A review of entropy fixes as applied to Roe's linearization

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Abstract

The class of entropy fixes originated by Harten and Hyman, as applied to Roe's linearization of convex hyberbolic equations, is reformulated within a unified framework. By adopting a complementary viewpoint with respect to that of Harten, the entropy fix is recast in terms of expressions involving either the propagation speeds or the speed differences. The proposed formulation allows to analyze and compare several versions of this class of entropy fix and to elaborate some new variants thereof. In addition, this framework accommodates an interpretation of the HLL schemes which leads to the concept of a positivity preserving entropy fix and to a solution-dependent correction to Roe's scheme so as to assure positive solutions.

Keywords: Numerical solution of conservation laws, Non-linear hyperbolic equations, Entropy fix

Contents

1 Introduction

Weak solutions of initial value problems for hyperbolic conservation laws are nonunique. To select the physically relevant unique solution, additional criteria have to be considered, which are mostly referred to as *entropy conditions*—a denomination originated from gas dynamics. In particular, a weak form of the entropy condition may be used as a mathematical tool to verify that a given numerical method converges necessarily to the physically unique weak solution [12, p. 39]. As an example, it can be proved that weak solutions obtained by means of a Godunov type method satisfy the entropy condition, assuming that entropy-satisfying solutions to the Riemann problems are used [12, p. 143]. This condition is verified by the original Godunov method [6], which considers the exact solution of each Riemann problem, but may be violated by one of the most popular flux difference splitting schemes based on Approximate Riemann Solvers (ARS), namely that formulated by Roe [17]. In order to make this scheme to satisfy the entropy condition, it must be properly modified: such a correction is usually indicated as *entropy fix*.

The first proposals for such an entropy fix were put forward in the mid 80's by Harten and co-workers. Their approach is quite general: the entropy fix is considered as a means to guarantee that the numerical scheme produces the entropic solution to the hyperbolic problem, defined as the limit of the solution of the corresponding viscous problem when the viscosity and the thermal conductivity vanish (vanishing viscosity solution). This can be accomplished by writing the upwind scheme so as to put into evidence its numerical dissipation matrix, and by operating on this matrix to assure a non-zero viscous contribution to the numerical flux. This viewpoint is not limited to ARS-based schemes, but may be applied to any method written in dissipation form [8]. The Harten [8] and Harten-Hyman [9] formulations of the entropy have been widely accepted by the CFD community, mainly because of the simplicity of their implementation in existing codes based on Roe's scheme.

A slightly different viewpoint, that has led to a different form of the entropy fix for Roe's ARS, has been adopted by LeVeque [12]. Here the entropy fix is intended as a specific cure for the failure of Roe's scheme in case of transonic rarefactions. Roe himself in [17, p. 370] mentioned this kind of difficulty in a final remark discussing the entropy conditions. LeVeque's approach, although limited to Roe's linearization, develops a physical insight by extending some considerations valid in the scalar case to the general case of hyperbolic systems. In the scalar case the representation of a rarefaction fan by means of a single discontinuity leads to difficulties only under the condition of transonic rarefaction: otherwise the introduction of rarefaction shocks is admissible, because the resulting scheme selects the physically relevant solution anyway. According to this viewpoint, the entropy fix avoids representing a transonic rarefaction in the linearized problem by means of a single discontinuity and replaces the rarefaction fan by an alternative description. An advantage of LeVeque's approach is, in our opinion, that it suggests to apply the entropy correction in a selective way, and specifically only on the acoustic waves. On the contrary, the original Harten and Hyman approach implicitly suggests to act on all the eigenvalues of the system and therefore to correct also the contact discontinuity, where the addition of an artificial dissipation may be detrimental to the accuracy of the results.

To better understand these approaches and appreciate their differences, it seems worthwhile to establish a general framework encompassing the three aforementioned versions of the Harten entropy fix as well as the entropy fix due to LeVeque. In this work we will show that it is possible to develop a unitary setting suitable for analyzing all these entropy fixes and for clarifying the relationships existing among them.

The main purpose of this work is to provide an analysis of these widely used tools so that they may be appreciated in their actual significance, rather than being used merely as "black boxes" with blind attitude. We do not pretend to make a complete review of all the entropy fixes proposed in conjunction with different solvers. In fact, we will concentrate on the Harten's family of entropy fix, which has become very popular for its usefulness in correcting Roe's scheme. Other proposals, like for example the one due to Osher [16], are not considered herein.

It is worth remarking that such an analysis will prove useful to unveil the connection between the idea of entropy fix and some recently proposed positivity preserving schemes, like the class of HLL methods [10, 2, 3]. This will lead to the concept of positivity preserving entropy fix and to a solution-dependent correction of Roe's linearization in Jacobian form that assures positive solutions.

The paper is organized as follows. In Section 2 we briefly recall the main concepts of Roe's linearization, as they are used in the subsequent analysis. Then, in Section 3, we derive a general formulation of the entropy fix starting from LeVeque's viewpoint of breaking a single transonic wave into two waves connected by a linear variation of the unknown. In Section 4 we show how the different methods belonging to Harten's family of entropy fix, as well as LeVeque's method, fall within the derived general framework, and we propose a new variant of them. In Section 5 we report some numerical results of 1D test cases. In Section 6 we show how also the HLL schemes fall within the general formulation presented and we introduce the concept of positivity preserving entropy fix. We end with some concluding remarks.

2 Roe's linearization

As well known, Roe's method [17] is a Godunov-type scheme based on the approximate solution of a Riemann problem at each interface separating pair of neighbouring cells of the spatial discretization.

Let's consider the hyperbolic problem consisting in the one-dimensional system of conservation laws in the form:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} = \boldsymbol{0}, \tag{2.1}$$

where $\boldsymbol{u} = \boldsymbol{u}(x,t)$ is the vector of the conservative variables and $\boldsymbol{f}(\boldsymbol{u})$ is the corresponding flux vector. We assume that the set of states $\boldsymbol{u}$ has dimension p, with p an integer ≥ 1. A Riemann problem with left and right states $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$ is defined by the initial condition:

$$\boldsymbol{u}(x,0) = \begin{cases} \boldsymbol{u}_L & \text{if } x < 0 \\ \boldsymbol{u}_R & \text{if } x > 0\,. \end{cases} \tag{2.2}$$

The Roe solver replaces the original Riemann problem with a linearized problem of the form:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \hat{\boldsymbol{A}}(\boldsymbol{u}_L,\boldsymbol{u}_R)\frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{0}, \tag{2.3}$$

together with the initial condition (2.2). The matrix $\hat{\boldsymbol{A}}(\boldsymbol{u}_L,\boldsymbol{u}_R)$ is called a Roe linearization if it satisfies the following properties [17, p. 358]:

i) $\hat{\boldsymbol{A}}(\boldsymbol{u}_L,\boldsymbol{u}_R)$ has real eigenvalues and a corresponding set of eigenvectors that form a basis of $\mathbb{R}^p$,

ii) $\hat{\boldsymbol{A}}(\boldsymbol{u},\boldsymbol{u}) = \boldsymbol{A}(\boldsymbol{u})$ and $\hat{\boldsymbol{A}}(\boldsymbol{u}_L,\boldsymbol{u}_R) \to \boldsymbol{A}(\boldsymbol{u})$ smoothly for $\boldsymbol{u}_L,\ \boldsymbol{u}_R \to \boldsymbol{u}$,

iii) $\hat{\boldsymbol{A}}(\boldsymbol{u}_L,\boldsymbol{u}_R)(\boldsymbol{u}_R - \boldsymbol{u}_L) = \boldsymbol{f}(\boldsymbol{u}_R) - \boldsymbol{f}(\boldsymbol{u}_L)$.

We remark that in general such a linearization is not uniquely defined.

The (first order accurate) numerical flux of Roe's scheme is given by:

$$\boldsymbol{F}^{\text{Roe}}(\boldsymbol{u}_L,\boldsymbol{u}_R) = \frac{1}{2}[\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k=1}^{p} |\hat{a}_k|\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,, \tag{2.4}$$

where $\hat{a}_k = \hat{a}_k(\boldsymbol{u}_L,\boldsymbol{u}_R)$ and $\hat{\boldsymbol{r}}_k = \hat{\boldsymbol{r}}_k(\boldsymbol{u}_L,\boldsymbol{u}_R)$ are respectively the eigenvalues and the eigenvectors of the Roe matrix $\hat{\boldsymbol{A}} = \hat{\boldsymbol{A}}(\boldsymbol{u}_L,\boldsymbol{u}_R)$. The coefficients $\hat{\chi}_k$ are the components of the jump $\boldsymbol{u}_R - \boldsymbol{u}_L$ on these eigenvectors, namely,

$$\boldsymbol{u}_R - \boldsymbol{u}_L = \sum_{k=1}^{p} \hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,, \tag{2.5}$$

so that, by defining $\left|\hat{\boldsymbol{A}}\right| = \hat{\boldsymbol{R}}\left|\hat{\boldsymbol{\Lambda}}\right|\hat{\boldsymbol{R}}^{-1}$, with $\left|\hat{\boldsymbol{\Lambda}}\right| = \text{diag}(|\hat{a}_1|, |\hat{a}_2|, \ldots, |\hat{a}_p|)$ and $\hat{\boldsymbol{R}}$ the matrix of the right eigenvectors $\hat{\boldsymbol{r}}_k$, we also have

$$\left|\hat{\boldsymbol{A}}(\boldsymbol{u}_L,\boldsymbol{u}_R)\right|(\boldsymbol{u}_R - \boldsymbol{u}_L) = \sum_{k=1}^{p} |\hat{a}_k|\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,. \tag{2.6}$$

With this notation the numerical flux assumes the standard form:

$$\boldsymbol{F}^{\text{Roe}}(\boldsymbol{u}_L,\boldsymbol{u}_R) = \frac{1}{2}[\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\left|\hat{\boldsymbol{A}}(\boldsymbol{u}_L,\boldsymbol{u}_R)\right|(\boldsymbol{u}_R - \boldsymbol{u}_L)\,. \tag{2.7}$$

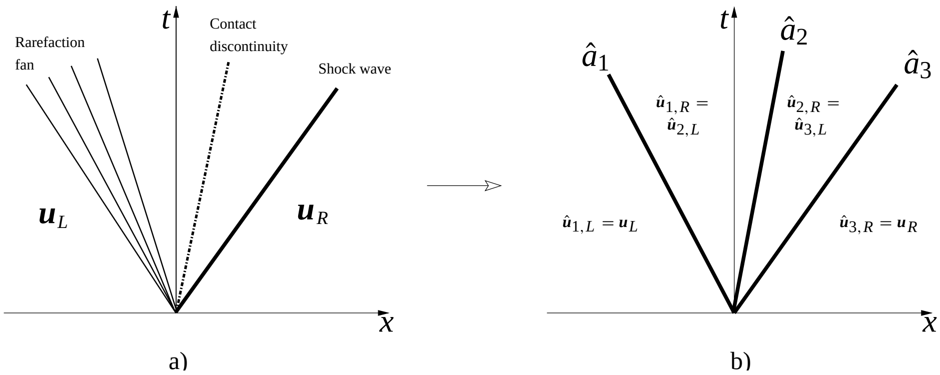

Figure 1: Solution of the Riemann problem: a) exact; b) Roe's linearization

2.1 Entropy violations

It is well known that Roe's linearization may lead to non-entropic weak solutions of the governing equations, due to the approximation of the exact solution of the Riemann problem with $\boldsymbol{\omega}^{\text{Roe}}(\frac{x}{t};\, \boldsymbol{u}_L, \boldsymbol{u}_R)$, made of constant states separated by discontinuities (Fig. 1) :

$$\boldsymbol{\omega}^{\text{Roe}}\big(\tfrac{x}{t}\,;\, \boldsymbol{u}_L, \boldsymbol{u}_R\big) = \begin{cases} \hat{\boldsymbol{u}}_0 = \boldsymbol{u}_L & \text{if} \quad x < \hat{a}_1\, t \\ \hat{\boldsymbol{u}}_1 & \text{if} \;\; \hat{a}_1\, t < x < \hat{a}_2\, t \\ \dots & \\ \hat{\boldsymbol{u}}_k & \text{if} \;\; \hat{a}_k\, t < x < \hat{a}_{k+1}\, t \\ \dots & \\ \hat{\boldsymbol{u}}_p = \boldsymbol{u}_R & \text{if} \;\; x > \hat{a}_p\, t\,. \end{cases} \tag{2.8}$$

This may be easily seen when the initial discontinuity of the Riemann problem satisfies the Rankine–Hugoniot condition for some propagation speed σ, namely:

$$\boldsymbol{f}(\boldsymbol{u}_R) - \boldsymbol{f}(\boldsymbol{u}_L) = \sigma\,(\boldsymbol{u}_R - \boldsymbol{u}_L)\,. \tag{2.9}$$

In such a case, because of property iii), the solution $\boldsymbol{\omega}^{\text{Roe}}(\frac{x}{t};\, \boldsymbol{u}_L, \boldsymbol{u}_R)$ consists in a single discontinuity of speed σ and strength $(\boldsymbol{u}_R - \boldsymbol{u}_L)$, but may not represent the correct weak solution. Entropy conditions have been formulated in different forms as additional conditions to the initial value problem to select the physically relevant, vanishing viscosity, solution, see, *e.g.*, [11] and [12]. Following Lax [11, p. 32], the entropy condition may be expressed as:

$$\psi(\boldsymbol{u}_R) - \psi(\boldsymbol{u}_L) - \sigma\,[\eta(\boldsymbol{u}_R) - \eta(\boldsymbol{u}_L)] \le 0\,, \tag{2.10}$$

where $\eta(\boldsymbol{u})$ is a convex entropy function and $\psi(\boldsymbol{u})$ the associated entropy flux for the considered hyperbolic system.

If $\boldsymbol{u}_R$ and $\boldsymbol{u}_L$ satisfy both (2.9) and (2.10), $\boldsymbol{\omega}^{\text{Roe}}(\frac{x}{t}; \boldsymbol{u}_L, \boldsymbol{u}_R)$ coincides with the exact solution. If, however, the pair $\boldsymbol{u}_R$,$\boldsymbol{u}_L$ satisfies (2.9) but not the entropy condition (2.10), the exact solution evolves in time (possibly as a rarefaction fan and some discontinuities), while $\boldsymbol{\omega}^{\text{Roe}}(\frac{x}{t}; \boldsymbol{u}_L, \boldsymbol{u}_R)$ still consists in the propagation of the initial discontinuity, and this is not an entropic solution.

We need to observe here that a correct approximation $\boldsymbol{F}^{\text{Roe}}(\boldsymbol{u}_L, \boldsymbol{u}_R)$ may still be obtained despite the presence of entropy violating discontinuities in $\boldsymbol{\omega}^{\text{Roe}}(\frac{x}{t}; \boldsymbol{u}_L, \boldsymbol{u}_R)$. In particular, it is easily checked in the scalar case that Roe's numerical flux coincides with the (entropy satisfying) Godunov flux, except when the exact solution consists of a transonic rarefaction. This consideration was expressed by Roe in [17, p. 370] with reference to the Euler equations for a polytropic ideal gas.

2.2 Entropy corrections

The above observations lead to a natural way of modifying Roe's linearization to ensure entropic solutions: when the exact solutions is a transonic rarefaction, we need to substitute the entropy violating discontinous approximation with some other approximation able to satisfy the weak form of the entropy condition. We denote this viewpoint as LeVeque's viewpoint, and we will use it in the following chapter to derive a general form of the Harten and Hyman entropy fix.

A different viewpoint, denoted here as Harten's one, is suggested by the expression (2.7) of the numerical flux, where $\left|\hat{\boldsymbol{A}}(\boldsymbol{u}_L, \boldsymbol{u}_R)\right|$ can be regarded as the viscosity matrix

$$\boldsymbol{Q}_\varepsilon = \left|\hat{\boldsymbol{A}}(\boldsymbol{u}_L, \boldsymbol{u}_R)\right| . \tag{2.11}$$

In the basis of the eigenvectors $\hat{\boldsymbol{r}}_k$, the viscosity matrix is diagonal, namely,

$$\boldsymbol{Q}'_\varepsilon = \text{diag}(|\hat{a}_1|, |\hat{a}_2|, \ldots, |\hat{a}_p|) . \tag{2.12}$$

This way of writing the viscosity matrix indicates that, to obtain a method guaranteeing the entropic solution, in the sense of vanishing viscosity solution, it is necessary to prevent any eigenvalue $\hat{a}_k$ from becoming zero. As a consequence, the entropy fix must modify the eigenvalues when they become too small.

Both viewpoints lead however to the same result—an easily implementable modification of Roe's flux. With the Harten and Hyman entropy fix, Roe's numerical flux is written in the following form:

$$\boldsymbol{F}^{\text{(e.f.)}}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{1}{2}[\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k=1}^{p} q(\hat{a}_k)\, \hat{\chi}_k\, \hat{\boldsymbol{r}}_k , \tag{2.13}$$

where the function $q(\hat{a}_k)$ is a suitable modification of the function $|\hat{a}_k|$.

3 General formulation of the entropy fix

We introduce here a general formulation of the entropy fix applied to Roe's scheme. This formulation comprises different versions of entropy fix due to Harten and Hyman, both the original ones and a version presented by LeVeque: as will be shown, the latter provides also the guidelines for the physical interpretation of the former.

The methods proposed by Harten and Hyman are defined as follows:

1. Consider the solution of the linearized Riemann problem (2.3) with left state $\boldsymbol{u}_L$ and right state $\boldsymbol{u}_R$: the k-th elementary wave, corresponding to the eigenvalue $\hat{a}_k(\boldsymbol{u}_L, \boldsymbol{u}_R)$ for $1 \leq k \leq p$, separates the constant states

$$\hat{\boldsymbol{u}}_{k,L} = \boldsymbol{u}_L + \sum_{j=1}^{k-1} \hat{\chi}_j \, \hat{\boldsymbol{r}}_j \qquad \text{and} \qquad \hat{\boldsymbol{u}}_{k,R} = \hat{\boldsymbol{u}}_{k,L} + \hat{\chi}_k \, \hat{\boldsymbol{r}}_k \, ,$$

   so that $\hat{\boldsymbol{u}}_{1,L} = \boldsymbol{u}_L$, $\hat{\boldsymbol{u}}_{k+1,L} = \hat{\boldsymbol{u}}_{k,R}$ for $1 \leq k \leq p-1$, and $\hat{\boldsymbol{u}}_{p,R} = \boldsymbol{u}_R$ (Fig. 1). We have marked all the intermediate states of the linear Riemann problem by a caret, writing $\hat{\boldsymbol{u}}_{k,L}$ and $\hat{\boldsymbol{u}}_{k,R}$, to explicitly distinguish all the elements associated with the Roe matrix $\hat{\boldsymbol{A}}$.

2. To each k-th elementary wave we associate two propagation velocities $a_{k,L}$ and $a_{k,R}$ (Fig. 2). If the k-th wave is approximating a rarefaction, $a_{k,L}$ and $a_{k,R}$ represent an estimate of the propagation velocity of the first and the last perturbation component of the fan, going from left to right. As a consequence, the condition identifying the occurrence of a transonic rarefaction for the k-th wave will be expressed by:

$$a_{k,L} < 0 < a_{k,R} \, .$$

3. Once recognized the presence of a transonic rarefaction, it is necessary to introduce a suitable representation of it which will replace the single discontinuity occurring in Roe's linearized problem. The equivalent substitute of a transonic rarefaction proposed by Harten and Hyman consists in two discontinuities with an intermediate state (Fig. 3). In the following we will consider first the formulation of the entropy fix which results from assuming a *constant* intermediate state (Subsection 3.1) and then a more general formulation based on a *linearly variable* intermediate state (Subsections 3.2 and 3.3).

Clearly, different choices of the velocities $a_{k,L}$ and $a_{k,R}$ and of the kind of representation for the transonic rarefaction lead to different types of entropy fixing.

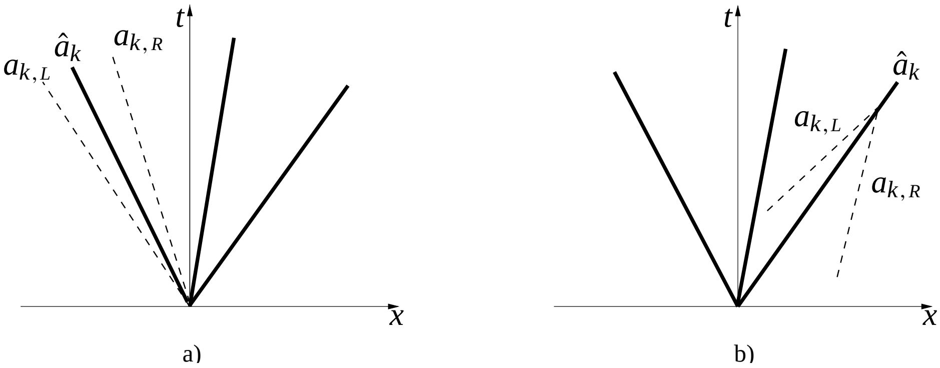

Figure 2: Propagation velocities: a) case $a_{k,L} < a_{k,R}$; b) case $a_{k,L} > a_{k,R}$

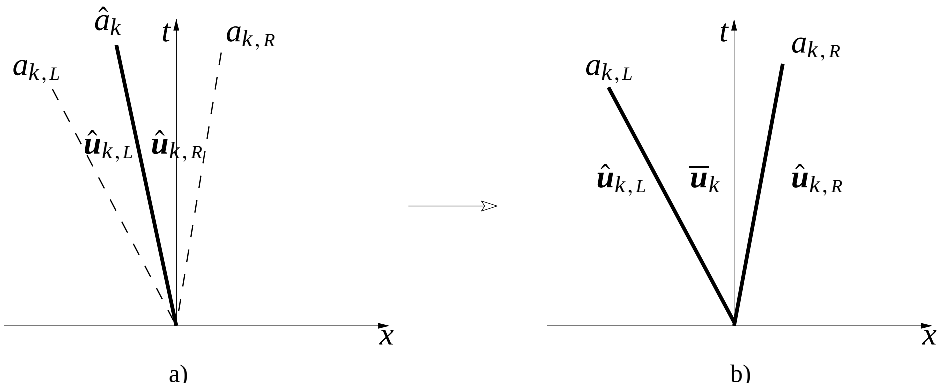

Figure 3: Transonic rarefaction: a) single Roe's wave; b) equivalent two-wave split

3.1 Constant intermediate state

Whenever for the k-th wave the condition $a_{k,L} < 0 < a_{k,R}$ is satisfied (with $a_{k,L}$ and $a_{k,R}$ to be suitably defined), the rarefaction wave is assumed to be replaced by *two discontinuities* propagating with velocities $a_{k,L}$ and $a_{k,R}$, with an intermediate constant state $\overline{\boldsymbol{u}}_k$. We therefore have the solution

$$\boldsymbol{u}_k^{\mathrm{c}}\big(\tfrac{x}{t}\big) = \begin{cases} \hat{\boldsymbol{u}}_{k,L} & \text{if } \ x < a_{k,L}\, t \\ \overline{\boldsymbol{u}}_k & \text{if } \ a_{k,L}\, t < x < a_{k,R}\, t \\ \hat{\boldsymbol{u}}_{k,R} & \text{if } \ x > a_{k,R}\, t\,, \end{cases} \tag{3.1}$$

The state $\overline{\boldsymbol{u}}_k$ is determined by imposing that the resulting modified Roe's solver be consistent with the integral form of the conservation law, namely,

$$(a_{k,R} - a_{k,L})\overline{\boldsymbol{u}}_k = (\hat{a}_k - a_{k,L})\hat{\boldsymbol{u}}_{k,L} + (a_{k,R} - \hat{a}_k)\hat{\boldsymbol{u}}_{k,R}\,,$$

which gives immediately the intermediate state

$$\overline{\boldsymbol{u}}_k = \frac{(\hat{a}_k - a_{k,L})\hat{\boldsymbol{u}}_{k,L} + (a_{k,R} - \hat{a}_k)\hat{\boldsymbol{u}}_{k,R}}{a_{k,R} - a_{k,L}}\,. \tag{3.2}$$

Then, the two jumps between the intermediate state $\overline{\boldsymbol{u}}_k$, and the left and right states across the k-th wave are expressed in the form:

$$\overline{\boldsymbol{u}}_k - \hat{\boldsymbol{u}}_{k,L} = \frac{a_{k,R} - \hat{a}_k}{a_{k,R} - a_{k,L}}\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k = \chi_{k,L}\,\hat{\boldsymbol{r}}_k\,,$$

$$\hat{\boldsymbol{u}}_{k,R} - \overline{\boldsymbol{u}}_k = \frac{\hat{a}_k - a_{k,L}}{a_{k,R} - a_{k,L}}\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k = \chi_{k,R}\,\hat{\boldsymbol{r}}_k\,,$$

where we have introduced the coefficients

$$\chi_{k,L} = \frac{a_{k,R} - \hat{a}_k}{a_{k,R} - a_{k,L}}\,\hat{\chi}_k \qquad \text{and} \qquad \chi_{k,R} = \frac{\hat{a}_k - a_{k,L}}{a_{k,R} - a_{k,L}}\,\hat{\chi}_k\,,$$

that, once multiplied by $\hat{\boldsymbol{r}}_k$, represent the left and right fraction of the solution jump $\hat{\chi}_k\,\hat{\boldsymbol{r}}_k$ associated with the k-th wave. In fact we have

$$\begin{aligned}\hat{\chi}_k\,\hat{\boldsymbol{r}}_k &= \hat{\boldsymbol{u}}_{k,R} - \hat{\boldsymbol{u}}_{k,L} \\ &= (\overline{\boldsymbol{u}}_k - \hat{\boldsymbol{u}}_{k,L}) + (\hat{\boldsymbol{u}}_{k,R} - \overline{\boldsymbol{u}}_k) \\ &= \chi_{k,L}\,\hat{\boldsymbol{r}}_k + \chi_{k,R}\,\hat{\boldsymbol{r}}_k\,.\end{aligned}$$

On this basis, we obtain that, when the k-th wave is a transonic rarefaction, in the expression of Roe's numerical flux the term $|\hat{a}_k|\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k$ will be replaced by the sum

$$|a_{k,L}|\,\chi_{k,L}\,\hat{\boldsymbol{r}}_k + |a_{k,R}|\,\chi_{k,R}\,\hat{\boldsymbol{r}}_k\,.$$

So, for $a_{k,L} < 0 < a_{k,R}$, the quantity $|\hat{a}_k|$ in the term $|\hat{a}_k|\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k$ of Roe's flux will be substituted by

$$-a_{k,L}\frac{a_{k,R}-\hat{a}_k}{a_{k,R}-a_{k,L}}+a_{k,R}\frac{\hat{a}_k-a_{k,L}}{a_{k,R}-a_{k,L}}=\frac{(a_{k,R}+a_{k,L})\hat{a}_k-2a_{k,R}\,a_{k,L}}{a_{k,R}-a_{k,L}}\,.$$

As a consequence, we come to the following expression of the numerical flux, with a corrected treatment of the transonic rarefaction,

$$\boldsymbol{F}^{(c)}(\boldsymbol{u}_L,\boldsymbol{u}_R)=\frac{1}{2}[\,\boldsymbol{f}(\boldsymbol{u}_L)+\,\boldsymbol{f}(\boldsymbol{u}_R)]-\frac{1}{2}\sum_{k=1}^{p}q^{c}(\hat{a}_k)\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,, \tag{3.3a}$$

where

$$q^{c}(\hat{a}_k)=\begin{cases}\dfrac{(a_{k,R}+a_{k,L})\hat{a}_k-2a_{k,R}\,a_{k,L}}{a_{k,R}-a_{k,L}} & \text{if } a_{k,L}<0<a_{k,R}\\ |\hat{a}_k| & \text{otherwise}\,.\end{cases} \tag{3.3b}$$

3.1.1 Entropy condition

Following Harten [9] (see also [5]) we want to demonstrate, at least for the scalar case, that the modified solution (3.1) does satisfy an entropy condition. Let us consider the scalar equation:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \tag{3.4a}$$

for a convex flux $f(u)$, with initial conditions

$$u(x,0)=\begin{cases}u_L & \text{if } x<0\\ u_R & \text{if } x>0\,.\end{cases} \tag{3.4b}$$

such as $a_L = f'(u_L) < 0 < f'(u_R) = a_R$. The exact solution of the Riemann problem (3.4) is:

$$\omega\left(\tfrac{x}{t}\right)=\begin{cases}u_L & \text{if } x<a_L\,t\\ (f')^{-1}(x) & \text{if } a_L\,t<x<a_R\,t\\ u_R & \text{if } x>a_R\,t\,.\end{cases} \tag{3.5}$$

and represents the case of a *transonic* rarefaction, in which the original Roe scheme would fail.

The scalar ($p = 1$) version of the approximate solution (3.1), based on a constant intermediate state $\overline{u}$, is:

$$u^{c}\left(\tfrac{x}{t}\right)=\begin{cases}u_L & \text{if } x<a_{1,L}\,t\\ \overline{u} & \text{if } a_{1,L}\,t<x<a_{1,R}\,t\\ u_R & \text{if } x>a_{1,R}\,t\,.\end{cases} \tag{3.6}$$

The entropy condition may be satisfied by (3.6) provided we chose $a_{1,L}$,$a_{1,R}$ such as:

$$a_{1,L} \le f'(u_L) = a_L\,; \qquad\qquad a_{1,R} \ge f'(u_R) = a_R\,.$$

Therefore, let us assume $a_{1,L} = a_L$, $a_{1,R} = a_R$ and rewrite (3.6) as:

$$u^{c}\left(\tfrac{x}{t}\right) = \begin{cases} u_L & \text{if } x < a_L\,t \\ \overline{u} & \text{if } a_L\,t < x < a_R\,t \\ u_R & \text{if } x > a_R\,t\,. \end{cases} \tag{3.7}$$

We need to show that (3.7) is consistent with the integral form of the entropy condition [12, p. 38]:

$$\int_0^{\tau} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\frac{\partial \eta(u)}{\partial t} + \frac{\partial \psi(u)}{\partial x}\right] dx\,dt \le 0\,, \tag{3.8}$$

where $\eta(u)$ is a convex entropy function and $\psi(u)$ is the associated flux function for equation (3.4a), h and τ being the space and time discretization intervals. For $u = u^{c}\left(\frac{x}{t}\right)$ this inequality is equivalent to

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \eta\left(u^{c}\left(\tfrac{x}{\tau}\right)\right)\,dx \le \frac{h}{2}\left[\eta(u_L) + \eta(u_R)\right] - \tau\left[\psi(u_R) - \psi(u_L)\right]. \tag{3.9}$$

Considering the piecewise constant function (3.7), we may first write the integral on the left side of (3.9) as:

$$\begin{aligned}\int_{-\frac{h}{2}}^{\frac{h}{2}} \eta\left(u^{c}\left(\tfrac{x}{\tau}\right)\right)\,dx = &\left(a_L\,\tau + \frac{h}{2}\right)\eta(u_L) \\ &+ (a_R - a_L)\tau\;\eta(\overline{u}) + \left(\frac{h}{2} - a_R\tau\right)\eta(u_R)\,.\end{aligned} \tag{3.10}$$

We may then observe that $\overline{u}$ has been obtained imposing the consistency with the integral form of the conservation law. This implies, since a_L and a_R are the limiting physical propagation speeds, that $\overline{u}$ is equal to the average value of the exact solution (3.5) in the interval $(a_L\tau, a_R\tau)$:

$$\overline{u} = \frac{1}{(a_R - a_L)\tau} \int_{a_L\tau}^{a_R\tau} \omega\left(\tfrac{x}{\tau}\right)\,dx\,. \tag{3.11}$$

By (3.11) Jensen's inequality gives:

$$\begin{aligned}\eta(\overline{u}) &= \eta\left(\frac{1}{(a_R - a_L)\tau} \int_{a_L\tau}^{a_R\tau} \omega\left(\tfrac{x}{\tau}\right)\,dx\right) \\ &\le \frac{1}{(a_R - a_L)\tau} \int_{a_L\tau}^{a_R\tau} \eta\left(\omega\left(\tfrac{x}{\tau}\right)\right)\,dx\,.\end{aligned} \tag{3.12}$$

Combining (3.10) and (3.12) we obtain:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \eta\left(u^{c}\left(\tfrac{x}{\tau}\right)\right)$$

$$\leq \left(a_L\,\tau + \frac{h}{2}\right)\eta(u_L) + \int_{a_L\tau}^{a_R\tau} \eta\left(\omega\left(\tfrac{x}{\tau}\right)\right)\,dx + \left(\frac{h}{2} - a_R\tau\right)\eta(u_R) =$$

$$\leq \int_{-\frac{h}{2}}^{\frac{h}{2}} \eta\left(\omega\left(\tfrac{x}{\tau}\right)\right)\,dx\,.$$

where in the last passage we have used the properties of the solution (3.5). From this last relation, considering that the exact solution $\omega\left(\frac{x}{t}\right)$ satisfies the integral form of the entropy condition:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \eta\left(\omega\left(\tfrac{x}{\tau}\right)\right)\,dx \leq \frac{h}{2}\left[\eta(u_L) + \eta(u_R)\right] - \tau\left[\psi(u_R) - \psi(u_L)\right],$$

we immediately obtain (3.9).

□

3.2 Linearly variable intermediate state

Supposing that a transonic rarefaction occurs for the k-th wave, we now introduce a linearly variable intermediate state in the range $a_{k,L} < \frac{x}{t} < a_{k,R}$ as follows:

$$\boldsymbol{u}_k^{l}\left(\tfrac{x}{t}\right) = \overline{\boldsymbol{u}}_k + \left(\frac{x}{t} - \frac{a_{k,L} + a_{k,R}}{2}\right)\frac{\sigma_k}{a_{k,R} - a_{k,L}}\left(\hat{\boldsymbol{u}}_{k,R} - \hat{\boldsymbol{u}}_{k,L}\right), \tag{3.13}$$

where $\overline{\boldsymbol{u}}_k$ is given by (3.2) and the parameter σ_k has to be suitably defined. It is easy to check that the solution $\boldsymbol{u}_k^{l}(\frac{x}{t})$ satisfies the consistency with the integral form of the conservation law.

The linear solution $\boldsymbol{u}_k^{l}(\frac{x}{t})$ given by (3.13) leads to the following expression of $q(\hat{a}_k)$:

$$q^{l}(\hat{a}_k) = \frac{(a_{k,R} + a_{k,L})\hat{a}_k - 2a_{k,R}\,a_{k,L}}{a_{k,R} - a_{k,L}} + \frac{a_{k,R}\,a_{k,L}}{a_{k,R} - a_{k,L}}\,\sigma_k\,. \tag{3.14}$$

In fact, let's assume that for a given Riemann problem the velocities $a_{k,L}$ and $a_{k,R}$ satisfy the condition identifying the transonic rarefaction for the k-th wave, namely, $a_{k,L} < 0 < a_{k,R}$. We now define a new approximate Riemann solver by means of a function $\boldsymbol{\omega}^{l}(\frac{x}{t}; \boldsymbol{u}_L, \boldsymbol{u}_R)$ which is obtained by introducing the linearly variable solution $\boldsymbol{u}_k^{l}(\frac{x}{t})$ between the propagation waves $a_{k,L}$ and $a_{k,R}$ in the standard Roe solver, *i.e.* we have:

$$\boldsymbol{\omega}^{l}\left(\tfrac{x}{t}\,;\,\boldsymbol{u}_L, \boldsymbol{u}_R\right) = \boldsymbol{u}_k^{l}\left(\tfrac{x}{t}\right), \qquad a_{k,L} < \tfrac{x}{t} < a_{k,R}\,. \tag{3.15}$$

(Here we assume that $\hat{a}_{k-1} < a_{k,L}$ and $a_{k,R} < \hat{a}_{k+1}$.) We can compute the corresponding numerical flux by means of:

$$\boldsymbol{F}^{(l)}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \boldsymbol{f}(\boldsymbol{u}_L) - \frac{1}{\tau}\int_{-\frac{h}{2}}^{0} \boldsymbol{\omega}^{l}\big(\tfrac{x}{\tau}; \boldsymbol{u}_L, \boldsymbol{u}_R\big)\, dx + \frac{h}{2\tau}\,\boldsymbol{u}_L\,, \tag{3.16}$$

where τ is the time step and h is the spatial mesh size. First, let's consider the integral appearing in the last relation:

$$\begin{aligned}\int_{-\frac{h}{2}}^{0} \boldsymbol{\omega}^{l}\big(\tfrac{x}{\tau}; \boldsymbol{u}_L, \boldsymbol{u}_R\big)\, dx &= \int_{-\frac{h}{2}}^{\hat{a}_1\tau} \boldsymbol{u}_L\, dx + \sum_{j=1}^{k-2}\left(\int_{\hat{a}_j\tau}^{\hat{a}_{j+1}\tau} \hat{\boldsymbol{u}}_j\, dx\right)\\ &\quad + \int_{\hat{a}_{k-1}\tau}^{a_{k,L}\,\tau} \hat{\boldsymbol{u}}_{k-1}\, dx + \int_{a_{k,L}\,\tau}^{0} \boldsymbol{u}_k^{l}\big(\tfrac{x}{\tau}\big)\, dx\,,\end{aligned} \tag{3.17}$$

where, for $1 \le j \le p$, $\hat{\boldsymbol{u}}_j = \boldsymbol{u}_L + \sum_{i=1}^{j-1} \hat{\chi}_i\, \hat{\boldsymbol{r}}_i$ denotes the j-th Roe's state. An easy computation gives, for the second term appearing in (3.17):

$$\sum_{j=1}^{k-2}\left(\int_{\hat{a}_j\tau}^{\hat{a}_{j+1}\tau} \hat{\boldsymbol{u}}_j\, dx\right) = \tau\,(\hat{a}_{k-1} - \hat{a}_1)\,\boldsymbol{u}_L + \tau\sum_{j=1}^{k-2} (\hat{a}_{k-1} - \hat{a}_j)\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j\,. \tag{3.18}$$

From (3.17) and (3.18) we get:

$$\begin{aligned}&\int_{-\frac{h}{2}}^{0} \boldsymbol{\omega}^{l}\big(\tfrac{x}{\tau}; \boldsymbol{u}_L, \boldsymbol{u}_R\big)\, dx\\ &\quad= \big(\hat{a}_1\tau + \frac{h}{2}\big)\boldsymbol{u}_L + \tau\,(\hat{a}_{k-1} - \hat{a}_1)\,\boldsymbol{u}_L + \tau\,\hat{a}_{k-1}\sum_{j=1}^{k-2} \hat{\chi}_j\,\hat{\boldsymbol{r}}_j - \tau\sum_{j=1}^{k-2} \hat{a}_j\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j\\ &\qquad + \tau\,(a_{k,L} - \hat{a}_{k-1})\,\hat{\boldsymbol{u}}_{k-1} + \int_{a_{k,L}\,\tau}^{0} \boldsymbol{u}_k^{l}\big(\tfrac{x}{\tau}\big)\, dx\\ &\quad= \frac{h}{2}\boldsymbol{u}_L + \tau\,\hat{a}_{k-1}\left(\boldsymbol{u}_L + \sum_{j=1}^{k-2} \hat{\chi}_j\,\hat{\boldsymbol{r}}_j - \hat{\boldsymbol{u}}_{k-1}\right) - \tau\sum_{j=1}^{k-2} \hat{a}_j\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j\\ &\qquad + \tau\,a_{k,L}\,\hat{\boldsymbol{u}}_{k-1} + \int_{a_{k,L}\,\tau}^{0} \boldsymbol{u}_k^{l}\big(\tfrac{x}{\tau}\big)\, dx\,.\end{aligned} \tag{3.19}$$

Introducing the last relation in (3.16) we have:

$$\begin{aligned}\boldsymbol{F}^{(l)}(\boldsymbol{u}_L, \boldsymbol{u}_R) &= \boldsymbol{f}(\boldsymbol{u}_L) + \hat{a}_{k-1}\,\hat{\chi}_{k-1}\,\hat{\boldsymbol{r}}_{k-1} + \sum_{j=1}^{k-2} \hat{a}_j\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j\\ &\quad - a_{k,L}\,\hat{\boldsymbol{u}}_{k-1} - \frac{1}{\tau}\int_{a_{k,L}\,\tau}^{0} \boldsymbol{u}_k^{l}\big(\tfrac{x}{\tau}\big)\, dx\,.\end{aligned} \tag{3.20}$$

Now, by adding and subtracting the quantity

$$\frac{1}{2}\,[\,\boldsymbol{f}(\boldsymbol{u}_R) - \boldsymbol{f}(\boldsymbol{u}_L)] = \frac{1}{2}\sum_{j=1}^{p} \hat{a}_j\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j$$

from (3.20) we obtain:

$$\begin{aligned}\boldsymbol{F}^{(l)}(\boldsymbol{u}_L,\boldsymbol{u}_R) = \frac{1}{2}\,[\,\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] + \sum_{j=1}^{k-1} \hat{a}_j\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j - a_{k,L}\,\hat{\boldsymbol{u}}_{k-1} \\ - \frac{1}{\tau}\int_{a_{k,L}\,\tau}^{0} \boldsymbol{u}_k^{l}\big(\tfrac{x}{\tau}\big)\,dx - \frac{1}{2}\sum_{j=1}^{p} \hat{a}_j\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j\,.\end{aligned} \tag{3.21}$$

Finally, taking into account that

$$\hat{a}_j < 0 \ \text{ if } \ j < \qquad \text{and} \qquad \hat{a}_j > 0 \ \text{ if } \ j > k\,,$$

we have:

$$\begin{aligned}\boldsymbol{F}^{(l)}(\boldsymbol{u}_L,\boldsymbol{u}_R) = \frac{1}{2}\,[\,\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{j=1}^{k-1} |\hat{a}_j|\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j \\ - a_{k,L}\,\hat{\boldsymbol{u}}_{k-1} - \frac{1}{\tau}\int_{a_{k,L}\,\tau}^{0} \boldsymbol{u}_k^{l}\big(\tfrac{x}{\tau}\big)\,dx - \frac{1}{2}\,\hat{a}_k\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k \\ - \frac{1}{2}\sum_{j=k+1}^{p} |\hat{a}_j|\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j\,.\end{aligned} \tag{3.22}$$

Let's now evaluate the integral appearing in (3.22), using the definition (3.13) and reminding that $\hat{\boldsymbol{u}}_{k,R} - \hat{\boldsymbol{u}}_{k,L} = \hat{\chi}_k\,\hat{\boldsymbol{r}}_k$:

$$\begin{aligned}-\frac{1}{\tau}\int_{a_{k,L}\,\tau}^{0} \boldsymbol{u}_k^{l}\big(\tfrac{x}{\tau}\big)\,dx &= a_{k,L}\,\overline{\boldsymbol{u}}_k + \left(\frac{a_{k,L}^2}{2} - \frac{a_{k,L}+a_{k,R}}{2}\,a_{k,L}\right)\frac{\sigma_k}{a_{k,R}-a_{k,L}}\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k \\ &= a_{k,L}\,\overline{\boldsymbol{u}}_k - \frac{a_{k,R}\,a_{k,L}}{2}\,\frac{\sigma_k}{a_{k,R}-a_{k,L}}\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,.\end{aligned} \tag{3.23}$$

Noticing that $\hat{\boldsymbol{u}}_{k,L} = \hat{\boldsymbol{u}}_{k-1}$ and $\hat{\boldsymbol{u}}_{k,R} = \hat{\boldsymbol{u}}_k$, by substituting (3.23) in (3.22) we obtain:

$$\begin{aligned}\boldsymbol{F}^{(l)}(\boldsymbol{u}_L,\boldsymbol{u}_R) = \frac{1}{2}\,[\,\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{j=1}^{k-1} |\hat{a}_j|\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j \\ - \frac{1}{2}\left[\frac{(a_{k,R}+a_{k,L})\hat{a}_k - 2a_{k,R}\,a_{k,L}}{a_{k,R}-a_{k,L}} + \frac{a_{k,R}\,a_{k,L}}{a_{k,R}-a_{k,L}}\,\sigma_k\right]\hat{\chi}_k\,\hat{\boldsymbol{r}}_k \\ - \frac{1}{2}\sum_{j=k+1}^{p} |\hat{a}_j|\,\hat{\chi}_j\,\hat{\boldsymbol{r}}_j\,.\end{aligned} \tag{3.24}$$

The last equation proves that, if we introduce in Roe's solver the linearly variable solution (3.13) for $a_{k,L} < \frac{x}{t} < a_{k,R}$, being $a_{k,L} < 0 < a_{k,R}$, the function $|\hat{a}_k|$ appearing in the expression of Roe's numerical flux is replaced by the function $q^1(\hat{a}_k)$ given by (3.14).

□

On the basis of the above result, we can now write a general formulation of the entropy fix in terms of propagation velocities, expressed by the following form of the numerical flux:

$$\boldsymbol{F}^{(1)}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{1}{2}[\,\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k=1}^{p} q_a^1(\hat{a}_k)\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,, \tag{3.25a}$$

where

$$q_a^1(\hat{a}_k) = \begin{cases} \dfrac{(a_{k,R}+a_{k,L})\hat{a}_k - 2a_{k,R}\,a_{k,L}}{a_{k,R}-a_{k,L}} + \dfrac{a_{k,R}\,a_{k,L}}{a_{k,R}-a_{k,L}}\,\sigma_k & \\ & \text{if } \; a_{k,L} < 0 < a_{k,R} \\ |\hat{a}_k| & \text{otherwise}\,. \end{cases} \tag{3.25b}$$

Of course, if we set $\sigma_k = 0$ we recover the formulation (3.3) based on the assumption of a constant intermediate state as a particular case.

The subscript 'a' used in the above relations has been introduced to distinguish the expression (3.25) from the one to be introduced in the next subsection.

3.3 General formulation in terms of velocity differences

The general scheme (3.25), comprising both a linearly variable and a constant (for $\sigma_k = 0$) intermediate state, is expressed in terms of propagation speeds a, but it can also be reformulated in terms of velocity *differences* as follows. Let us first define the two (always nonnegative) quantities:

$$\delta_{k,L} = \max\{0,\ \hat{a}_k - a_{k,L}\} \qquad \text{and} \qquad \delta_{k,R} = \max\{0,\ a_{k,R} - \hat{a}_k\}\,. \tag{3.26}$$

Assuming that $\hat{a}_k$ belongs to the interval $(a_{k,L}, a_{k,R})$ with $a_{k,L} < a_{k,R}$, definition (3.26) implies that the condition $a_{k,L} < 0 < a_{k,R}$ is equivalent to the condition $-\delta_{k,R} < \hat{a}_k < \delta_{k,L}$. In such a situation we have $\delta_{k,L} = \hat{a}_k - a_{k,L}$ and $\delta_{k,R} = a_{k,R} - \hat{a}_k$, so that we can write:

$$a_{k,L} = \hat{a}_k - \delta_{k,L} \qquad \text{and} \qquad a_{k,R} = \hat{a}_k + \delta_{k,R}\,.$$

Introducing these relations in the expression of $q^1(\hat{a}_k)$ given by (3.14), we have:

$$q^1(\hat{a}_k) = \frac{\sigma_k\,\hat{a}_k^2 - (1-\sigma_k)(\delta_{k,R}-\delta_{k,L})\hat{a}_k + (2-\sigma_k)\delta_{k,L}\,\delta_{k,R}}{\delta_{k,L}+\delta_{k,R}}\,. \tag{3.27}$$

It follows that the general formulation (3.25b) of the entropy fix with a linearly variable intermediate state can be recast in terms of the velocity *differences* $\delta_{k,L}$ and $\delta_{k,R}$, as:

$$\boldsymbol{F}^{(\mathrm{l})}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{1}{2}[\, \boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k=1}^{p} q_\delta^{\mathrm{l}}(\hat{a}_k)\, \hat{\chi}_k\, \hat{\boldsymbol{r}}_k\,, \tag{3.28a}$$

where

$$q_\delta^{\mathrm{l}}(\hat{a}_k) = \begin{cases} \dfrac{\sigma_k\, \hat{a}_k^2 - (1-\sigma_k)(\delta_{k,R} - \delta_{k,L})\hat{a}_k + (2-\sigma_k)\delta_{k,L}\,\delta_{k,R}}{\delta_{k,L} + \delta_{k,R}}\,, \\ \qquad\qquad\qquad\qquad \text{if } \; -\delta_{k,R} < \hat{a}_k < \delta_{k,L} \\ |\hat{a}_k| \qquad\qquad\qquad\quad \text{otherwise.} \end{cases} \tag{3.28b}$$

The expression (3.27) allows us to recognize, better than (3.25b), the type of modification of the function $|\hat{a}_k|$ accomplished by an entropy fix in the neighbourhood of the origin. As we have already said, each entropy fix has to modify the eigenvalue $\hat{a}_k$ when it becomes too small, so as to prevent its vanishing. Formulation (3.28) helps regarding the entropy fix as a means to select the vanishing viscosity solution, in the spirit of Harten's original interpretation.

We can easily see from (3.27) that for $-\delta_{k,R} < \hat{a}_k < \delta_{k,L}$ the function $|\hat{a}_k|$ is replaced by a parabolic function, which degenerates in a straight line when the intermediate state is constant, *i.e.* for $\sigma_k = 0$. We also notice that the function $q_\delta^{\mathrm{l}}(\hat{a}_k)$ is continuous. The parameter σ_k must satisfy the condition $\sigma_k \geq 0$ in order to have a curve under the straight line corresponding to the case of a constant intermediate state, and therefore to reduce the amount of artificial numerical viscosity. Moreover, we require that the slope of $q_\delta^{\mathrm{l}}(\hat{a}_k)$ in $\hat{a}_k = \delta_{k,L}$ is ≤ 1 and the slope in $\hat{a}_k = -\delta_{k,R}$ is ≥ -1. From (3.27) it follows immediately the condition for σ_k:

$$\sigma_k \leq \frac{2\min(\delta_{k,L}, \delta_{k,R})}{\delta_{k,L} + \delta_{k,R}}\,.$$

4 Review of entropy fixes

In this section we recall the different versions of Harten and Hyman entropy fix and we show how they fit within the general framework derived above.

4.1 First entropy fix of Harten and Hyman

In [9] Harten and Hyman present an entropy fix formulated as follows[1]:

$$\boldsymbol{F}^{(\mathrm{HH1})}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{1}{2}[\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k=1}^{p} q^{\mathrm{HH1}}(\hat{a}_k)\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,, \tag{4.1a}$$

where

$$q^{\mathrm{HH1}}(\hat{a}_k) = \begin{cases} \delta_k & \text{if } |\hat{a}_k| < \delta_k \\ |\hat{a}_k| & \text{if } |\hat{a}_k| \geq \delta_k\,, \end{cases} \tag{4.1b}$$

with

$$\delta_k = \max\{0,\ \hat{a}_k - a_k(\boldsymbol{u}_L),\ a_k(\boldsymbol{u}_R) - \hat{a}_k\}\,, \tag{4.1c}$$

$a_k(\boldsymbol{u})$ being the k-th eigenvalue of the matrix $\boldsymbol{A}(\boldsymbol{u}) = \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial \boldsymbol{u}}$.

Whenever $a_k(\boldsymbol{u}_L) \geq \hat{a}_k \geq a_k(\boldsymbol{u}_R)$, *i.e.*, if the k-th elementary wave is a substitute for a wave that cannot be a rarefaction, we observe that $\delta_k = 0$ and $q^{\mathrm{HH1}}(\hat{a}_k) = |\hat{a}_k|$.

The Harten and Hyman scheme (4.1) may be shown equivalent to a particular case of the general formulations presented in the previous section. It is convenient to consider the form (3.28b) in terms of velocity differences: by inspection, scheme (4.1) is equivalent to (3.28b) provided we set $\sigma_k = 0$ and

$$\delta_{k,L} = \delta_{k,R} = \delta_k\,. \tag{4.2a}$$

This corresponds to the choice

$$a_{k,L} = \hat{a}_k - \delta_k \qquad \text{and} \qquad a_{k,R} = \hat{a}_k + \delta_k \tag{4.2b}$$

to be used in form (3.25b). Being $\sigma_k = 0$, the intermediate state introduced in the transonic rarefaction situation for the k-th wave is given by the constant solution $\overline{\boldsymbol{u}}_k$ expressed by (3.2). Moreover, due to definitions (4.2b), $\overline{\boldsymbol{u}}_k$ is equal to the arithmetic mean of the left and right states $\hat{\boldsymbol{u}}_{k,L}$ and $\hat{\boldsymbol{u}}_{k,R}$:

$$\overline{\boldsymbol{u}}_k = \frac{\hat{\boldsymbol{u}}_{k,L} + \hat{\boldsymbol{u}}_{k,R}}{2}\,, \tag{4.3}$$

since $(\hat{a}_k - a_{k,L}) = (a_{k,R} - \hat{a}_k) = \delta_k$ and $(a_{k,R} - a_{k,L}) = 2\delta_k$. It should be emphasized that this method does not require to evaluate the states $\hat{\boldsymbol{u}}_{k,L}$ and $\hat{\boldsymbol{u}}_{k,R}$ explicitly.

From expression (4.1b) the type of modification performed on the eigenvalues $\hat{a}_k$ is realized immediately: for $|\hat{a}_k| < \delta_k$ the function $|\hat{a}_k|$ is replaced by a constant function equal to δ_k, as shown in Figure 4.

[1] Here we assume suitable convexity hypothesis which in the scalar case consist in adopting a convex flux function $f(u)$. For a system of equations of a problem which requires the specification of an equation of state, we expect a *convex* equation of state, as occurs for the Euler equations. In this connection, see also [4] and [15].

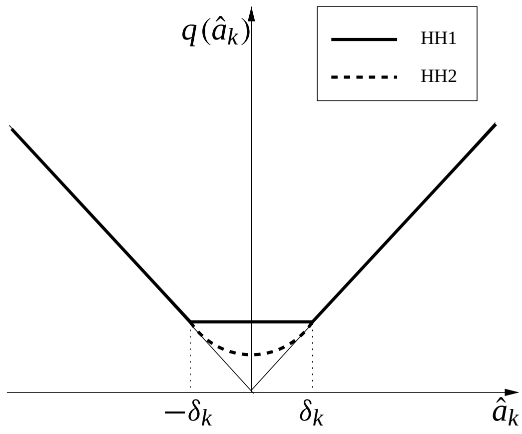

Figure 4: First and second entropy fix of Harten and Hyman

Remark 4.1 We observe that the choice (4.2b) for $a_{k,L}$ and $a_{k,R}$ may produce a simultaneous action on several waves, in the case of systems of equations, *i.e*. the condition $a_{k,L} < 0 < a_{k,R}$ can be verified for more than one value of k, while in the exact solution of the given Riemann problem only one *transonic* rarefaction is possible at most.

4.2 LeVeque entropy fix

LeVeque in [12, Chap. 14, Sec. 2.2] presents a formulation of the entropy fix which explicitly recognizes the occurence of a transonic rarefaction. LeVeque attributes the method to Harten and Hyman, but it is slightly different and therefore we prefer to denote it as LeVeque's method.

LeVeque entropy fix acts only when a situation of transonic rarefaction is detected for the k-wave of Roe method, *i.e.*, when:

$$a_k(\hat{\boldsymbol{u}}_{k,L}) < 0 < a_k(\hat{\boldsymbol{u}}_{k,R}) \tag{4.4}$$

where we have, for $1 \le k \le p$,

$$\hat{\boldsymbol{u}}_{k,L} = \boldsymbol{u}_L + \sum_{j=1}^{k-1} \hat{\chi}_j\, \hat{\boldsymbol{r}}_j \qquad \text{and} \qquad \hat{\boldsymbol{u}}_{k,R} = \hat{\boldsymbol{u}}_{k,L} + \hat{\chi}_k\, \hat{\boldsymbol{r}}_k\,, \tag{4.5}$$

so that, in particular, $\hat{\boldsymbol{u}}_{1,L} = \boldsymbol{u}_L$ and $\hat{\boldsymbol{u}}_{p,R} = \boldsymbol{u}_R$, as usual in Roe scheme. When condition (4.4) is satisfied for wave $k = r$, the original Roe method is modified by the following numerical flux:

$$\begin{aligned} \boldsymbol{F}^{(\mathrm{LV})}(\boldsymbol{u}_L, \boldsymbol{u}_R) &= \boldsymbol{f}(\boldsymbol{u}_L) + \sum_{k \neq r} \mathcal{N}(\hat{a}_k)\, \hat{\chi}_k\, \hat{\boldsymbol{r}}_k + a_r(\hat{\boldsymbol{u}}_{r,L})\, \frac{a_r(\hat{\boldsymbol{u}}_{r,R}) - \hat{a}_r}{a_r(\hat{\boldsymbol{u}}_{r,R}) - a_r(\hat{\boldsymbol{u}}_{r,L})}\, \hat{\chi}_r\, \hat{\boldsymbol{r}}_r \\ &= \boldsymbol{f}(\boldsymbol{u}_R) - \sum_{k \neq r} \mathcal{P}(\hat{a}_k)\, \hat{\chi}_k\, \hat{\boldsymbol{r}}_k - a_r(\hat{\boldsymbol{u}}_{r,R})\, \frac{\hat{a}_r - a_r(\hat{\boldsymbol{u}}_{r,L})}{a_r(\hat{\boldsymbol{u}}_{r,R}) - a_r(\hat{\boldsymbol{u}}_{r,L})}\, \hat{\chi}_r\, \hat{\boldsymbol{r}}_r\,, \end{aligned} \tag{4.6}$$

where we introduced the operators $\mathcal{P}(\cdot)$ and $\mathcal{N}(\cdot)$, defined as follows[2]:

$$\mathcal{P}(\alpha) \overset{\Delta}{=} \text{positive part of } \alpha \,,$$
$$\mathcal{N}(\alpha) \overset{\Delta}{=} \text{negative part of } \alpha \,,$$

for any real α, which will be used throughout. We prefer using the operators $\mathcal{P}(\cdot)$ and $\mathcal{N}(\cdot)$ instead of employing the notation with the superscripts '$+$' or '$-$', appended to a variable to represent respectively the maximum or the minimun between zero and the value of the considered variable. The notation adopted in this work represents a given quantity as the result of the action of an operator upon an argument.

Avearging the two alternative equivalent expressions of the numerical flux given by (4.6), we get:

$$\begin{aligned} \boldsymbol{F}^{(\mathrm{LV})}(\boldsymbol{u}_L, \boldsymbol{u}_R) =& \frac{1}{2}[\,\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k\neq r} |\hat{a}_k|\, \hat{\chi}_k\, \hat{\boldsymbol{r}}_k \\ & - \frac{1}{2}\frac{[a_r(\hat{\boldsymbol{u}}_{r,R}) + a_r(\hat{\boldsymbol{u}}_{r,L})]\,\hat{a}_r - 2a_r(\hat{\boldsymbol{u}}_{r,R})\,a_r(\hat{\boldsymbol{u}}_{r,L})}{a_r(\hat{\boldsymbol{u}}_{r,R}) - a_r(\hat{\boldsymbol{u}}_{r,L})}\,\hat{\chi}_r\,\hat{\boldsymbol{r}}_r \,, \end{aligned} \tag{4.7}$$

or, in more compact form,

$$\boldsymbol{F}^{(\mathrm{LV})}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{1}{2}[\,\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k=1}^{p} q^{\mathrm{LV}}(\hat{a}_k)\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k \,, \tag{4.8a}$$

with

$$q_a^{\mathrm{LV}}(\hat{a}_k) = \begin{cases} \dfrac{[a_k(\hat{\boldsymbol{u}}_{k,R}) + a_k(\hat{\boldsymbol{u}}_{k,L})]\,\hat{a}_k - 2a_k(\hat{\boldsymbol{u}}_{k,R})\,a_k(\hat{\boldsymbol{u}}_{k,L})}{a_k(\hat{\boldsymbol{u}}_{k,R}) - a_k(\hat{\boldsymbol{u}}_{k,L})} & \\ \qquad\qquad\qquad \text{if } \ a_k(\hat{\boldsymbol{u}}_{k,L}) < 0 < a_k(\hat{\boldsymbol{u}}_{k,R}) & \\ |\hat{a}_k| \qquad\qquad \text{otherwise}\,. & \end{cases} \tag{4.8b}$$

This function is coincident with that given by (3.25b) with $\sigma_k = 0$, provided in the latter we define

$$a_{k,L} = a_k(\hat{\boldsymbol{u}}_{k,L}) \qquad \text{and} \qquad a_{k,R} = a_k(\hat{\boldsymbol{u}}_{k,R})\,. \tag{4.9}$$

The intermediate state is constant and is now given by:

$$\overline{\boldsymbol{u}}_k = \frac{[\hat{a}_k - a_k(\hat{\boldsymbol{u}}_{k,L})]\hat{\boldsymbol{u}}_{k,L} + [a_k(\hat{\boldsymbol{u}}_{k,R}) - \hat{a}_k]\hat{\boldsymbol{u}}_{k,R}}{a_k(\hat{\boldsymbol{u}}_{k,R}) - a_k(\hat{\boldsymbol{u}}_{k,L})}\,.$$

[2]Strictly speaking, $\mathcal{P}(\alpha)$ is the *nonnegative* part of α and $\mathcal{N}(\alpha)$ is the *nonpositive* part of α.

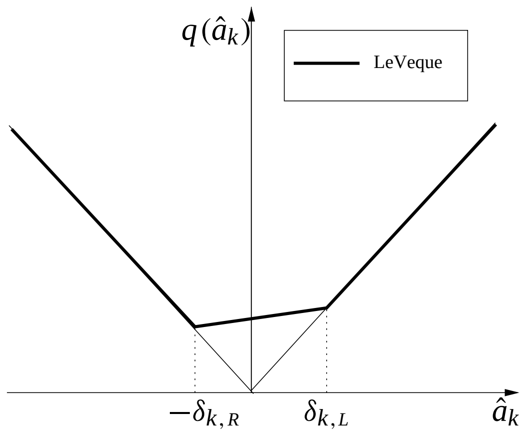

Figure 5: LeVeque's entropy fix

Using the definitions $a_{k,L}$ and $a_{k,R}$ above, LeVeque's entropy fix can also be expressed in terms of velocity differences, which gives the following equivalent expression of the function $q^{\text{LV}}(\hat{a}_k)$:

$$q_\delta^{\text{LV}}(\hat{a}_k) = \begin{cases} \dfrac{(\delta_{k,L} - \delta_{k,R})\hat{a}_k + 2\delta_{k,L}\,\delta_{k,R}}{\delta_{k,L} + \delta_{k,R}} & \text{if } \; -\delta_{k,R} < \hat{a}_k < \delta_{k,L} \\ |\hat{a}_k| & \text{otherwise} \end{cases}$$

with

$$\delta_{k,L} = \max\{0,\ \hat{a}_k - a_k(\hat{\boldsymbol{u}}_{k,L})\} \qquad \text{and} \qquad \delta_{k,R} = \max\{0,\ a_k(\hat{\boldsymbol{u}}_{k,R}) - \hat{a}_k\}\,.$$

For $-\delta_{k,R} < \hat{a}_k < \delta_{k,L}$ the function $|\hat{a}_k|$ is replaced by a straight line with slope $\frac{\delta_{k,L}-\delta_{k,R}}{\delta_{k,L}+\delta_{k,R}}$, and passing through $\left(0,\ \frac{2\delta_{k,L}\,\delta_{k,R}}{\delta_{k,L}+\delta_{k,R}}\right)$. See Figure 5.

Original form of LeVeque entropy fix

It is worth noting that, with the aim of generalizing the computation of wave speed for algorithmic easiness, LeVeque [12] expressed his entropy fix in the following form:

$$\begin{aligned} \boldsymbol{F}^{(\text{LV})}(\boldsymbol{u}_L, \boldsymbol{u}_R) &= \boldsymbol{f}(\boldsymbol{u}_L) + \sum_{k=1}^{p} a^{\text{Neg}}(\hat{a}_k)\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k \\ &= \boldsymbol{f}(\boldsymbol{u}_R) - \sum_{k=1}^{p} a^{\text{Pos}}(\hat{a}_k)\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,, \end{aligned} \tag{4.10}$$

where negative and positive wave speeds have been introduced in the form:

$$a^{\text{Neg}}(\hat{a}_k) = \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))\,\frac{\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) - \hat{a}_k}{\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) - \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))}\,,$$

$$a^{\text{Pos}}(\hat{a}_k) = \mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R}))\,\frac{\hat{a}_k - \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))}{\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) - \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))}\,.$$

Averaging again the two alternative equivalent expressions of the numerical flux, we get:

$$\boldsymbol{F}^{(\text{LV})}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{1}{2}[\,\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k=1}^{p} q^{\text{LV}}(\hat{a}_k)\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,, \tag{4.11a}$$

where

$$q^{\text{LV}}(\hat{a}_k) = \frac{[\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) + \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))]\,\hat{a}_k - 2\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R}))\,\mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))}{\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) - \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))}\,. \tag{4.11b}$$

Actually, this relation must be completed by adding the condition that prevents the vanishing of denominator, $\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) \neq \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))$, a situation that can happen only if $\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) = \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L})) = 0$, *i.e.* if $(a_k(\hat{\boldsymbol{u}}_{k,L}) \geq 0) \wedge (a_k(\hat{\boldsymbol{u}}_{k,R}) \leq 0)$, hence never in the transonic rarefaction case. We can therefore impose $q^{\text{LV}}(\hat{a}_k) = |\hat{a}_k|$ if $\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) = \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))$.

The expression of $q^{\text{LV}}(\hat{a}_k)$ taking into account the last comment assumes the explicit form:

$$q_a^{\text{LV}}(\hat{a}_k) = \begin{cases} \dfrac{[\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) + \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))]\,\hat{a}_k - 2\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R}))\,\mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))}{\mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) - \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))} \\ \qquad\qquad\qquad\qquad \text{if } \mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) \neq \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L})) \\ |\hat{a}_k| \qquad\qquad\qquad \text{if } \mathcal{P}(a_k(\hat{\boldsymbol{u}}_{k,R})) = \mathcal{N}(a_k(\hat{\boldsymbol{u}}_{k,L}))\,. \end{cases} \tag{4.12}$$

It can be shown easily that expression (4.12) is still equivalent to the general form (3.25b) with the choice (4.9) and $\sigma_k = 0$.

Comparison of the first method of Harten and Hyman with LeVeque's method

As we have seen, both the first entropy fix of Harten and Hyman and the entropy fix of LeVeque introduce a constant intermediate state, while they use different definitions of the propagation velocities $a_{k,L}$ and $a_{k,R}$. This implies a different kind of intervention criterion and a different action on the modification of the function $|\hat{a}_k|$. The differences between the two methods can be summarized as follows:

- The intervention criterion of the first entropy fix of Harten and Hyman is based only on the extreme states $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$, while LeVeque's method uses also the information of the intermediate states $\hat{\boldsymbol{u}}_{k,L}$ and $\hat{\boldsymbol{u}}_{k,R}$, which are different, in general, from $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$.

- The action of the entropy fix of Harten and Hyman introduces a higher numerical viscosity than LeVeque's method. This is illustrated clearly in Figures 4 and 5, which show the modifications of the function $|\hat{a}_k|$ accomplished by the two methods and by a second entropy fix due to Harten and Hyman to be described below.

4.3 Second entropy fix of Harten and Hyman

In [9, p. 266], in a note, Harten and Hyman present an entropy fix formulated as follows:

$$\boldsymbol{F}^{(\mathrm{HH2})}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{1}{2}[\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k=1}^{p} q^{\mathrm{HH2}}(\hat{a}_k)\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,, \tag{4.13a}$$

where

$$q^{\mathrm{HH2}}(\hat{a}_k) = \begin{cases} \dfrac{1}{2}\left(\dfrac{\hat{a}_k^2}{\delta_k} + \delta_k\right) & \text{if } |\hat{a}_k| < \delta_k \\ |\hat{a}_k| & \text{if } |\hat{a}_k| \geq \delta_k\,, \end{cases} \tag{4.13b}$$

with

$$\delta_k = \max\{0,\ \hat{a}_k - a_k(\boldsymbol{u}_L),\ a_k(\boldsymbol{u}_R) - \hat{a}_k\}\,. \tag{4.13c}$$

This second method of Harten and Hyman differs from the first one (scheme (4.1)) only in the type of variation of the intermediate state, which here consists in a linear transition between $\hat{\boldsymbol{u}}_{k,L}$ and $\hat{\boldsymbol{u}}_{k,R}$ for $a_{k,L} < \frac{x}{t} < a_{k,R}$, matching continuously with the end values $\hat{\boldsymbol{u}}_{k,L}$ and $\hat{\boldsymbol{u}}_{k,R}$; the linearly variable intermediate state is defined by Harten and Hyman as follows:

$$\boldsymbol{u}_k^{\mathrm{l}}\left(\frac{x}{t}\right) = \hat{\boldsymbol{u}}_{k,L} + \frac{\frac{x}{t} - a_{k,L}}{a_{k,R} - a_{k,L}}\,(\hat{\boldsymbol{u}}_{k,R} - \hat{\boldsymbol{u}}_{k,L})\,. \tag{4.14}$$

Exactly as in the first method HH1, the limiting velocities in the second method HH2 are defined by:

$$a_{k,L} = \hat{a}_k - \delta_k \qquad \text{and} \qquad a_{k,R} = \hat{a}_k + \delta_k\,, \tag{4.15}$$

and we have $\delta_{k,L} = \delta_{k,R} = \delta_k$. Moreover, also in this method, there is no need to calculate the states $\hat{\boldsymbol{u}}_{k,L}$ and $\hat{\boldsymbol{u}}_{k,R}$ explicitly.

It is an easy computation to see that scheme (4.13) is equivalent to (3.25), provided we use definitions (4.15) and we set $\sigma_k = 1$.

Expression (4.13b) shows that, in the second Harten and Hyman entropy fix, the function $|\hat{a}_k|$ for $-\delta_k < \hat{a}_k < \delta_k$ is replaced by a parabolic function with vertex in $\left(0, \frac{\delta_k}{2}\right)$, as seen in Figure 4.

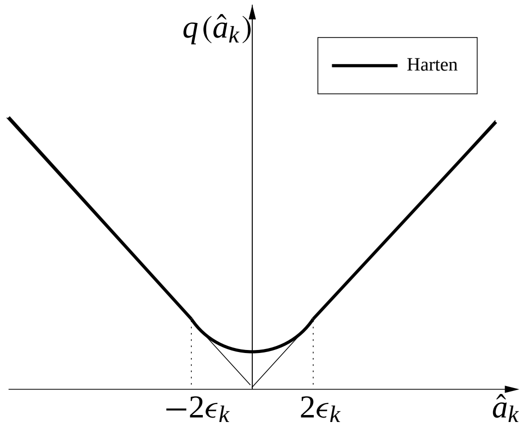

Figure 6: Harten's entropy fix

4.4 Harten entropy fix

In [8] Harten proposes the following entropy fix:

$$\boldsymbol{F}^{(\mathrm{H})}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{1}{2}[\,\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k=1}^{p} q^{\mathrm{H}}(\hat{a}_k)\,\hat{\chi}_k\,\hat{\boldsymbol{r}}_k\,, \tag{4.16a}$$

where

$$q^{\mathrm{H}}(\hat{a}_k) = \begin{cases} \dfrac{\hat{a}_k^2}{4\varepsilon_k} + \varepsilon_k & \text{if } |\hat{a}_k| < 2\varepsilon_k \\ |\hat{a}_k| & \text{if } |\hat{a}_k| \geq 2\varepsilon_k\,. \end{cases} \tag{4.16b}$$

The parameter ε_k is, for each k, a positive constant value, which Harten suggests to choose in general in the interval $(0, \frac{1}{2})$.

We notice that the function $q^{\mathrm{H}}(\hat{a}_k)$ has the form of the function $q^{\mathrm{HH2}}(\hat{a}_k)$ given by (4.13b), the difference between the two schemes consisting in the definition of δ_k: the function $\delta_k = \varphi(\hat{a}_k, a_k(\boldsymbol{u}_L), a_k(\boldsymbol{u}_R))$ defined in (4.13c) is replaced here by a constant value given by $2\varepsilon_k$: $\delta_k = 2\varepsilon_k$. On the basis of the second method of Harten and Hyman described in 4.3, we can interpret Harten's scheme as the result of the introduction of a linear transition between $\hat{\boldsymbol{u}}_{k,L}$ and $\hat{\boldsymbol{u}}_{k,R}$ for $a_{k,L} < \frac{x}{t} < a_{k,R}$, if $a_{k,L} < 0 < a_{k,R}$. For this entropy fix the intervention criterion and the linearly variable intermediate state are defined by means of the relations:

$$a_{k,L} = \hat{a}_k - 2\varepsilon_k \qquad \text{and} \qquad a_{k,R} = \hat{a}_k + 2\varepsilon_k\,. \tag{4.17}$$

Choosing the above definition of $a_{k,L}$ and $a_{k,R}$, and setting $\sigma_k = 1$, allows us to obtain scheme (4.16) from the general formulation (3.25).

As easily seen from (4.16b), like in the second method of Harten and Hyman, the function $|\hat{a}_k|$ is modified by introducing a parabolic function for $|\hat{a}_k| < 2\varepsilon_k$, with minimum value in $(0, \varepsilon_k)$. See Figure 6.

It is worth noticing that, differently from all the other methods presented so far, in the Harten method there is no condition aimed at avoiding the action of the entropy fix in the case of shocks. This feature follows from the fact that the value of ε_k is a *constant* independent of the solution. On the contrary, in the second method of Harten and Hyman, as well as in the first one, the action of the entropy fix is prevented if a shock is estimated to occur in correspondence to the k-th wave, since in this case $\delta_k = 0$. Therefore, it is concluded that for Harten's method it is necessary an explanation coming from the second interpretation of the role of the entropy fix, as Harten pointed out in [8] introducing this scheme as a means to prevent the vanishing of the numerical viscosity.

4.5 An extension of LeVeque entropy fix

The general formulation of the entropy fix presented in section 3 allows us to derive an extension of LeVeque's method. Assuming the same definitions $a_{k,L} = a_k(\hat{\boldsymbol{u}}_{k,L})$ and $a_{k,R} = a_k(\hat{\boldsymbol{u}}_{k,R})$ introduced by LeVeque, we consider a linear intermediate state in the general form (3.13), rewritten here for convenience, for $a_k(\hat{\boldsymbol{u}}_{k,L}) < \frac{x}{t} < a_k(\hat{\boldsymbol{u}}_{k,R})$,

$$\boldsymbol{u}_k^1\left(\frac{x}{t}\right) = \overline{\boldsymbol{u}}_k + \left(\frac{x}{t} - \frac{a_k(\hat{\boldsymbol{u}}_{k,L}) + a_k(\hat{\boldsymbol{u}}_{k,R})}{2}\right) \frac{\sigma_k\,(\hat{\boldsymbol{u}}_{k,R} - \hat{\boldsymbol{u}}_{k,L})}{a_k(\hat{\boldsymbol{u}}_{k,R}) - a_k(\hat{\boldsymbol{u}}_{k,L})}\,, \tag{4.18}$$

with now the slope σ_k subjected to the condition:

$$0 \le \sigma_k \le \frac{2\min(\delta_{k,L},\, \delta_{k,R})}{\delta_{k,L} + \delta_{k,R}}\,, \tag{4.19}$$

where $\delta_{k,L} = \max\{0,\ \hat{a}_k - a_k(\hat{\boldsymbol{u}}_{k,L})\}$ and $\delta_{k,R} = \max\{0,\ a_k(\hat{\boldsymbol{u}}_{k,R}) - \hat{a}_k\}$.

The formulation of this extended LeVeque entropy fix is given by the function $q_a^1(\hat{a}_k)$ appearing in (3.25) under the definitions $a_{k,L} = a_k(\hat{\boldsymbol{u}}_{k,L})$ and $a_{k,R} = a_k(\hat{\boldsymbol{u}}_{k,R})$, and σ_k satisfying (4.19). The corresponding form in terms of velocity differences (3.28b) is rewritten here

$$q_\delta^{\mathrm{LVM}}(\hat{a}_k) = \begin{cases} \dfrac{\sigma_k\,\hat{a}_k^2 - (1-\sigma_k)(\delta_{k,R} - \delta_{k,L})\hat{a}_k + (2-\sigma_k)\delta_{k,L}\,\delta_{k,R}}{\delta_{k,L} + \delta_{k,R}} & \\ & \text{if } \ -\delta_{k,R} < \hat{a}_k < \delta_{k,L}\,, \\ |\hat{a}_k| & \text{otherwise.} \end{cases} \tag{4.20}$$

Clearly, for $\sigma_k = 0$ we recover the original LeVeque entropy fix. In order to minimize the introduction of numerical viscosity, we suggest the following definition of σ_k:

$$\sigma_k = \frac{2\min(\delta_{k,L},\, \delta_{k,R})}{\delta_{k,L} + \delta_{k,R}}\,, \tag{4.21}$$

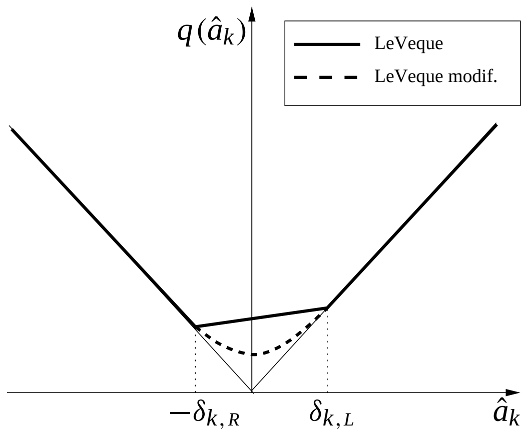

Figure 7: Modification of LeVeque's entropy fix

which is always ≤ 1. Assuming the above choice of σ_k means imposing continuity for the first derivative of $q(\hat{a}_k)$ at $\hat{a}_k = \delta_{k,L}$, if $\delta_{k,L} \geq \delta_{k,R}$, or at $\hat{a}_k = -\delta_{k,R}$, if $\delta_{k,R} \geq \delta_{k,L}$. See Figure 7.

Remark 4.2 In the scalar case, definition (4.21) is equivalent to select a slope for the linear variation of the intermediate state corresponding to the minimum between $|\overline{u} - u_L|$ and $|u_R - \overline{u}|$.

All the entropy fixes examined so far fit nicely in the general formulation 3.25. Their differences may be better appreciated looking at Table 1 which summarizes the various fixes.

4.6 The scalar case

The comparison between the different methods just described is particularly simple in the scalar case, since there is only one eigenvalue. We assume a convex function $f(u)$ and we consider a Riemann problem whose exact solution consists in a transonic rarefaction. The propagation velocities of the boundary waves of the rarefaction fan are $a(u_L)$ and $a(u_R)$, where $a(u) = f'(u)$. Roe's scheme represents any solution, including the rarefaction of interest here (Fig. 8), by means of a single discontinuity propagating at a velocity given by:

$$\hat{a} = \frac{f(u_R) - f(u_L)}{u_R - u_L}\,.$$

As already seen, the entropy fix replaces the exact rarefaction by an artificial rarefaction based on some estimated values a_L and a_R for the propagation velocity of the bounding waves. Let us consider first in some detail the two methods based on a constant intermediate state, here recalled for convenience. We have:

$$F^{(l)}(u_L, u_R) = \frac{1}{2}[f(u_L) + f(u_R)] - \frac{1}{2}\sum_{k=1}^{p} q_a^l(\hat{a}_k)\,\hat{\chi}_k\,\hat{\mathbf{r}}_k$$

where

$$q_a^l(\hat{a}_k) = \begin{cases} \dfrac{(a_{k,R}+a_{k,L})\hat{a}_k - 2a_{k,R}\,a_{k,L}}{a_{k,R}-a_{k,L}} + \dfrac{a_{k,R}\,a_{k,L}}{a_{k,R}-a_{k,L}}\sigma_k & \text{if } a_{k,L} < 0 < a_{k,R} \\ |\hat{a}_k| & \text{otherwise} \end{cases}$$

SCHEME	$a_{k,L}$	$a_{k,R}$	σ_k	Notes
HH1	$\hat{a}_k - \delta_k$	$\hat{a}_k + \delta_k$	0	$\delta_k = \max\{0, \hat{a}_k - a_k(u_L), a_k(u_R) - \hat{a}_k\}$
HH2	$\hat{a}_k - \delta_k$	$\hat{a}_k + \delta_k$	1	as above
Harten	$\hat{a}_k - 2\varepsilon_k$	$\hat{a}_k + 2\varepsilon_k$	1	$0 < \varepsilon_k < 0.5$
LeVeque	$a_k(\hat{\mathbf{u}}_{k,L})$	$a_k(\hat{\mathbf{u}}_{k,R})$	0	
LeVeque m.	$a_k(\hat{\mathbf{u}}_{k,L})$	$a_k(\hat{\mathbf{u}}_{k,R})$	$\frac{2\min(\delta_{k,L},\delta_{k,R})}{\delta_{k,L}+\delta_{k,R}}$	$\delta_{k,L} = \max\{0,\ \hat{a}_k - a_k(\hat{\mathbf{u}}_{k,L})\}$ $\delta_{k,R} = \max\{0,\ a_k(\hat{\mathbf{u}}_{k,R}) - \hat{a}_k\}$

Table 1: *Summary of different entropy fixes*

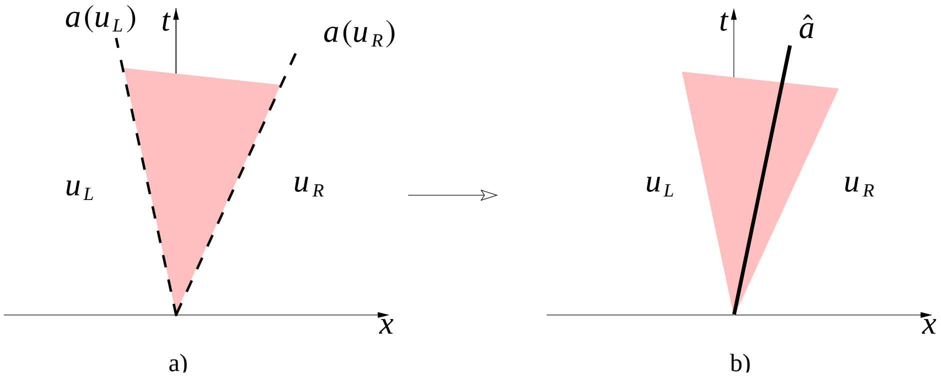

Figure 8: Transonic rarefaction in the scalar case: a) exact solution; b) Roe's linearization

First method of Harten and Hyman:

$$q^{\mathrm{HH1}}(\hat{a}) = \begin{cases} \delta & \text{if } |\hat{a}| < \delta \\ |\hat{a}| & \text{if } |\hat{a}| \geq \delta\,, \end{cases}$$

with $\delta = \max\{0,\ \hat{a} - a(u_L),\ a(u_R) - \hat{a}\}\,.$

LeVeque's method:

$$q^{\mathrm{LV}}(\hat{a}) = \begin{cases} \dfrac{(\delta_L - \delta_R)\,\hat{a} + 2\delta_L\,\delta_R}{\delta_L + \delta_R} & \text{if } \ -\delta_R < \hat{a} < \delta_L \\ |\hat{a}| & \text{otherwise}\,, \end{cases}$$

with $\delta_L = \max\{0,\ \hat{a} - a(u_L)\}$ and $\delta_R = \max\{0,\ a(u_R) - \hat{a}\}$.

LeVeque's method (Fig. 9) assumes an artificial rarefaction with a constant intermediate state but of the same spread of the exact one by defining:

$$a_L = a(u_L) \qquad \text{and} \qquad a_R = a(u_R)\,.$$

In this method the modification of the wavespeed $\hat{a}$ (eigenvalue) occurs only in connection with an actual transonic rarefaction and the intermediate state $\overline{u}$ between the two discontinuities a_L and a_R is defined by imposing the conservation, to give:

$$\overline{u} = \frac{[\hat{a} - a(u_L)]u_L + [a(u_R) - \hat{a}]u_R}{a(u_R) - a(u_L)}\,.$$

By contrast, the first method of Harten and Hyman (fig. 10) constructs an artificial rarefaction which is symmetrical with respect to the discontinuity of velocity $\hat{a}$, taking from both sides of this discontinuity an angle equal to the bigger of the two that is formed by the bounding waves of the fan and the discontinuity itself, according to the following definitions:

$$a_L = \hat{a} - \delta \qquad \text{and} \qquad a_R = \hat{a} + \delta\,,$$

with

$$\delta = \max\{0,\ \hat{a} - a(u_L),\ a(u_R) - \hat{a}\}\,.$$

Again, when the entropy fix acts, it introduces two discontinuities, one coincident with a boundary wave of the fan, while the other external to the fan, in the general case. Here, as we have already seen, the definition of a_L and a_R leads to a particularly simple expression of the intermediate state $\overline{u}$:

$$\begin{aligned} \overline{u} &= \frac{(\hat{a} - a_L)u_L + (a_R - \hat{a})u_R}{a_R - a_L} \\ &= \frac{(\hat{a} - \hat{a} + \delta)u_L + (\hat{a} + \delta - \hat{a})u_R}{\hat{a} + \delta - \hat{a} + \delta} \\ &= \frac{u_L + u_R}{2}\,. \end{aligned}$$

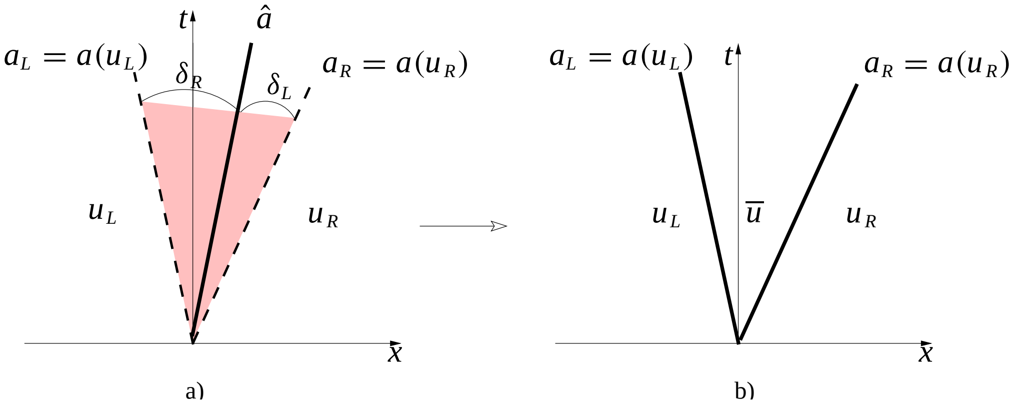

Figure 9: LeVeque's entropy fix applied to a transonic rarefaction in the scalar case

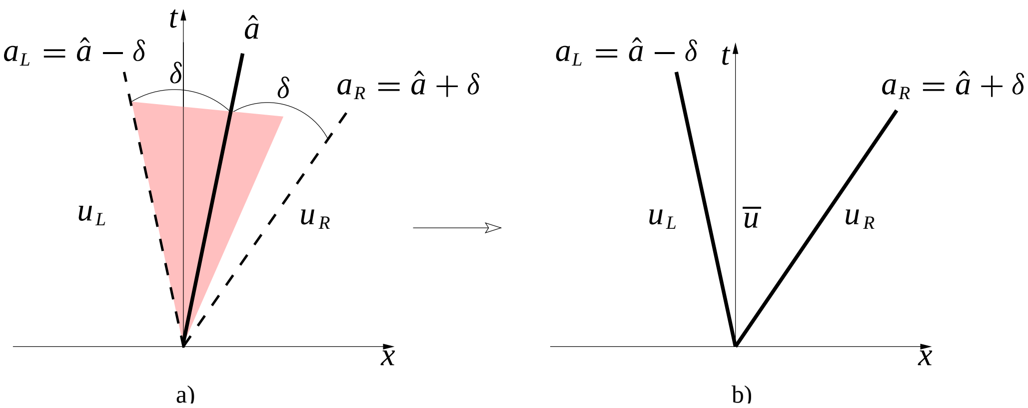

Figure 10: First entropy fix of Harten and Hyman applied to a transonic rarefaction in the scalar case

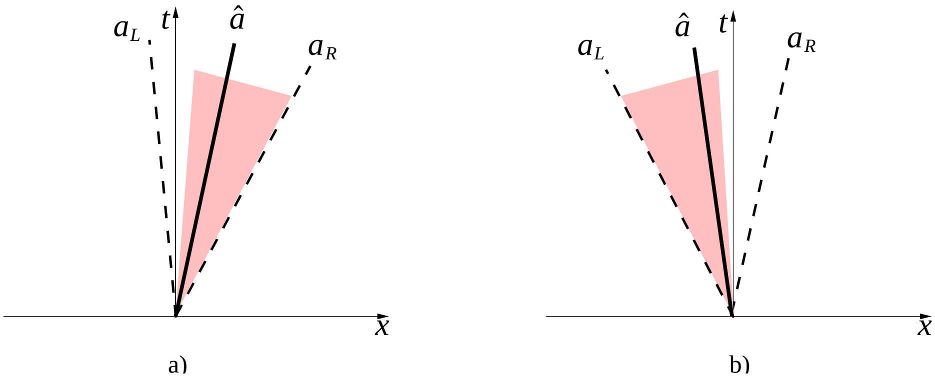

Figure 11: Action of the first entropy fix of Harten and Hyman in the case of a non transonic rarefaction

As anticipated, the first entropy fix of Harten and Hyman can make transonic a nontransonic rarefaction, introducing an error which, on the contrary, never occurs in LeVeque's entropy fix. The transformation of a nontransonic rarefaction in a transonic one occurs either when (Fig. 11a):

$$\begin{cases} a(u_L) > 0 \\ \hat{a} > a(u_L) \\ a(u_R) > 2\,\hat{a} \end{cases} \tag{a}$$

so that $a_L = \hat{a} - \delta < 0$, or when (fig. 11b):

$$\begin{cases} a(u_R) < 0 \\ \hat{a} < a(u_R) \\ a(u_L) > 2\,\hat{a} \end{cases} \tag{b}$$

so that $a_R = \hat{a} + \delta > 0$.

Let us consider now all the other schemes, namely:

Second method of Harten and Hyman:

$$q^{\mathrm{HH2}}(\hat{a}) = \begin{cases} \dfrac{1}{2}\left(\dfrac{\hat{a}^2}{\delta} + \delta\right) & \text{if } |\hat{a}| < \delta \\ |\hat{a}| & \text{if } |\hat{a}| \geq \delta\,, \end{cases}$$

with $\delta = \max\{0,\ \hat{a} - a(u_L),\ a(u_R) - \hat{a}\}$.

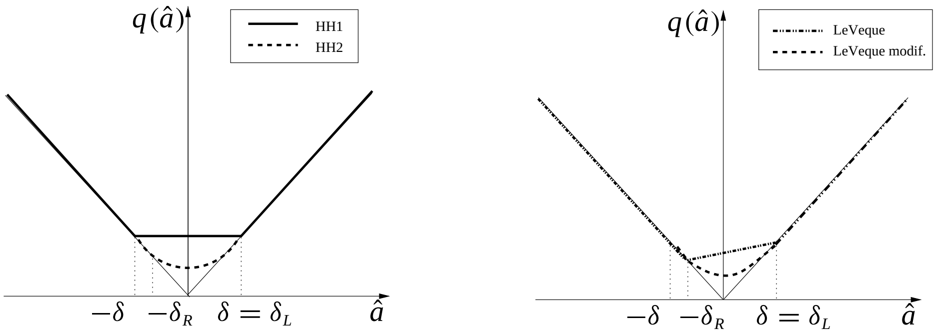

Figure 12: Comparison of different entropy fixes in the scalar case

Modified LeVeque's method:

$$q^{\text{LVM}}(\hat{a}) = \begin{cases} \dfrac{\sigma\,\hat{a}^2 - (1-\sigma)(\delta_R - \delta_L)\,\hat{a} + (2-\sigma)\,\delta_L\,\delta_R}{\delta_L + \delta_R} & \text{if } \; -\delta_R < \hat{a} < \delta_L \\ |\hat{a}| & \text{otherwise}\,, \end{cases}$$

with $\delta_L = \max\{0,\ \hat{a} - a(u_L)\}$, $\delta_R = \max\{0,\ a(u_R) - \hat{a}\}$ and

$$\sigma = \frac{2\min(\delta_L, \delta_R)}{\delta_L + \delta_R}\,.$$

We can easily compare the modification of the function $|\hat{a}|$ accomplished by the various methods (Fig. 12), and, as a consequence, the different level of numerical viscosity introduced. It is clear that the minimum amount of artificial viscosity is introduced by the entropy fixes in which the rarefaction is represented by a linearly variable intermediate state.

Remark 4.3 In the particular case of the inviscid Burgers' equation, the first entropy fix of Harten and Hyman is coincident with the original LeVeque scheme. This fact is due to the expression of Burgers' flux $f(u) = \frac{1}{2}u^2$, which leads to $a(u) = u$. It follows that any rarefaction is symmetrical with respect to the discontinuity representing the fan in Roe's scheme and propagating with velocity

$$\hat{a} = \hat{a}(u_L, u_R) = \frac{f(u_R) - f(u_L)}{u_R - u_L} = \frac{\frac{1}{2}u_R^2 - \frac{1}{2}u_L^2}{u_R - u_L} = \frac{u_R + u_L}{2}\,.$$

In fact we have:

$$\begin{aligned} \hat{a} - f'(u_L) = \hat{a} - a(u_L) &= \frac{u_R + u_L}{2} - u_L = \frac{u_R - u_L}{2} \\ &= u_R - \frac{u_R + u_L}{2} = a(u_R) - \hat{a} = f'(u_R) - \hat{a}\,. \end{aligned}$$

□

Furthermore, with Burgers' flux the second entropy fix of Harten and Hyman is coincident with the modified LeVeque entropy fix, for which, according to (4.21), $\sigma = 1$. In this case, both fixes make the Roe scheme equivalent to Godunov method.

Remark 4.4 If the exact solution consists in a shock, *i.e.* if we have $a(u_R) \leq \hat{a} \leq a(u_L)$ (assuming a convex function $f(u)$), then $\delta = \delta_L = \delta_R = 0$. In such a case, the condition identifying the transonic rarefaction, namely $|\hat{a}| < \delta$ for the first and second method of Harten and Hyman, and $-\delta_R < \hat{a} < \delta_L$ for LeVeque's method, cannot be satisfied. Therefore we deduce that the entropy fixes considered here don't act in the case of shock.

5 Numerical results

We present in this section two test problems showing the action of the different entropy fixes.

Problem 1

This numerical experiment consists in the solution of a Riemann problem for the inviscid Burgers' equation with initial data:

$$u_L = -0.5 \qquad \text{and} \qquad u_r = 1.0\,.$$

The domain $x = (0, 1)$ is discretized with 100 intervals. The solutions are computed with a constant time step of 0.5×10^{-2}, corresponding to a maximum CFL number of 0.5, and are shown at $t = 0.3$.

Figure 13 shows the failure of Roe's scheme when no entropy fix is used.

For Burgers' equation, as we proved in Subsection 4.6, the results obtained with the first entropy fix of Harten and Hyman (HH1) and with LeVeque entropy fix (LV) are the same. Both schemes adequately compute the transonic rarefaction of this test case (Figure 14).

The schemes based on a linearly variable intermediate state, namely the second entropy fix of Harten and Hyman (HH2) and the modified LeVeque entropy fix (LVm), shown in the right part of Figure 14, coincide for Burgers' flux with the original Godunov method making use of an exact Riemann solver (Figure 15, left), as already observed. They all feature a "dog-leg" or "entropy glitch" at the sonic point [7]. This behavior implies an incorrect spreading rate of the rarefaction, in spite of the fact that the entropy condition is satisfied in this case (as may be proved for Godunov scheme), because these schemes lack sufficient numerical viscosity when the wave speed is close to zero. The small expansion shock visible near $x = 0.5$ is however of magnitude $\mathcal{O}(h)$ and vanishing as the grid is refined, cf.

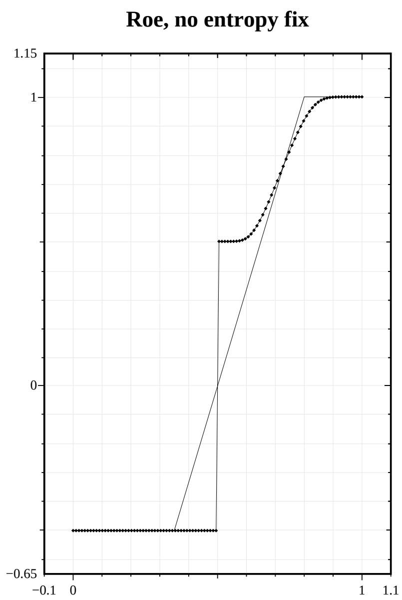

Figure 13: Problem 1. Burgers' equation solved by Roe method with no entropy fix.

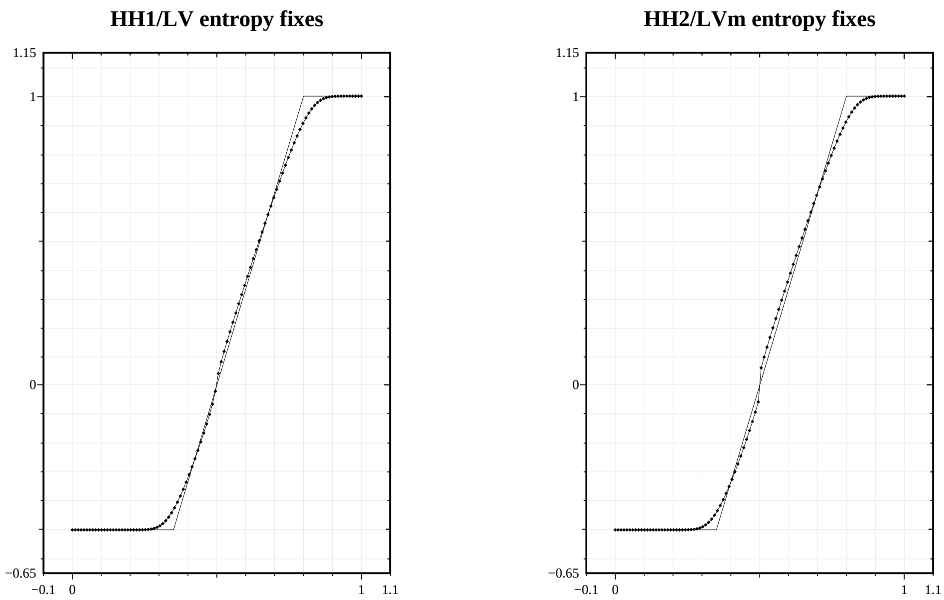

Figure 14: Problem 1: Burgers' equation solved by Roe method. Left: first Harten and Hyman entropy fix = LeVeque entropy fix. Right: second Harten and Hyman entropy fix = modified LeVeque entropy fix.

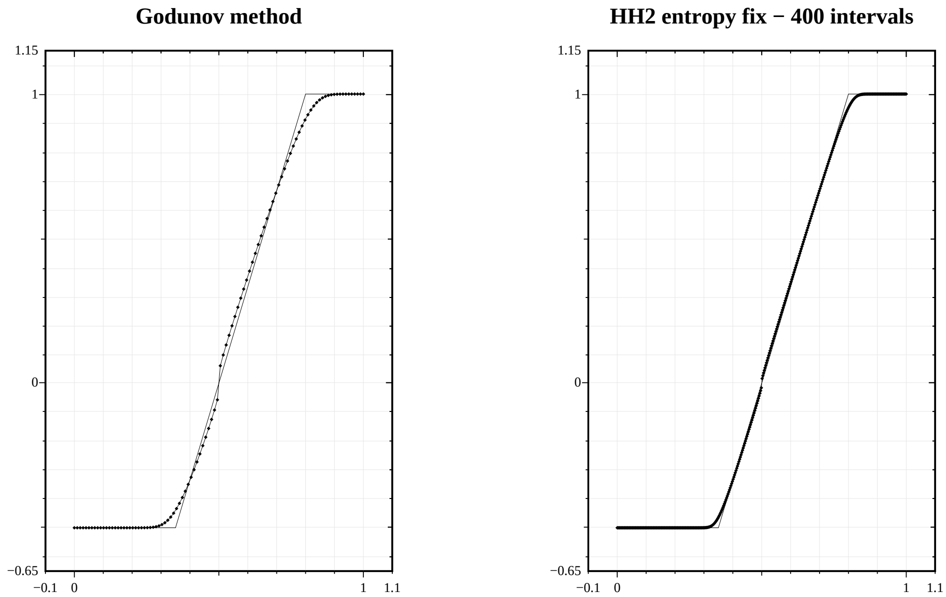

Figure 15: Problem 1: Burgers' equation. Left: Godunov method. Right: second Harten and Hyman entropy fix with 400 discretization intervals.

also [13]: the right part of Figure 15 shows the results obtained with HH2 using 400 discretization intervals.

Finally, Figure 16 shows the results obtained with Harten entropy fix (4.16) for several values of ε. We observe that this method fails if ε is not sufficiently high, while a greater ε leads to a larger numerical dissipation. This fact comes from the fixed value ε, no modulation dependent on the specific local Riemann problem being allowed.

Problem 2

This numerical experiment is proposed by LeVeque in [14]. We resolve a Riemann problem for the one-dimensional Euler equations for a polytropic ideal gas. The initial data $\boldsymbol{w} = (\rho, v, P)^{\mathrm{T}}$, expressing the values of density, velocity and pressure, are:

$$\boldsymbol{w}_L = \begin{pmatrix} 3.0 \\ 0.9 \\ 3.0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{w}_R = \begin{pmatrix} 1.0 \\ 0.9 \\ 1.0 \end{pmatrix}$$

We use again 100 discretization intervals, a constant time step of 0.2×10^{-2} (maximum CFL of 0.52) and report the solutions at $t = 0.14$.

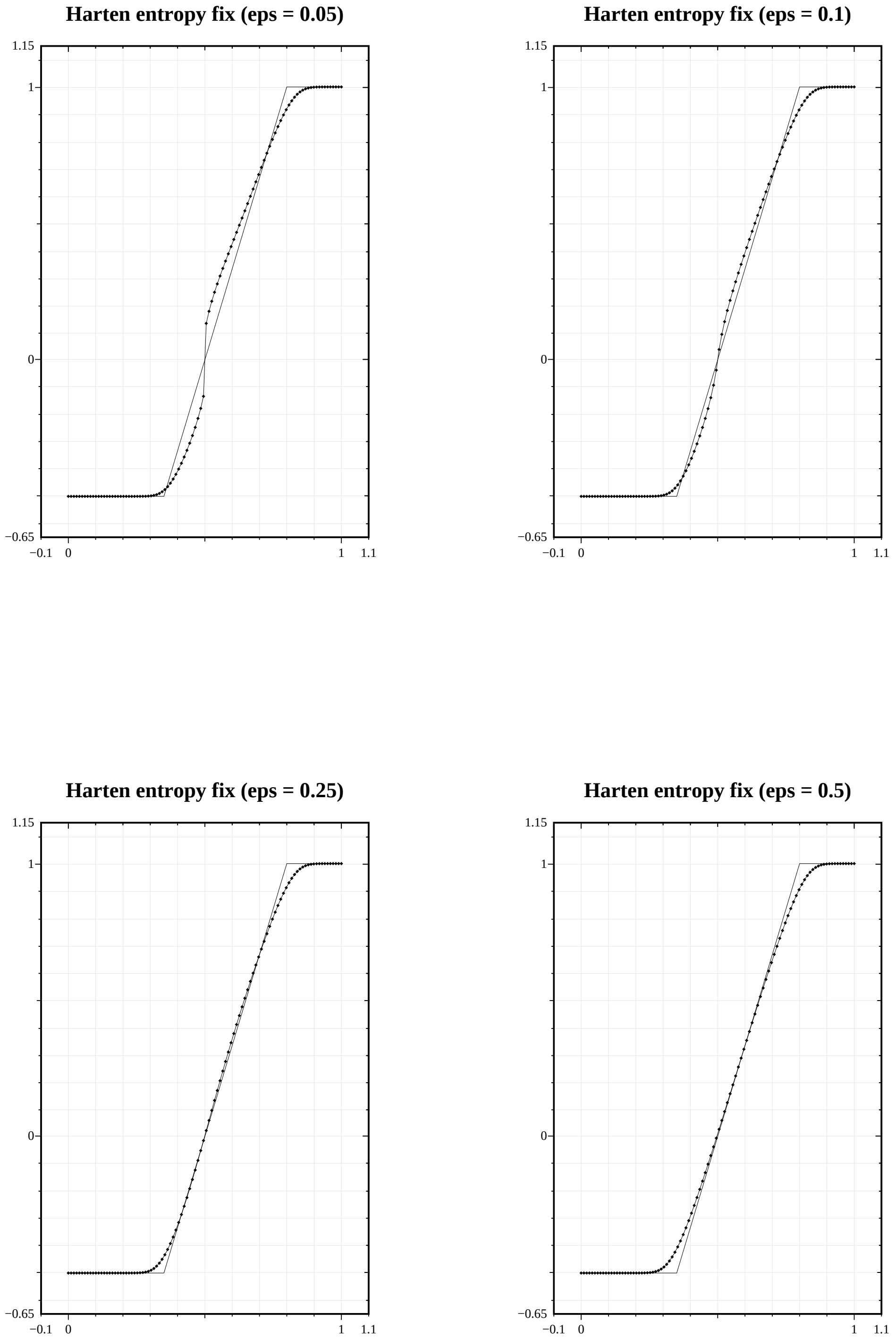

Figure 16: Problem 1: Burgers' equation solved by Roe method with Harten entropy fix and several ε values

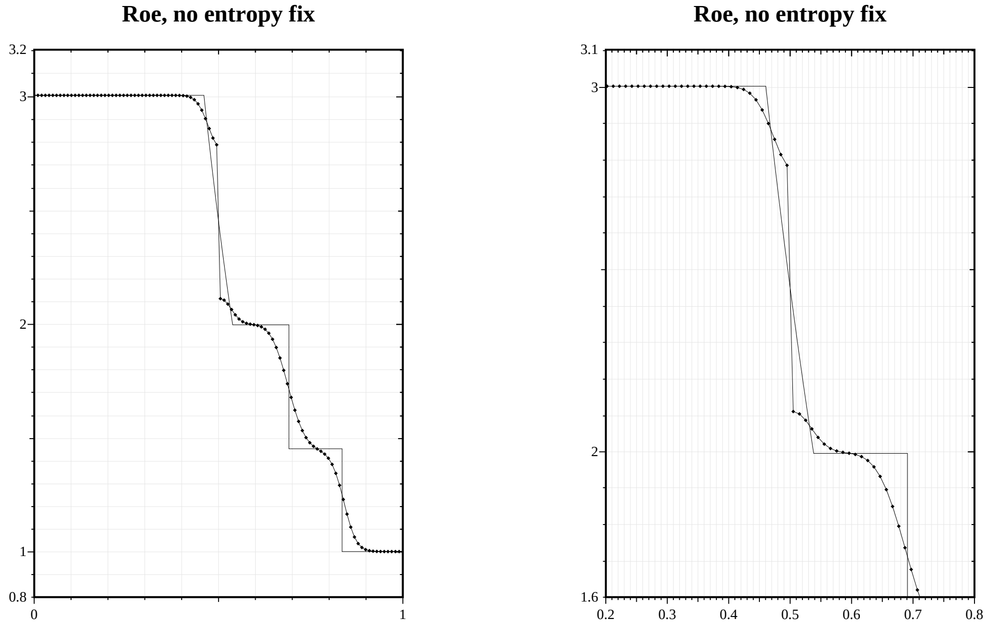

Figure 17: Problem 2. Density distribution: Roe's method without entropy fix.

The solution of this problem consists in a *transonic* rarefaction, a contact discontinuity, and a shock, as shown in Figure 17 where the results obtained with Roe's scheme without entropy fix are compared with the exact solution. We will concentrate the analysis on the transonic rarefaction, which appears enlarged in the right of Figure 17 and in all the remaining figures.

For the Euler system observations similar to the scalar case can be done. Only HH1 and LeVeque's entropy fixes (Figure 18) are able to avoid the generation of a spurious expansion shock. All the other schemes (Figure 19) yield an entropy glitch of the same magnitude of that computed with Godunov method. However, this unphysical discontinuity represents only an error of $\mathcal{O}(h)$, as shown in Figure 20 for the particular case of the LVm method.

6 Entropy fix in presence of strong rarefactions

When solving the Euler equations in presence of strong rarefactions, which imply low density regions, the classical Roe linearization based Jacobian matrix $\hat{\boldsymbol{A}}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \boldsymbol{A}(\hat{\boldsymbol{u}}(\boldsymbol{u}_L, \boldsymbol{u}_R)) = \boldsymbol{A}^{(\mathrm{r})}(\hat{v}, \hat{h}^{\mathrm{t}})$, where $\hat{v}$ and $\hat{h}^{\mathrm{t}}$ are the Roe-averaged velocity and total entalpy, may fail. In fact nonphysical states, with negative density or internal energy or both, can be computed. The original Roe scheme has not enough degrees of freedom to impose together positivity and consistency with the conservation laws, as demonstrated in [3], at least for a certain class of symmetrical

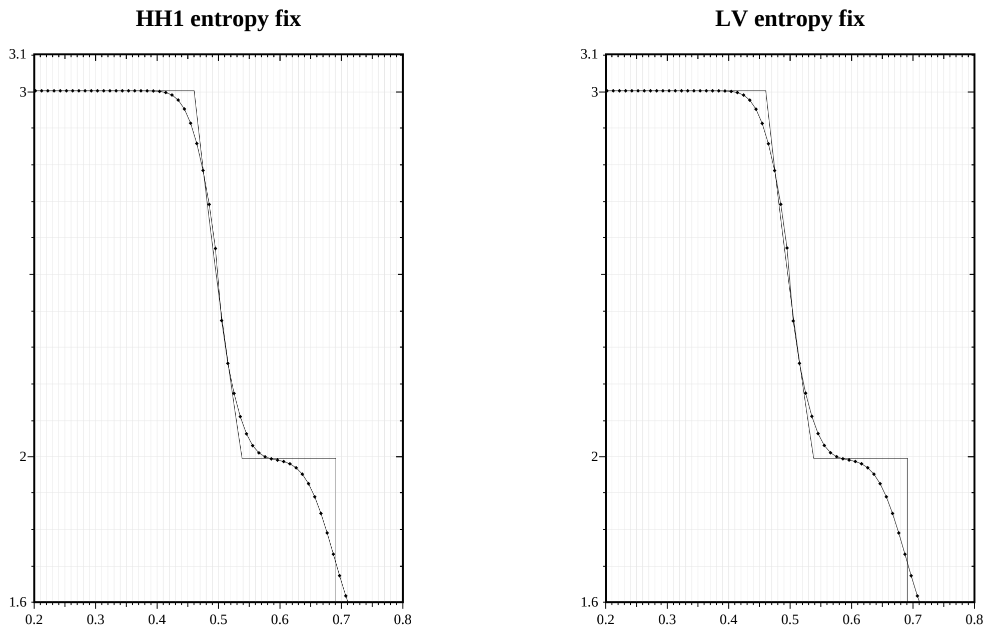

Figure 18: Problem 2. Density distribution: HH1 and LV entropy fixes.

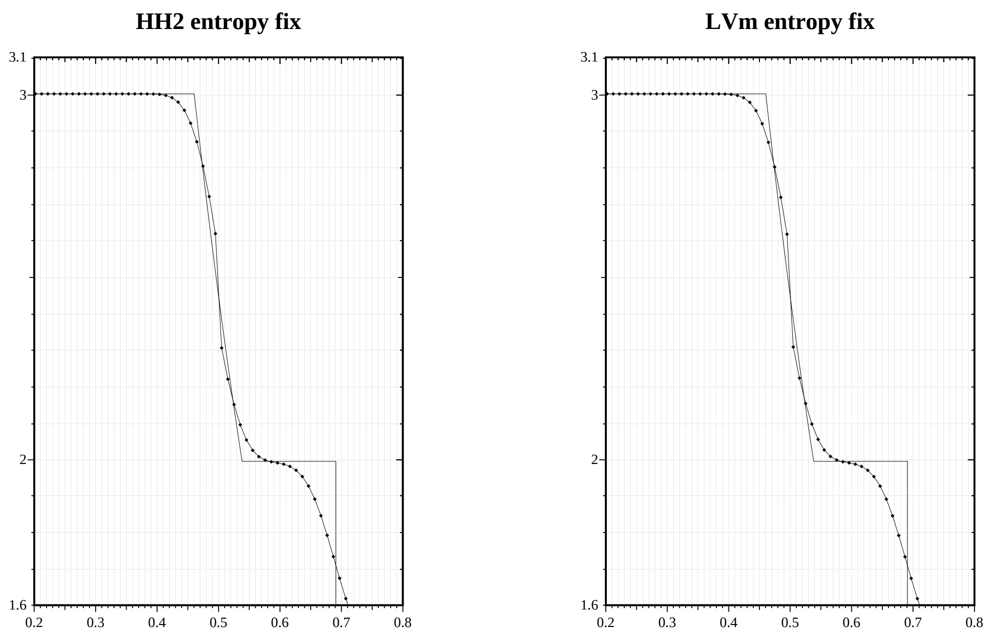

Figure 19: Problem 2. Density distribution: HH2 and LVm entropy fixes.

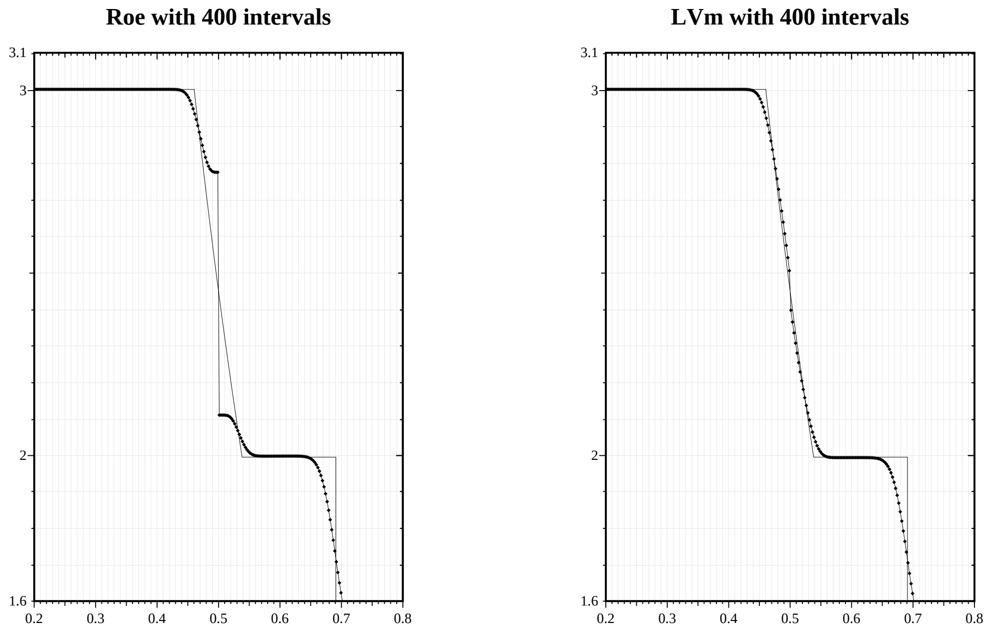

Figure 20: Problem 2. Density distribution on a grid with 400 points: Roe's method without entropy fix and with LVm entropy fix.

Riemann problems. The difficulty is avoided by resorting to a positivity preserving approximation, as the HLLE scheme [2, 3]. Unfortunately the HLLE method is characterized by a numerical dissipation larger than in Roe's scheme, particularly near contact discontinuities. To overcome this drawback in recent years some modifications of the original HLLE scheme have been proposed [3, 18]. An alternative approach is suggested by Dubroca [1], that introduces the required degrees of freedom by modifying the classical Jacobian-based Roe's linearization .

We suggest here a different approach. The starting point is the observation that the lack of positivity of Roe's scheme has been explained [3, p. 285] as the consequence of the underestimation of the physical value of the minimum and maximum signal velocity by the approximate Riemann solver. From the general formulation presented in Section 3, the entropy fix may be seen as operating a correction of the propagation velocities: therefore it may in some cases act also to maintain the positivity of the solution. In other words, it is possible to increase the degrees of freedom of the Roe method through the general formulation of the entropy fix, so as to enforce the positivity of the scheme. It will be shown in the next subsection that the proposed general formulation encompasses also the HLLE scheme, that may then interpreted as a *positivity preserving* entropy fix.

6.1 The HLLE scheme revisited

The HLLE Riemann solver, as proposed in [2], may be formulated in conservation form, with the numerical flux function given by:

$$\boldsymbol{F}^{(\mathrm{HLLE})}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \begin{cases} \boldsymbol{f}(\boldsymbol{u}_L) & \text{if } b_L > 0\,, \\ \dfrac{b_R\, \boldsymbol{f}(\boldsymbol{u}_L) - b_L\, \boldsymbol{f}(\boldsymbol{u}_R)}{b_R - b_L} + \dfrac{b_L\, b_R}{b_R - b_L}(\boldsymbol{u}_R - \boldsymbol{u}_L) & \text{if } b_L < 0 < b_R \\ \boldsymbol{f}(\boldsymbol{u}_R) & \text{if } b_R < 0\,, \end{cases} \tag{6.1}$$

where b_L and b_R are approximations to the smallest and the largest physical signal velocities. As proposed in [3], these velocities can be defined by:

$$b_L = \min\left(\hat{a}_1, v_L - c_L\right) \qquad \text{and} \qquad b_R = \max\left(\hat{a}_3, v_R + c_R\right)\,, \tag{6.2}$$

where $\hat{a}_1$ and $\hat{a}_3$ are the smallest and largest eigenvalues of the standard Roe matrix, while v and c denotes respectively the fluid velocity and the sound speed.

By defining $b^+ = \max(0, b_R) = \mathcal{P}(b_R)$ and $b^- = \min(0, b_L) = \mathcal{N}(b_L)$ to retain the original Einfeldt's notation, the above flux function may be rewritten in the more compact form:

$$\boldsymbol{F}^{(\mathrm{HLLE})}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{b^+\, \boldsymbol{f}(\boldsymbol{u}_L) - b^-\, \boldsymbol{f}(\boldsymbol{u}_R)}{b^+ - b^-} + \frac{b^-\, b^+}{b^+ - b^-}(\boldsymbol{u}_R - \boldsymbol{u}_L)\,. \tag{6.3}$$

Alternatively, the HHLE scheme may rewritten so as to put into evidence its viscosity matrix, as [2]:

$$\boldsymbol{F}^{(\mathrm{HLLE})}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{1}{2}[\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\boldsymbol{Q}(\boldsymbol{u}_L, \boldsymbol{u}_R)(\boldsymbol{u}_R - \boldsymbol{u}_L)\,, \tag{6.4}$$

where

$$\boldsymbol{Q}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{b^+ + b^-}{b^+ - b^-}\hat{\boldsymbol{A}} - \frac{2\, b^-\, b^+}{b^+ - b^-}\boldsymbol{I}\,. \tag{6.5}$$

Notice that when b_L and b_R have the same sign (supersonic flow on both the left and right states), the HHLE flux reduces to Roe's flux.

Pushing forward the connection between Roe's and HLLE flux function, from (6.4) and (6.5) the latter may be finally written as:

$$\boldsymbol{F}^{(\mathrm{HLLE})}(\boldsymbol{u}_L, \boldsymbol{u}_R) = \frac{1}{2}[\boldsymbol{f}(\boldsymbol{u}_L) + \boldsymbol{f}(\boldsymbol{u}_R)] - \frac{1}{2}\sum_{k=1}^{p} q^{(\mathrm{HLLE})}(\hat{a}_k)\, \hat{\chi}_k\, \hat{\boldsymbol{r}}_k\,, \tag{6.6}$$

where the function $q^{(\mathrm{HLLE})}(\hat{a}_k)$ is given by:

$$q^{(\mathrm{HLLE})} = \frac{\left(b^+ + b^-\right)\hat{a}_k - 2\, b^-\, b^+}{b^+ - b^-}\,. \tag{6.7}$$

It may be clear now the connection between the HLLE scheme and the general formulation of the entropy fix presented in section 3. In fact the flux (6.6) is written as a Roe flux modified by an entropy fix, and the modification (6.7) reduces to the general formulation (3.25) if we set $\sigma_k = 0$ and

$$a_{k,L} = b^- \qquad \text{and} \qquad a_{k,R} = b^+ \qquad \forall k\,. \tag{6.8}$$

Remark 6.1 From the general formulation (3.25) we may recover also the HLLEM scheme [3, p. 284] if we set $\sigma_1 = \sigma_3 = 0$ and $\sigma_2 = 2\hat{\delta}$, with $\hat{\delta} = \frac{\hat{c}}{\hat{c}+|\bar{v}|}$, being $\hat{c} = c(\hat{h}^{\mathrm{t}}, \hat{v})$ and $\bar{v} = \frac{b_L+b_R}{2}$.

6.2 A positivity preserving entropy fix with low dissipation

Placing the HLLE scheme in the same setting of the classical entropy fix formulations supports the introduction of the idea of *positivity preserving* entropy fix and also suggests how to correct Roe's scheme to impose the positivity. The general formulation (3.25) is found to be a useful and simple tool to benefit from the properties of the different methods considered here in order to guarantee:

i) consistency with the entropy condition,

ii) positivity,

iii) low numerical dissipation.

Indeed, we can suitably define the quantities $a_{k,L}$, $a_{k,R}$ and σ_k depending on the local solution to assure the aforementioned properties.

For fixed values of $a_{k,L}$ and $a_{k,R}$, an increase in the slope σ_k implies a lower numerical dissipation. Nevertheless, it cannot be proved that the entropy condition will be satisfied for values of σ_k different than zero. Therefore, in the following we restrict our analysis to the case $\sigma_k = 0$, $\forall k$, for simplicity.

We start distinguishing the case in which Roe's intermediate states $\hat{\boldsymbol{u}}_1 = \boldsymbol{u}_L + \hat{\chi}_1\,\hat{\boldsymbol{r}}_1$ and $\hat{\boldsymbol{u}}_2 = \boldsymbol{u}_L + \hat{\chi}_1\,\hat{\boldsymbol{r}}_1 + \hat{\chi}_2\,\hat{\boldsymbol{r}}_2$ are physically admissible from the case in which one of them or both are not. The condition discriminating the two cases consists in checking the positivity of the density and internal energy of states $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$.

If the two computed intermediate states are physical, the positivity of the solution is naturally preserved, and we correct Roe's numerical flux by means of (3.25) only to avoid entropy violations, as usual. In such a case, for $a_{k,L}$ and $a_{k,R}$ we use the definitions (4.9), as in LeVeque's method. According to the physical interpretation of the entropy fix suggested by LeVeque, this choice of the propagating velocities allows a better approximation of the exact solution of the Riemann problem and is found to introduce the lowest level of numerical viscosity, with respect to the other possible definitions of $a_{k,L}$ and $a_{k,R}$, for $\sigma_k = 0$.

If on the contrary negative values of density or internal energy or both are detected, definition (4.9) for the propagation velocities cannot be used, since they depend at least on one not physically admissible state. Moreover, in this case we need to define $a_{k,L}$ and $a_{k,R}$ so as to force a suitable enlargement of the numerical signal velocities, thus avoiding the underestimation of the limiting physical velocities caused by Roe's approximate solver. Following the HLLE idea, we use definition (6.8) for the propagation velocities. This choice guarantees both consistency with entropy condition and positivity, as demonstrated in [3]. We remark that, if we use a nonzero value for the parameter σ_2, still having $\sigma_1 = \sigma_3 = 0$ and the same definition (6.8) of $a_{k,L}$ and $a_{k,R}$, it is in principle possible to find out sufficient conditions on σ_2 guaranteeing positivity. These conditions are presently under investigation.

The proposed version of entropy fix proves to be a positivity preserving correction of Roe's scheme that allows an easy implementation and requires an additional computation of no relevant cost with respect to Roe's method augmented by LeVeque's entropy fix [12].

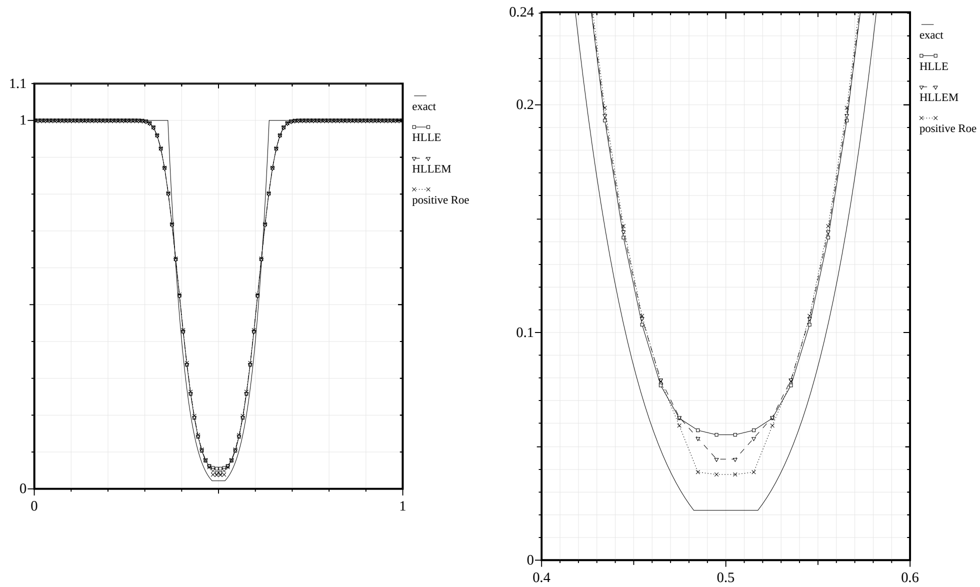

Figure 21: Strong rarefaction test problem [3]. Computed densities for Roe method augmented with the proposed entropy fix in comparison with the exact solution and the solutions by HLLE/M methods.

6.3 Numerical results

Figure 21 compares the first order numerical results obtained with the presented method and the HLLE/M methods for a Riemann problem proposed in [3], consisting in two symmetric rarefactions. The initial data are

$$\boldsymbol{w}_L = \begin{pmatrix} 1.0 \\ -2.0 \\ 0.4 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{w}_R = \begin{pmatrix} 1.0 \\ 2.0 \\ 0.4 \end{pmatrix}$$

with $\boldsymbol{w} = (\rho, v, P)^{\mathrm{T}}$.

The computation is run with 100 discretization intervals and a constant time step of 0.2×10^{-2} (maximum CFL of 0.54), while the solutions are reported at $t = 0.05$.

The proposed entropy fix allows resolving without difficulties this strong rarefaction test, which causes the failure of Roe's classical scheme, and is found to be slightly less dissipative than the HLLE scheme.

A slightly different test case was also considered, that consists of a Riemann problem obtained from the former, by replacing the value of the pressure of the left state with $P_\ell = 2$. In such a case the problem is non-symmetric.

Figure 22 shows the solutions computed with the same discretization as in the previous case. The HLLEM method and – to a lesser extent – the present method, being less dissipative than the HLLE scheme, feature a small undershoot with respect to the exact solution, which does not prevent, however, to compute a positive solution.

In solving Riemann problems different from those implying low density regions, the presented method preserves all the properties of Roe's scheme.

7 Conclusions

Different versions of the Harten and Hyman entropy fix, as applied to Roe's scheme, are analysed in this work from two complementary viewpoints. On the one side, assuming the original Harten approach, which considers the entropy fix as a means to select the physically relevant weak solution, corresponding to the vanishing viscosity solution, we were able to easily compare the different methods in terms of the numerical viscosity they introduce. On the other side, adopting the approach suggested by LeVeque we may look at the action of the entropy fix as a specific correction to Roe's linearization in case of a transonic rarefaction: the single wave that represents the rarefaction fan in Roe's method is split in two waves connected by an intermediate solution state. In this way it has been possible to derive a very general formulation that encompasses all the different versions of this class of entropy fix.

Furthermore, the general formulation has allowed to develop a new scheme: motivated by the appeal of symmetry, LeVeque's formulation of the entropy fix has been extended to the case of a linearly variable intermediate state.

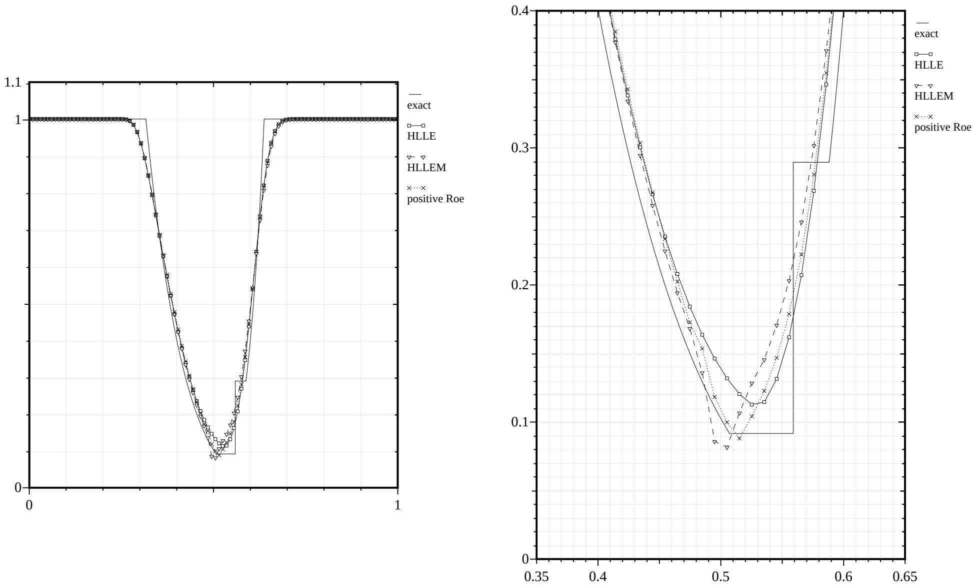

Figure 22: Strong rarefaction problem [3] modified so as to obtain a non-symmetric solution.

Finally, we take advantage of the introduced unitary framework to reconsider the HLLE scheme as special kind of entropy fix as applied to Roe's scheme. This leads to the concept of a positivity preserving entropy fix and to the formulation of a mixed scheme that automatically switches from the classical Roe scheme with entropy fix to the HLLE scheme by simply choosing the form of entropy fix applied. The proposed hybrid scheme is positivity preserving by construction, when applied to a first-order method, as it has also been proven by our simple numerical experiments. It allows an easy implementation and requires an additional computation of practically no cost with respect to Roe's method incorporating LeVeque's entropy fix.

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