\cdot(\mathbf{w}\otimes\mathbf{n})\,\mathrm{d}S\Big\} \qquad (\mathbf{x}\in\Gamma_a,\ \mathbf{w}\in H^{1/2}_0(\Gamma_a))$$

$$\mathcal{D}_\Gamma := \mathbf{n}\otimes\boldsymbol{\nabla} - \boldsymbol{\nabla}\otimes\mathbf{n} \quad \text{(surface curl operator)}$$



Asymptotic behaviour of $\mathbf{v}^a$ for $a \to 0$

- Coordinate scaling: $\boldsymbol{\xi} = \mathbf{z} + a\bar{\boldsymbol{\xi}}, \quad \mathbf{x} = \mathbf{z} + a\bar{\mathbf{x}}$, $\mathrm{d}S_\xi = a^2\,\mathrm{d}S_{\bar{\xi}}$, $\boldsymbol{\nabla}_\xi = a^{-1}\boldsymbol{\nabla}_{\bar{\xi}}$
- Scaling properties of $\mathcal{G}(\boldsymbol{\xi},\mathbf{x};\omega)$ (by homogeneity properties of $\mathcal{G}(\boldsymbol{\xi}-\mathbf{x})$):

$$\mathcal{G}(\boldsymbol{\xi},\mathbf{x};\omega) = a^{-1}\mathbf{G}(\bar{\boldsymbol{\xi}}-\bar{\mathbf{x}}) + O(1)$$

$$\boldsymbol{\nabla}_1\mathcal{G}(\boldsymbol{\xi},\mathbf{x};\omega) = a^{-2}\boldsymbol{\nabla}\mathbf{G}(\bar{\boldsymbol{\xi}}-\bar{\mathbf{x}}) + O(1)$$

- Ansatz: $\boxed{\mathbf{v}_a(\boldsymbol{\xi}) = a\mathbf{V}(\bar{\boldsymbol{\xi}}) + o(a) \quad \text{on } \Gamma_a}$
- Leading contribution to components of traction integral equation found as:

$$\boxed{\mathcal{L}^\omega_a\big[\,[\![\mathbf{v}_a]\!]\,\big](\mathbf{x}) = a^0\mathcal{L}^{0,\infty}_{\bar{\Gamma}}[\mathbf{V}](\bar{\mathbf{x}}) + o(a^0)} \quad \text{and} \quad \boxed{\mathbf{t}[\mathbf{u}](\mathbf{x}) = \boldsymbol{\sigma}[\mathbf{u}](\mathbf{z})\cdot\mathbf{n}(\bar{\mathbf{x}}) + o(a^0)}$$

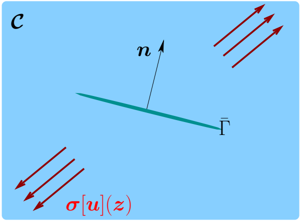
where $\mathcal{L}^{0,\infty}_{\bar{\Gamma}}$ is the **elastostatic** traction integral operator for the **normalized** crack $\bar{\Gamma}$ embedded in an **unbounded** medium:

$$\mathcal{L}^{0,\infty}_{\bar{\Gamma}}[\mathbf{W}](\bar{\mathbf{x}}) := \mathbf{n}(\bar{\mathbf{x}})\cdot\mathcal{C} : \left\{ \displaystyle{\int\!\!\!\!\!\!-}_{\Gamma_a(z)} \mathcal{D}_{\bar{\Gamma}}\mathbf{W}\cdot(\mathcal{C} : \boldsymbol{\nabla}\mathbf{G}(\cdot - \mathbf{x}))\,\mathrm{d}S \right\}$$

Hence $\mathbf{V}$ solves the **normalized elastostatic exterior crack problem** in $\mathbb{R}^3 \setminus \bar{\Gamma}$

$$\boxed{\mathcal{L}^{0,\infty}_{\bar{\Gamma}}[\mathbf{V}](\bar{\mathbf{x}}) = -\boldsymbol{\sigma}[\mathbf{u}](\mathbf{z})\cdot\mathbf{n}(\bar{\mathbf{x}}) \qquad (\bar{\mathbf{x}} \in \bar{\Gamma})}$$

Asymptotic behaviour of $\mathbf{v}^a$ for $a \to 0$



$$\mathbf{u}_a(\xi) = \mathbf{u}(\xi) + a\mathbf{V}(\bar{\xi}) + o(a)$$

- $\mathbf{V}(\bar{\xi}) + \boldsymbol{\nabla}\mathbf{u}(\mathbf{z}).\bar{\xi}$ solution to **elastostatic** problem: **crack under remote uniform stress** $\mathcal{C}:\boldsymbol{\nabla}\mathbf{u}(\mathbf{z})$
- $\mathbf{V}$ depends **linearly** on constant tensor $\mathbf{E} = \boldsymbol{\nabla}\mathbf{u}$: write $\mathbf{V}[\mathbf{E}]$ by notational analogy to inclusion case

Asymptotic behaviour of $\mathbf{v}_a$ away from the crack:

$$\mathbf{v}_a(\mathbf{x}) = \Big[\boldsymbol{\nabla}\mathbf{u}(\mathbf{z}):\mathcal{A}:\boldsymbol{\nabla}_1\mathcal{G}(\mathbf{z},\mathbf{x};\omega)\Big]a^3 + o(a^3)$$

with (fourth-order) **polarization tensor** $\mathcal{A} = \mathcal{A}(\mathcal{C},\bar{\Gamma})$ defined by

$$\mathcal{A}:\mathbf{E} = \mathcal{C}:\Big\{\int_{\bar{\Gamma}} [\![\mathbf{V}[\mathbf{E}]]\!](\bar{\xi}) \otimes \mathbf{n}(\bar{\xi})\,\mathrm{d}S_{\bar{\xi}}\Big\} \qquad (\forall\mathbf{E} \in \mathbb{R}^{3\times 3}_{\text{sym}})$$

Property: $\mathcal{A}(\mathcal{C},\bar{\Gamma})$ has major and minor symmetries.

Bellis, B, (2011, in preparation)

Polarization tensor

Polarization tensor for a penny-shaped crack with normal $\mathbf{n}$: closed-form expression (with $\mathbf{e}_1, \mathbf{e}_2$ an orthonormal basis of the crack plane)

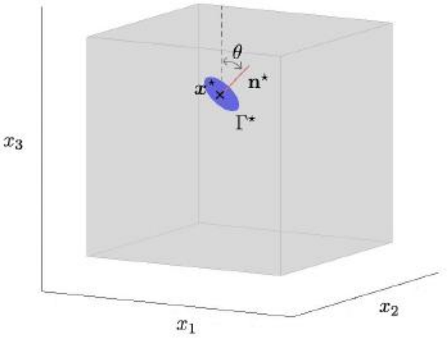
$$\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{A}}(\mathbf{n}) = \frac{8(1-\nu)}{3\mu(2-\nu)}\big[\mathbf{e}_i \otimes \mathbf{n} \otimes \mathbf{e}_i \otimes \mathbf{n} + \mathbf{e}_i \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{e}_i - \nu \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n}\big]$$

- Sound-hard penny-shaped 'crack' (screen) in acoustic medium (Amstutz, Dominguez 2008):

$$\mathbf{A} = \mathbf{A}(\mathbf{n}) = \frac{8}{3}\mathbf{n} \otimes \mathbf{n}$$

Bellis, MB, 2011 (in preparation)

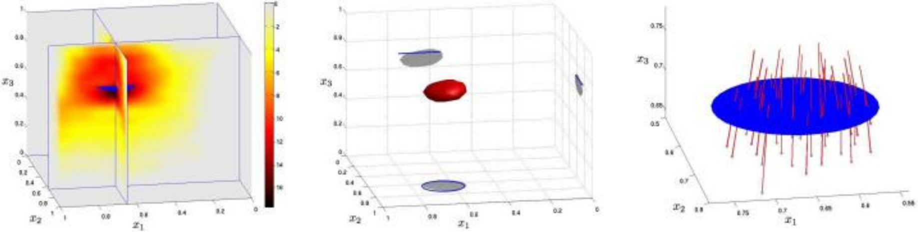
Example (3D, FEM, time-domain elasticity, crack in cube)



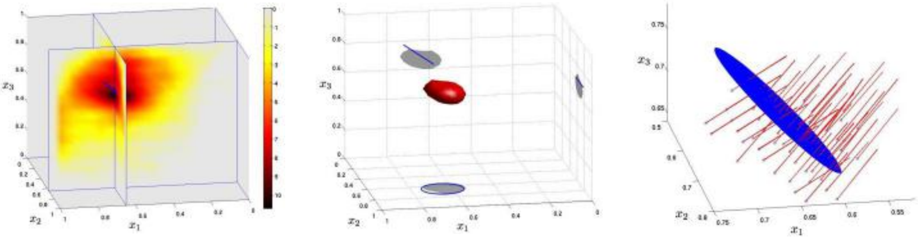
- $S_N = \partial\Omega$; Normal uniform compressive load $\mathbf{t} = -H(t)\mathbf{e}_3$ on top surface
- $S_{obs} = \partial\Omega$
- Cast3m (CEA) FEM toolbox
- Unconditionally-stable Newmark time integration

Example (3D, FEM, time-domain elasticity, crack in cube)

Horizontal penny-shaped crack:

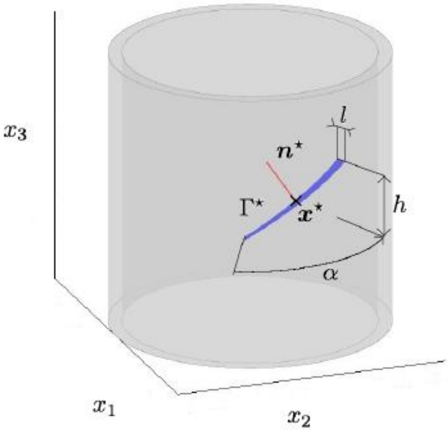


Inclined ($\pi/4$) penny-shaped crack:



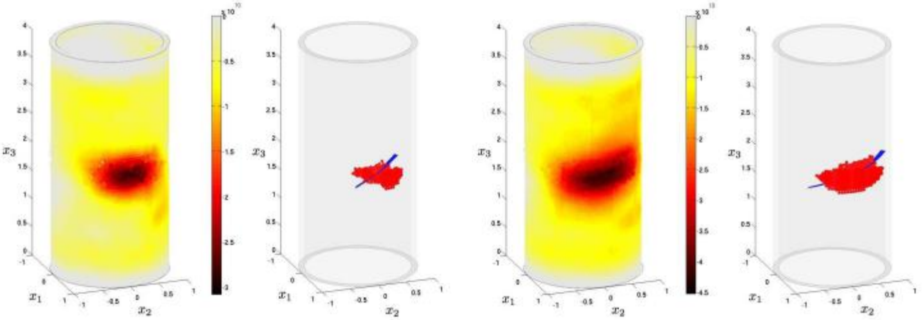
Bellis, MB, *in preparation* (2011)

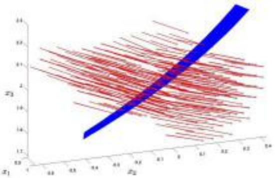
Example (3D, FEM, time-domain elasticity, crack in cylindrical shell)



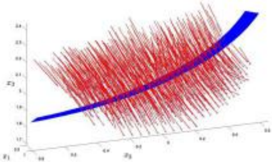
- External radius 1, thickness 0.1, height 4.
- $S_{\mathrm{N}} = \partial\Omega$; Normal compressive loads on square patches on external surface
- $S_{\mathrm{obs}} = \{r = 1,\ 0 \le x_3 \le 4\}$ (external lateral surface)
- Cast3m (CEA) FEM toolbox
- Unconditionally-stable Newmark time integration

Example (3D, FEM, time-domain elasticity, crack in cylindrical shell)



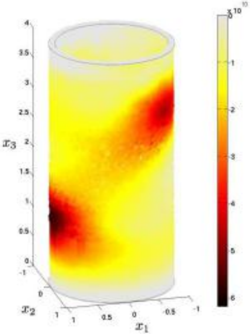


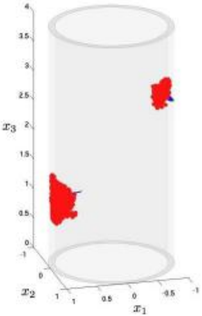
$\alpha = \pi/4$

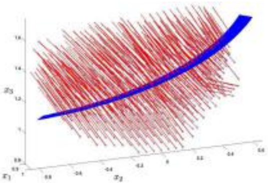


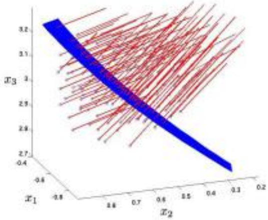
$\alpha = \pi/2$

Example (3D, FEM, time-domain elasticity, dual crack in cylindrical shell)

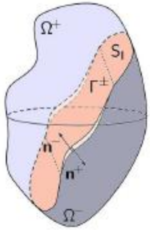








Example (3D elastic, time-domain FEM, interface crack identification)



- **Bi-material** domain $\Omega = \Omega^+ \cup \Omega^-$ with interface S_I

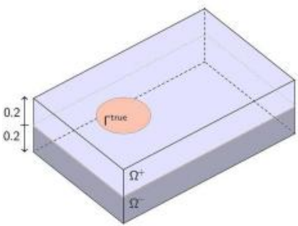
  $\rightarrow$ Interface crack $\Gamma \subset S_I$
- Number of applications
  - *Delamination cracks* in composite materials
  - *Fatigue* behaviors of laminates
  - *Matrix-matrix* or *fibre-matrix* interface *debonding*
  - ...
- **Specificities**: Interface location and orientation are known beforehand
- Derivation of specific **polarization tensor** for the **elastic** case

$$\rightarrow \boldsymbol{\mathcal{A}}^{\sigma} = \frac{8}{3}\pi^2\kappa(1+\kappa^2)(\alpha^2-\beta^2)\left\{\frac{1}{\beta}\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n} + \frac{1}{(\alpha-\delta)\pi\kappa(1+\kappa^2)+\beta}\left[\boldsymbol{n}\otimes\boldsymbol{e}_i+\boldsymbol{e}_i\otimes\boldsymbol{n}\right]\otimes\left[\boldsymbol{n}\otimes\boldsymbol{e}_i+\boldsymbol{e}_i\otimes\boldsymbol{n}\right]\right\}$$

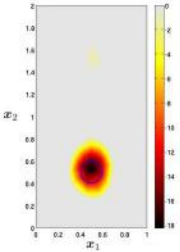
with constants $\alpha, \beta, \delta, \kappa$ depending on isotropic elastic material constants $\nu^{\pm}, \mu^{\pm}$

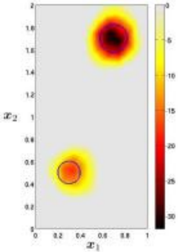
Example (3D elastic, time-domain FEM, interface crack identification)

- FEM-based time domain 3D simulations in *stiff/soft* bi-material domain
- Gaussian time distribution of compressional loading on top face
- Adimensionalization w.r.t. longitudinal wave velocity, $T = 1$
- Observation on top face



- Topological derivative $\mathbb{T}(\cdot, T) \leqslant 0$ at interface





- Extension to multilayered domains
- Study of other type of interface, e.g. fiber reinforced composites

Bellis, B, *in preparation* (2011)

Higher-order topological sensitivity: motivation

Computation of topological sensitivity of $\mathcal{J}$

- Non-iterative;
- Computationally faster than a minimization-based inversion algorithm; (2 forward solutions governed by same linear field equations)
- Yields **qualitative** results, since the approximation

$$\mathcal{J}(\Omega_a) \approx \mathcal{J}(\Omega) + a^D |\mathcal{B}| \mathcal{T}(\mathbf{z}) \qquad (\Omega \subset \mathbb{R}^D)$$

  cannot be minimized

Next step: several possibilities:

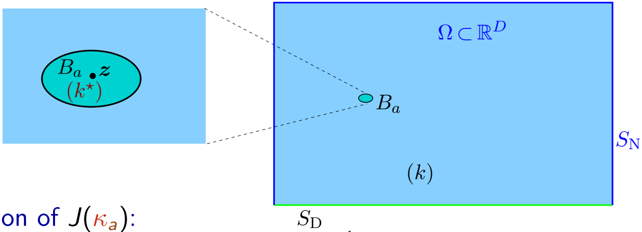
(a) Do nothing, i.e. accept the qualitative results as sufficiently informative;

(b) Compute "good" initial guess for minimization-based inversion from information given by field $\mathcal{T}(\mathbf{z})$ (currently under investigation);

(c) **Higher-order topological expansion** $\longrightarrow$ polynomial approximation of cost function $\mathcal{J}$:

- Much faster to compute than $\mathcal{J}$;
- Lends itself to minimization w.r.t. defect size a

Higher-order expansion of $J(\kappa_a)$



- Expansion of $J(\kappa_a)$:
$$J(\kappa_a) = J(\kappa) + \mathbb{J}'(u; v^a) + \frac{1}{2}\mathbb{J}''(u; v^a, v^a) + o(\|v^a\|^2_{2,\partial\Omega})$$
- Since $\boxed{|v^a| = O(a^D) \text{ on } S_{\rm obs}}$, seek $\boxed{O(a^{2D})}$ expansion of $J(a)$.
(order $2D$: natural choice for least-squares cost functionals)
- 'Direct' approach: expand $v^a\,|_{S_{\rm obs}}$ (outer expansion) to order $O(a^{2D})$:
$$v^a(\xi) = a^3 V_1(\xi) + a^4 V_2(\xi) + a^5 V_3(\xi) + a^6 V_4(\xi) + o(a^6) \qquad (\xi \in S_{\rm obs})$$
Expansions in terms of Green's functions, to any order in a (Ammari, Kang 04)
- **Adjoint solution approach**: computationally more efficient in the context of cost function minimization.

B, *Inverse Problems* (2008), *Int. J. Solids Struct.* (2009), *Eng. Anal. Bound. Elem.* (2010)

$O(a^{2D})$ expansion of $J(\kappa_a)$

(i) Expansion of $J(\kappa_a)$ (with $v^a = u^a - u$):

$$J(\kappa_a) = J(\kappa) + \mathbb{J}'(u; v^a) + \frac{1}{2}\mathbb{J}''(u; v^a, v^a) + o(\|v^a\|^2_{2,\partial\Omega})$$

(ii) Adjoint solution (again):

$$\text{Find } \hat{u} \in W_0, \qquad \langle \hat{u}, w \rangle^{\kappa}_{\Omega} = \mathbb{J}'(u; w), \quad \forall w \in W_0$$

(iii) Rewrite expansion of $J(a)$ invoking **reciprocity** between v^a and $\hat{u}$:

$$\mathbb{J}'(u; v^a) = -\langle \hat{u}, u^a \rangle^{\Delta\kappa}_{B_a}$$

(iv) Expansion of $J(\kappa_a)$ becomes

$$J(\kappa_a) = J(\kappa) - \langle \hat{u}, u^a \rangle^{\Delta\kappa}_{B_a} + \frac{1}{2}\mathbb{J}''(u; v^a, v^a) + o(\|v^a\|^2_{2,\partial\Omega})$$

- $v^a = O(a^D)$ on $\partial\Omega$, hence $O(a^{2D})$ contribution to $\mathbb{J}''(u; v^a, v^a)$ known from leading contribution of outer expansion of v^a;
- $O(a^{2D})$ expansion of $\langle \hat{u}, u^a \rangle^{\Delta\kappa}_{B_a}$ requires $O(a^{D+1})$ expansion of v^a in B_a (since $\langle \hat{u}, u^a \rangle^{\Delta\kappa}_{B_a} = O(a^D)$ while $v^a = O(a)$ in B_a)

$O(a^{2D})$ expansion of $J(\kappa_a)$ (Laplace, penetrable inclusion)

Governing integral equation / representation for perturbation $v^a = u^a - u$:

$$\mathcal{L}_a[v^a](\mathbf{x}) = -\langle u, \mathcal{G}(\cdot, \mathbf{x})\rangle_{B_a}^{\Delta\kappa}$$

1. Introduce scaled coordinates and set ansatz for inner expansion of v^a:

$$v^a(\mathbf{x}) = \sum_{p=1}^{D+1} a^p V_p(\bar{\mathbf{x}}) \quad \Longrightarrow \quad \boldsymbol{\nabla} v^a(\mathbf{x}) = \sum_{p=1}^{D+1} a^{p-1} \boldsymbol{\nabla} V_p(\bar{\mathbf{x}}) \qquad (\mathbf{x} \in B_a = \mathbf{z} + a\bar{\mathbf{x}})$$

2. Substitute above ansatz into (and use scaled coordinates in) integral equation, then expand the result in powers of a, to obtain governing integral equations for $V_1, \ldots, V_{D+1}$.
3. Evaluate $O(a^{2D})$ expansion of $\langle \hat{u}, u^a\rangle_{B_a}^{\Delta\kappa}$;
4. Evaluate $O(a^{2D})$ expansion of $\mathbb{J}''(u; v^a, v^a)$ by invoking outer expansion $v^a(\mathbf{x}) = a^D W(\mathbf{x}, \mathbf{z}) + o(a^D)$ of v^a on $\partial\Omega$.

$O(a^4)$ expansion of $J(\kappa_a)$ (Laplace, penetrable inclusion, 2D)

For a **circular** penetrable inclusion B_a, one has

$$J(\kappa_a) = J(\kappa) + \mathcal{T}_2(\mathbf{z})a^2 + \mathcal{T}_3(\mathbf{z})a^3 + \mathcal{T}_4(\mathbf{z})a^4 + o(a^4)$$

$$\mathcal{T}_2(\mathbf{z}) = 2\eta\pi\kappa\boldsymbol{\nabla} u(\mathbf{z})\cdot\boldsymbol{\nabla}\hat{u}(\mathbf{z})$$

$$\mathcal{T}_3(\mathbf{z}) = 0$$

$$\mathcal{T}_4(\mathbf{z}) = \frac{\eta}{2}\pi\kappa\boldsymbol{\nabla}^2 u(\mathbf{z}):\boldsymbol{\nabla}^2\hat{u}(\mathbf{z}) + (2\eta\pi)^2\kappa\boldsymbol{\nabla} u(\mathbf{z})\cdot\boldsymbol{\nabla}_{12}\mathcal{G}_{\text{c}}(\mathbf{z},\mathbf{z})\cdot\boldsymbol{\nabla}\hat{u}(\mathbf{z}) + \mathbb{J}''(u; W, W)$$

where $\mathcal{G}_{\text{c}} := \mathcal{G} - G$ is the complementary Green's function and

$$W(\mathbf{x},\mathbf{z}) = 2\pi\eta\boldsymbol{\nabla}\mathcal{G}(\mathbf{x},\mathbf{z})\cdot\boldsymbol{\nabla} u(\mathbf{z}), \quad \eta = \frac{\kappa - \kappa^\star}{\kappa + \kappa^\star}$$

- $\mathcal{T}_2(\mathbf{z})$ is the usual topological derivative;
- Generalization for arbitrary shape $\mathcal{B}$ also available (not shown);
- Above result valid only for a **single** small inclusion; generalization to finitely-spaced multiple small inclusion available (see also Bonnaillie, Dambrine, Tordeux, Vial (2009) for a-dependent inclusion spacing);
- $\mathcal{T}_{D+1}(\mathbf{z}) = 0$ for *any* centrally-symmetric shape $\mathcal{B}$;

$O(a^4)$ expansion of $\mathcal{J}(a)$ (2D, Laplace, crack)

For a **straight**, perfectly insulated small crack (location $\mathbf{z}$, normal $\mathbf{n}$, length $2a$), one has

$$J(a) = J(0) + \mathcal{T}_2(\mathbf{z})a^2 + \mathcal{T}_3(\mathbf{z})a^3 + \mathcal{T}_4(\mathbf{z})a^4 + o(a^4)$$

$$\mathcal{T}_2(\mathbf{z};\mathbf{n}) = \kappa\pi[u_{,n}\hat{u}_{,n}](\mathbf{z})$$

$$\mathcal{T}_3(\mathbf{z};\mathbf{n}) = 0$$

$$\mathcal{T}_4(\mathbf{z};\mathbf{n}) = \frac{\kappa\pi}{8}\big[u_{,n\tau}\hat{u}_{,n\tau} + u_{,n\tau\tau}\hat{u}_{,n} + u_{,n}\hat{u}_{,n\tau\tau}\big](\mathbf{z}) + \kappa\pi^2\big(\mathbf{n}\cdot\boldsymbol{\nabla}_1\boldsymbol{\nabla}_2\mathcal{G}_{\text{c}}(\mathbf{z},\mathbf{z})\cdot\mathbf{n}\big)[u_{,n}\hat{u}_{,n}](\mathbf{z}) + \mathbb{J}''(u;W,W)$$

$$W(\mathbf{x},\mathbf{z}) = 2\pi\kappa(\boldsymbol{\nabla}_2$$

$$Gcal(\mathbf{x},\mathbf{z})\cdot\mathbf{n})u_{,n}(\mathbf{z})$$

B, *Eng. Anal. Bound. Elem.* (2010)

$O(a^6)$ expansion of $J(m_a, n_a)$ (3-D, Helmholtz, penetrable scatterer)

$$J(a) = J(0) + \underbrace{\mathcal{T}_3(\mathbf{z})a^3 + \mathcal{T}_4(\mathbf{z})a^4 + \mathcal{T}_5(\mathbf{z})a^5 + \mathcal{T}_6(\mathbf{z})a^6}_{J_6(a;\mathbf{z})} + o(a^6)$$

where, for (i) least-squares misfit function and (ii) centrally-symmetric scatterers:

$$\mathcal{T}_3(\mathbf{z}) = \text{Re}\big[\,\boldsymbol{\nabla} u \cdot \mathbf{A}_{11} \cdot \boldsymbol{\nabla} \hat{u} - |\mathcal{B}|\,(1-\eta)k^2 u\hat{u}\,\big](\mathbf{z}) \qquad \text{'Topological derivative'}$$

$$\mathcal{T}_4(\mathbf{z}) = 0$$

$$\begin{aligned}\mathcal{T}_5(\mathbf{z}) = \frac{1}{2}\text{Re}\Big[&\boldsymbol{\mathcal{I}}_2 : \boldsymbol{\nabla}^2\big[\,(1-\beta)\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \hat{u} - (1-\eta)k^2 u\hat{u}\,\big] + \boldsymbol{\nabla} \hat{u} \cdot \boldsymbol{\mathcal{B}}_{11} \cdot \boldsymbol{\nabla} u \\ &+ \big(\,\boldsymbol{\nabla}^3 u \,\vdots\, \mathbf{A}_{31} \cdot \boldsymbol{\nabla} \hat{u} + 2\boldsymbol{\nabla}^2 u : \mathbf{A}_{22} : \boldsymbol{\nabla}^2 \hat{u} + \boldsymbol{\nabla}^3 \hat{u} \,\vdots\, \mathbf{A}_{31} \cdot \boldsymbol{\nabla} u\,\big)\Big](\mathbf{z})\end{aligned}$$

$$\mathcal{T}_6(\mathbf{z}) = \frac{1}{6}\text{Re}\Big[\,G_4 \cdot \mathbf{A}_{11} \cdot \boldsymbol{\nabla} \hat{u} - 3(1-\eta)F_3|\mathcal{B}|\hat{u}\,\Big](\mathbf{z}) + \mathbb{J}''(u; W, W)$$

with

$\mathbf{A}_{11}$, $\mathbf{A}_{22}$, $\mathbf{A}_{31}$, $\boldsymbol{\mathcal{B}}_{11}$, $\boldsymbol{\mathcal{I}}_2$: **known** constant tensors (in particular, $\mathbf{A}_{11} = \mathbf{A}$),
F_3, G_4: **known** functions that involve $G_c = G - G$

B (2011, in preparation)

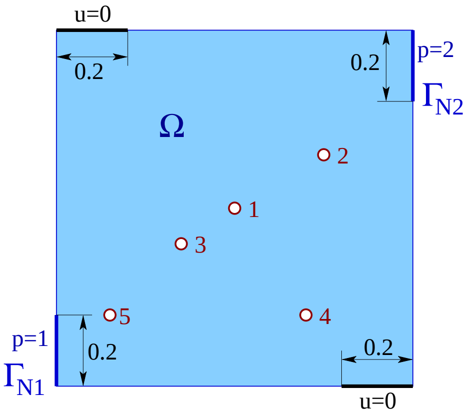
$O(a^6)$ expansion of $J(m_a, n_a)$ (3-D, Helmholtz, penetrable scatterer)

For a **spherical** scatterer:

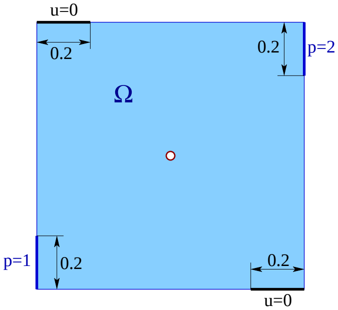
$$
\begin{aligned}
\boldsymbol{\mathcal{I}}_2 &= \frac{4\pi}{9}\mathbf{I} \\
\mathbf{A}_{11} &= 4\pi\frac{1-\beta}{2+\beta}\mathbf{I} \\
\boldsymbol{\mathcal{B}}_{11} &= \frac{1-\beta}{(2+\beta)^2}\,\frac{8\pi}{15}\big[5+\eta\big]\mathbf{I} \\
\mathbf{A}_{22} &= \frac{8\pi}{15}\,\frac{(1-\beta)^2}{3+2\beta}\boldsymbol{\mathcal{I}}_4 + \frac{4\pi}{45}\Big[(5\beta+1)\beta\gamma^4 - 12\eta + \frac{13+17\beta}{3+2\beta}\Big]\mathbf{I}\otimes\mathbf{I} \\
\mathbf{A}_{31} &= \frac{4\pi}{45}\,\frac{1-\beta}{2+\beta}\big[3-\beta-2\eta\big]\big(2\boldsymbol{\mathcal{I}}_4+\mathbf{I}\otimes\mathbf{I}\big)
\end{aligned}
$$

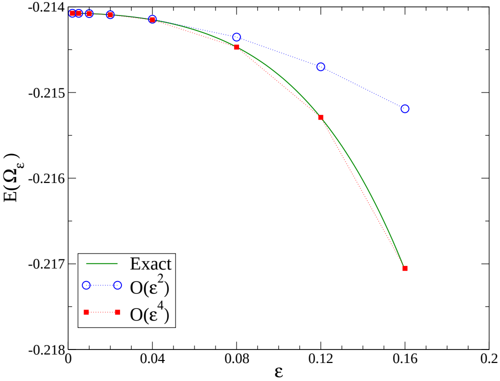
Example 1: expansion of potential energy (2-D Laplace)

$$E_{\text{pot}}(\Omega_a) = -\int_{\Gamma_{\text{N1}}} u^a \,\text{d}\Gamma - 2\int_{\Gamma_{\text{N2}}} u^a \,\text{d}\Gamma$$

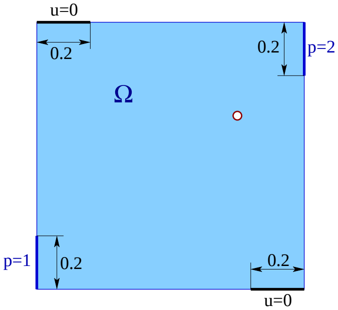


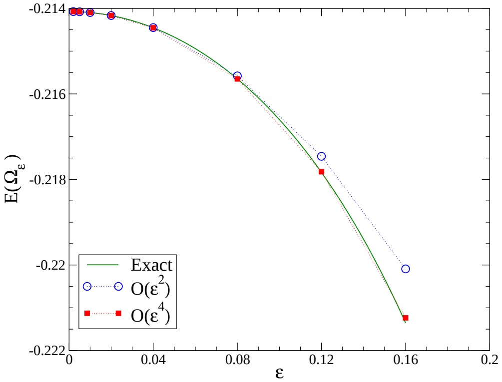
Example 1: expansion of potential energy (2-D Laplace, circular hole, $\beta = 0$)



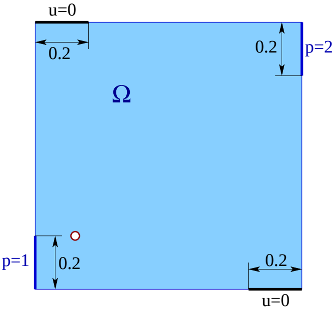


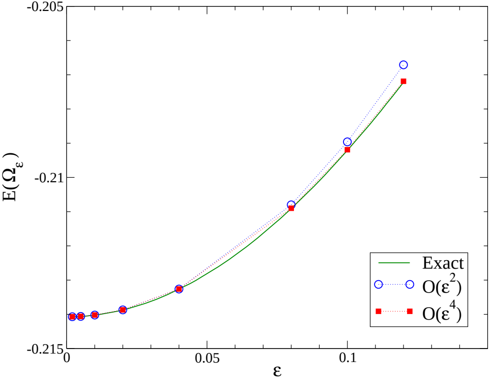
Example 1: expansion of potential energy (2-D Laplace, circular inclusion, $\beta = 0.6$)





Example 1: expansion of potential energy (2-D Laplace, circular inclusion, $\beta = 5$)

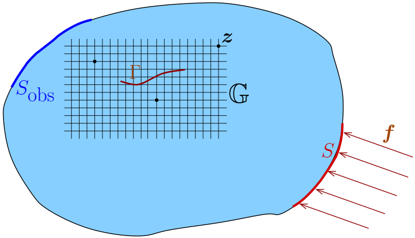




Approximate global crack search based on high-order expansion of $\mathcal{J}$

Polynomial (in a) approximation of $\mathbb{J}(u^a)$:

$$\boxed{\mathbb{J}(u^a) = J_{2D}(a;\mathbf{z},\mathbf{n}) + o(a^6)}, \qquad J_{2D}(a;\mathbf{z},\mathbf{n}) := \mathbb{J}(u) + \sum_{p=D}^{2D} \mathcal{T}_p(\mathbf{z},\mathbf{n})a^p$$

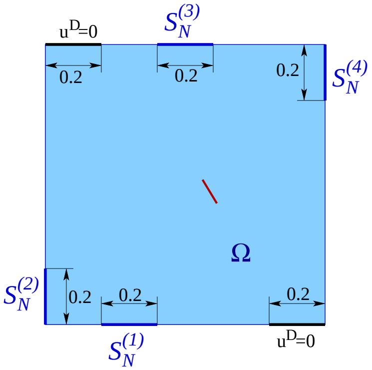


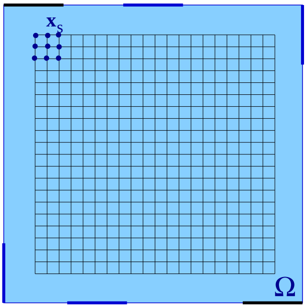
- Introduce fine search grid $\mathbb{G}$ in region of interest;
- Define $\big(\ell(\mathbf{z}), \mathbf{n}(\mathbf{z})\big) = \arg\min_{a,n} J_{2D}(a;\mathbf{z}), \quad J_{2D}^{\text{min}}(\mathbf{z}) = \min_{a} J_{2D}(a;\mathbf{z})$
- Best estimate of crack defined by

$$\boxed{\mathbf{x}^{\text{est}} = \arg\min_{\mathbf{z}\in\mathbb{G}} J_{2D}^{\text{min}}(\mathbf{z}) \qquad \ell^{\text{est}} = R(\mathbf{x}^{\text{est}}) \qquad \mathbf{n}^{\text{est}} = \mathbf{n}(\mathbf{x}^{\text{est}})}$$

Example 2: crack identification (2-D Laplace)

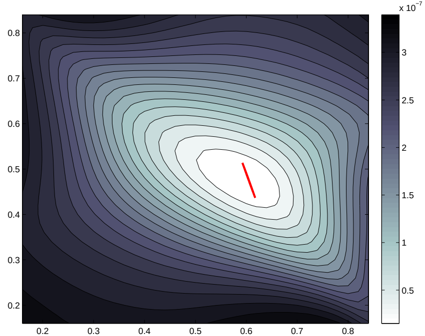
$$J(a) = \frac{1}{2}\int_{S_N} |u^a - u_{obs}|^2 \, d\Gamma$$
$$= J(0) + a^2\mathcal{T}_2(\mathbf{z},\mathbf{n}) + a^3\mathcal{T}_3(\mathbf{z},\mathbf{n}) + a^4\mathcal{T}_4(\mathbf{z},\mathbf{n}) + o(a^4)$$



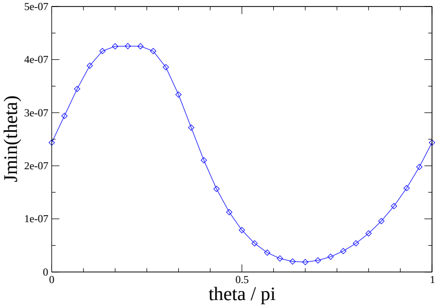


Example 2: crack identification (2-D Laplace)

True crack: straight (length 0.04, orientation $\theta_{\text{true}} = 7\pi/10$)



$J^{\text{min}}(\mathbf{z}, \theta^{\text{true}})$



$J^{\text{min}}(\mathbf{z}^{\text{est}}, \theta)$

Influence of orientation θ of trial crack

- Polarization tensor with correct normal $\mathbf{n} = \big(\cos(2\pi/3), \sin(2\pi/3)\big)$
- Length reconstruction: relative error 0.01 (20% data noise)

B, *Eng. Anal. Bound. Elem.* (2011)

Further reading I

- Allaire, G., de Gournay, F., Jouve, F., Toader, A.-M..
  Structural optimization using topological and shape sensitivity via a level-set method.
  Control and Cybernetics, **34**:59–80 (2005).
- Ammari, H., Kang, H.
  Polarization and moment tensors with applications to inverse problems and effective medium theory. Applied Mathematical Sciences, Vol. 162. Springer-Verlag (2007).
- Ammari, H., Kang, H., Nakamura, G., Tanuma, K.
  Complete asymptotic expansions of solutions of the system of elastostatics in the presence of an inclusion of small diameter and detection of an inclusion. *J. Elast.*, **67**:97–129 (2002).
- Ammari, H and. Garnier, J., Jugnon, V., Kang, H.
  Stability and resolution analysis for a topological derivative based imaging functional (2011).
- Amstutz, S., Dominguez, N.
  Topological sensitivity analysis in the context of ultrasonic non-destructive testing.
  Eng. Anal. Bound. Elem., **32**:936–947 (2008).

Further reading II

- Amstutz, S., Horchani, I., Masmoudi, M.
  Crack detection by the topological gradient method. *Control and Cybernetics*, **34**:81–101 (2005).
- Bellis, C., Bonnet, M.
  A FEM-based topological sensitivity approach for fast qualitative identification of buried cavities from elastodynamic overdetermined boundary data. *Int. J. Solids Struct.*, **47**:1221–1242 (2010).
- Ben Abda, A., Hassine, M., Jaoua, M., Masmoudi, M.
  Topological sensitivity analysis for the location of small cavities in Stokes flow. *SIAM J. Contr. Opt.*, **48**:2871–2900 (2009).
- Beretta, E., Bonnetier, E., Francini, E., Mazzucato, A. L.
  An asymptotic formula for the displacement field in the presence of small anisotropic elastic inclusions. Preprint, `arXiv:1105.4111v1` (2011).
- Bonnet, M.
  Topological sensitivity for 3D elastodynamic and acoustic inverse scattering in the time domain. *Comp. Meth. Appl. Mech. Eng.*, **195**:5239–5254 (2006).

Further reading III

- Bonnet, M.
  Higher-order topological sensitivity for 2-D potential problems. Application to fast identification of inclusions. *Int. J. Solids Struct.*, **46**:2275–2292 (2009).
- Bonnet, M.
  Fast identification of cracks using higher-order topological sensitivity for 2-D potential problems. *Eng. Anal. Bound. Elem.*, **35**:223–235 (2011).
- Bonnet, M., Guzina, B. B.
  Sounding of finite solid bodies by way of topological derivative. *Int. J. Num. Meth. Eng.*, **61**:2344–2373 (2004).
- Céa, J., Garreau, S., Guillaume, P., Masmoudi, M.
  The shape and topological optimization connection. *Comp. Meth. Appl. Mech. Eng.*, **188**:703–726 (2001).
- Dominguez, N., Gibiat, V., Esquerré, Y.
  Time domain topological gradient and time reversal analogy: an inverse method for ultrasonic target detection. *Wave Motion*, **42**:31–52 (2005).
- Eschenauer, H. A., Kobelev, V. V., Schumacher, A.
  Bubble method for topology and shape optimization of structures. *Structural Optimization*, **8**:42–51 (1994).

Further reading IV

- Feijóo, G. R.
  A new method in inverse scattering based on the topological derivative. *Inverse Problems*, **20**:1819–1840 (2004).
- Garreau, S., Guillaume, P., Masmoudi, M.
  The topological asymptotic for PDE systems: the elasticity case. *SIAM J. Contr. Opt.*, **39**:1756–1778 (2001).
- Guillaume, P., Sid Idris, K.
  The topological asymptotic expansion for the Dirichlet problem. *SIAM J. Contr. Opt.*, **41**:1042–1072 (2002).
- Guzina, B. B., Bonnet, M.
  Topological derivative for the inverse scattering of elastic waves. *Quart. J. Mech. Appl. Math.*, **57**:161–179 (2004).
- Guzina, B. B., Chikichev, I.
  From imaging to material identification: a generalized concept of topological sensitivity. *J. Mech. Phys. Solids*, **55**:245–279 (2007).
- Il'in, A. M.
  Matching of asymptotic expansions of solutions of boundary value problems. American Mathematical Society (1992).

Further reading V

- Masmoudi, M., Pommier, J., Samet, B.
  The topological asymptotic expansion for the Maxwell equations and some applications. *Inverse Problems*, **21**:547–564 (2005).
- Maz'ya, V., Nazarov, S. A., Plamenevskii, B. A.
  Asymptotic theory of elliptic boundary value problems under a singular perturbation of the domains (vols. 1 and 2). Birkhaüser (2000).
- Nazarov, S. A., Sokolowski, J.
  Asymptotic analysis of shape functionals. *Journal de Mathématiques Pures et Appliquées*, **82**:125–196 (2003).
- Nazarov, S. A., Sokolowski, J., Specovius-Neugebauer, M.
  Polarization matrices in anisotropic heterogeneous elasticity. *Asymptotic Analysis*, **68**:189–221 (2010).
- Nemitz, N., Bonnet, M.
  Topological sensitivity and FMM-accelerated BEM applied to 3D acoustic inverse scattering. *Eng. Anal. Bound. Elem.*, **32**:957–970 (2008).
- Novotny, A. A., Feijóo, R. A., Taroco, E., Padra, C..
  Topological sensitivity analysis. *Comp. Meth. Appl. Mech. Eng.*, **192**:803–829 (2003).

Further reading VI

- Silva, M., Matalon, M., Tortorelli, D. A.
  Higher order topological derivatives in elasticity. *Int. J. Solids Struct.*, **47**:3053–3066 (2010).
- Sokolowski, J., Zochowski, A.
  Topological derivatives of shape functionals for elasticity systems. *Mechanics Based Design of Structures and Machines*, **21**:331–349 (2001).
- Sokolowski, J., Zochowski, A.
  Topological derivatives for optimization of plane elasticity contact problems. *Eng. Anal. Bound. Elem.* (2008, in press).
- Vogelius, M. S., Volkov, D.
  Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter. *M2AN Math. Model. Num. Anal.*, **34**:723–748 (2000)