

Convergence of approximations of stochastic optimization problems subject to measurability constraints.

Pierre Carpentier and SOWG¹

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¹**Systems and Optimization Working Group:**

K. Barty, J.-P. Chancelier, G. Cohen, M. de Lara, A. Dallagi & C. Strugarek



Outline of the presentation

- 1 Introduction
- 2 Counterexample
- 3 Convergence theorem
- 4 Conclusions



- 1 Introduction
 - Problem statement
 - Strong convergence topology of σ -fields
 - A two-stage convergence result
- 2 Counterexample
- 3 Convergence theorem
- 4 Conclusions



Prototype Problem

Stochastic optimization problem under consideration:

$$V(\xi, \mathcal{F}) = \min_{\mathbf{u} \in L^2(\Omega, \mathcal{A}, \mathbb{P}; U)} \mathbb{E}[j(\mathbf{u}, \xi)] , \quad (1a)$$

$$\text{subject to } \mathbf{u} \text{ } \mathcal{F}\text{-measurable} . \quad (1b)$$

- $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space,
- ξ a random variable on Ξ (**noise**),
- \mathbf{u} a random variable on U (**control**),
- \mathcal{F} a sub σ -field of \mathcal{A} (**observation**, usually generated by \mathbf{y}).



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$$\min \mathbb{E} \left[\sum_{t=0}^{T-1} L_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \xi_{t+1}) + K(\mathbf{x}_T) \right] ,$$

$$\text{subject to } \begin{cases} \mathbf{x}_0 &= f_0(\xi_0) \\ \mathbf{x}_{t+1} &= f_{t+1}(\mathbf{x}_t, \mathbf{u}_t, \xi_{t+1}) \end{cases} ,$$

$$\mathbf{u}_t \text{ } \sigma(\xi_0, \dots, \xi_t)\text{-measurable} .$$



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$$\text{subject to } \mathbf{u} \text{ } \mathcal{F}\text{-measurable} . \quad (1b)$$

In order to obtain a tractable approximation of problem (1),

- ① the expectation in (1a) must be discretized,
- ② and the σ -field \mathcal{F} in (1b) must be discretized.

These two discretizations are a priori **independent**.

The first discretization is somewhat traditional (Monte Carlo), whereas the last one is not so well-known. . .



Strong convergence topology of σ -fields (Neveu)

Coarsest topology such that conditional expectation is continuous w.r.t. the σ -field:

$$\lim_{n \rightarrow +\infty} \mathcal{F}_n = \mathcal{F} \iff \lim_{n \rightarrow +\infty} \|\mathbb{E}[f | \mathcal{F}_n] - \mathbb{E}[f | \mathcal{F}]\|_{L^1} = 0 \quad \forall f \in L^1.$$

Main properties of the strong topology (Cotter)

- The strong convergence topology is metrizable.
- The set of σ -fields generated by a finite partition is dense.
- If $\mathbf{y}_n \xrightarrow{\mathbb{P}} \mathbf{y}$ and $\sigma(\mathbf{y}_n) \subset \sigma(\mathbf{y})$, then $\sigma(\mathbf{y}_n) \rightarrow \sigma(\mathbf{y})$.

Based on these notions, Barty has proposed a discretization scheme in order to approximate problem (1).



Discretization scheme

- 1 Approximate \mathcal{F} by \mathcal{F}_k generated by a **finite partition** of Ω :

$$V(\xi, \mathcal{F}_k) = \min_{\mathbf{u} \text{ } \mathcal{F}_k\text{-measurable}} \mathbb{E}[j(\mathbf{u}, \xi)] .$$

- 2 Approximate ξ by a **finitely valued** random variable ξ_n :

$$V(\xi_n, \mathcal{F}_k) = \min_{\mathbf{u} \text{ } \mathcal{F}_k\text{-measurable}} \mathbb{E}[j(\mathbf{u}, \xi_n)] .$$

Convergence theorem (Barty)

- 1 **Information structure discretization error:**

$$|V(\xi, \mathcal{F}) - V(\xi, \mathcal{F}_k)| \longrightarrow 0 \text{ as } \mathcal{F}_k \longrightarrow \mathcal{F} \text{ strongly.}$$

- 2 **Mean computation discretization error:**

$$|V(\xi, \mathcal{F}_k) - V(\xi_n, \mathcal{F}_k)| \longrightarrow 0 \text{ as } \xi_n \longrightarrow \xi \text{ in distribution.}$$

Discretization scheme independent in ξ and \mathcal{F} ?



- 1 Introduction
- 2 **Counterexample**
 - Formulation and exact solution
 - Discretization scheme
 - Approximated solution
 - What is wrong?
- 3 Convergence theorem
- 4 Conclusions



Formulation

- \mathbf{x} and \mathbf{w} : independent uniformly distributed random variables on $[-1, 1]$ (**initial state** and **noise**),
- \mathbf{u} : random variable based on the observation of \mathbf{x} (**control**),
- $\mathbf{z} = \mathbf{x} + \mathbf{u} + \mathbf{w}$ (**final state**).
- The problem is formulated on $([-1, 1]^2, \mathcal{B}_{[-1,1]^2}, \mu)$:

$$\min_{\mathbf{u} \text{ } \sigma(\mathbf{x})\text{-measurable}} \mathbb{E} [\epsilon \mathbf{u}^2 + \mathbf{z}^2] . \quad (2)$$

Exact resolution using dynamic programming

$$u^\#(x) = -\frac{x}{1+\epsilon} \quad \text{and} \quad J^\# = V((\mathbf{x}, \mathbf{w}), \mathcal{F}) = \frac{1}{3} \left(1 + \frac{\epsilon}{1+\epsilon} \right) ,$$

with $\mathcal{F} = \mathcal{B}_{[-1,1]} \otimes \{\emptyset, [-1, 1]\}$: sub σ -field generated by \mathbf{x} .



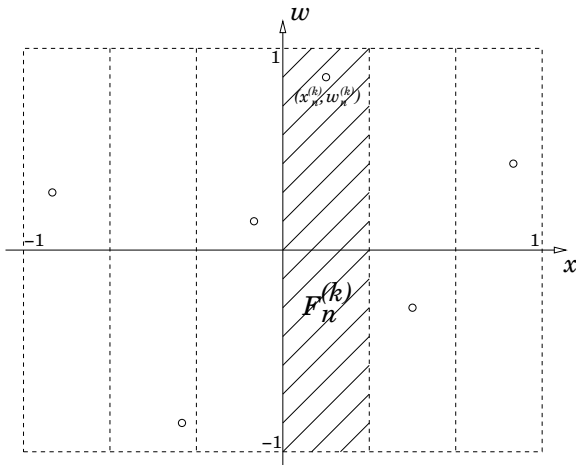


Figure: Partition of $[-1, 1]^2$ and associated sample.



Information

Let $n \in \mathbb{N}^*$. Let $(F_n^{(1)}, \dots, F_n^{(n)})$ be a partition of $[-1, 1]^2$, with

$$F_n^{(k)} = \left(\frac{2(k-1)}{n} - 1, \frac{2k}{n} - 1 \right] \times [-1, 1].$$

Let \mathcal{F}_n be the sub σ -field generated by $(F_n^{(1)}, \dots, F_n^{(n)})$.

- $(\mathcal{F}_n)_{n \in \mathbb{N}}$ strongly converges to \mathcal{F} ,
- \mathbf{u} is \mathcal{F}_n -measurable $\iff \mathbf{u}$ is constant over each $F_n^{(k)}$
 $\iff \mathbf{u}(x, w) = \sum_{k=1}^n u_n^{(k)} \mathbb{I}_{F_n^{(k)}}(x, w).$



Random variable

Let $(\zeta_n)_{n \in \mathbb{N}}$ be a deterministic sequence of elements in $[-1, 1]^2$ such that the associated sequence of empirical probability laws *narrowly converges* to μ . For $n \in \mathbb{N}^*$ and $k \in \{1, \dots, n\}$, let

$$(x_n^{(k)}, w_n^{(k)}) = \left(\frac{2k-1}{n} - 1 + \frac{\zeta_{k,1}}{n}, \zeta_{k,2} \right),$$

and define the approximation $(\mathbf{x}_n, \mathbf{w}_n)$ of (\mathbf{x}, \mathbf{w}) by

$$(\mathbf{x}_n, \mathbf{w}_n) = \sum_{k=1}^n (x_n^{(k)}, w_n^{(k)}) \mathbb{I}_{F_n^{(k)}}(\mathbf{x}, \mathbf{w}).$$

- $(\mathbf{x}_n, \mathbf{w}_n)_{n \in \mathbb{N}}$ converges in distribution to (\mathbf{x}, \mathbf{w}) ,
- $(\mathbf{x}_n, \mathbf{w}_n)$ is constant over each subset $F_n^{(k)}$.



Approximated problem

$$\min_{(u_n^{(1)}, \dots, u_n^{(n)}) \in \mathbb{R}^n} \sum_{k=1}^n \int_{F_n^{(k)}} \left(\epsilon (u_n^{(k)})^2 + (x_n^{(k)} + u_n^{(k)} + w_n^{(k)})^2 \right) \mu(dx dw).$$

Approximated solution

$$\hat{u}_n^{(k)} = -\frac{x_n^{(k)} + w_n^{(k)}}{1 + \epsilon} \quad \text{and} \quad \hat{J}_n = V((\mathbf{x}_n, \mathbf{w}_n), \mathcal{F}_n) \longrightarrow \frac{2}{3} \left(\frac{\epsilon}{1 + \epsilon} \right).$$

Approximated feedback in (2)

$$\hat{\mathbf{u}}_n(x, w) = -\sum_{k=1}^n \frac{x_n^{(k)} + w_n^{(k)}}{1 + \epsilon} \mathbb{I}_{F_n^{(k)}}(x, w) \quad \rightsquigarrow \quad \mathbb{E} \left[\epsilon \hat{\mathbf{u}}_n^2 + \mathbf{z}^2 \right] \longrightarrow \frac{2}{3}.$$

Discretization fails to asymptotically give the optimal solution.




Same notions of convergence as in the two-stage procedure, but

- \mathcal{F} and ξ are *independently approximated*:

this “diagonal” discretization makes possible to solve each open-loop subproblem using a *unique* sample of the random variable (a poor way to compute conditional expectations).

- The convergence notion used for ξ is *weak*:

$\{(\mathbf{x}_n, \mathbf{w}_n)\}_{n \in \mathbb{N}}$ does not converge *in probability* to (\mathbf{x}, \mathbf{w}) .

Note that with an stronger convergence notion (e.g. in L^2), each open-loop subproblem will be correctly approximated. 

Question: can we expect a diagonal convergence when using a stronger convergence notion for the random variable?



- 1 Introduction
- 2 Counterexample
- 3 Convergence theorem**
 - Notations
 - Theorem
 - Sketch of proof
 - Remarks
- 4 Conclusions



We go back to problem (1):

- $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, and \mathcal{F} a sub σ -field of \mathcal{A} ,
- \mathcal{X} is the space of integrable Ξ -valued random variables with the topology of convergence in probability \mathbb{P} ,
- \mathcal{U} is the space $L^p(\Omega, \mathcal{A}, \mathbb{P}; U)$ with $p \in [1, +\infty)$,
- $\Delta(\mathcal{F})$ is the subset of \mathcal{F} -measurable random variables of \mathcal{U} .

Let j be a normal integrand on $U \times \Xi$, and let J be the associated integral functional on $\mathcal{U} \times \mathcal{X}$: $J(\mathbf{u}, \xi) = \mathbb{E}[j(\mathbf{u}, \xi)]$.

$$V(\xi, \mathcal{F}) = \min_{\mathbf{u} \in \Delta(\mathcal{F})} J(\mathbf{u}, \xi) .$$



Theorem

Under the following assumptions:

H1 J is a continuous function on $\mathcal{U} \times \mathcal{X}$,

H2 $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ strongly converges to \mathcal{F} ,

H3 $\forall n \in \mathbb{N}, \mathcal{F}_n \subset \mathcal{F}$,

H4 $\{\xi_n\}_{n \in \mathbb{N}}$ converges to ξ in probability,

H5 $\{J(\cdot, \xi_n)\}_{n \in \mathbb{N}}$ uniformly converges to $J(\cdot, \xi)$ on $\Delta(\mathcal{F})$:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall \mathbf{u} \in \Delta(\mathcal{F}), |J(\mathbf{u}, \xi_n) - J(\mathbf{u}, \xi)| \leq \epsilon,$$

the convergence of the optimal costs holds true:

$$\lim_{n \rightarrow +\infty} V(\xi_n, \mathcal{F}_n) = V(\xi, \mathcal{F}). \quad (3)$$



$$\limsup_{n \rightarrow +\infty} V(\xi_n, \mathcal{F}_n) \leq V(\xi, \mathcal{F})$$

- $\forall \mathbf{u} \in \Delta(\mathcal{F})$, define $\mathbf{u}_n = \mathbb{E}[\mathbf{u} \mid \mathcal{F}_n]$. Then, $\mathcal{F}_n \rightarrow \mathcal{F} \implies \mathbf{u}_n \rightarrow \mathbf{u}$.
The set-valued mapping Δ is thus lsc.
- J being usc, we conclude that the marginal function V is also usc.

$$\liminf_{n \rightarrow +\infty} V(\xi_n, \mathcal{F}_n) \geq V(\xi, \mathcal{F})$$

- From $J(\mathbf{u}, \xi_n) = J(\mathbf{u}, \xi) + (J(\mathbf{u}, \xi_n) - J(\mathbf{u}, \xi))$, we obtain:

$$\min_{\mathbf{u} \in \Delta(\mathcal{F}_n)} J(\mathbf{u}, \xi_n) \geq \min_{\mathbf{u} \in \Delta(\mathcal{F}_n)} J(\mathbf{u}, \xi) + \min_{\mathbf{u} \in \Delta(\mathcal{F}_n)} (J(\mathbf{u}, \xi_n) - J(\mathbf{u}, \xi)) .$$

- Using $\mathcal{F}_n \subset \mathcal{F} \implies \Delta(\mathcal{F}_n) \subset \Delta(\mathcal{F})$, we deduce:

$$V(\xi_n, \mathcal{F}_n) \geq V(\xi, \mathcal{F}) + \min_{\mathbf{u} \in \Delta(\mathcal{F})} (J(\mathbf{u}, \xi_n) - J(\mathbf{u}, \xi)) .$$

- The conclusion is then a consequence of **H5**.



- ① Same result with:

$\Delta(\mathcal{F}) = \{ \mathbf{u} \in \mathcal{U}, \mathbf{u} \text{ } \mathcal{F} \text{-measurable, } \mathbf{u}(\omega) \in U^{\text{ad}} \text{ } \mathbb{P}\text{-as} \} ,$
 U^{ad} being a closed convex subset of U .

- ② If \mathcal{F} is generated by a Y -valued random variable \mathbf{y} , \mathcal{F}_n may be constructed thanks to a *quantification operator* q_n on Y :

$$\mathcal{F}_n = \sigma(q_n \circ \mathbf{y}) .$$

- ③ Assumption **H5** is implied by:

$$\exists \alpha > 0, \forall \mathbf{u} \in U, \forall (\xi, \xi') \in \Xi \times \Xi, |j(\mathbf{u}, \xi) - j(\mathbf{u}, \xi')| \leq \alpha \|\xi - \xi'\|_{\Xi} .$$

- ④ Theorem does not make use of the right tool for convergence analysis, namely **epi-convergence**.



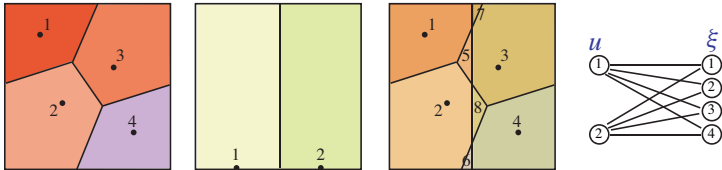
Conclusions

- 1 Another sight on stochastic approximation.
- 2 Scenario trees are not built-in in stochastic programming.
- 3 Comparison with Pennanen work.



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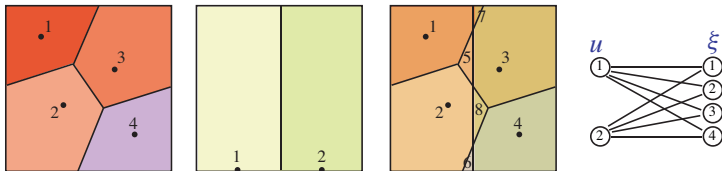
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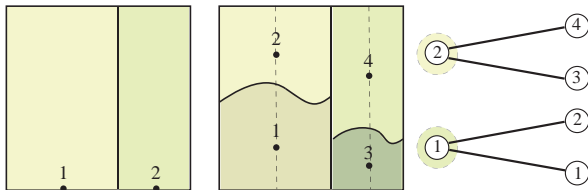
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K. Barty

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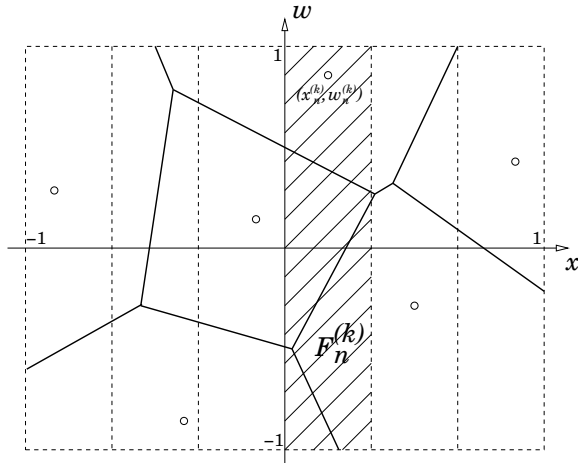


Figure: Partition of $[-1, 1]^2$ and Voronoi cells. ◀

