Projection-based Polygonality Measurement
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Abstract—Measuring the degree to which a shape resembles a polygon (referred to as “polygonality”) is a difficult problem due to the intrinsic diversity in the form and distortion of shapes caused by digitization and similarity transformation. This paper proposes a generic approach for this problem by performing the measurement in the projection space where the Radon image of some primitive shapes which compose the shape becomes apparent. The obtained measures take value in \((0, 1]\) with \(1\) corresponds to analytical polygons. They are robust to additive noise, boundary distortion and is invariant to similarity transformation. The new framework generalizes existing polygonal measures such as triangularity and quadrangularity. In addition, the ability to estimate polygon’s geometric quantities in the projection space allows approximating a shape by analytical polygons. The efficiency of the proposed approach is demonstrated through a number of experiments on both synthetic and real datasets.

Index Terms—Polygonality measurement, Radon transform, geometric quantity estimation, polygonal approximation

I. INTRODUCTION

Object recognition is an important task in digital image analysis. For many years, a large amount of digital images has been created and there is a need for automated methods to classify objects that appear in images in order to better understand images’ contents. Since directly comparing objects using pixel intensities is of high complexity, object comparison is often carried out by mapping objects of interest onto feature spaces. The mapping operation generally relies on object’s appearance properties such as texture, color, and shape. Since shape information has high discrimination power, it has been widely used for object description and classification [1].

There exist two main directions in using shape information for recognition tasks. The first uses different types of transforms to describe shapes. Popular methods are based on Fourier transform [2], Hough and Radon transforms [3], [4], and image moments [5], [6]. Methods in this direction can provide high dimensional feature vectors and are suitable for generic applications such as shape retrieval [7]. The second direction measures or estimates geometric properties of shapes. Descriptors in this direction usually have clear geometric interpretation and can solve some specific problems such as circular separability [8], sigmoidality measurement [9], or curvature estimation [10]. For simplicity, methods in this direction are henceforth called shape measurement methods.

Even though a number of methods have been proposed, shape measurement remains a difficult problem in order to efficiently and rapidly characterize shapes’ geometric properties. Rosin [11] pointed out three main difficulties in defining shape measures. The first is the robustness of the measures against variation among shapes of the same class. The second is the insensitivity of the measures to similarity transformation such as translation, rotation, and scaling. And the last is the invariance of the measures to other transformations such as affine transformation on triangles or aspect ratio adjustment on rectangles. Since triangle, quadrangle, and other polygonal shapes often appear in images, there is a need for efficient methods to measure shape’s polygonality (i.e., the degree to which a shape is polygonal) and to estimate some geometric quantities (e.g., edge lengths and vertex angles) of the hypothetical polygon. Although methods do exist to measure the similarity between a shape and a specific polygonal type, to our knowledge there exists no generic approach for the polygonality measurement and geometric quantity estimation problems.

We present in this paper a novel and generic approach to measure the polygonality of shapes. Our approach exploits some beneficial geometric properties of polygons in the projection space. The main idea is to decompose a shape into a set of primitive shapes, each can be quantitatively characterized by using its projection at a certain direction. The obtained measures take value in \((0, 1]\) and are robust to additive noise, boundary distortion, and similarity transformation. We also present an approach to estimate in the projection space the edge lengths and vertex angles of polygons to facilitate the approximation of a shape by analytical polygons. It should be noted here that projection data have been used for some shape measurement problems such as geometrical feature extraction [12], straight line and curved shape primitive extraction [13], triangularity measurement [11], orientation [14] and convex-set perimeter [15] estimation. However, our approach exploits projection data in a totally different way and generalizes existing polygonal measures such as triangularity and quadrangularity. Our approach can be applied to complex shapes composed of several closed contours or shapes having very noisy boundaries. For these cases, it is well-known that boundary-based methods such as directly polygonalizing shapes’ boundaries by polygons or approximating shapes’ boundaries by bounding boxes cannot be used.

II. RELATED WORKS

There exist in the literature two main groups of methods for shape measurement. The first measures the similarity between a shape and a geometrically predefined shape such as a circle, an ellipse, a triangle, or a rectangle. And the second estimates some geometric properties of shapes such as orientation, symmetry, and rectilinearity.

A. Similarity measurement

1) Circularity: Circularity or compactness estimates how a shape is similar to a circle. In his pioneer work, Rosenfeld
[16] proposed to use the square of shape’s perimeter divided by its area, $P^2/A$, as a non-compactness measure. A related quantity, $4\pi A/P^2$, can also be used as a compactness measure. Other works define circularity based on different geometric properties of circles. For example, Haralick [17] considered the statistics of the random variable that represents the distance from the shape’s centroid to its perimeter; Proffitt [18] used the Fourier transform of an ordered sequence of boundary points; Stojmenovic and Nayak [19] exploited the linearity of the polar coordinates of boundary points when plotted in the Cartesian coordinates; Zunic et al. [20] used Hu moments; Fisk [8] and Roussillon et al. [21] took advantage of the separating circle problem; Nguyen et al. [22] measured the similarity between polygons; and Kim [23] utilized the shape’s convex hull.

2) Ellipticity: Ellipticity determines how far a shape is from an ellipse. Proffitt [18] fit an ellipse to the shape and then defined measure based on the Fourier transform of the distances between the ellipse’s and shape’s boundary points. Peura and Iivarinen [24] measured ellipticity by using elliptic variance. Tabbone et al. [25] instead used the $R$-transform, and Rosin [11] used affine moment invariants to measure ellipticity. Recently, a family of ellipticity measures, each of whom is distinguished by an axis length ratio, was proposed by Aktas and Zunic [26] to classify galaxy images.


4) Triangularity: Triangularity measures how a shape is similar to a triangle. Rosin [11] proposed a number of approaches to measure triangularity based on moments, triangle approximation, the horizontal and vertical projections of shapes, and the minimum-area bounding triangle.

B. Geometric property measurement

1) Orientation: Measuring shape’s orientation is an important topic in shape analysis [29]. The classical approach [30] defined orientation as the axis of the least second moment of inertia. Zunic et al. [31] presented a measure based on the degree that a shape has a dominant orientation. Zunic and Rosin [32] determined orientation by maximizing the total length of all line segments inside shape that have the same direction. The orientation of multi-component shapes can also be determined by additionally using boundary information [33]. Other approaches are based on curvature weighted gradient [34], boundary information [35], and image moments [36].

2) Symmetry: Symmetry detection determines whether a shape possesses reflective and rotational symmetries. Yip introduced a number of approaches using Hough or Fourier transform [37], [38], [39]. Xiao et al. [40] used the phase congruency calculated from Gabor wavelets to detect rotational symmetry. A method presented in [41] detects both types of symmetry using angular correlation computed from pseudopolar Fourier transform. And image moments can also be used for symmetry detection [36], [42].

3) Rectilinearity: Rectilinearity determines whether a shape is similar to a polygon whose corner angles are either $\frac{\pi}{3}$ or $\frac{2\pi}{3}$. Besides the sum of the differences of each corner angle from multiples of $\frac{\pi}{3}$, Zunic and Rosin [43] proposed two other measures by characterizing polygons using their perimeters in the sense of $\ell_1$ and $\ell_2$ metrics.

III. THEORETICAL FOUNDATIONS

A. Radon transform

Let $f \in \mathbb{R}^2$ be a 2D function and $L(\theta, \rho) = \{x \in \mathbb{R}^2 | x \cdot n(\theta) = \rho\}$ be a straight line in $\mathbb{R}^2$, where $\theta$ is the angle $L$ makes with the y axis, $n(\theta) = (\cos \theta, \sin \theta)$, and $\rho$ is the radial distance from the origin to $L$. The Radon transform [44] of $f$, denoted as $R_f$, is a functional defined on the space of lines $L(\theta, \rho)$ by calculating the line integral along each line as $R_f(\theta, \rho) = \int f(x) \delta(\rho - x \cdot n(\theta)) \, dx$. In the field of shape analysis and recognition, the function $f$ is constrained to take value 1 if $x \in D$ and 0 otherwise, where $D$ is the domain of the binary shape represented by $f$. Radon transform is robust to additive noise [45] and has some beneficial geometric properties that have been exploited in a number of pattern analysis and recognition problems [46]:

- Periodic with respect to $\theta$ with period 2$\pi$: $R_f(\theta, \rho) = R_f(\theta + 2\pi k, \rho)$, $\forall k \in \mathbb{Z}$.
- Semi-symmetric: $R_f(\theta, \rho) = R_f(\theta \pm \pi, -\rho)$.
- A translation of $f$ by a distance $x_0$ results in a shift in the variable $\rho$ of $R_f$: $R_f(\theta, \rho) \rightarrow R_f(\theta, \rho - x_0 \cdot n(\theta))$.
- A rotation of $f$ by an angle $\theta_0$ implies a circular shift in the variable $\theta$ of $R_f$: $R_f(\theta, \rho) \rightarrow R_f(\theta + \theta_0, \rho)$.
- A homogeneous scaling of $f$ by a factor $\alpha$ results in scalings in the variable $\rho$ and the amplitude of $R_f$: $R_f(\theta, \rho) \rightarrow \frac{1}{\alpha} R_f(\theta, \alpha \rho)$.

B. Notations and definitions

Let $f \in \mathbb{R}^2$ be a 2D function that represents a finite shape $D$. For each projection direction $\theta$, the radial distances $\rho_1^f$ and $\rho_2^f$ are respectively defined as $\rho_1^f = \inf \rho \mid R_D(\theta, \rho) > 0$ and $\rho_2^f = \sup \rho \mid R_D(\theta, \rho) > 0$. The “profile” of $D$ in the direction $\theta$, denoted as $C_D^\theta$, is defined as $R_D(\theta, \rho_1^f, \rho_2^f)$. More precisely, $C_D^\theta(\rho - \rho_1^f) = R_D(\theta, \rho)$, $\forall \rho \in [\rho_1^f, \rho_2^f]$. The two extremities in the boundary of $D$ that correspond to $\rho_1^f$ and $\rho_2^f$ are denoted as $E_1^\theta$ and $E_2^\theta$, respectively. An extremity is called “simple” if it is a point or a line segment. The boundary segments of $D$ that are limited by these two extremities are named $B_{12}^\theta$ and $B_{21}^\theta$. Figure 1a illustrates the value of $\rho_1^f$ and $\rho_2^f$, the position of $E_1^\theta$ and $E_2^\theta$, and the extend of $B_{12}^\theta$ and $B_{21}^\theta$ using a sample shape. The three possible configurations of simple extremities are given in Figures 1a–1c, and an example of non-simple extremities is provided in Figure 1d. Figure 2 illustrates two profiles of a dog shape. Since a shape that has non-simple extremities in a certain direction can always be decomposed into a set of shapes each of which has simple extremities in that direction, we hereafter consider only shapes that have simple extremities.

A shape $D$ is called “primitive in the direction $\theta$” (denoted as $\theta$-primitive) if and only if its extremities $E_1^\theta$ and $E_2^\theta$ are...
Theorem 1. If a shape $D$ is $\theta$-primitive then \( \exists k_1, k_2 \in \mathbb{R} \) such that \( k_1^2 + k_2^2 > 0 \) and \( \mathcal{R}_{\mathcal{D}}(\theta, \rho) = k_1 \rho + k_2, \forall \rho \in [\rho_1', \rho_2'] \).

Proof. Without loss of generality, we present here a proof for a simple $\theta$-primitive triangle as illustrated in Figure 5a. Since \( \mathcal{R}_{\mathcal{D}}(\theta, \rho) \) with \( \rho \in [\rho_1', \rho_2'] \) is the length of the line segment of direction $\theta$ that has ending points in $B_{12}^{\rho}$ and $B_{21}^{\rho}$ of $D$. Using the intercept theorem leads to
\[
\frac{\mathcal{R}_{\mathcal{D}}(\theta, \rho)}{\mathcal{R}_{\mathcal{D}}(\theta, \rho_1')} = \frac{\rho - \rho_2'}{\rho_2' - \rho_1'},
\]
then
\[
\mathcal{R}_{\mathcal{D}}(\theta, \rho) = \frac{\mathcal{R}_{\mathcal{D}}(\theta, \rho_1')} {\rho_2' - \rho_1'} \rho + \frac{\mathcal{R}_{\mathcal{D}}(\theta, \rho_1')} {\rho_2' - \rho_1'} \rho_2'.
\]

Corollary 2. If a shape $D$ is $\theta$-primitive, $C_{\mathcal{D}}^\theta$ is a line segment.

Lemma 3. If a convex polygon $\mathcal{P}$ can be decomposed into a set of $\theta$-primitive shapes, then each $\theta$-primitive shape cannot have a vertex of $\mathcal{P}$ in its $B_{12}^{\rho} \cup B_{21}^{\rho} \setminus \{E_1^\rho, E_2^\rho\}$.

Proof. Without loss of generality, assuming that $B_{12}^{\rho} \setminus \{E_1^\rho, E_2^\rho\}$ of a composing $\theta$-primitive shape $\mathcal{D}$ contains a vertex $P_i$ of $\mathcal{P}$. $B_{12}^{\rho}$ is then not straight since it is composed of at least two line segments from the two edges of $\mathcal{P}$ that meet at $P_i$. A non-straight $B_{12}^{\rho}$ violates the assumption that $\mathcal{D}$ is $\theta$-primitive.

Theorem 4. An $n$-sided polygon $\mathcal{P}$ can be decomposed into $n - 1$ maximally $\theta$-primitive shapes if there exist no line of direction
that passes through two vertices, and into a maximum of $n - 2$ maximally $\theta$-primitive shapes otherwise.

**Proof.** Let’s consider $E_1^\theta$ and $E_2^\theta$ of $\mathcal{P} = P_0 P_1 \ldots P_{n-1}$, they must be either edges or vertices of $\mathcal{P}$. Denoting $\mathcal{V} = \{P_i\}_{i=1}^{n-1} \setminus \{E_1^\theta, E_2^\theta\}$ then $|\mathcal{V}| = n - 2$ in the case of Figure 1a, $n - 3$ in Figure 1b, and $n - 4$ in Figure 1c. For each vertex $P_i$, its projection in the direction $\theta$ onto the boundary of $\mathcal{P}$ is denoted as $P_i$ (Figure 5b). Let $L$ be the line of direction $\theta$ passing through $E_1^\theta$ and $E_2^\theta$ and $\mathcal{T} = \{P_i\}_{i=1}^{n-1} \setminus \{E_1^\theta, E_2^\theta\}$ be an reordered version of $\mathcal{V}$ so that $d(P_{i_{1-1}}, L) \geq d(P_{i_{1}}, L)$, where $d(P_{i_{1}}, L)$ is the distance from $P_{i_{1}}$ to $L$. Considering now the $m + 1$ composing parts of $\mathcal{P}$ that are sandwiched between $P_{i_k}$ and $P_{i_{k-1}}$ and $P_{i_k}$, $P_{i_{k-1}}$ and $E_2^\theta$. It can be seen that when $d(P_{i_{1-1}}, L) = d(P_{i_{1}}, L)$, the composing part of $\mathcal{P}$ between $P_{i_{1-1}}$, $P_{i_{1}}$, and $P_{i_{k}}$ degenerates to a line segment of direction $\theta$ having ending points $P_{i_{1}}$ and $P_{i_{k}}$. For each non-degenerated composing part of $\mathcal{P}$, it is a maximally $\theta$-primitive shape due to Lemma 3. In addition, since $\max m = \max |\mathcal{V}| = n - 2$, the maximum number of maximally $\theta$-primitive shapes that compose $\mathcal{P}$ is $n - 1$. ■

**Corollary 5.** $C_{\theta}^\mathcal{D}$ of a polygonal shape $\mathcal{D}$ is composed of $N^\theta(\mathcal{D})$ line segments.

**Theorem 6.** For an $n$-sided polygon $\mathcal{P}$, $N(\mathcal{P}) = n - 1$.

**Proof.** From Theorem 4, $N^\theta(\mathcal{P}) \leq n - 1$. Since the number of pairs of vertices of $\mathcal{P}$ is $\binom{n}{2} = \frac{n(n-1)}{2}$ and the cardinality of $[0, \pi)$ is infinite, there exists an infinite number of projection directions that do not coincide with any of these $\frac{n(n-1)}{2}$ directions. Thus, $N(\mathcal{P}) = \max_{\theta \in [0, \pi)} N^\theta(\mathcal{P}) = n - 1$. ■

**Lemma 7.** Let $L_1$ and $L_2$ be the two line segments that connect the ending points of $B_{12}^\theta$ and $B_{21}^\theta$ of a shape $\mathcal{D}$. $L$ is a line of direction $\theta$ that cuts $B_{12}^\theta$, $B_{21}^\theta$, $L_1$, $L_2$ at $Q$, $Q'$, $Q''$, $Q'''$, respectively. If $C_{\theta}^\mathcal{D}$ is a line segment, then $|QQ'| = |Q'Q''|$.

**Proof.** Without loss of generality, let’s consider the shape $\mathcal{D}$ in Figure 5c. Since $C_{\theta}^\mathcal{D}$ is a line segment, $|QQ'| = R_{\mathcal{D}}(\theta | q)$ is linearly related with $\rho$. From the boundary conditions $R_{\mathcal{D}}(\theta | p_1) = P_0 P_1$ and $R_{\mathcal{D}}(\theta | p_2) = P_2 P_3$, we have $|QQ'| = \frac{n-\rho}{n-\rho} P_0 P_1 + \frac{n-\rho}{n-\rho} P_2 P_3$. In addition, the intercept theorem gives $|Q'Q''| = \frac{n-\rho}{n-\rho} P_0 P_1 + \frac{n-\rho}{n-\rho} P_2 P_3$. This means that $|QQ'| = |Q'Q''|$, or $QQ'$ and $Q'Q''$ have the same length. ■

**Corollary 8.** The total area of the regions enclosed by $L_1$ and $B_{12}^\theta$ is equal to that of the regions enclosed by $L_2$ and $B_{21}^\theta$.

**Corollary 9.** When $C_{\theta}^\mathcal{D}$ of a shape $\mathcal{D}$ is a line segment, $B_{12}^\theta$ is a line segment $\iff B_{21}^\theta$ is a line segment.

**Lemma 10.** For a shape $\mathcal{D}$, if $\sharp(C_{\theta}^\mathcal{D}) \geq 1$ and $B_{12}^\theta$ is composed of line segments then $B_{21}^\theta$ is also composed of line segments.

**Proof.** When $\sharp(C_{\theta}^\mathcal{D}) = k = 1$, the proof is trivial by using Lemma 7. For $k > 1$, $\mathcal{D}$ can be decomposed into $k$ pseudo $\theta$-primitive or $\theta$-primitive shapes $\{S_i\}_{i=1}^k$ so that $\sharp(C_{\theta}^\mathcal{D}) = 1$. For each $S_i$, since $B_{12}^\theta(S_i) \subset B_{12}^\theta(\mathcal{D})$, $S_i$ in turn can be also decomposed into a set of pseudo $\theta$-primitive or $\theta$-primitive shapes $\{S_{ij}\}$ such that $B_{12}^\theta(S_{ij})$ is a line segment. By applying Corollary 9 on each $S_{ij}$, $B_{21}^\theta(S_{ij})$ is a line segments. Thus, $B_{21}^\theta(\mathcal{D}) = \cup_{i,j} B_{21}^\theta(S_{ij})$ is composed of line segments. ■

**Lemma 11.** A shape $\mathcal{D}$ is polygonal if $\max_{\theta \in [0, \pi]} \sharp(C_{\theta}^\mathcal{D}) = k (k \geq 2)$ and $\exists \theta_0$ such that $E_{\theta_0}^\mathcal{D}$ or $E_{\theta_0}^{-\mathcal{D}}$ is a line segment.

**Proof.** Without loss of generality, suppose that $E_{\theta_0}^\mathcal{D}$ is a line segment as in Figure 5d. For any $\theta \neq \theta_0$, let $E_{\theta}^\mathcal{D}$ be the projection of $E_{\theta_0}^\mathcal{D}$ onto the boundary of $\mathcal{D}$ in the direction $\theta$. Since $E_{\theta_0}^\mathcal{D}$ is a line segment and $\sharp(C_{\theta_0}^\mathcal{D}) \leq k$, Lemma 10 requires that $E_{\theta}^\mathcal{D}$ is composed of line segments. By using a set of project directions $\{\theta_i\}_i$ such that $\cup_{i} E_{\theta}^\mathcal{D} \cup E_{\theta_0}^\mathcal{D}$ totally covers $\mathcal{D}$’s boundary, it can be seen that the boundary of $\mathcal{D}$ is composed of line segments. Thus, $\mathcal{D}$ is a polygon. ■

**Lemma 12.** When $\max_{\theta \in [0, \pi]} \sharp(C_{\theta}^\mathcal{D}) = n - 1 (n \geq 3)$, a shape $\mathcal{D}$ can always be decomposed into a triangle and a shape $\mathcal{D}'$ such that $\max_{\theta \in [0, \pi]} \sharp(C_{\theta}^{\mathcal{D}'}) \leq n$.

**Proof.** If $\exists \theta_0$ such that $E_{\theta_0}^\mathcal{D}$ or $E_{\theta_0}^{-\mathcal{D}}$ is a line segment, $\mathcal{D}$ is then an $n$-sided polygon due to Lemma 11 and the proof becomes trivial. Consider now the direction $\theta_0$ where $E_{\theta_0}^\mathcal{D}$ is a simple point $P_2$ as in Figure 5e. Since $\sharp(C_{\theta_0}^\mathcal{D}) \leq n - 1$, $\mathcal{D}$ can be decomposed into a maximum of $n - 1$ regions $\{R_{k_0}^\mathcal{D}\}_k$ such that $\sharp(C_{\theta_0}^\mathcal{D}) = 1$ by cutting it using lines of direction $\theta_0$. Assuming that $E_{\theta_0}^\mathcal{D} \subset R_1^\mathcal{D}$ and let $\mathcal{D}' = \cup_{k>1} R_{k_0}^\mathcal{D}$. The condition $\sharp(C_{\theta_0}^\mathcal{D}) = 1$ requires that $R_1^\mathcal{D}$ be either $\theta_0$-primitive or pseudo $\theta_0$-primitive. If $R_1^\mathcal{D}$ is $\theta_0$-primitive, it is a triangle and $\max_{\theta \in [0, \pi]} \sharp(C_{\theta}^{\mathcal{D}'}) = 2$. In addition, because the difference between two piece-wise linear functions is also a piece-wise linear function, it is not difficult to show that $\max_{\theta \in [0, \pi]} \sharp(C_{\theta}^{\mathcal{D}'}) \leq n$ and the proof is done.

When $R_1^\mathcal{D}$ is pseudo $\theta_0$-primitive (illustrated as the shape...
$P_0P_1P_2$ in Figure 5e), the condition $\|C^q_{R_0}\| = 1$ additionally requires that the boundary of $D$ is not differentiable at $P_2$. This means that there exists an infinite number of tangent lines of $D$ at $P_2$. Let’s consider one of these lines that has direction $\theta \neq \theta_0$, $P_2$ is thus $E^q_{R_0}$. Again, $D$ can be cut into a maximum of $n - 1$ regions $\{R^q_{i,k}\}$ by using lines of direction $\theta$ where $R^q_{i,k}$ contains $P_2$. Without loss of generality, supposing that $R^q_{i,k}$ is the shape $P^q_{i,k}P_2$. For any two points $X$ and $Y$ on the boundary of $D$, denoting $\mathcal{A}(D, XY)$ as the area of the region enclosed by the boundary segment $XY$ of $D$ and the line segment $XY$. Assuming that $P_0P_2$ first cuts $P_0P_2$ and $P_0P_2$ at $Q$ and $Q'$, respectively (Figure 5e). Due to Lemma 7, $P_0P_2$ must cut $P_1P_2$ at $Q = Q'$ such that $\mathcal{A}(D, P_0Q) = \mathcal{A}(D, P_1Q)$ and $\mathcal{A}(D, P_0Q') = \mathcal{A}(D, P_1Q')$, leading to $\mathcal{A}(D, P_0Q) = \mathcal{A}(D, P_0Q')$. However, since $\theta \neq \theta_0$ then $\mathcal{A}(D, P_0Q) > \mathcal{A}(D, P_0Q')$, the aforementioned assumption is invalid and $P_0P_2$ cannot cut $P_2$ as illustrated in Figure 5f. In this case, using Corollary 8 leads to $\mathcal{A}(D, P_0Q) = \mathcal{A}(D, P_1Q') = \mathcal{A}(D, P_2Q')$, meaning that $P_0P_2 \equiv P^q_{0}P_2 \equiv P^q_{0}P_2 \equiv P^q_{0}P_2$ and $P_0P_2 \equiv P_2P_2$. In other words, $P_0P_1P_2$ is a triangle and the proof is done.

**Theorem 13.** A shape $D$ is an $n$-sided polygon $(n \geq 3)$ if $\max_{\theta \in [0, \pi]} \frac{\|C^q_{D}\|}{\|C^q_{\theta}\|} = n - 1$.

**Proof.** Lemma 12 allows decomposing $D$ into a triangle $T$ and a shape $D'$ where $\max_{\theta \in [0, \pi]} \frac{\|C^q_{D'}\|}{\|C^q_{\theta}\|} \leq n$. Since the boundary segment of $D'$ that is common with $T$ is straight, $D'$ is a polygon due to Lemma 11. Thus, $D = T \cup D'$ is also a polygon. The condition $\max_{\theta \in [0, \pi]} \frac{\|C^q_{D}\|}{\|C^q_{\theta}\|} = n - 1$ together with Theorem 6 require that $D$ has $n$ sides.

**IV. POLYGONALITY MEASUREMENT**

The theoretical framework developed in the previous section can be used to determine how a shape $D$ is similar to an $n$-sided polygon $P$. Based on Corollary 5 and Theorems 6, 13, we will compute the polygonal measure for $D$ by relying on the assumption that the more $D$ is similar to a polygon, the more its $C^q_D$ profile looks like a set of line segments. Thus, the polygonal measure will be defined based on the error in the polygonalization of $C^q_D$. It should be noted here that by using the first two properties of the Radon transform in Section III-A, we only need to perform the projection at $\theta \in [0, \pi]$.

**A. $C^q_D$ polygonalization**

A number of polygonalization methods have been proposed in the literature by solving one of two popular problems. The $\min(\varepsilon)$ problem approximates a curve by a polygon of a fixed number of vertices with a minimal error $\varepsilon$. And the $\min(\tilde{\varepsilon})$ problem looks for a polygon of least number of vertices $\tilde{\varepsilon}$ with a given upper-bound approximation error. Since finding solutions to these global optimization problems is often computationally expensive, sub-optimal solutions are often used in practice.

The divide-and-conquer strategy [47] is employed to solve the $\min(\varepsilon)$ problem in order to polygonize $C^q_D$. For simplicity, the approximation error is defined as the maximal distance between a point on $C^q_D$ and its projection onto the approximated polygon of $C^q_D$ as illustrated in Figure 6a. Given a curve $P_0P_1$ ended by $P_0$ and $P_1$, a point $P_2$ on $P_0P_1$ is determined such that its perpendicular distance to the line passing through $P_0$ and $P_1$ is maximal. The new vertex $P_3$ and the corresponding distance are then denoted as $V(P_0P_1)$ and $e(P_0P_1)$, respectively. This vertex-determining process is recursively performed on the two sub-curves $P_0P_2$ and $P_2P_1$ until the maximal distance of all sub-curves is less than a threshold. The two initial ending points and all the determined vertices together form the approximated polygon of $C^q_D$. The above procedure is listed in Algorithm 1.

![Figure 6. Polygonalization of $C^q_D$](image)

**Algorithm 1 Polygonalization of $C^q_D$.**

**Input:** $C^q_D$ – curve $\mathcal{C} = P_aP_b$, $n$ – expected number of line segments $(n \geq 2)$, $\varepsilon$ – maximal accepted error

**Output:** $sList$ – list of line segments, $\{e_m(\zeta)\}_{m=2}^n$ – maximal errors in approximating $\zeta$ by $m$ line segments, $\zeta_m(\zeta)$ – number of necessary line segments to represent $C^q_D$ with $\varepsilon$

**Initialization:**

1. $pList = \{V(P_0P_1)\}$, $eList = \{e(\tilde{P}_0\tilde{P}_1)\}$, $sList = \{(P_a, P_b)\}$

**Incremental polygonalization:**

1. for $m=2$ to $n$ do
2. Determine $e_M = \max(eList)$, $i_M = \arg\max(eList)$
3. Get $P_2 = pList(i_M)$, $(P_0, P_3) = sList(i_M)$
4. Determine $V(P_0P_2)$ and $e(P_0P_2)$ from $P_0P_2$; $V(P_2P_1)$ and $e(P_2P_1)$ from $P_2P_1$
5. Add $V(P_0P_2)$ and $V(P_2P_1)$ to $pList$; $e(P_0P_2)$ and $e(P_2P_1)$ to $eList$; $(P_0, P_2)$ and $(P_2, P_1)$ to $sList$
6. $e_m(\zeta) = e_M$
7. end for

**Determination of the number of line segments:**

1. Find $k \in [2, n-1]$ such that $(e_k(\zeta) - \varepsilon)(e_{k+1}(\zeta) - \varepsilon) < 0$
2. $\zeta_m(\zeta) = k$

By relying on Theorem 13, the quality of approximating $C^q_D$ by $n - 1$ line segments is assessed in order to help determining the similarity between $D$ and an $n$-sided polygon $P$. It is evident from Corollary 5 and Theorem 6 that the smaller the approximation error is, the more similar $D$ and $P$ are. Considering Figure 6b where $P_0P_1$ is a segment of $C^q_D$ that has been approximated by a line segment $P_0P_1$. By denoting $A(P_0P_1, P_1P_1)$ the area of the region enclosed by $P_0P_1$ and $P_1P_1$ and $A(H_kH_{k+1}P_1P_1)$ the area of the trapezoid $H_kH_{k+1}P_1P_1$, the error in the approximation of $P_0P_1$ by $P_0P_1$ is defined as $q_k = \frac{A(P_0P_1, P_1P_1)}{A(H_kH_{k+1}P_1P_1)}$. Let
Algorithm 2 Similarity between a shape $\mathcal{D}$ and an arbitrary $n$-sided polygon.

**Input:** $\mathcal{D}$ — arbitrary shape, $n$ — number of sides, $\Theta$ — set of projection directions  
**Output:** $M_n(\mathcal{D})$ — polygonal measure

```
1: for $\theta_i \in \Theta$ do
2:    Polyomalize $C^\theta_D$ by $n-1$ line segments (Algorithm 1)
3:    for $j = 2 \ldots (n-1)$ do
4:        Calculate $q_j$ and $w_j$ as in Section IV-A
5:    end for
6:    $Q_n(C^\theta_D) = \sum_{i=2}^{n-1} q_i w_i$
7: end for
8: $M_n(\mathcal{D}) = 1 - \frac{1}{|\Theta|} \sum_{\theta_i \in \Theta} Q_n(C^\theta_D)
```

$w_i = \frac{H_{i+1}}{\sum_{i=1}^{n-1} H_i H_{i+1}}$ be the weighting factor for the line segment $P_i P_{i+1}$. The error in the polygonalization of $C^\theta_D$ is then defined as $Q_n(C^\theta_D) = \sum_{i=1}^{n-1} w_i q_i$, i.e., the weighted sum of all errors obtained when approximating $P_i P_{i+1}$ by $P_i P_{i+1}$ ($i = 1, \ldots, n-1$).

B. Polygonal measurement of shapes

1) Definitions: It can be seen that $Q_n(C^\theta_D)$ defined above takes values in the range $[0, 1]$ with 0 corresponds a perfect polygonalization. $Q_n(C^\theta_D)$ can be used to measure the similarity between a shape $\mathcal{D}$ and an arbitrary $n$-sided polygon. In addition, the theoretical results in Section III-C suggest that we need to combine $Q_n(C^\theta_D)$ at a set of projection directions $\Theta = \{\theta_i | \theta_i \in [0, \pi]\}$, in order to have reliable measures defined as follows.

**Definition 14.** Let $Q_n(\mathcal{D}) = \{Q_n(C^\theta_D)\}_{\theta_i \in \Theta}$ be the set of polygonalization errors for a shape $\mathcal{D}$. The similarity between $\mathcal{D}$ and an n-sided polygon is defined as $M_n(\mathcal{D}) = 1 - \frac{1}{|\Theta|} \sum_{\theta_i \in \Theta} Q_n(C^\theta_D)$.

From the properties of polygonalization, it can be seen that $M_n(\mathcal{D})$ takes value $\sim 1$ when $\mathcal{D}$ is a k-sided polygon, denoted as $\mathcal{P}_k$, $\forall k \in [3, n]$. In other words, $M_n$ views a $\mathcal{P}_k$ as an $\mathcal{P}_n$. In order to overcome this ambiguity, we rely on the following observation to determine $n$: if $\mathcal{D}$ is similar to a polygon, its order (i.e., the number of vertices) is the integer $n$ such that $M_{n+1}(\mathcal{D}) \approx 1$ and $M_n(\mathcal{D}) \ll 1$.

**Definition 15.** For a shape $\mathcal{D}$, let $\varepsilon$ be maximal accepted error in approximating each segment $P_i P_{i+1}$ of $C^\theta_D$ by $P_i P_{i+1}$. Let $\mathcal{N}_\varepsilon(\mathcal{D}) = \{n_{\theta_i} = \varepsilon(C^\theta_D)\}_{\theta_i \in \Theta}$ be the set of the numbers of line segments obtained from the polygonalization of $(C^\theta_D)_{\theta_i \in \Theta}$. The order of the approximated polygon of $\mathcal{D}$ is defined as $n_{\varepsilon}(\mathcal{D}) = \max n_{\theta_i}(\mathcal{N}_\varepsilon(\mathcal{D}) + 1)$.

Two algorithms are introduced to measure the polygonality of a shape $\mathcal{D}$. Algorithm 3 first determines the polygonal order $n = n_{\varepsilon}(\mathcal{D})$ as in Definition 15 and then calls Algorithm 2 to compute the similarity score between $\mathcal{D}$ and a $\mathcal{P}_n$.

2) Properties:

- Robustness to additive noise and distorted boundary: Due to the use of Radon transform, a degradation to shapes such as salt & pepper noise addition or boundary distortion only results in a small change in their $C^\theta_D$ profiles. As an example, Figure 7 shows that $C_{\mathcal{D}}$ only changes slightly when the triangle boundary is distorted. Thus, the proposed polygonal measures are robust to small degradation.

- Robustness to similarity transformation: From the last three properties of Radon transform in Section III-A, the $C^\theta_D$ of a function $f$ can be shown to have the following properties.
  - A translation of $f$ by a distance $x_0$ does not change $C^\theta_{f(x_0)}$.
  - A rotation of $f$ by an angle $\theta_0$ results in a shift in the index $\theta$ of $C^\theta_{f(x_0)}$ by a distance $\theta_0$: $C^\theta_{f(x_0)}(\rho) \rightarrow C^\theta_{f(x_0)}(\theta + \theta_0)$.
  - A scaling of $f$ by factor $\alpha$ results in the scaling in the variable $\rho$ and the amplitude of $C^\theta_{f(x_0)}(\rho)$.

In order to make the proposed polygonal measures invariant to rotation, we consider $C^\theta_D$ at equidistant samples of $\theta \in [0, \pi]$. Scaling invariance is obtained by a simple normalization in the definition of $q_i$ in Section IV-A. Thus, the proposed polygonal measures are invariant to translation, rotation, and scaling.

C. Dealing with noisy shapes embedded in noisy background

Our approach is region-based and possesses distinct capability in dealing with noisy shapes embedded in noisy background that boundary-based approaches cannot easily have. This is because sophisticated means are often needed to segment contours that represent noisy shapes from noise background, while some simple operations to segment the shapes out of the background seem sufficient for our approach. Even thought the segmented shapes usually have distorted boundary and additive salt & pepper noise, our approach is robust to these types of degradation as presented in the previous section. This simple extension is illustrated in Figure 8 where a closing with a disk-based structuring element followed by a simple thresholding are applied to the noisy image in Figure 8a (SNR=1). Our approach can be applied directly to the image in Figure 8c.
Algorithm 4 Geometric quantity estimation.

Input: \( \mathcal{P} \) – input polygon, \( n \) – number of sides, \( \Delta \rho \) – averaging width

Output: \( \mathcal{L} \) – set of edge lengths, \( \mathcal{A} \) – set of vertex angles

1: Compute \( B(\theta) = \text{mean}_{\rho \in [\rho_1^\theta, \rho_1^\theta + \Delta \rho]} \mathcal{R}_\mathcal{P}(\theta, \rho) \)
2: Find \( \Theta = \{ \theta_i \mid \frac{\partial B}{\partial \theta} = 0, \frac{\partial^2 B}{\partial \theta^2} < 0 \} \)
3: if \( |\Theta| \geq n \) then
4: Sort \( \Theta \) incrementally
5: Remove the first \( |\Theta| - n \) elements from \( \Theta \)
6: \( \mathcal{L} = \{ B(\theta_i) \mid \theta_i \in \Theta \} \)
7: \( \mathcal{A} = \{ \pi - (\theta_{i+1} - \theta_i) \mid \theta_i \in \Theta \} \) with \( \theta_{n+1} = \theta_1 + 2\pi \)
8: else
9: Warning: \( \mathcal{P} \) has less than \( n \) sides
10: end if

V. GEOMETRIC QUANTITY ESTIMATION

Assuming that a shape \( \mathcal{D} \) has been classified as an \( n \)-sided polygon \( \mathcal{P} \) using the approach presented in Section IV, this section shows that the edge lengths and vertex angles of \( \mathcal{P} \) can also be estimated directly in the projection space. Note that if \( \theta \) is the direction of one edge of \( \mathcal{P} \), the edge must be \( E_1^\theta \) or \( E_2^\theta \) of \( \mathcal{P} \) and \( \mathcal{R}_\mathcal{P}(\theta, \rho_1^\theta) \geq 0 \) or \( \mathcal{R}_\mathcal{P}(\theta, \rho_2^\theta) \geq 0 \). On the contrary, when \( \theta \) is not the direction of any edge of \( \mathcal{P} \), \( E_1^\theta \) and \( E_2^\theta \) are then simple points representing two vertices of \( \mathcal{P} \), meaning that \( \mathcal{R}_\mathcal{P}(\theta, \rho_1^\theta) \simeq 0 \) and \( \mathcal{R}_\mathcal{P}(\theta, \rho_2^\theta) \simeq 0 \). These observations lead to the following two propositions that will be used to estimate the edge lengths and vertex angles of \( \mathcal{P} \).

- A direction \( \theta \) coincides with the direction of an edge of \( \mathcal{P} \) if and only if \( \mathcal{R}_\mathcal{P}(\theta, \rho_1^\theta) \geq 0 \) or \( \mathcal{R}_\mathcal{P}(\theta, \rho_2^\theta) \geq 0 \). The positive projection value is the edge length, if it exists.
- There exists a set of exactly \( n \) directions \( \{ \theta_i \in [0, \pi] \}_{i=1}^n \) such that \( \mathcal{R}_\mathcal{P}(\theta_i, \rho_1^\theta) \geq 0 \) or \( \mathcal{R}_\mathcal{P}(\theta_i, \rho_2^\theta) \geq 0 \), \( \forall i \in [1, n] \). In addition, the set \( \mathcal{A} = \{ \pi - (\theta_{i+1} - \theta_i) \}_{i=1}^n \) where \( \theta_{n+1} = \theta_1 + 2\pi \) defines the \( n \) vertex angles of \( \mathcal{P} \).

These propositions pave the way for an algorithm to estimate the edge lengths and vertex angles of an arbitrary polygon (Algorithm 4). The basic idea is to consider \( \mathcal{R}_\mathcal{P}(\theta, \rho_1^\theta) \) as a function of \( \theta \) and then detect its local maxima. In practice, \( \mathcal{R}_\mathcal{P}(\theta, \rho_1^\theta) \) is computed as \( \text{mean}_{\rho \in [\rho_1^\theta, \rho_1^\theta + \Delta \rho]} \mathcal{R}_\mathcal{P}(\theta, \rho) \) to make it robust to noise, where \( \Delta \rho = 4 \) is determined experimentally. Figure 9 illustrates geometric quantity estimation using a quadrangle \( \mathcal{P} \). The values of the four highest local maxima in the profile \( \mathcal{R}_\mathcal{P}(\theta, \rho_1^\theta) \) in Figure 9b represent the four edge lengths of \( \mathcal{P} \) in Figure 9a. The distance between each two neighboring maxima is the angle at the vertex formed by the corresponding two edges.

Now if we have a shape \( \mathcal{D} \) and want to approximate it by an analytical \( n \)-sided polygon, we can simply construct this polygon by detecting exactly \( n \) highest local maxima from the \( \mathcal{R}_\mathcal{P}(\theta, \rho_1^\theta) \) profile and then estimating its edge lengths and vertex angles using Algorithm 4. Since \( n \) is an input parameter and can take any value \( \geq 3 \), the approximated polygon can have any number of sides. In other words, we can construct a set of analytical polygons of different number of sides and each polygon in the set is an approximation of \( \mathcal{D} \).

VI. EXPERIMENTATIONS

A number of experiments have been carried out to validate the performance of our polygonal measurement and geometric quantity estimation methods on some synthetic and real datasets. This section shows that the proposed polygonal measures can be computed at a relatively coarse angular sampling step. They can also be used as features in shape classification problems.

A. Behaviors of the proposed measures on synthetic shapes

A number of synthetic shapes have been designed to investigate the behaviors of the proposed polygonal measures in the presence of deformation. For simplicity, we only give results for two simple measures, triangularity and quadrangularity, obtained by using \( n = 3, 4 \) as the input to Algorithm 2, respectively. For polygons of a higher number of sides such as pentagons and hexagons, the behaviors of the proposed measures on them have been observed to be similar. Two types of deformation on the shape boundary are considered: spike addition (Figure 10) and noise addition (Figure 11). In these figures, the caption under each shape contains its measured values for triangularity (left) and quadrangularity (right).

Figure 10 shows that the proposed measures decrease when more spikes are added to the shape boundary. Similarly in
Figure 11. Triangularity and quadrangularity on synthetic shapes with additive noise along their boundary. The shapes in the bottom row are from [28]. For each shape type, the measured values decrease when more noise is added to the shape boundary, except for triangularity on non-triangular shapes.

Figure 11, these measures also take lower values when the shape boundary is degraded with additive noise of higher intensity. However, it can be seen that the impact of these deformations on the proposed measures is not significant. This is because the proposed measures do not change substantially and still keep their discriminative power when the shape boundary is deformed. In addition, the results in these figures also confirm the observation in Section IV that an $n$-sided polygonal measure of an $m$-sided polygon takes a value close to 1 for $m \leq n$ and a much lower value for $m > n$. As an example, quadrangularity of a triangle gives 0.989 (Figure 10a) whereas triangularity of a quadrangle gives only 0.626 (Figure 10c).

B. Behaviors of the proposed measures on real shapes

The proposed polygonal measures have also been evaluated using datasets that contain object shapes segmented from real images. Quadrangularity is calculated on shapes in the demids and fish datasets (Figures 12 and 13) whereas triangularity is calculated on shapes from the Sharvit’s dataset [48] (Figure 14). The calculated quadrangularity measures are given in the caption below each shape; and those for triangularity are encapsulated in Table I. Assuming that each shape is represented by its quadrangularity or triangularity, it can be seen that the inner distance between shapes of the same class is sufficiently smaller than the intra distance between shapes from different classes. Thus, the proposed polygonal measures can be used as features in shape recognition problems.

C. Influence of the angular sampling step

In order to investigate the influence of $\theta$ sampling step in Radon transform to the proposed polygonal measures, we calculated triangularity and quadrangularity for the four shapes in the last row of Figure 11 at a number of sampling steps: $\Delta \theta = 1, 2, 4, 8, 16, 32$ (degree). The obtained values are given in Tables II and III for triangularity and quadrangularity, respectively. It can be seen that the proposed polygonal measures do not change very much when $\Delta \theta$ increases from 1

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<th>Row 2</th>
<th>Row 3</th>
<th>Row 4</th>
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Figure 12. Quadrangularity of some desmids. The inner distance between shapes of the same class is sufficiently smaller than the intra distance between shapes from different classes.

Figure 13. Quadrangularity of some fishes. The inner distance between shapes of the same class is sufficiently smaller than the intra distance between shapes from different classes.

Figure 14. Some shapes from the Sharvit’s dataset.
### Table II

**Triangularity at Different Sampling Steps.**

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<th>Shape</th>
<th>$\Delta \theta$ (degree)</th>
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<td>Figure 11k</td>
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### Table III

**Quadrangularity at Different Sampling Steps.**

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D. Accuracy of geometric quantity estimation

Two synthetic datasets of 100 triangles $T = \{T_k\}_{k=1}^{100}$ and 100 quadrangles $Q = \{Q_k\}_{k=1}^{100}$ have been created with random vertex angles and edge lengths as illustrated in Figure 15. For each triangle $T_k$, let $A_k = \{a_i^k\}_{i=1}^{n=3}$ and $L_k = \{l_i^k\}_{i=1}^{n=3}$ be the sets that contain its actual vertex angles and edge lengths. The geometric quantities of all triangles are then estimated using Algorithm 4 in Section V. The estimated values of these quantities are respectively stored in the sets $\tilde{A}_k = \{\tilde{a}_i^k\}_{i=1}^{n=3}$ and $\tilde{L}_k = \{\tilde{l}_i^k\}_{i=1}^{n=3}$ for the triangle $T_k$.

![Figure 15. Sample images from the synthetic triangle and quadrangle datasets.](image)

Assuming the values in each of the sets $A_k$, $L_k$, $\tilde{A}_k$, $\tilde{L}_k$ where $k = 1, \ldots, 100$ have been sorted incrementally, we use the following metrics to evaluate the accuracy of the geometric quantity estimation algorithm for the triangle dataset:

$$
\varepsilon_a^T = \frac{1}{100} \sum_{k=1}^{100} \max_i \{|a_i^k - \tilde{a}_i^k|\}, \quad \varepsilon_l^T = \frac{1}{100} \sum_{k=1}^{100} \max_i \{|l_i^k - \tilde{l}_i^k|\}.
$$

The above procedure is repeated to compute $\varepsilon_a^Q$ and $\varepsilon_l^Q$ for the quadrangle dataset with $n = 4$. The values of $\varepsilon_a^T$, $\varepsilon_l^T$, $\varepsilon_a^Q$, and $\varepsilon_l^Q$ are given in Table IV. It can be seen that the geometric quantities of triangles and quadrangles have been accurately estimated with only $\sim 1\%$ estimation error. This means that the proposed algorithm is reliable and can be used to estimate the vertex angles and edge lengths of polygonal shapes.

### Table IV

**Geometric Quantity Estimation Errors.**

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<tr>
<th></th>
<th>$\varepsilon_a^T$</th>
<th>$\varepsilon_l^T$</th>
<th>$\varepsilon_a^Q$</th>
<th>$\varepsilon_l^Q$</th>
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E. Noisy polygon classification

In order to evaluate the usefulness of the proposed polygonal measures in polygonal classification problems, a synthetic dataset of 100 polygons has been created with random number of sides (from 3 to 6), vertex angles, and edge lengths. These synthetic polygonal images are then degraded by additive noise of three different levels to obtain three datasets of noisy polygonal shapes. Some images from these datasets are given in Figure 16. Algorithm 3 is used to compute the polygonal measures, which are then thresholded by the value 0.9 to classify these noisy polygons into their polygonal types. The polygonal measures for images in Figure 16 are given in Table V. The classification rates on the three datasets are 98%, 94%, and 89%, respectively. Thus, our proposed polygonal measures can be used as features in polygonal classification problems. To facilitate comparison with an existing projection-based approach, we program Rosin’s triangularity measurement method [11, Section 4.3] and give the calculated measures for the shapes in Figure 16 in the last column of Table V. It can be seen that even though Rosin’s triangularity measure is also robust to noise, it seems less effective than our method in pattern classification problems because it returns high score for some non-triangular shapes (i.e., Figures 16c, 16f, 16g, 16k).

F. Shape classification

We investigate in this experiment the applicability of the proposed polygonal measures in real shape classification...
problems by using the MPEG-7 dataset [1], which contains 70 classes with 20 shape images in each class. Figure 17 shows some representative shapes from the MPEG-7 dataset. We compute the first 18 polygonal measures, from triangularity (3-sided polygon) to icosagonality (20-sided polygon), from each shape and use them in the shape’s feature vector. The training set contains the first 19 shapes of each class, and the testing set contains the remaining one shape per class. Neighborhood classifier is used for simplicity. Table VI provides the classification rates obtained at different feature vector size (i.e., different number of polygonal measures in the feature vector). It can be seen that, while using only triangularity measure results in a low classification rate (15.7%), a combination of polygonal measures leads to a relatively high classification rate, reaching 72.9% by using only the first 14 measures and a simple classifier. These results demonstrate that our proposed polygonal measures can be used to describe shapes in classification problems. It should also be noted that calculating the first $k$ polygonal measures has practically the same complexity as calculating the $k^{th}$ polygonal measure only (see Algorithm 1). In practical applications, we suggest to use about 10 first polygonal measures in the shape’s feature vector to balance performance and complexity.

### VII. Conclusions

We have presented in this paper a generic framework which exploits some beneficial geometric properties of polygon in the projection space to measure the polygonality of shapes. Based on this framework, a family of polygonal measures that takes triangularity and quadrangularity as special cases has been proposed. The proposed measures take value in $[0, 1]$. They are robust to additive noise, boundary distortion and is invariant to similarity transformation. In addition, the proposed framework allows estimating the edge lengths and vertex angles of polygons, thus permits approximating a shape by analytical polygons. The efficiency of the proposed approach in measuring polygonality and in estimating geometric quantities of shapes has been demonstrated through a number of experiments. This framework might also be used for some particular quadrangles such as rectangle and parallelogram and for other problems such as polygon detection [49]. Future work will explore further in this direction and investigate the utility of the proposed approach in a number of applications [50], [51].

### Acknowledgements

We are grateful to Dr. Paul L. Rosin for kindly providing us the desmids and fish datasets and to Dr. Antoine Manzanera for fruitful discussions.

### References


**Table V**

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**Table VI**

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![Figure 17. Some representative shapes from the MPEG-7 dataset.](image-url)