A Roe-type scheme with low Mach number preconditioning for a two-phase compressible flow model with pressure relaxation

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Abstract

We describe two-phase compressible flows by a hyperbolic six-equation single-velocity two-phase flow model with stiff mechanical relaxation. In particular, we are interested in the simulation of liquid-gas mixtures such as cavitating flows. The model equations are numerically approximated via a fractional step algorithm, which alternates between the solution of the homogeneous hyperbolic portion of the system through Godunov-type finite volume schemes, and the solution of a system of ordinary differential equations that takes into account the pressure relaxation terms. When used in this algorithm, classical schemes such as Roe's or HLLC prove to be very efficient to simulate the dynamics of transonic and supersonic flows. Unfortunately, these methods suffer from the well known difficulties of loss of accuracy and efficiency for low Mach number regimes encountered by upwind finite volume discretizations. This issue is particularly critical for liquid-gas mixtures due to the large and rapid variation in the flow of the acoustic impedance. To cure the problem of loss of accuracy at low Mach number, in this work we apply to our original Roe-type scheme for the two-phase flow model the Turkel's preconditioning technique studied by Guillard–Viozat [Computers & Fluids, 28, 1999] for the Roe's scheme for the classical Euler equations. We present numerical results for a two-dimensional liquid-gas channel flow test that show the effectiveness of the resulting Roe-Turkel method for the two-phase system.

Keywords: Two-phase compressible flows, mechanical relaxation, liquid-gas mixtures, finite volume schemes, Riemann solvers, low Mach number preconditioning.
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1 Introduction

We describe compressible two-phase flows by a variant [8] of the hyperbolic six-equation single-velocity two-phase flow model with stiff pressure relaxation of Saurel–Petitpas–Berry [9]. In particular, we are interested in the simulation of liquid-gas

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mixtures such as cavitating flows, which are relevant in many engineering applications. We numerically approximate the two-phase model equations by a fractional step algorithm, which alternates between the solution of the homogeneous hyperbolic portion of the system through wave propagation finite volume schemes based on Riemann solvers, and the solution of a system of ordinary differential equations that takes into account the pressure relaxation terms. For the solution of the homogeneous system we have presented in [8] a HLLC-type scheme and an original Roe-type scheme. These numerical methods for the considered two-phase flow model have proven to be very efficient to simulate wave propagation phenomena and shocks in transonic and supersonic flows. Unfortunately, these schemes suffer from the well known difficulties of loss of accuracy and efficiency at low Mach number regimes encountered by upwind finite volume discretizations for compressible flows. This issue is particularly critical for liquid-gas mixtures, since in these flows the Mach number may range from very low values in the nearly incompressible liquid medium to very large values in the gas and liquid-gas mixture regions. A classical strategy to cure the loss of accuracy for vanishing Mach number of finite volume Godunov-type schemes consists in correcting the numerical dissipation term of the spatial discretization of the convective portion of the system by applying a suitable preconditioning matrix, e.g. [10, 2]. In the present work we extend to our Roe-type scheme for the two-phase flow model the low Mach number Turkel's preconditioning technique presented by Guillard–Viozat [2] for the Roe's scheme for the Euler equations. Numerical results show the effectiveness of the resulting Roe-Turkel method for the two-phase system. This paper is organized as follows. We begin by recalling in Section 2 the six-equation two-phase flow model with stiff mechanical relaxation [8]. In Section 3 we describe the basic finite volume wave propagation scheme based on a Roe-type Riemann solver used for the numerical solution of the two-phase equations. Some properties of the asymptotic behavior of solutions of the continuous two-phase flow model in the low Mach number limit are briefly discussed in Section 4. In Section 5 we illustrate our novel Roe-Turkel scheme for the two-phase system. Some numerical experiments are finally presented in Section 6.

2 The two-phase flow model with pressure relaxation

The six-equation single-velocity two-phase flow model with stiff mechanical relaxation of Saurel–Petitpas–Berry [9] can be written in the following form [8], which uses the equations for the total energy of the two phases instead of the equations for the phasic internal energies employed in the classical version [9]:

$$
\begin{aligned}
&\partial_t \alpha_1 + \vec{u} \cdot \nabla \alpha_1 = \mu \left(p_1 - p_2\right), && \text{(1a)}\\
&\partial_t \left(\alpha_1 \rho_1\right) + \nabla \cdot \left(\alpha_1 \rho_1 \vec{u}\right) = 0, && \text{(1b)}\\
&\partial_t \left(\alpha_2 \rho_2\right) + \nabla \cdot \left(\alpha_2 \rho_2 \vec{u}\right) = 0, && \text{(1c)}\\
&\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla \left(\alpha_1 p_1 + \alpha_2 p_2\right) = 0, && \text{(1d)}\\
&\partial_t \left(\alpha_1 E_1\right) + \nabla \cdot \left(\alpha_1 E_1 \vec{u} + \alpha_1 p_1 \vec{u}\right) + \Sigma\left(q, \nabla q\right) = -\mu p_{\mathrm{I}} \left(p_1 - p_2\right), && \text{(1e)}\\
&\partial_t \left(\alpha_2 E_2\right) + \nabla \cdot \left(\alpha_2 E_2 \vec{u} + \alpha_2 p_2 \vec{u}\right) - \Sigma\left(q, \nabla q\right) = \mu p_{\mathrm{I}} \left(p_1 - p_2\right), && \text{(1f)}
\end{aligned}
$$

where the non-conservative term Σ appearing in the phasic total energy equations is $\Sigma(q, \nabla q) = \vec{u} \cdot (Y_1 \nabla(\alpha_2 p_2) - Y_2 \nabla(\alpha_1 p_1))$, with q denoting the vector of the system unknowns. In the above system α_k is the volume fraction of phase k, $k = 1, 2$ ($\alpha_1 + \alpha_2 =$

1), ρ_k is the phasic density, p_k is the phasic pressure, and E_k is the phasic total energy, $E_k = \mathcal{E}_k + \rho_k \frac{|\vec{u}|^2}{2}$, where $\mathcal{E}_k$ is the phasic internal energy per unit volume. The mixture density is $\rho = \alpha_1\rho_1 + \alpha_2\rho_2$, and $\vec{u}$ denotes the flow velocity vector. Moreover, $Y_k = \frac{\alpha_k\rho_k}{\rho}$ denotes the mass fraction of phase k. The source terms appearing in (1a), (1e), and (1f) model mechanical relaxation. In these terms $\mu > 0$ represents the pressure relaxation parameter and p_{I} is the interface pressure, $p_{\mathrm{I}} = \frac{Z_2 p_1 + Z_1 p_2}{Z_1 + Z_2}$, where $Z_k = \rho_k c_k$ is the acoustic impedance of phase k, and c_k is the phasic sound speed. We assume an infinite-rate pressure relaxation with $\mu \to \infty$, therefore mechanical equilibrium is reached instantaneously. The closure of system (1) is obtained through the specification of an equation of state for each phase $p_k = p_k(\mathcal{E}_k, \rho_k)$, $k = 1, 2$. Here we will consider species governed by the stiffened gas (SG) equation of state $p_k(\mathcal{E}_k, \rho_k) = (\gamma_k - 1)\mathcal{E}_k - \gamma_k\pi_k - (\gamma_k - 1)\eta_k\rho_k$, where γ_k, π_k, and η_k are material-dependent parameters. The mixture internal energy per unit volume is $\mathcal{E} = \alpha_1\mathcal{E}_1 + \alpha_2\mathcal{E}_2$. The latter relation, by using the isobaric assumption $p_1 = p_2 = p$ in the energy laws $\mathcal{E}_k(p_k, \rho_k)$, $k = 1, 2$, gives the mixture equation of state, which determines implicitly the mixture pressure law $p = p(\mathcal{E}, \rho_1, \rho_2, \alpha_1)$: $\mathcal{E} = \alpha_1\mathcal{E}_1(p, \rho_1) + \alpha_2\mathcal{E}_2(p, \rho_2)$. In the case of the SG EOS we find an explicit expression for p. The two-phase model system (1) is hyperbolic, and its associated speed of sound is the *frozen* mixture sound speed $c_{\mathrm{f}} = \sqrt{Y_1 c_1^2 + Y_2 c_2^2}$. The phasic sound speeds c_k can be expressed as $c_k = \sqrt{\kappa_k h_k + \chi_k}$, where $\kappa_k = \frac{\partial p_k(\mathcal{E}_k, \rho_k)}{\partial \mathcal{E}_k}$, $\chi_k = \frac{\partial p_k(\mathcal{E}_k, \rho_k)}{\partial \rho_k}$, and $h_k = \frac{\mathcal{E}_k + p_k}{\rho_k}$ is the phasic specific enthalpy. Total enthalpies will be denoted with $H_k = h_k + \frac{|\vec{u}|^2}{2}$. System (1) can be written in compact form as

$$\partial_t q + \nabla \cdot \mathcal{F}(q) + \sigma(q, \nabla q) = \psi(q). \tag{2a}$$

We will consider the 2-dimensional case with $\vec{u} = (u, v)$ and $\nabla = (\partial_x, \partial_y)$. Then we have

$$q = [\alpha_1, \alpha_1\rho_1, \alpha_2\rho_2, \rho u, \rho v, \alpha_1 E_1, \alpha_2 E_2]^{\mathrm{T}}, \quad \mathcal{F}(q) = [f^{(x)}(q), f^{(y)}(q)]^{\mathrm{T}}, \tag{2b}$$

$$f^{(x)} = [0, \alpha_1\rho_1 u, \alpha_2\rho_2 u, \rho u^2 + (\alpha_1 p_1 + \alpha_2 p_2), \rho u v, \alpha_1 (E_1 + p_1) u, \alpha_2 (E_2 + p_2) u]^{\mathrm{T}}, \tag{2c}$$

$$f^{(y)} = [0, \alpha_1\rho_1 v, \alpha_2\rho_2 v, \rho u v, \rho v^2 + (\alpha_1 p_1 + \alpha_2 p_2), \alpha_1 (E_1 + p_1) v, \alpha_2 (E_2 + p_2) v]^{\mathrm{T}}, \tag{2d}$$

$$\sigma = [\vec{u} \cdot \nabla\alpha_1, 0, 0, 0, 0, \Sigma, -\Sigma]^{\mathrm{T}}, \ \psi = [\Psi, 0, 0, 0, 0, -p_{\mathrm{I}}\Psi, p_{\mathrm{I}}\Psi]^{\mathrm{T}}, \ \Psi = \mu(p_1 - p_2). \tag{2e}$$

Above we have put into evidence the conservative contribution $\nabla \cdot \mathcal{F}(q)$ and the non-conservative term $\sigma(q, \nabla q)$ in the convective portion of the system. The vector $\psi(q)$ contains the mechanical relaxation source terms. Let us finally remark that the sum of the two equations for the phasic total energies (1e), (1f), recovers a conservation law for the mixture total energy E with the form $\partial_t E + \nabla \cdot ((E + \alpha_1 p_1 + \alpha_2 p_2)\vec{u}) = 0$.

3 Numerical solution

To numerically solve the two-phase system (2) we use a fractional step technique, where we alternate between the solution of the homogeneous hyperbolic system $\partial_t q + \nabla \cdot \mathcal{F}(q) + \sigma(q, \nabla q) = 0$ and the solution of the system of ordinary differential equations $\partial_t q = \psi(q)$, which takes into account the pressure relaxation terms. The homogeneous system is solved by the wave propagation method described below. The system of ODEs $\partial_t q = \psi(q)$ is solved in the limit of instantaneous mechanical relaxation $\mu \to \infty$, and this step drives the phasic pressures to an equilibrium value $p_1 = p_2 = p$. The solution procedure is detailed in [8]. In the relaxation step the partial densities, the mixture momentum, and the mixture internal energy remain constant. The values at mechanical equilibrium of the volume fraction α_1 and of the pressure p are used to update the solution.

3.1 Wave propagation schemes

To approximate the homogeneous portion of the system in (2), we employ the wave propagation methods of LeVeque [5], which are a class of finite volume Godunov-type schemes to approximate hyperbolic systems of partial differential equations. We consider the two-dimensional case and we assume here for simplicity a spatial discretization on a Cartesian grid with cells of uniform size Δx and Δy in the x and y directions, respectively. We denote by $q_{i,j}$ the approximate discrete solution of the system at the cell (i, j), $i, j \in \mathbb{Z}$. The two-dimensional first order wave propagation algorithm [5] has the following spatial discrete form

$$\frac{dq_{i,j}}{dt} + \frac{1}{\Delta x}\left(\mathcal{A}^{+}\Delta Q_{i-1/2,j} + \mathcal{A}^{-}\Delta Q_{i+1/2,j}\right) + \frac{1}{\Delta y}\left(\mathcal{B}^{+}\Delta Q_{i,j-1/2} + \mathcal{B}^{-}\Delta Q_{i,j+1/2}\right) = 0. \quad (3)$$

Here $\mathcal{A}^{\pm}\Delta Q$ and $\mathcal{B}^{\pm}\Delta Q$ are the fluctuations arising from plane wave Riemann problems between adjacent cells in the x and y directions, respectively, cf. [5]. To define these quantities, a Riemann solver must be supplied. We recall below the Roe-type Riemann solver that we have presented in [8] for the considered two-phase model. Concerning the integration in time of (3), here we use a standard explicit forward Euler method since in the present work we focus on the inaccuracies that arise at low Mach number in the spatial discretization.

3.1.1 Roe-type solver

We consider the approximation of a two-dimensional plane wave Riemann problem in the x direction for the two-phase model (2), hence the approximate Riemann solution of a one-dimensional system $\partial_t q+\partial_x f(q)+\sigma(q, \partial_x q) = 0$, with q, $f(q) = f^{(x)}(q)$, as in (2b), (2c), and $\sigma(q, \partial_x q) = [u\partial_x\alpha_1, 0, 0, 0, 0, \Sigma, -\Sigma]^{\mathrm{T}}$, where $\Sigma(q, \partial_x q) = u(Y_1\partial_x(\alpha_2 p_2) - Y_2\partial_x(\alpha_1 p_1))$. Note that the eigenvalues of this system are given by $\lambda_{1,7} = u \mp c_{\mathrm{f}}$, $\lambda_\xi = u$, $\xi = 2, \ldots, 6$. Following the classical Roe's approach we define an approximate solution for a Riemann problem for this system with left and right initial data q_ℓ and q_r by using the exact solution to a Riemann problem for a linearized system $\partial_t q + \hat{A}(q_\ell, q_r)\partial_x q = 0$. The constant coefficient matrix $\hat{A} = \hat{A}(q_\ell, q_r)$ (Roe matrix) is an averaged version of the matrix $A(q)$ of the system written in quasi-linear form, $\partial_t q + A(q)\partial_x q = 0$. This Roe matrix is defined in order to guarantee conservation for the quantities that are physically conserved, namely $\alpha_k\rho_k$, $k = 1, 2$, $\rho\vec{u}$, and E. The Riemann solution structure of this Roe-type solver consists of 7 waves $\mathcal{W}^\xi$ moving at speeds s^ξ that correspond to the eigenstructure of the Roe matrix: $\mathcal{W}^\xi = \hat{\zeta}_\xi\hat{r}_\xi$, $s^\xi = \hat{\lambda}_\xi$, $\xi = 1, \cdots, 7$, where $\hat{r}_\xi$ and $\hat{\lambda}_\xi$ are the Roe eigenvectors and eigenvalues, respectively, and $\hat{\zeta}_\xi$ are the coefficients of the projection of the jump $q_r - q_\ell$ onto the basis of the Roe eigenvectors, $q_r - q_\ell = \sum_{\xi=1}^{7} \hat{\zeta}_\xi\hat{r}_\xi$. The definition of the Roe eigenstructure for the two-phase flow model for the case of the stiffened gas EOS is reported in Appendix A. With the Riemann solution structure $\{\mathcal{W}^\xi_{i+1/2,j}, s^\xi_{i+1/2,j}\}_{\xi=1,\ldots,7}$ computed for plane wave Riemann problems in the x direction for each interface $x_{i+1/2,j}$ between cells (i, j), $(i+1, j)$ (here $q_\ell = q_{i,j}$ and $q_r = q_{i+1,j}$), the fluctuations $\mathcal{A}^{\pm}\Delta Q_{i+1/2,j}$ in (3) are obtained as $\mathcal{A}^{\pm}\Delta Q_{i+1/2,j} = \sum_{\xi=1}^{7}(s^\xi_{i+1/2,j})^{\pm}\mathcal{W}^\xi_{i+1/2,j}$, $s^+ = \max(s, 0)$, $s^- = \min(s, 0)$. The computation of the fluctuations $\mathcal{B}^{\pm}\Delta Q_{i,j+1/2}$ associated to the y direction is analogous.

4 The two-phase model in the low Mach number limit

Let us recall [9] that the physical flow model that corresponds to the 6-equation two-phase model in the limit $\mu \to \infty$ of instantaneous mechanical equilibrium is the reduced single-pressure 5-equation model of Kapila *et al.* [3]. This 5-equation two-phase model is characterized by the Wood's *equilibrium* mixture sound speed c_{W}, defined by $\frac{1}{\rho c_{\mathrm{W}}^2} = \frac{\alpha_1}{\rho_1 c_1^2} + \frac{\alpha_2}{\rho_2 c_2^2}$, and we have $c_{\mathrm{W}} \leq c_{\mathrm{f}}$ (sub-characteristic condition). The behavior of solutions of the 5-equation model as the Mach number tends to zero was first studied by Murrone–Guillard in [6], by using asymptotic expansions in powers of the Mach number $M_{\mathrm{W}*}$, where here $M_{\mathrm{W}*} = \frac{|\vec{u}_*|}{c_{\mathrm{W}*}}$ is a reference Mach number based on the Wood's sound speed. In particular, the authors in [6] show that, analogously to the results found for the Euler equations, at the continuous level in the low Mach number limit pressure fluctuations scale with the square of the Mach number, $p(\vec{x}, t) = p^{(0)}(t) + M_{\mathrm{W}*}^2 p^{(2)}(\vec{x}, t) + \ldots$. Moreover, continuing the analysis of [6], we can deduce, by assuming constant pressure at open boundaries and particle paths coming from regions with constant volume fraction, that volume fractions α_k also support perturbations of order $M_{\mathrm{W}*}^2$ as $M_{\mathrm{W}*} \to 0$. These results for the 5-equation model characterize then the asymptotic behavior at the continuous level for vanishing Mach number of solutions of the 6-equation model with $\mu \to \infty$. Analogously to the case of the Euler equations, discrete approximations via upwind finite volume methods of the 5-equation two-phase model support pressure perturbations of the wrong magnitude (order $M_{\mathrm{W}*}$) in the low Mach number limit. This difficulty arises both for numerical methods based on direct discretizations of the 5-equation model [6] and for schemes based on approximations of the 6-equation model with instantaneous mechanical relaxation as in our case. Let us remark that it is useful to analyze also the low Mach number behavior of the continuous homogeneous 6-equation non-equilibrium model, since our numerical method based on the fractional step algorithm employs discretizations of this homogeneous system. The results are again analogous to the results inferred for the Euler equations and for the 5-equation model, except that for the homogeneous 6-equation model the effective pressure $p_{\mathrm{m}} = \alpha_1 p_1 + \alpha_2 p_2$ plays the role of p: $p_{\mathrm{m}}(\vec{x}, t) = p_{\mathrm{m}}^{(0)}(t) + M_{\mathrm{f}*}^2 p_{\mathrm{m}}^{(2)}(\vec{x}, t) + \ldots$ as $M_{\mathrm{f}*} \to 0$, where $M_{\mathrm{f}*}$ is a reference Mach number based on the frozen (non-equilibrium) mixture sound speed c_{f}. Note that we have $M_{\mathrm{f}*} = \frac{c_{\mathrm{W}}}{c_{\mathrm{f}}} M_{\mathrm{W}*} \leq M_{\mathrm{W}*}$ since $c_{\mathrm{W}} \leq c_{\mathrm{f}}$.

5 A Roe-Turkel scheme for the two-phase flow model

To cure the accuracy problem at low Mach number of the Roe-type scheme for our two-phase flow model we extend the Turkel's preconditioning technique proposed by Guillard–Viozat [2] for the Roe's scheme for the Euler equations. This approach consists in correcting at low Mach number the numerical viscosity matrix associated to the spatial discretization of the convective portion of the system by applying a suitable preconditioning matrix P. In particular, Guillard and Viozat use Turkel's preconditioning matrix [10], which is defined for the Euler equations in terms of the entropic variables $\varphi = [p, \vec{u}, s]^{\mathrm{T}}$, where s denotes the entropy, as $P^{\varphi} = \mathrm{diag}(\beta^2, I_d, 1)$. Here I_d is the identity matrix $\in \mathbb{R}^{d \times d}$, where d is the spatial dimension. The parameter $\beta \leq 1$ is of the order of the local Mach number (or of a reference Mach number) for subsonic flows and $= 1$ otherwise. To begin with, we rewrite suitably the fluctuations that are used in the wave propagation algorithm in order to make explicit the numerical dissipation contribution. For simplicity, we shall restrict here the illustration of our approach to the quantities in the numerical scheme associated to plane wave Riemann problems in the x direction, and we will omit the subscript j identifying grid cells in the y direction. We obtain the following expressions of $\mathcal{A}^{\pm}\Delta Q_{i+1/2}$, equivalent to the expressions reported in Section 3.1.1:

$$\mathcal{A}^{\pm}\Delta Q_{i+1/2} = \tfrac{1}{2}\Delta\tilde{f}_{i+1/2} \pm \tfrac{1}{2}\mathcal{V}\Delta Q_{i+1/2}, \tag{4}$$

where $\Delta\tilde{f}_{i+1/2} = \sum_{\xi=1}^{7} s_{i+1/2}^{\xi}\mathcal{W}_{i+1/2}^{\xi} = \sum_{\xi=1}^{7}\hat{\zeta}_\xi\hat{\lambda}_\xi\hat{r}_\xi = \hat{A}_{i+1/2}(q_{i+1} - q_i)$, $\hat{A}_{i+1/2} = \hat{A}(q_i, q_{i+1})$, and the components of this quantity coincide with the jumps in the physical fluxes for the conserved variables. The numerical dissipation term is given by

$$\mathcal{V}\Delta Q_{i+1/2} = \sum_{\xi=1}^{7}|s_{i+1/2}^{\xi}|\mathcal{W}_{i+1/2}^{\xi} = \sum_{\xi=1}^{7}\hat{\zeta}_\xi|\hat{\lambda}_\xi|\hat{r}_\xi = |\hat{A}_{i+1/2}|(q_{i+1} - q_i), \tag{5}$$

where we have omitted for simplicity grid indexes for $\hat{r}$ and $\hat{\lambda}$. The viscosity matrix that characterizes the Roe-type solver for the two-phase flow model has a form analogous to the single-phase case: $|\hat{A}_{i+1/2}| = \hat{R}_{i+1/2}|\hat{\Lambda}_{i+1/2}|\hat{R}_{i+1/2}^{-1}$, where $\hat{R} = [\hat{r}_1, \ldots, \hat{r}_7]$ and $\hat{\Lambda} = \mathrm{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_7)$. Now we correct the numerical dissipation term by replacing the original viscosity matrix $|\hat{A}|$ in (5) with the preconditioned one $P^{-1}|P\hat{A}|$, as in [2]. To this aim, we need to define a suitable Turkel-type preconditioning matrix $P(q)$. The observations in Section 4 suggest that the preconditioning factor β^2 should act on the equation for the effective pressure $p_\mathrm{m} = \alpha_1 p_1 + \alpha_2 p_2$, which plays the equivalent role of the pressure p in the Euler system. Moreover, as for the Euler equations, we should modify at low Mach number only the characteristic acoustic fields, and preserve unaltered interface waves. To this end, we have chosen a Turkel-type preconditioner for our two-phase system of the form $P^\varphi = \mathrm{diag}(\beta^2, I_d, 1, 1, 1, 1)$ in terms of the variables $\varphi = [p_\mathrm{m}, \vec{u}, s_1, s_2, Y_1, \alpha_1]^\mathrm{T}$, where s_k is the entropy of phase k. Note that preconditioning does not affect the advection equations that govern the volume fractions, mass fractions and phasic entropies. Finally, we obtain $P(q) = \frac{\partial q}{\partial\varphi}P^\varphi\frac{\partial\varphi}{\partial q}$ and the preconditioned dissipation term is given by

$$\mathcal{V}^P\Delta Q_{i+1/2} = P_{i+1/2}^{-1}|P_{i+1/2}\hat{A}_{i+1/2}|(q_{i+1} - q_i) = \sum_{\xi=1}^{7}\hat{\zeta}_\xi^P|\hat{\lambda}_\xi^P|\tilde{\hat{r}}_\xi^P = \sum_{\xi=1}^{7}|s^{P\xi}|\mathcal{W}^{P\xi}, \tag{6}$$

where we have introduced the preconditioned waves and speeds $\mathcal{W}^{P\xi} = \hat{\zeta}_\xi^P\tilde{\hat{r}}_\xi^P$, $s^{P\xi} = \hat{\lambda}_\xi^P$. Here $\tilde{\hat{r}}_\xi^P = P_{i+1/2}^{-1}\hat{r}_\xi^P$, $\hat{\lambda}_\xi^P$ and $\hat{r}_\xi^P$ are the eigenvalues and the eigenvectors of the matrix $P_{i+1/2}\hat{A}_{i+1/2}$, and $\hat{\zeta}_\xi^P$ are the coefficients of the decomposition $q_{i+1} - q_i = \sum_{\xi=1}^{7}\hat{\zeta}_\xi^P\hat{r}_\xi^P$. The expression of the preconditioned Roe eigenstructure results to be a natural extension of the one derived by Guillard–Viozat [2] for the Euler equations. We have $s^{P\xi} = s^\xi = \hat{u}$ for $\xi = 2, \ldots, 6$ and $\sum_{\xi=2}^{6}\mathcal{W}^{P\xi} = \sum_{\xi=2}^{6}\mathcal{W}^\xi$, and only the acoustic fields $\xi = 1, 7$ are modified. Below we summarize the quantities of the dissipation term that are corrected at low Mach number, namely the eigenvalues of the Roe matrix $\hat{\lambda}_{1,7}$, the wave strengths $\hat{\zeta}_{1,7}$, and the Roe eigenvectors components $\hat{r}_{1,7}^{(l)}$, $l = 4, 6, 7$:

$$\hat{\lambda}_{1,7} = \hat{u}_{1,7} \mp \hat{c}_\mathrm{f} \quad\rightarrow\quad \hat{\lambda}_{1,7}^P = \tfrac{1}{2}\left((1+\beta^2)\hat{u} \mp \sqrt{X_\beta}\right) \tag{7a}$$

$$\hat{\zeta}_{1,7} = \tfrac{1}{2\hat{c}_\mathrm{f}}\left(\tfrac{\Delta p_\mathrm{m}}{\hat{c}_\mathrm{f}} \mp \hat{\rho}\Delta u\right) \quad\rightarrow\quad \hat{\zeta}_{1,7}^P = \tfrac{1}{\sqrt{X_\beta}}\left(\tfrac{\Delta p_\mathrm{m}}{\mp(\hat{\lambda}_{1,7}^P - \hat{u}\beta^2)} \mp \hat{\rho}\Delta u\right) \tag{7b}$$

$$\hat{r}_{1,7}^{(4)} = \hat{u} \mp \hat{c}_\mathrm{f} \quad\rightarrow\quad \tilde{\hat{r}}_{1,7}^{P(4)} = \hat{u} + (\hat{\lambda}_{1,7}^P - \hat{u}\beta^2) \tag{7c}$$

$$\hat{r}_{1,7}^{(5+k)} = \widehat{Y_k H_k} \mp \widetilde{uY_k}\hat{c}_\mathrm{f} \quad\rightarrow\quad \tilde{\hat{r}}_{1,7}^{P(5+k)} = \widehat{Y_k H_k} + \widetilde{uY_k}(\hat{\lambda}_{1,7}^P - \hat{u}\beta^2),\ k = 1, 2, \tag{7d}$$

with $X_\beta = ((1-\beta^2)\hat{u})^2 + (2\beta\hat{c}_\mathrm{f})^2$. The Roe-Turkel scheme uses then the expression of $\mathcal{A}^{\pm}\Delta Q$ in (4) with $\mathcal{V}\Delta Q_{i+1/2}$ replaced by the preconditioned dissipation term in (6). In practice, in the implementation of the algorithm we still use the standard expression of $\mathcal{A}^{\pm}\Delta Q$ in Sec. 3.1.1 for the equation for the volume fraction α_1, since this equation is not affected by preconditioning. Let us remark that the discrete equations for the mixture variables of the Roe-Turkel scheme for the two-phase model recover formally the discrete equations of the Roe-Turkel scheme for the Euler equations with p_{m} replacing p. Then the analysis of [2] can be extended to the two-phase model, and, at least under suitable assumptions, we can show that solutions of the two-phase preconditioned discrete equations support perturbations of the effective pressure p_{m} of order $M_{\mathrm{f}*}^2$ in the low Mach number limit $M_{\mathrm{f}*} \leq M_{\mathrm{W}*} \to 0$, as in the continuous case [7]. Since the pressure relaxation procedure in the fractional step algorithm does not alter the scaling of the variables with the Mach number, discrete solutions of the two-phase model in the limit of mechanical equilibrium also possess the correct magnitude of pressure fluctuations.

6 Numerical test: liquid-gas channel flow

We present some numerical results obtained with our new Roe-Turkel scheme for the two-phase flow model (1), showing some comparison with the standard Roe-type scheme. We have set $\beta = \min(\max(\epsilon, \tilde{M}), 1)$, with $\epsilon = 10^{-10}$, and $\tilde{M}$ is a local Mach number defined through the Roe-averaged quantities. We perform a two-phase liquid-gas flow numerical experiment analogous to channel flow tests considered by several authors in the literature, e.g. [2, 6]. We simulate a flow of liquid water initially containing a uniformly distributed small amount of gas, $\alpha_{g0} = 10^{-3}$, in a channel of length = 4 m and height = 1 m with a bump defined by $y = (1 - \cos((x-1)\pi))/10$, if $x \in [1, 3]$, and $y = 0$ otherwise. We set an inlet pressure $p_0 = 10^6$ Pa and inlet phasic densities $\rho_{l0} = 890.27$ kg/m^3, $\rho_{g0} = 4.88$ kg/m^3, for liquid and gas. We impose an inlet velocity $\vec{u}_0 = (u_0, 0)$ and an outlet pressure p_0. The values $u_0, p_0, \rho_{l0}, \rho_{g0}$ are also used for the initial conditions. The EOS parameters for the two phases are $\gamma_l = 2.35$, $\pi_l = 10^9$ Pa, $\eta_l = -1167 \times 10^3$ J/kg, $\gamma_g = 1.43$, $\pi_g = 0$, $\eta_g = 2030 \times 10^3$ J/kg. We perform simulations for four different decreasing values of the inlet velocity: $u_0 = 20$, 10, 5, 2 m/s. With this setup the reference inlet Mach number M_0 of the equilibrium mixture for the four cases is $M_0 = \frac{u_0}{c_{\mathrm{W}0}} = 0.0199, 0.0099, 0.0049, 0.0019$. The reference equilibrium sound speed is $c_{\mathrm{W}0} = 1.00017 \times 10^3$ m/s. The frozen sound speed is $c_{\mathrm{f}0} = 1.62551 \times 10^3$ m/s. The two-phase flow in this problem is expected to reach a subsonic stationary regime characterized by a small decrease of the pressure and a slight increase of the gas volume fraction in correspondence of the channel restriction [4], with a symmetric configuration with respect to the vertical axis at mid-length in the channel. Simulations are performed with both the Roe's scheme and the Roe-Turkel scheme over the computational domain $[0, 4] \times [0, 1]\,\mathrm{m}^2$ with a grid of 100×25 cells. We stop the simulations at a time at which stationary conditions are approximately attained. We begin by displaying in Fig. 1 a comparison of the results obtained with the Roe's scheme and with the Roe-Turkel scheme for the case with higher Mach number $M_0 = 0.0199$. We observe that the correct solution behavior is captured only when Turkel's preconditioning is activated. In Fig. 2 we show the results for the Mach number and the gas volume fraction obtained by the Roe-Turkel scheme for the test case with smaller Mach number $M_0 = 0.0019$. Moreover, in the same figure we compare the profiles of the computed Mach number and of the normalized pressure p/p_0 at the upper and lower boundaries, together with the average numerical solution over the channel height, with the exact steady solution that can be obtained for the quasi-one-dimensional two-phase flow model [4]. The preconditioned method is able to capture accurately the correct features of the two-phase flow.

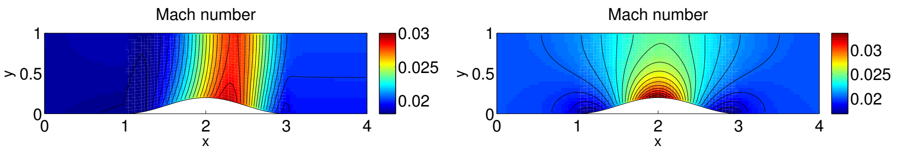

Figure 1: Mach number M_W at approximately stationary conditions computed by the Roe's scheme (left) and the Roe-Turkel scheme (right) for the channel flow test with $M_0 = 0.0199$.

To assess the accuracy of the preconditioned method we have also computed the values of the pressure fluctuations $\delta p_{\max} = \frac{p_{\max}-p_{\min}}{p_0}$ and of the volume fraction fluctuations $\delta\alpha_{g\,\max} = \alpha_{g\,\max} - \alpha_{g\,\min}$ in the flow domain for decreasing reference Mach number for both the Roe-Turkel scheme and the Roe's scheme. We plot in Fig. 3 the values of $\delta p_{\max}$ and $\delta\alpha_{g\,\max}$ versus M_0 for the two schemes. We can observe that for the Roe-Turkel scheme perturbations correctly scale with the square of the Mach number M_0, in agreement with the theoretical results.

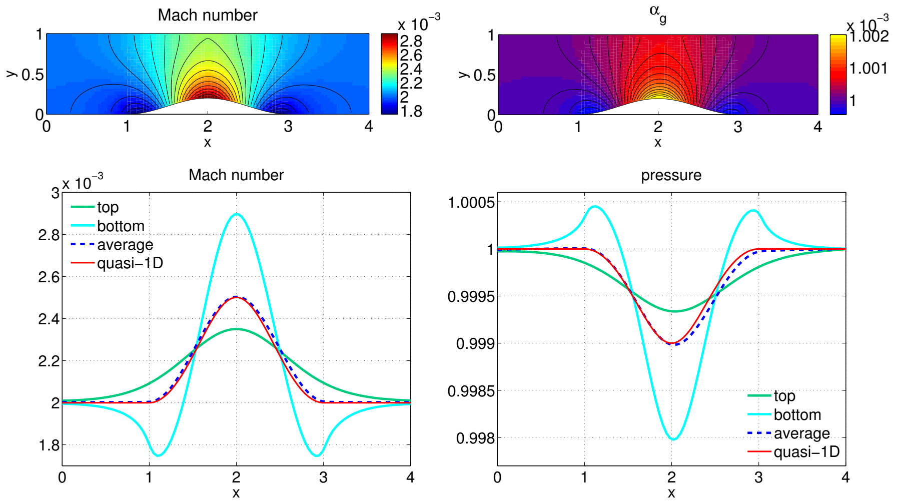

Figure 2: Results computed by the Roe-Turkel method for the channel flow test with $M_0 = 0.0019$. Above: Mach number M_W (left) and gas volume fraction α_g (right). Below: computed profiles of the Mach number (left) and of normalized pressure p/p_0 (right) at the upper and lower boundaries, and average numerical solution over the vertical section, compared to the exact steady quasi-one-dimensional two-phase flow solution.

7 Conclusions

Following the work in [2] for the Euler equations, we have derived an original low Mach number Turkel-type [10] preconditioning technique for a Roe-type scheme for the two-phase compressible flow model with pressure relaxation that we have studied in [8]. Numerical experiments show the effectiveness of the preconditioned method for the two-phase system in curing the accuracy difficulties encountered at low Mach

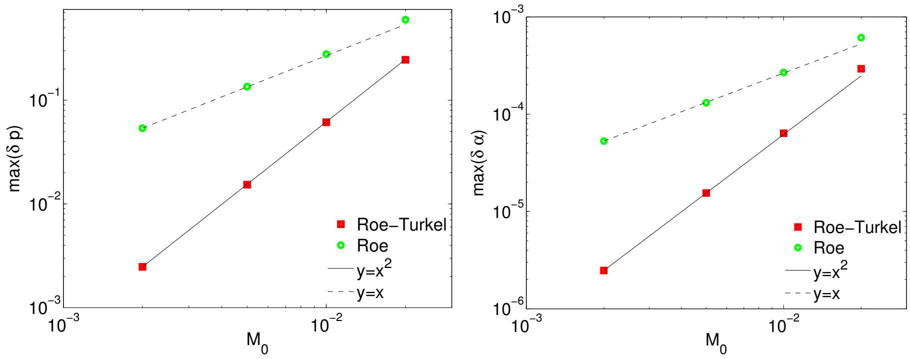

Figure 3: Log-log plot of the pressure fluctuations $\delta p_{\max}$ (left) and of the volume fraction fluctuations $\delta\alpha_{g\,\max}$ (right) versus the Mach number M_0 computed by the Roe-Turkel scheme and by the Roe's scheme for the set of channel flow tests performed with different values of M_0.

number by standard upwind finite volume discretizations. A more detailed analysis and additional numerical tests will be presented in [7]. We recall that preconditioning techniques cure the accuracy problem for vanishing Mach number but they suffer from a very severe time step stability restriction when explicit time discretizations are used [1]. For practical applications, preconditioning is typically combined with implicit time integration. This will be studied in future work.

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A Roe eigenstructure for the two-phase flow model

We write here the expression of the eigenstructure of the Roe matrix $\hat{A}(q_\ell, q_r)$ for a plane wave Riemann problem with data q_ℓ, q_r in the x direction for the 2-dimensional two-phase flow model. First, let us introduce the following averaged quantities:

$$\hat{u} = \frac{u_\ell\sqrt{\rho_\ell} + u_r\sqrt{\rho_r}}{\sqrt{\rho_\ell} + \sqrt{\rho_r}}, \quad \hat{v} = \frac{v_\ell\sqrt{\rho_\ell} + v_r\sqrt{\rho_r}}{\sqrt{\rho_\ell} + \sqrt{\rho_r}}, \quad \hat{\rho} = \sqrt{\rho_\ell\rho_r}, \tag{8a}$$

$$\hat{Y}_k = \frac{Y_{k\,\ell}\sqrt{\rho_\ell} + Y_{k\,r}\sqrt{\rho_r}}{\sqrt{\rho_\ell} + \sqrt{\rho_r}}, \quad \widehat{Y_kH_k} = \frac{(Y_kH_k)_\ell\sqrt{\rho_\ell} + (Y_kH_k)_r\sqrt{\rho_r}}{\sqrt{\rho_\ell} + \sqrt{\rho_r}}, \tag{8b}$$

$$\widehat{uY_k} = \frac{(uY_k)_\ell\sqrt{\rho_\ell} + (uY_k)_r\sqrt{\rho_r}}{\sqrt{\rho_\ell} + \sqrt{\rho_r}}, \quad \widehat{vY_k} = \frac{(vY_k)_\ell\sqrt{\rho_\ell} + (vY_k)_r\sqrt{\rho_r}}{\sqrt{\rho_\ell} + \sqrt{\rho_r}}, \tag{8c}$$

$$\widetilde{uY_k} = \tfrac{1}{2}\left(\hat{u}\hat{Y} + \widehat{uY_k}\right), \quad \widetilde{vY_k} = \tfrac{1}{2}\left(\hat{v}\hat{Y} + \widehat{vY_k}\right), \quad k = 1, 2, \quad \hat{\mathcal{K}} = \tfrac{\hat{u}^2+\hat{v}^2}{2}, \tag{8d}$$

$$\hat{c}_{\mathrm{f}} = \sqrt{\widehat{Y_1c_1^2} + \widehat{Y_2c_2^2}}, \quad \widehat{Y_kc_k^2} = \kappa_k\left(\widehat{Y_kH_k} - \hat{\mathcal{K}}\,\hat{Y}_k\right) + \chi_k\hat{Y}_k\,, \quad k = 1, 2, \tag{8e}$$

where $\kappa_k = (\gamma_k - 1)$ and $\chi_k = -(\gamma_k - 1)\eta_k$. The Roe eigenvalues are found as

$$\hat{\lambda}_1 = \hat{u} - \hat{c}_{\mathrm{f}}\,, \quad \hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_4 = \hat{\lambda}_5 = \hat{\lambda}_6 = \hat{u}\,, \quad \hat{\lambda}_7 = \hat{u} + \hat{c}_{\mathrm{f}}\,, \tag{9}$$

and the corresponding matrix of the Roe right eigenvectors $\hat{R} = [\hat{r}_1, \dots, \hat{r}_7]$ is

$$\hat{R} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hat{Y}_1 & 0 & 0 & 1 & 0 & 0 & \hat{Y}_1 \\ \hat{Y}_2 & 0 & 1 & 0 & 0 & 0 & \hat{Y}_2 \\ \hat{u} - \hat{c}_\mathrm{f} & 0 & \hat{u} & \hat{u} & 0 & 0 & \hat{u} + \hat{c}_\mathrm{f} \\ \hat{v} & 0 & 0 & 0 & 0 & 1 & \hat{v} \\ \widehat{Y_1 H_1} - \widetilde{uY_1}\,\hat{c}_\mathrm{f} & -\frac{\kappa_2}{\kappa_1} & -\frac{\chi_2}{\kappa_1} + \frac{\kappa_2}{\kappa_1}\hat{\mathcal{K}} - \hat{v}\frac{\tilde{V}}{\kappa_1} & -\frac{\chi_1}{\kappa_1} + \hat{\mathcal{K}} - \hat{v}\frac{\tilde{V}}{\kappa_1} & \frac{\Pi_1 - \Pi_2}{\kappa_1} & \frac{\tilde{V}}{\kappa_1} & \widehat{Y_1 H_1} + \widetilde{uY_1}\,\hat{c}_\mathrm{f} \\ \widehat{Y_2 H_2} - \widetilde{uY_2}\,\hat{c}_\mathrm{f} & 1 & 0 & 0 & 0 & 0 & \widehat{Y_2 H_2} + \widetilde{uY_2}\,\hat{c}_\mathrm{f} \end{pmatrix}, \tag{10}$$

where $\tilde{V} = \kappa_1\,\widetilde{vY_1} + \kappa_2\,\widetilde{vY_2}$ and $\Pi_k = \gamma_k \pi_k$, $k = 1, 2$. The coefficients $\hat{\zeta}_\xi$, $\xi = 1, \dots 7$, of the Roe eigen-decomposition $q_r - q_\ell = \sum_{\xi=1}^{7} \hat{\zeta}_\xi \hat{r}_\xi$, are given by

$$\hat{\zeta}_1 = \frac{\Delta p_\mathrm{m} - \hat{c}_\mathrm{f}\,\hat{\rho}\,\Delta u}{2\hat{c}_\mathrm{f}^2}, \quad \hat{\zeta}_7 = \frac{\Delta p_\mathrm{m} + \hat{c}_\mathrm{f}\,\hat{\rho}\,\Delta u}{2\hat{c}_\mathrm{f}^2}, \quad \hat{\zeta}_6 = \hat{\rho}\Delta v + \hat{v}\left(\Delta\rho - \frac{\Delta p_\mathrm{m}}{\hat{c}_\mathrm{f}^2}\right), \tag{11a}$$

$$\hat{\zeta}_2 = \Delta(\alpha_2 E_2) - \frac{\Delta p_\mathrm{m}}{\hat{c}_\mathrm{f}^2}\widehat{Y_2 H_2} - \hat{\rho}\,\widetilde{uY_2}\,\Delta u \tag{11b}$$

$$= -\frac{\Delta p_\mathrm{m}}{\hat{c}_\mathrm{f}^2}\widehat{Y_2 H_2} + \hat{\mathcal{K}}\Delta(\alpha_2\rho_2) + \Delta(\alpha_2 \mathcal{E}_2) + \hat{\rho}\,\widetilde{vY_2}\,\Delta v, \tag{11c}$$

$$\hat{\zeta}_3 = \Delta(\alpha_2\rho_2) - \hat{Y}_2\frac{\Delta p_\mathrm{m}}{\hat{c}_\mathrm{f}^2}, \quad \hat{\zeta}_4 = \Delta(\alpha_1\rho_1) - \hat{Y}_1\frac{\Delta p_\mathrm{m}}{\hat{c}_\mathrm{f}^2}, \quad \hat{\zeta}_5 = \Delta\alpha_1. \tag{11d}$$

where $\Delta(\cdot) \equiv (\cdot)_r - (\cdot)_\ell$ and $p_\mathrm{m} = \alpha_1 p_1 + \alpha_2 p_2$.

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