A multipole Galerkin boundary element method for acoustics

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Abstract

A fast multilevel multipole (FMM) algorithm is derived for the Helmholtz equation and adopted to the symmetric Galerkin boundary element method (BEM) for acoustics. The FMM allows to evaluate a matrix–vector product of the BEM with a computational cost of \( O(N \log^2 N) \), thus leading to a significant reduction of computation time and memory requirements compared to standard BEM formulations. This allows the simulation of large scale acoustic models. The performance of the algorithm is demonstrated on the example of sound radiation from an L-shaped domain with BE discretizations of up to 45,000 elements.

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1. Introduction

The application of the boundary element method (BEM) for structural acoustics suffers from the high computational cost. The simulation of sound radiation from flexible vibrating structures at higher frequencies must be performed on a fine discretization. As a rule of thumb, six linear elements per wavelength are required to obtain an accuracy of a few percent which is typical for many engineering applications. Using the BEM leads to fully populated system matrices and thus to a numerical cost of \( O(N^2) \) in the number of unknowns \( N \). This cost cannot be handled, even for a moderate number of unknowns, by standard implementations. The BEM can be accelerated by employing a multilevel multipole scheme to approximate the fundamental solution at some distance from the source point. The resulting complexity of the algorithm is \( O(N \log^2 N) \), and thus, much faster than the traditional BEM.

The multipole method was originally developed by Greengard and Rokhlin [8] for the fast simulation of large particle fields in physics. The theory for diagonal translation operators in 3D acoustics was developed as well by Rokhlin [14]. A very detailed description of the multipole method for the Helmholtz equation was presented by Epton and Dembart [5]. Rahola [13] examined the error committed by truncating the series solution. The concept of the multipole algorithm is described very clearly by Gyure and Stalzer [9], whereas a state of the art review on multipole methods for various applications is presented by Nishimura [11]. Particularly for the large-scale simulation of electromagnetic fields, integral equations accelerated by the fast multipole algorithm are today’s method of choice and most of the developments cited above were investigated in the context of Maxwell’s equation. For a detailed survey on the fast solution of integral equations for Maxwell’s equation, it is referred to the review article by Chew et al. [3].

The application of the multipole algorithm to accelerate the BEM is triggered by the similarity between the potential in particle simulations and the fundamental solution in the kernels of the boundary integral operators. Giebermann [7] compared two fast summation methods in his thesis, the panel clustering method and the multipole method, and their application on BEM for scattering problems. More lately Of [12] developed a Multipole BEM algorithm for Laplace’s equation and elasticity problems. Schneider [15] applied the multipole expansion to standard collocation BEM for the Helmholtz equation. In the paper at hand, the multipole algorithm is used to accelerate a symmetric Galerkin BEM for acoustics. The symmetric Galerkin BEM in contrast to the standard collocation approach was first proposed by Sirtori [16].
Since then a lot of research work was spent on the development of the method and it is referred to the extensive literature review by Bonnet et al. [2]. The Galerkin approach was chosen for the presented work since it is favorable compared to standard collocation BEM for large systems which is the scope of the multipole accelerated BEM.

While the multipole algorithm for the Helmholtz equation is covered extensively in literature, the application on the Galerkin BEM for structural acoustics is new. The scope of the paper at hand is to investigate the adaptability of the theoretical results to engineering practice. In the first section of the paper, the symmetric Galerkin BEM is derived for acoustics. Thereafter, the multipole algorithm and its application on the BEM is described. The paper concludes with a numerical example to demonstrate the efficiency of the proposed scheme. The parameters of the examples are chosen to represent typical dimensions, frequencies and precisions as they are encountered in engineering structural-acoustics applications.

2. Symmetric Galerkin BEM for acoustics

The pressure \( u \) in an acoustic field is governed by the Helmholtz equation
\[
\nabla^2 u(x) + k^2 u(x) = 0,
\]
where \( k = \omega/c \) is the acoustic circular wavenumber. The relation to the wavelength \( \lambda \) is given by \( k = 2\pi/\lambda \). Here, the Helmholtz equation is solved in an exterior domain \( \Omega \), which is the complement to the open set \( \Omega \subset \mathbb{R}^3 \) with boundary \( \Gamma = \partial \Omega \). The unit normal \( n \) to \( \Gamma \) is defined to point outwards from \( \Omega \). Neumann boundary conditions are prescribed as
\[
\frac{\partial u(x)}{\partial n} = g(x), \quad x \in \Gamma,
\]
corresponding to the surface velocity of a vibrating structure. Furthermore, the Sommerfeld radiation condition has to be fulfilled, i.e.
\[
\left| \frac{\partial u}{\partial r} - iku \right| \leq \frac{C}{r^2} \quad \text{at } r \to \infty.
\]

The fundamental solution \( u^*(x, y) \) for the Helmholtz equation is given by
\[
u^*(x, y) = \frac{1}{4\pi} \frac{e^{ikr}}{r},
\]
where \( r = |x - y| \) is the Euclidean distance between the field or receiver point \( x \) and the source or load point \( y \). Weighting the Helmholtz equation with the fundamental solution and applying Green’s second theorem yields the representation formula
\[
u(y) = -\int_\Gamma u^*(x, y) \frac{\partial u(x)}{\partial n} \, ds_x + \int_\Gamma \frac{\partial u^*(x, y)}{\partial n_x} u(x) ds_x,
\]
y \in \Omega_e.

For more details on the derivation of BEMs, it is referred to Gaul et al. [6]. A boundary integral formulation for the exterior Neumann problem is obtained by taking the normal derivative on the smooth boundary, \( \Omega_e \equiv y \to \Gamma \).

\[
\frac{\partial u(y)}{\partial n} = \frac{1}{2} \frac{\partial u(y)}{\partial n} - \int_\Gamma \frac{\partial u^*(x, y)}{\partial n_x} \frac{\partial u(x)}{\partial n} ds_x + \int_\Gamma \frac{\partial^2 u^*(x, y)}{\partial n_x \partial n_y} u(x) ds_x, \quad y \in \Gamma.
\]

In operator notation, Eq. (6) is written as
\[
(Dv)_{y} = (-\frac{1}{2} \Delta - K')g(y), \quad y \in \Gamma,
\]
with the Neumann boundary condition \( \partial u/\partial n = g(x) \), the identity operator \( I \) and the adjoint double layer potential \( K'\)
\[
(K'g)(y) := \int_\Gamma \frac{\partial u^*(x, y)}{\partial n_x} g(x) ds_x.
\]

The kernel of the hyper-singular operator
\[
(Dv)_{y} := -\int_\Gamma \frac{\partial^2 u^*(x, y)}{\partial n_x \partial n_y} u(x) ds_x
\]
exhibits a strong singularity \( (1/|x - y|^3) \) and is thus not an integrable function. However, the hyper-singular operator can be computed using regularization techniques.

Solving the exterior Neumann problem is equivalent to finding the solution \( u \in H^{1/2}(\Gamma) \) of the variational formulation
\[
\int_\Gamma Duv \, ds_y = \int_\Gamma (-\frac{1}{2} \Delta - K')gv \, ds_y
\]
for all \( v \in H^{1/2}(\Gamma) \), yielding the Galerkin formulation. Triangularization of the boundary \( \Gamma \) yields an approximated solution \( u_h \) on the boundary element space of piecewise linear basis functions \( \phi_i \)
\[
u_h(x) = \sum_{i=1}^n u_i \phi_i(x),
\]
where \( u_i \) are nodal pressure values. The given Neumann data is interpolated on the boundary triangularization with constant basis functions \( \phi_k \)
\[
g_h(x) = \sum_{k=1}^m g_k \phi_k(x).
\]

Using isoparametric test functions \( \phi_i \), the matrix representations of the discrete boundary integral operators
are given by
\begin{align}
D_h[i, j] &= \int_D D\psi_i \psi_j \, dx,
\end{align}
(13)
\begin{align}
K_h'[i, k] &= \int_D K'\psi_i \psi_j \, dx,
\end{align}
(14)
\begin{align}
I_h[i, k] &= \int_D \psi_i \psi_j \, dx,
\end{align}
leading to the matrix equation
\begin{align}
D_h u_h &= ( - \frac{1}{c^2} I_h - K_h') g_h.
\end{align}
(15)

3. Multipole expansion for the Helmholtz equation

As \( D_h \) is fully populated, computing and storing the matrix for large scale problems is not feasible. The key idea of the multipole BEM is to combine the effect of sources far away from a field point in a far-field term using the multipole expansion whereas for nearby sources standard BEM evaluations are used. In this way, the fast multilevel multipole (FMM) algorithm realizes the matrix–vector product \( D_h u_h \) with a numerical cost that depends quasi-linearly on the number of unknowns.

The application of a multipole expansion scheme in the BEM for the Helmholtz equation is straightforward due to the following fact: the fundamental solution \( (4) \) with respect to two points \( x, y \in \mathbb{R}^3 \) obeys the wave function \( \Phi(x, y) \) except for a constant factor of \( 1/4\pi \)
\begin{align}
\Phi(x, y) = \frac{e^{ik|x-y|}}{|x-y|},
\end{align}
where \( h_0^{(1)}(x) \) denotes the Hankel function of first order. The wave function describes the acoustic field at the field point \( x \) that is emitted by a monopole source of intensity \( q = 1 \) at the load point \( y \). In the original FMM for particle fields, this acoustic source would correspond to a single particle.

The idea is to evaluate the wave function induced by a number of sources in a distant point \( x \) not directly as in standard BEM but indirectly by combining sources in groups and using series expansions. A vital identity for this approach is Gegenbauer’s addition theorem \([1]\) for the Hankel functions. The wave function is expressed as a function of the vectors \( d = z - y \) and \( D = x - z \). The expansion center is \( z \) as depicted in Fig. 1. Later, the expansion center will be the center of a boundary element cluster.

For the two vectors \( d \) and \( D \) the addition theorem reads
\begin{align}
\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|D+d|}}{|D+d|} - i k \sum_{l=0}^\infty (2l+1)j_l(kD)h_l^{(1)}(kd)P_l(\hat{D}\cdot\hat{d})d,
\end{align}
(16)
\begin{align}
|D| > |d|.
\end{align}
can be computed as
\[
\Phi(x_b) = \frac{ik}{4\pi} \int_{S^2} e^{ik(x_b - z_2) \cdot s} M_L(s, z_2 - z_1) \sum_{a = 1}^A e^{ik(z_2 - y_a) \cdot s} q_a \, ds.
\]

(20)

The field of the sources \( q_a \) is represented in \( z_2 \) by the translation operators \( M_L(s, z_2 - z_1) \) and translated to the field point \( x_b \) close to \( z_2 \) by simple multiplication with \( e^{ik(x_b - z_2) \cdot s} \). The translation operators do not depend on the location of the single sources, but only on the distance vector between the expansion centers of the clusters. Thus, leading to favorable numerical complexity for the FMM BEM as discussed in Section 4.

For simplicity of later references, the far-field signature \( F \) and near-field signature \( N \) are defined as follows
\[
F(s) = \sum_{a = 1}^A e^{ik(z_2 - y_a) \cdot s} q_a,
\]

(21)

\[
N(s) = M_L(s, z_2 - z_1)F(s).
\]

(22)

### 4. Multipole BEM

When adapting the multipole scheme to BEM, a source refers to an integration point of a boundary element. To take advantage of the fast evaluation of the fundamental solution by the translation operators \( M_L \), it is required to cluster the boundary elements in groups. This can either be done on one level resulting in a single-level computational scheme or, as in the present paper, on several levels of clusters resulting in a multilevel scheme. A hierarchical tree with levels \( \ell = 0, \ldots, m \) containing clusters of elements is introduced. The first cluster on the highest level \( \ell = 0 \) is a parallelepiped containing all elements. Sons of a cluster are created by bisection of the cluster’s parallelepiped. In contrast to many publications on multipole algorithms, where oct trees are employed, binary trees were chosen for the level hierarchy to yield more flexibility for practical applications. Elements as a whole are assigned to a son and the diameter of a cluster \( d_\ell \) is defined as the maximum distance between two boundary element nodes in the cluster.

Attributes of a cluster are center and diameter, as well as a list of clusters in the near-field and clusters of the interaction set. The clusters in the near-field (\( N \)) of a cluster are all those clusters on the same level whose distance is below a specific threshold. All other sons of the father’s near-field clusters join the interaction set (I). The information transfer in the multilevel scheme is displayed in Fig. 3. The contribution of a source in the hatched cluster on the lowest level is distributed by the following steps:

- On the lowest level the contribution to the near-field clusters is computed directly by standard BEM evaluations.

The goal of the multilevel multipole scheme is to achieve a numerical complexity of \( N \log^2 N \) in the number of unknowns \( N \). Thus, on the lowest level, the number of elements in the near-field of each cluster must be proportional to \( \log^2 N \). This results in \( n_m \sim N/\log^2 N \) clusters with diameter \( d_m \sim \log N/sN \) on the lowest level \( \ell = m \). On the higher levels, number and diameter of the clusters behave as \( n_\ell \sim n_m 4^{(\ell-m)^2/3} \) and \( d_\ell \sim d_m 2^{(\ell-m)^2/3} \), respectively. The total number of levels is clearly of the order \( \log(N) \).

Furthermore, for an estimate of the numerical complexity of the proposed algorithm, the expansion length on each level has to be taken into account. The required expansion length \( L_\ell \) depends on the cluster diameter \( d_\ell \) and the wavenumber \( k \). Coifman et al. [4] give the semi-empirical fit
\[
L_\ell(kd_\ell) = kd_\ell + \log_{10}(kd_\ell + \pi),
\]

(23)

where \( p \) specifies the required precision. The parameter \( p \) does not translate directly to the number of accurate digits, since the error in the truncated multipole expansion does not only depend on the cluster size but also on the distance between two clusters where the translation operator is applied. A detailed discussion of the truncation error is given in the paper by Koc et al. [10].

A consequence of the choice (23) is that the expansion length changes from level to level. As the far-field signatures are translated to the father and later, during the downward cycle, the near-field signatures are translated to the sons, efficient routines for interpolating and filtering, respectively, are necessary. This is not possible in the diagonal form of the multipole expansion (17), but one has to use the original expansion (16). An algorithm based on fast Fourier transforms is developed by Gyure and Stalzer.
It scales as \( n_{L} L_{c}^{2} \log L_{c} \) on each level, yielding a total complexity of \( O(N \log^2 N) \).

Due to the dependence on the wavenumber, the computational cost of the FMM is discussed for keeping the ratio \( k/\sqrt{N} \) constant. This situation corresponds to the engineering rule of thumb that a constant number of elements per wavelength results in a constant discretization error. Choosing the expansion length according to Eq. (23) and assuming \( k/\sqrt{N} = \text{const} \) yields the following estimates:

- On the lowest level of the cluster tree, the sources will be combined to the far-field signature of the cluster using Eq. (21). The numerical complexity is \( \mathcal{O}(NL_{m}^{2}) = \mathcal{O}(N(kd_{m})^{2}) = \mathcal{O}(N \log^2 N) \), as each element has to be considered in each of the \( 2L_{m}^{2} \) discrete far-field directions.
- Transformation of far-field signature of each cluster to its interaction list by applying the translation operators \( M_{L} \).

\[
\mathcal{O}(n_{L} L_{c}^{2} \log N) = \mathcal{O}(n_{L}(kd_{L})^{2} \log N) = \mathcal{O}(k^{2} \log N) = \mathcal{O}(N \log N).
\]

- Translation of the far-field signature to the center of the father cluster, interpolation if necessary.

\[
\mathcal{O}(n_{L} L_{c}^{2} \log L_{c} \log N) = \mathcal{O}(N \log^{2} N).
\]

- In the downward cycle, translation and filtering of the near-field signatures.

\[
\mathcal{O}(n_{L} L_{c}^{2} \log L_{c} \log N) = \mathcal{O}(N \log^{2} N).
\]

- On the lowest level, translation of the near-field signature function each quadrature point and recovering the solution.\( \mathcal{O}(NL_{m}^{2}) = \mathcal{O}(N \log^2 N) \).
- The near-field of a cluster contains at most \( \mathcal{O}(\log^{2} N) \) elements, so that the evaluation of all near-field entries scales as \( \mathcal{O}(N \log^{2} N) \).

The FMM algorithm accomplishes the targeted numerical complexity of \( N \log^{2} N \) for computations on refined meshes for increasing frequencies.

5. Realization of BEM operators

The FMM BEM algorithm does not assemble the complete BEM matrices from Section 2, but allows a fast computation of the matrix–vector products of Eq. (14). That allows the solution using an iterative solver for the linear system of equations. The following section discusses how the discrete BEM operators \( K'_{h} \) and \( D_{h} \) are adopted to the FMM matrix–vector product.

One row of the matrix–vector product \( (K'_{h}g_{h})_{j} \) for the adjoint double layer potential at node \( j \) yields

\[
(K'_{h}g_{h})_{j} = NF + \sum_{i \in \mathcal{E}(j)} \sum_{n=1}^{N_{i}} \Delta_{i} \omega_{i} \varphi_{i/\tau}(x_{i,n}) n_{x,i_{n}} \cdot \nabla_{x,i_{n}} \sum_{k=1}^{N} q_{k,m} u^{*}(x_{k,m},x_{i,n})
\]  

where \( NF \) denotes the near-field and \( FF \) the far-field share. \( E(j) \) is the set of elements \( \tau \) belonging to node \( j \), \( \varphi_{i/\tau} \) the linear shape function restricted to the element \( \tau_{i} \), \( N_{i} \) is the number of quadrature points on an element, \( \Delta_{i} \) its Jacobian and \( \omega \) the integration weights. The term involving the fundamental solution \( u^{*} \) results in

\[
u^{*}(x_{k,m},x_{i,n}) = \frac{1}{4\pi} \frac{e^{i|x_{k,m}-x_{i,n}|}}{|x_{k,m}-x_{i,n}|}
\]

and is substituted by the far-field signature function (21). The intensity \( q_{k,m} \) of the source at point \( x_{k,m} \) is

\[
q_{k,m} = \Delta_{i} \omega_{k,m} (g_{h})_{i} \psi_{k}(x_{k,m}),
\]

using the constant shape function \( \psi_{k} \) on element \( \tau_{k} \). The normal derivative of the far-field share is considered during the recovery of the solution by applying the normal derivative on Eq. (20), yielding

\[
\frac{\partial \phi(x_{i,n})}{\partial n_{x,i_{n}}} = -\frac{k^{2}}{4\pi} \int_{\Omega} n_{x,i_{n}} \cdot e^{i|x_{s}-x_{i,n}|} N(s)ds.
\]

In the near-field, regularization techniques must be employed to compute the hyper-singular operator. However, in the far-field, i.e. in the multipole share, it can be computed directly. One row of the matrix–vector product \( (D_{h}u_{h})_{j} \) for the hyper-singular operator at node \( j \) yields

\[
(D_{h}u_{h})_{j} = NF + \sum_{i \in \mathcal{E}(j)} \sum_{n=1}^{N_{i}} \Delta_{i} \omega_{i} \varphi_{i/\tau}(x_{i,n}) n_{x,i_{n}} \cdot \nabla_{x,i_{n}} \left[ u^{*}(x_{k,m},x_{i,n}) \right]
\]

The normal derivative at the source point is considered when computing the far-field signature on the lowest level. Eq. (21) is replaced by its normal derivative

\[
F(s) = -ik \sum_{k=1}^{N_{c}} \sum_{m=1}^{N_{c}} n_{s,i_{m}} \cdot s e^{i(kz-s_{n,i_{m}})} q_{k,m}.
\]

After executing the FMM algorithm, the solution is recovered as in Eq. (27). The source strength \( q_{k,m} \) for
the hypersingular operator is computed as

\[ q_{k,m} = \Delta_k \omega_{k,m} \sum_{i \in K(k)} (u_h) \varphi(x_k, x_{k,m}), \]  

(30)

where \( K(k) \) are the nodes of element \( \tau_k \).

So far, the FMM algorithm provides a fast matrix-vector product for the discretized BEM operators. For the solution of Eq. (14), the Generalized Minimum Residual (GMRES) method is employed. Since the numerical efficiency of the multipole BEM algorithm depends heavily on the number of iterations required for the solution, pre-conditioning is mandatory. Steinbach and Wendland [17] propose pre-conditioners based on spectral equivalence properties of the integral operators in the BEM. For Neumann problems, as they are typical for applications in structural acoustics, the single layer potential is a suitable pre-conditioner. When using the discrete single layer potential \( V_h \), the pre-conditioner is scaled by \( M_h \) which corresponds to a standard mass matrix. The resulting pre-conditioner matrix is given by

\[ C = M_h V_h^{-1} M_h. \]  

(31)

For the application of the pre-conditioner, one application of the single layer potential and two applications of the inverse mass matrix are required. The single layer potential can be implemented using the multipole scheme as described before for the adjoint double layer potential and the hyper-singular operator. For the application of the inverse mass matrix a conjugate gradient iteration is implemented. Due to the single layer potential, the pre-conditioner doubles the computing cost per iteration. In return it guarantees an iteration count that is independent of the discretization, thus paying off for fine meshes as they are typical for the multipole BEM for acoustics in the medium-frequency range.

### 6. Numerical example

The FMM BEM as described before was implemented in C++ and the simulations presented in this section were performed on a standard desktop PC. As a numerical example, the acoustic radiation from an L-shaped domain is considered. The overall dimensions are \( 1 \text{ m} \times 1 \text{ m} \times 1 \text{ m} \) with an excavation of \( 0.5 \text{ m} \times 0.5 \text{ m} \times 1 \text{ m} \). The boundary conditions for the exterior Neumann problem are generated by applying a monopole source inside the domain and calculating the normal velocities on the surface of the L-shape. Fig. 4 shows the surface of the L-shape discretized with 2816 triangular elements and the surface velocity determined by a monopole source at 600 Hz located at \( 0.25 \text{ m} \times 0.25 \text{ m} \times 0.25 \text{ m} \).

The pressure values \( u_h^G \) of the monopole field. The \( L_2 \) Dirichlet error on the surface of the L-shape is defined as

\[ e^2 = \frac{\| u_h^G - u_h^G \|}{\| u_h^G \|}. \]  

(32)

In the first series of computations the convergence behavior of the multipole BEM algorithm is examined. The BE mesh is refined while the frequency is kept constant at \( f = 600 \text{ Hz} \), corresponding to a wavenumber of \( k = 2\pi f/c = 11.1 \text{ m}^{-1} \). The factor \( \lambda/h \) ranges from \( \lambda/h = 4.5 \) for the coarsest mesh with 704 elements to \( \lambda/h = 36 \) for the finest mesh with 45,056 elements. The pressure computed on the second mesh with 2816 elements is shown in Fig. 5.

The parameters of the multipole scheme are tuned in such a way that the error of the multipole BEM is...
near-field size as pointed out by Chew et al. [3]. For be noted that the achievable precision is limited by a small for the storage of the near-field matrices. However, it must the smallest possible, yielding low memory requirements for each dimension of the cluster. This near-field definition is compatible with the $N$-function and the condition number of the system of equations increases. The iteration count increases at higher frequencies, since the condition number of the system of equations increases. The 225 iterations at 1200 Hz are an exceptional high value, since the frequency is close to a resonance frequency of the corresponding interior problem where the hyper-singular operator exhibits a singularity as well for the interior as for the exterior problem. This situation can be circumvented by employing for example the Burton–Miller approach using a combination of the hyper-singular operator and the double layer potential. For the first three meshes, the computational cost for the pre-conditioner is about 5%. It is sufficient to increase the parameter $p$ defining the expansion length in Eq. (23) is set $p = 6$.

Table 2 The results of the FMM BEM simulations are documented. As expected, the numerical cost of a matrix–vector product is $C(N \log^2 N)$ as shown in Fig. 6. The iteration count increases at higher frequencies, since the condition number of the system of equations increases. The 225 iterations at 1200 Hz are an exceptional high value, since the frequency is close to a resonance frequency of the corresponding interior problem where the hyper-singular operator exhibits a singularity as well for the interior as for the exterior problem. This situation can be circumvented by employing for example the Burton–Miller approach using a combination of the hyper-singular operator and the double layer potential.

For the first three meshes, the computational cost for the iterative solution, i.e. for the multipole algorithm, increases quasi-linearly. It is sufficient to increase the parameter $p$ moderately. For the finest mesh, the target error of $10^{-3}$ is close to the best error achievable for the small near-field and low value of $kd$, thus, requiring a high value of $p$ leading to a rather high computational cost. In Fig. 6 the time per iteration is plotted against $N \log^2 N$ on logarithmic scales. Choosing $p = 14$ on the finest mesh results in approximately $C(N \log^2 N)$ computational cost, however, only yields a convergence ratio of 2.1 compared to the third mesh. Choosing $p = 16$ the error is reduced, but requiring a significantly higher time per iteration.

For the second series of simulations the ratio $\lambda/h$ is kept constant at nine elements per wavelength, i.e. the boundary element mesh is refined while the frequency increases. The frequency range from 300 to 2400 Hz corresponds to wavenumbers from $5.5 \, \text{m}^{-1}$ to $44 \, \text{m}^{-1}$. A constant ratio $\lambda/h$ guarantees a constant best approximation error and, for engineering practice, a constant BE error. For the computation on the L-shape, the choice of $\lambda/h = 9$ leads to an BE error of a few percent. The multipole parameters are chosen to give a total error of about 5%. In particular, the parameter $p$ defining the expansion length in Eq. (23) is set $p = 6$.

Table 1 Multipole BEM computations at 600 Hz

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<th>$t_{\text{near}}$ (s)</th>
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Fig. 6. Time per iteration for the FMM BEM algorithm.
Furthermore, other pre-conditioning techniques must be investigated that minimize the influence of the frequency on the iteration count.

7. Conclusion

For large scale models, standard BEM formulations are not applicable, since their numerical complexity scales as $N^2$. In the presented paper a FMM algorithm was derived for the Helmholtz equation and adopted for the symmetric Galerkin BEM. Estimates of the numerical complexity for the FMM, show that it allows to evaluate a matrix–vector product of the BEM with a computational cost of $N \log_2 N$ for simulations with a constant factor $l = h$ which is typical for engineering applications. This estimate was confirmed by the numerical example of sound radiation from an L-shaped domain. For mesh refinement at a constant frequency, the complexity is slightly higher, when increasing the expansion length to yield the convergence behavior of the BEM. For the solution of the system of equations GMRES with pre-conditioning by the single layer potential was employed, leading to iteration counts which are independent of the mesh size $h$. However, the number of iterations still depends considerably on the frequency, providing further research demands.

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References


Table 2

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<th>Mesh</th>
<th>Elements</th>
<th>Frequency (Hz)</th>
<th>$t_{\text{run}}$ (s)</th>
<th>Number of iterations</th>
<th>Time per iteration (s)</th>
<th>Ratio $c^2$</th>
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