A Relaxation Method for Modeling Two-Phase Shallow Granular Flows

Marica Pelanti and François Bouchut

Abstract. We present a relaxation approach for the numerical solution of a depth-averaged two-phase model describing the flow of a shallow layer of a mixture of solid granular material and fluid. A relaxation model is formulated by introducing auxiliary variables that replace the momenta in the spatial gradients of the original system. These new variables are governed by linear equations with coefficients that determine the eigenvalues of the relaxation model. The proposed relaxation strategy results in the definition of a particular approximate Riemann solver for the original model equations. Compared to a Roe-type Riemann solver that we have proposed in previous work, the new solver has the advantage of a certain degree of freedom in the specification of the wave speeds through the choice of the relaxation parameters. This flexibility can be exploited to obtain a more robust method than the Roe-type one in the treatment of wet/dry fronts. Some numerical experiments are presented to show the effectiveness of the proposed approach.

1. Introduction

We are interested in the numerical approximation of a depth-averaged model describing the motion of a mixture of solid granular material and interstitial fluid in the shallow flow assumption. The model system, presented in Section 2, follows the work of Pitman and Le [**PL05**], and consists of mass and momentum equations for the two phases, coupled together by both conservative and non-conservative terms involving the derivatives of the unknowns. The main interest for this model is its application to the simulation of geophysical gravitational flows such as landslides and debris flows, which typically contain both solid granular components and an interstitial fluid phase.

The considered model was first studied in [**PBMV08, PBM08**], where it was solved numerically by a finite volume scheme based on a Roe-type Riemann solver, which we recall in Sections 3-4. One disadvantage of this Roe-type method is that

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it may produce unphysical negative discrete values of the flow depth and of phase volume fractions. Positivity preservation of flow height is an important property for shallow flow numerical models for the treatment of interfaces between flow fronts and dry bed regions where the physical flow height vanishes (wet/dry fronts). For classical single-phase shallow water equations a large variety of positivity preserving Riemann solvers is available. Besides classical robust methods such as the exact Riemann solver and the HLL, HLLC solvers, let us mention some solvers obtained through relaxation strategies, such as Suliciu's solver (cf. [**Bou04**]) or the recent approach of [**BM08**]. However, it appears difficult to extend and apply the existing positivity preserving techniques for the single-phase case to our non-conservative two-phase model, due to the complexity of the model system and its Riemann solution structure. In an effort to build a more robust method than the Roe-type scheme, we have studied a new Riemann solver derived through a relaxation technique. The work was inspired by the recent approach of Berthon–Marche [**BM08**], although here we develop a new idea. The new relaxation solver has an added flexibility with respect to our previous Roe-type solver, thanks to the free relaxation parameters. This flexibility is exploited to obtain a scheme that handles more robustly vacuum states. The method is presented in Section 5, and in Section 6 we present some numerical experiments that show the effectiveness of the proposed approach. Some concluding remarks are written in Section 7.

2. The Two-Phase Shallow Granular Flow Model

We consider a shallow layer of a mixture of solid granular material and fluid over a horizontal surface. Solid and fluid components are assumed incompressible, with constant specific densities ρ_s and $\rho_f < \rho_s$, respectively. We denote with h the flow height and with φ the solid volume fraction, and we define the variables $h_s \equiv \varphi h$, $h_f \equiv (1-\varphi)h$. We will consider one-dimensional flow motion in the x direction, and we will indicate solid and fluid velocities with u_s, u_f, respectively. Phase momenta are given by $m_s = h_s u_s$ and $m_f = h_f u_f$. The flow can be modeled by the following system, consisting of mass and momentum equations for the two constituents:

$$\partial_t h_s + \partial_x m_s = 0\,, \tag{2.1a}$$

$$\partial_t m_s + \partial_x\Big(\tfrac{m_s^2}{h_s} + \tfrac{g}{2}h_s^2 + g\tfrac{1-\gamma}{2}h_s h_f\Big) + \gamma g h_s \partial_x h_f = \gamma F^{\mathrm{D}}, \tag{2.1b}$$

$$\partial_t h_f + \partial_x m_f = 0\,, \tag{2.1c}$$

$$\partial_t m_f + \partial_x\Big(\tfrac{m_f^2}{h_f} + \tfrac{g}{2}h_f^2\Big) + g\, h_f\, \partial_x h_s = -F^{\mathrm{D}}. \tag{2.1d}$$

Above, g is the gravity constant and $\gamma = \frac{\rho_f}{\rho_s} < 1$. Source terms on the right-hand side account for inter-phase drag forces $F^{\mathrm{D}} = D(h_s + h_f)(u_f - u_s)$, where D is a drag function. Drag effects in the model are important for maintaining flow conditions in the hyperbolic regime, as it will be clearer in the following. The two-phase model (2.1) is a variant of the two-phase debris flow model of Pitman and Le [**PL05**]. It was previously studied in [**PBMV08, PBM08**] in an extended form that included topography terms accounting for a variable bottom surface. The model system above differs from the original work of Pitman and Le [**PL05**] in the description of the fluid and mixture momentum balance, and, in contrast with

[**PL05**], has the property of recovering a conservative equation for the momentum of the mixture $m_{\rm m} = h_s u_s + \gamma h_f u_f$, which has the form $\partial_t m_{\rm m} + \partial_x f_{\rm m}(q) = 0$, with $f_{\rm m}(q) = h_s\, u_s^2 + \gamma\, h_f\, u_f^2 + \frac{g}{2}\left(h_s^2 + \gamma\, h_f^2\right) + g\,\frac{1+\gamma}{2}\, h_s\, h_f$. Let us also write the homogeneous system in quasi-linear form. Setting $q = (h_s, m_s, h_f, m_f)^{\rm T}$, we have:

$$\partial_t q + A(q)\partial_x q = 0\,, \tag{2.2a}$$

where

$$A(q) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -u_s^2 + gh_s + g\,\frac{1-\gamma}{2}\,h_f & 2u_s & g\,\frac{1+\gamma}{2}\,h_s & 0 \\ 0 & 0 & 0 & 1 \\ gh_f & 0 & -u_f^2 + gh_f & 2u_f \end{pmatrix}. \tag{2.2b}$$

2.1. Eigenvalues and Hyperbolicity. In general, simple explicit expressions of the eigenvalues λ_k, $k = 1, \ldots, 4$, of the matrix A of the system cannot be derived. In the particular case of equality of solid and fluid velocities, $u_f = u_s \equiv u$, the eigenvalues are real and distinct ($\varphi \neq 1$), and given by $\lambda_{1,4} = u \mp a$, and $\lambda_{2,3} = u \mp a\beta$, where we have introduced the quantities $a = \sqrt{gh}$ and $\beta = \sqrt{(1-\varphi)\frac{1-\gamma}{2}} < 1$. Other particular cases are: (i) $\varphi = 0$, for which the eigenvalues are $u_f \mp a$, $u_s \mp a\beta$, with $\beta = \sqrt{\frac{1-\gamma}{2}}$; (ii) $\varphi = 1$, for which we find the two distinct eigenvalues $u_s \mp a$ and the double eigenvalue u_f. For the general case ($h > 0$), in [**PBM08**] we proved the following :

Proposition 2.1. Matrix A has always at least two real eigenvalues $\lambda_{1,4}$, and moreover, the eigenvalues λ_k of A, $k = 1, \ldots, 4$, satisfy:

$$\min(u_f, u_s) - a \leq \lambda_1 \leq \Re(\lambda_2) \leq \Re(\lambda_3) \leq \lambda_4 \leq \max(u_f, u_s) + a\,, \tag{2.3}$$

where $\Re(\cdot)$ denotes the real part. Furthermore:

(i) If $|u_s - u_f| \leq 2a\beta$ or $|u_s - u_f| \geq 2a$ then all the eigenvalues are real. If these inequalities are strictly satisfied, and if $\varphi \neq 1$, then the eigenvalues are also distinct, and system (2.2) is strictly hyperbolic.

(ii) If $2a\beta < |u_s - u_f| < 2a$ then the internal eigenvalues $\lambda_{2,3}$ may be complex.

The result above shows that hyperbolicity holds at least for flow regimes characterized by differences of solid and fluid velocities sufficiently small. Based on this, it is understood that inter-phase drag forces act in favor of hyperbolic flow conditions, since they tend to drive phase velocities closer.

2.1.1. *Eigenvectors.* The right and left eigenvectors of the matrix A can be easily written in terms of the eigenvalues λ_k. For simplicity, here we assume $h_s, h_f > 0$. The right eigenvectors r_k, $k = 1, \ldots, 4$, can be expressed as $r_k = (1,\, \lambda_k,\, \xi_k, \xi_k\lambda_k)^{\rm T}$ with $\xi_k = \frac{(\lambda_k - u_s)^2 - g\left(h_s + \frac{1-\gamma}{2}h_f\right)}{g\,\frac{1+\gamma}{2}\,h_s} = \frac{gh_f}{(\lambda_k - u_f)^2 - gh_f}$. The left eigenvectors l_k of A can be taken as $l_k = \frac{n_k}{P'(\lambda_k)}$, where $P(\lambda)$ is the characteristic polynomial of A and $n_k = (\vartheta_{s,k}\,(\lambda_k - 2u_s),\, \vartheta_{s,k},\, \vartheta_f\,(\lambda_k - 2u_f),\, \vartheta_f)$, with $\vartheta_{s,k} = (\lambda_k - u_f)^2 - gh_f$ and $\vartheta_f = g\frac{1+\gamma}{2}h_s$. Here we have normalized the eigenvectors l_k so that $L = R^{-1}$, where R is the matrix with columns r_k, and L the matrix with rows l_k.

3. Wave-Propagation Finite Volume Methods

For the numerical solution of the two-phase model (2.1) we assume drag forces are strong enough for the flow to be in the hyperbolic regime. Here we focus on the solution of the homogeneous system, but inter-phase drag source terms can be included by employing the fractional step algorithm described in [**PBM08**]. The class of numerical schemes that we consider for the approximation of our model are finite volume methods based on Riemann solvers (Godunov-type schemes), cf. [**Tor97, LeV02**]. In fact, as we mentioned in the Introduction, the relaxation approach that we propose results in the definition of a particular Riemann solver for (2.2). See Section 5.

Let us consider a general hyperbolic system of the form $\partial_t q + A(q)\partial_x q = 0$, $q \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times m}$. A Riemann solver for the system provides a set of M_w waves $\mathcal{W}^k$ and corresponding speeds s^k that represent the (approximate) solution structure for a Riemann problem for the system. Denoting with q_ℓ and q_r the left and right Riemann data, the sum of the waves must recover the initial jump in the system variables: $\Delta q \equiv q_r - q_\ell = \sum_{k=1}^{M_w} \mathcal{W}^k$. Moreover, for conservative systems endowed with a flux function $\mathcal{F}(q)$, $\mathcal{F}'(q) = A(q)$, i.e. systems of the form $\partial_t q + \partial_x \mathcal{F}(q) = 0$, the initial flux jump must be recovered by the sum of the waves multiplied by the corresponding speeds: $\Delta f \equiv \mathcal{F}(q_\ell) - \mathcal{F}(q_r) = \sum_{k=1}^{M_w} s^k \mathcal{W}^k$. The quantities $\mathcal{Z}^k = s^k\mathcal{W}^k$ have the dimension of a flux, and we will call them *f-waves* following the nomenclature introduced in [**BLMR02**].

The updating formula of the resulting finite volume algorithm can be written in the following *wave-propagation form* [**LeV97, LeV02**] in terms of the f-waves $\mathcal{Z}^k_{i+1/2}$ and speeds $s^k_{i+1/2}$ arising from local Riemann problems with data Q_i^n, Q_{i+1}^n ($i \in \mathbb{Z}$ and $n \in \mathbb{N}$ are the indexes of the discretization in space and time):

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta Q_{i-1/2} + \mathcal{A}^-\Delta Q_{i+1/2}) - \frac{\Delta t}{\Delta x}(F^{\rm c}_{i+1/2} - F^{\rm c}_{i-1/2})\,, \tag{3.1a}$$

$$F^{\rm c}_{i+1/2} = \frac{1}{2}\sum_{k=1}^{M_w} \operatorname{sgn}\left(s^k_{i+1/2}\right)\left(1 - \frac{\Delta t}{\Delta x}\left|s^k_{i+1/2}\right|\right)\mathcal{Z}^{c,k}_{i+1/2}\,, \tag{3.1b}$$

where $\mathcal{A}^\mp\Delta Q$ are the *fluctuations* at cell interfaces,

$$\mathcal{A}^-\Delta Q_{i+1/2} = \sum_{k:s^k_{i+1/2}<0} \mathcal{Z}^k_{i+1/2} \qquad \text{and} \qquad \mathcal{A}^+\Delta Q_{i+1/2} = \sum_{k:s^k_{i+1/2}>0} \mathcal{Z}^k_{i+1/2}\,, \tag{3.1c}$$

and $F^{\rm c}_{i+1/2}$ are correction fluxes for second order resolution. $\mathcal{Z}^{c,k}_{i+1/2}$ are a modified version of $\mathcal{Z}^k_{i+1/2}$, obtained by applying to $\mathcal{Z}^k_{i+1/2}$ a limiter function, cf. [**LeV02**].

4. A Roe-Type Riemann Solver

In [**PBM08**] a Roe-type Riemann solver was presented for the solution of the two-phase flow model (2.2). The approximate solution structure for a Riemann problem for the system with data q_ℓ, q_r is obtained by solving exactly a Riemann problem for a linearized system $\partial_t q + \hat{A}(q_\ell, q_r)\partial_x q = 0$. The constant coefficient matrix $\hat{A}(q_\ell, q_r)$ is defined so as to guarantee conservation for the mass of each phase and for the momentum of the mixture. This can be satisfied by taking $\hat{A}$ as the original matrix $A(q)$ evaluated in an average state $\hat{q} = \hat{q}(\hat{h}_s, \hat{h}_f, \hat{u}_s, \hat{u}_f)$, where

$$\hat{h}_\theta = \frac{h_{\theta,\ell} + h_{\theta,r}}{2} \quad \text{and} \quad \hat{u}_\theta = \frac{\sqrt{h_{\theta,\ell}}\, u_{\theta,\ell} + \sqrt{h_{\theta,r}}\, u_{\theta,r}}{\sqrt{h_{\theta,\ell}} + \sqrt{h_{\theta,r}}}, \quad \theta = s, f\,. \tag{4.1}$$

Resulting waves and speeds for this solver are defined by $\mathcal{W}^k = \alpha_k \hat{r}_k$ and $s^k = \hat{\lambda}_k$, $k = 1, \dots, 4$, where $\{\hat{r}_k, \hat{\lambda}_k\}_{1 \le k \le 4}$ are the the eigenpairs of the Roe matrix $\hat{A}$ of our system, and α_k are the coefficients of the eigen-decomposition $\Delta q = \sum_{k=1}^{4} \alpha_k \hat{r}_k$. The Roe-type scheme of [**PBM08**] proved to be an efficient method for the solution of our model system. However, a drawback is that it may generate negative discrete values of the flow height and unphysical values of phase volume fractions.

5. A Relaxation Method

We present in this Section an approximate Riemann solver for (2.2) obtained by a relaxation technique (eg. [**JX95**]). We introduce two auxiliary variables ω_s and ω_f that are meant to be approximations of the momenta m_s and m_f and approach m_s and m_f as a relaxation time $\tau \to 0^+$. These relaxation variables ω_s and ω_f replace the momentum variables m_s and m_f in the spatial gradients of the original system and are governed by linearized forms of the momentum equations:

$$\partial_t \omega_s + \left(-\tilde{u}_s^2 + g\tilde{h}_s + g\tfrac{1-\gamma}{2}\tilde{h}_f\right)\partial_x h_s + 2\tilde{u}_s\,\partial_x \omega_s + g\tfrac{1+\gamma}{2}\tilde{h}_s\,\partial_x h_f = \tfrac{m_s - \omega_s}{\tau}\,, \tag{5.1a}$$

$$\partial_t \omega_f + g\tilde{h}_f\,\partial_x h_s + \left(-\tilde{u}_f^2 + g\tilde{h}_f\right)\partial_x h_f + 2\tilde{u}_f\,\partial_x \omega_f = \tfrac{m_f - \omega_f}{\tau}\,, \tag{5.1b}$$

where the linearization has been considered at an average state $\tilde{q}(\tilde{h}_s, \tilde{h}_f, \tilde{u}_s, \tilde{u}_f)$, and the source term on the right-hand side drives the relaxation process to equilibrium. Then the relaxation system has the form:

$$\partial_t h_s + \partial_x \omega_s = 0\,, \tag{5.2a}$$

$$\partial_t m_s + \partial_x\left(\tfrac{\omega_s^2}{h_s} + \tfrac{g}{2}h_s^2 + g\tfrac{1-\gamma}{2}h_s h_f\right) + \gamma g h_s \partial_x h_f = 0, \tag{5.2b}$$

$$\partial_t h_f + \partial_x \omega_f = 0\,, \tag{5.2c}$$

$$\partial_t m_f + \partial_x\left(\tfrac{\omega_f^2}{h_f} + \tfrac{g}{2}h_f^2\right) + g\,h_f\,\partial_x h_s = 0\,, \tag{5.2d}$$

$$\partial_t \omega_s + \left(-\tilde{u}_s^2 + g\tilde{h}_s + g\tfrac{1-\gamma}{2}\tilde{h}_f\right)\partial_x h_s + 2\tilde{u}_s\,\partial_x \omega_s + g\tfrac{1+\gamma}{2}\tilde{h}_s\,\partial_x h_f = \tfrac{m_s - \omega_s}{\tau}\,, \tag{5.2e}$$

$$\partial_t \omega_f + g\tilde{h}_f\,\partial_x h_s + \left(-\tilde{u}_f^2 + g\tilde{h}_f\right)\partial_x h_f + 2\tilde{u}_f\,\partial_x \omega_f = \tfrac{m_f - \omega_f}{\tau}\,. \tag{5.2f}$$

Formally, the system above recovers the original system in the limit $\tau \to 0^+$ (equilibrium limit). Here we will apply the *relaxed* scheme of [**JX95**], which consists in: (i) solving the relaxation system with no source term (propagation step), (ii) setting the relaxation variables equal to their equilibrium value at each time step (projection step). When such algorithm is used, the Riemann solution of the relaxation system results in the definition of an approximate Riemann solver for the original system, see e.g. [**LP01, Bou04**]. Since we will not be concerned with the relaxation source term, hereafter we will intend as relaxation system simply its homogeneous part.

5.1. Riemann Structure of the Relaxation Model. Let us note that our (homogeneous) relaxation model presents a decoupled sub-system for the variables $q^{\mathrm{R}} \equiv (h_s, \omega_s, h_f, \omega_f)^{\mathrm{T}}$. This sub-system corresponds to a linearized form of the original two-phase system (2.2), $\partial_t q^{\mathrm{R}} + \tilde{A}\,\partial_x q^{\mathrm{R}} = 0$, with a matrix $\tilde{A} = A(\tilde{q})$, where $A(q) \in \mathbb{R}^{4\times 4}$ is the matrix (2.2b), and $\tilde{q} = \tilde{q}(\tilde{h}_s, \tilde{h}_f, \tilde{u}_s, \tilde{u}_f)^{\mathrm{T}}$. The eigenvalues of the relaxation model are the four eigenvalues of this sub-system, that is the eigenvalues $\tilde{\lambda}_k = \lambda_k(\tilde{q})$, $k = 1, \ldots, 4$, of $\tilde{A} = A(\tilde{q})$, plus a zero eigenvalue with double algebraic multiplicity $\lambda^0 \equiv \lambda^0_1 = \lambda^0_2 = 0$. The eigenvectors associated to λ^0 are $r^{\mathrm{E}0}_1 = (0,1,0,0,0,0)^{\mathrm{T}}$ and $r^{\mathrm{E}0}_2 = (0,0,0,1,0,0)^{\mathrm{T}}$, while the eigenvectors corresponding to $\tilde{\lambda}_k$ can be written as

$$\tilde{r}^{\mathrm{E}}_k = \begin{pmatrix} 1 \\ \left(-\frac{\omega_s^2}{h_s^2} + g h_s + g\frac{1-\gamma}{2} h_f\right)\frac{1}{\tilde{\lambda}_k} + 2\frac{\omega_s}{h_s} + g\frac{1+\gamma}{2} h_s \frac{\tilde{\xi}_k}{\tilde{\lambda}_k} \\ \tilde{\xi}_k \\ g h_f \frac{1}{\tilde{\lambda}_k} + \left(-\frac{\omega_f^2}{h_f^2} + g h_f\right)\frac{\tilde{\xi}_k}{\tilde{\lambda}_k} + 2\frac{\omega_f}{h_f}\tilde{\xi}_k \\ \tilde{\lambda}_k \\ \tilde{\xi}_k\tilde{\lambda}_k \end{pmatrix},$$

where $\tilde{\xi}_k$ is the quantity ξ_k defined in Subsection 2.1 evaluated in $\tilde{q}$. Let us remark that all the characteristic fields are linearly degenerate.

5.1.1. *Riemann Invariants.* The variables $q^{\mathrm{R}} = (h_s, \omega_s, h_f, \omega_f)^{\mathrm{T}}$ are Riemann invariants across λ^0. Across $\tilde{\lambda}_k$:

$$\tilde{\xi}_k h_s - h_f = \text{const.}, \quad \tilde{\lambda}_k h_s - \omega_s = \text{const.}, \quad \tilde{\lambda}_k h_f - \omega_f = \text{const.}, \tag{5.3a}$$

$$\tilde{\lambda}_k m_s - \mathcal{S}_s(\omega_s, h_s, h_f) = \text{const.}, \quad \tilde{\lambda}_k m_f - \mathcal{S}_f(\omega_f, h_f) = \text{const.}, \tag{5.3b}$$

where $\mathcal{S}_s = \frac{\omega_s^2}{h_s} + \frac{1}{2} g h_s^2 + g\frac{1-\gamma}{2} h_s h_f + \frac{1}{2} g\gamma\tilde{\xi}_k h_s^2$ and $\mathcal{S}_f = \frac{\omega_f^2}{h_f} + \frac{1}{2} g h_f^2 + \frac{1}{2} g\frac{h_f^2}{\tilde{\xi}_k}$. Note that only the momentum variables m_s and m_f have a jump across $\lambda^0 = 0$.

5.2. Relaxation Riemann Solver. The exact solution of a Riemann problem for the relaxation system with left and right data q^{E}_ℓ, q^{E}_r, where $q^{\mathrm{E}} = (q, \omega_s, \omega_f)^{\mathrm{T}}$, defines an approximate Riemann solution for the original system with data q_ℓ, q_r, where $q = (h_s, m_s, h_f, m_f)^{\mathrm{T}}$. The solution for the variables $q^{\mathrm{R}} = (h_s, \omega_s, h_f, \omega_f)^{\mathrm{T}}$ is the solution of the linear system $\partial_t q^{\mathrm{R}} + \tilde{A}\partial_x q^{\mathrm{R}} = 0$. Let us denote with $\Delta_k(\cdot)$ the increments across the kth wave with speed $\tilde{\lambda}_k$, $k = 1, \ldots 4$, and let $\Delta(\cdot) \equiv (\cdot)_r - (\cdot)_\ell$. We have $\Delta_k q^{\mathrm{R}} = \alpha_k \tilde{r}_k$, where $\tilde{r}_k$ are the four eigenvectors of $\tilde{A}$, and α_k are the coefficients of the projection $\Delta q^{\mathrm{R}} = \sum_{k=1}^4 \alpha_k \tilde{r}_k$. By using the Riemann invariants, we then find the increments for the momenta m_s, m_f:

$$\tilde{\lambda}_k \Delta_k m_s = \Delta_k\left(\frac{\omega_s^2}{h_s} + \frac{g}{2} h_s^2 + g\frac{1-\gamma}{2} h_s\, h_f\right) + g\gamma\frac{h^{\mathrm{L}}_{s,k} + h^{\mathrm{R}}_{s,k}}{2}\Delta_k h_f\,, \tag{5.4a}$$

$$\tilde{\lambda}_k \Delta_k m_f = \Delta_k\left(\frac{\omega_f^2}{h_f} + \frac{g}{2} h_f^2\right) + g\frac{h^{\mathrm{L}}_{f,k} + h^{\mathrm{R}}_{f,k}}{2}\Delta_k h_s\,, \tag{5.4b}$$

where $(\cdot)^{\mathrm{L,R}}_k$ is used to denote the states to the left and to the right of the kth wave, $k = 1, \ldots 4$. The resulting approximate Riemann solver for the original system consists of six waves $\mathcal{W}^k$ moving at speeds s^k given by $s^k = \tilde{\lambda}_k$, $k = 1, \ldots, 4$, and $s^5 = s^6 = \lambda^0 = 0$. The wave structure can be written in terms of the f-waves $\mathcal{Z}^k = s^k \mathcal{W}^k$. We have $\mathcal{Z}^k = \tilde{\lambda}_k \Delta_k q$ for $k = 1, \cdots 4$, where $\tilde{\lambda}_k \Delta_k q$ is obtained through the relations reported above, and $\mathcal{Z}^5 = \mathcal{Z}^6 = 0$. Note that in the wave propagation algorithm (3.1) we only need to specify f-waves and speeds, and not the waves $\mathcal{W}^k$ themselves. This avoids computing the jump of the momenta across the zero eigenvalue λ^0, which would require knowledge of the order of $\tilde{\lambda}_k$ with respect to λ^0, and therefore a distinction between possible wave configurations.

5.3. Relaxation Parameters and Positivity. Physical consistency requires positivity preservation at the discrete level of the flow depth and of the solid and fluid volume fractions, that is we need $h_i^n \geq 0$ and $\varphi_i^n \in [0,1]$, or, equivalently, $h_{s,i}^n$, $h_{f,i}^n \geq 0$ (we recall that $h_s = \varphi h$, $h_f = (1-\varphi)h$). A sufficient condition (but not necessary) for the positivity of the numerical scheme is to guarantee that h_s, h_f are positive in all the intermediate states of the approximate Riemann solution. The idea here is to define the relaxation parameters of the average state $\tilde{q}(\tilde{h}_s, \tilde{h}_f, \tilde{u}_s, \tilde{u}_f)$ to fulfill this condition. When applied to single-phase shallow water equations, this approach leads to a positivity preserving scheme [**PB**]. Unfortunately, for the two-phase case the flexibility offered by the relaxation parameters does not seem enough to satisfy the intermediate state conditions for all the physically positive variables. Nonetheless, we can satisfy part of them, and at least fulfill the conditions for the intermediate values of $h = h_s + h_f$. Numerical experiments suggest that the resulting solver allows a robust modeling of a wide range of flow conditions involving dry bed zones.

We summarize here our results concerning positivity conditions and definitions of the relaxation averages $\tilde{(\cdot)}$, referring to [**PB**] for details and discussion. Motivated by our results for the single-phase case and by our analysis of the two-phase system eigenvalues, we suggest the following: (i) Take the relaxation average velocities as the Roe velocities in (4.1): $\tilde{u}_s = \hat{u}_s$, $\tilde{u}_f = \hat{u}_f$; (ii) Fix the ratio $\frac{\tilde{h}_s}{\tilde{h}_s+\tilde{h}_f} = \frac{\hat{h}_s}{\hat{h}_s+\hat{h}_f} \equiv \hat{\varphi}$, where $\hat{h}_s$, $\hat{h}_f$ are the Roe averages in (4.1); (iii) Let $\tilde{h} = \tilde{h}_s + \tilde{h}_f$ and $\tilde{a} = \sqrt{g\tilde{h}}$. We look for a sufficiently large value of the relaxation parameter $\tilde{a}$, with $\tilde{a} \geq \sqrt{g\hat{h}}$, $\hat{h} = \hat{h}_s + \hat{h}_f$, that allows to satisfy (achievable) positivity conditions. Then we take $\tilde{h} = \tilde{a}^2/g$, and we define $\tilde{h}_s = \hat{\varphi}\tilde{h}$, $\tilde{h}_f = (1-\hat{\varphi})\tilde{h}$.

Let us now consider positivity conditions for the intermediate states of the relaxation solver. We recall that h_s, h_f are invariant across the stationary wave with $\lambda^0 = 0$, therefore we have three intermediate states $k = 1, 2, 3$ to examine.

Case $\delta U \equiv |\tilde{u}_s - \tilde{u}_f| = 0$. In this case we have explicit expressions for the intermediate states and we can easily derive optimal bounds for $\tilde{a}$. Let us introduce

$$\hat{\beta} = \sqrt{(1-\hat{\varphi})\tfrac{1-\gamma}{2}}\,, \quad \hat{\Gamma} = \hat{\varphi}(1-\gamma) + 1 + \gamma\,, \quad h_\theta^{\triangle} = \sqrt{h_{\theta,\ell}\, h_{\theta,r}}\,,\ \theta = s, f\,,$$

$$B = h_s^{\triangle}\Delta u_s + \tfrac{1+\gamma}{2} h_f^{\triangle}\Delta u_f\,, \qquad C = (1-\hat{\varphi})h_s^{\triangle}\Delta u_s - \hat{\varphi}h_f^{\triangle}\Delta u_f\,,$$

$$K_\varphi = (1-\hat{\varphi})\Delta h_s - \hat{\varphi}\Delta h_f\,, \qquad K_s = \hat{\varphi}B + \tfrac{1+\gamma}{2}\tfrac{C}{\hat{\beta}}\,, \qquad K_f = (1-\hat{\varphi})B - \tfrac{C}{\hat{\beta}}\,,$$

and the notation $\bar{(\cdot)} \equiv \frac{(\cdot)_\ell + (\cdot)_r}{2}$, $(\cdot)_+ \equiv \max(0,(\cdot))$. We obtain the following positivity conditions for h_1, h_3:

$$\tilde{a} \geq \frac{B_+}{\hat{\Gamma}\min(D_1,D_3)} \equiv \tilde{a}_{1,3}\,, \quad \text{with} \quad D_{1,3} = \bar{h} \pm \tfrac{1-\gamma}{2}\tfrac{1}{\hat{\Gamma}}K_\varphi > 0\,, \tag{5.5a}$$

and the following conditions for positivity of h_{s2}, h_{f2}, and for positivity of all the intermediate states $h_{\theta k}$, $\theta = s, f$, $k = 1, 2, 3$, in the particular case $\Delta\varphi = 0$:

$$\tilde{a} \geq \frac{\max(\hat{\varphi} B_+, K_{s+})}{\hat{\Gamma} \bar{h}_s} \equiv \tilde{a}_{s2}\,, \qquad \tilde{a} \geq \frac{\max((1-\hat{\varphi}) B_+, K_{f+})}{\hat{\Gamma} \bar{h}_f} \equiv \tilde{a}_{f2}\,. \tag{5.5b}$$

Based on (5.5), we finally define $\tilde{a} = \max\Big(\sqrt{g\hat{h}}, \tilde{a}_{1,3}, \tilde{a}_{s2}, \tilde{a}_{f2}\Big)$.

Case $\delta U \neq 0$. In this case the intermediate states depend on $\tilde{a}$ through the eigenvalues $\tilde{\lambda}_k$, which are not explicitly available. Efficient analytical estimates for $\tilde{a}$ sufficiently large for positivity are difficult to derive, and here we prefer to apply a numerical iterative procedure. We use a first guess $\tilde{a}_{\delta U=0}$ computed through the formulas above for the case $\delta U = 0$, and we take iteratively $\tilde{a} = \tilde{a}_{\delta U=0} + j\, \mathrm{c}\, \delta U$, $j \in \mathbb{N}$, $\mathrm{c} \in \mathbb{R}_+$, increasing the counter j from 0 until positivity conditions are met, or until j reaches a fixed maximum number of iterations $N_{\max}$. In our numerical experiments we took $\mathrm{c} = 1$ and $N_{\max} = 3$.

Let us finally remark that although it seems appealing to take $\tilde{u}_s = \tilde{u}_f$ in the relaxation solver, this choice might lead to instabilities for certain flow conditions.

6. Numerical Experiments

We present here numerical results obtained with the proposed relaxation scheme for two problems involving vacuum states. Our scheme has been implemented by using the basic Fortran 77 routines of the CLAWPACK software [**LeV**]. In all the tests we take $\gamma = 1/2$ and $g = 1$.

Test 1. Spreading of a granular mass. We simulate the spreading of a granular mass on a horizontal surface. The mass is initially at rest, and the initial profiles of the flow height and of the solid volume fraction are defined by $h(x, 0) = 1$, if $x \in [-1, 1]$, $h(x, 0) = 0$ otherwise, and $\varphi(x, 0) = 0.3 + 0.4e^{-x^2}$. Inter-phase forces are not included in this experiment. We use 1000 grid cells over the computational domain $[-10, 10]$, with CFL = 0.9. Second order corrections with the Minmod limiter [**LeV02**] are applied, with the modification of the correction fluxes (3.1b) proposed in [**LG08**] to preserve the positivity property of the first-order scheme. Results are displayed in Figure 1, where we plot the profiles of the flow depth h (left) and of the solid volume fraction φ (right) at times $t = 0, 1, 2, 3, 4$. Let us remark that for this particular experiment the Roe-type scheme does not produce unphysical states, and we were able to compare the results of the two methods, noticing agreement.

Test 2. Dry bed generation. We perform a test in which we have an initial discontinuity at $x = 0$ (Riemann problem) with data $(h, \varphi, u_s, u_f) = (0.5, 0.3, -3, -3)$ on the left, and $(h, \varphi, u_s, u_f) = (0.7, 0.7, 3, 3)$ on the right. The solution for this problem consists of two opposite rarefactions that generate a dry bed region in between. This is a typical test for which the Roe-type solver of [**PBM08**] fails. We compute the solution for the case of infinitely large inter-phase drag forces. This is modeled numerically by forcing instantaneous phase velocity equilibrium through the fractional step algorithm for drag source terms described in [**PBM08**]. We use 200 grid cells, CFL = 0.9, and we apply second order corrections (Minmod limiter). Results at time $t = 1$ are displayed in Figure 2. On the left we show the flow height h and the variables h_s and h_f, on the right the momentum variables m_s, m_f and $m_{\mathrm{m}} = m_s + \gamma m_f$ (mixture momentum). We observe that our relaxation scheme is able to model the formation of the dry bed region with no generation

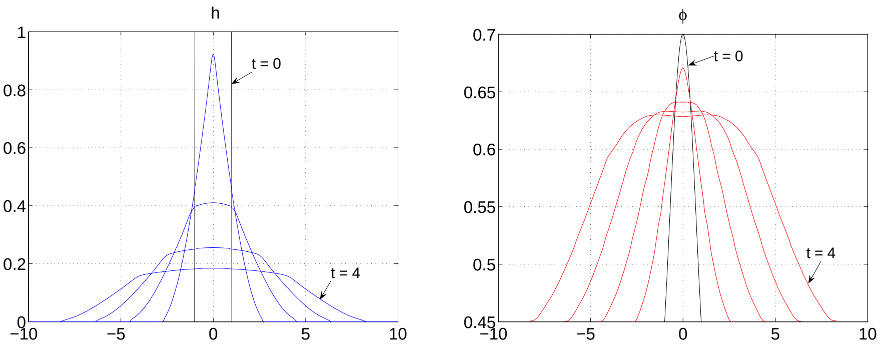

FIGURE 1. Spreading of a granular mass (no inter-phase drag).

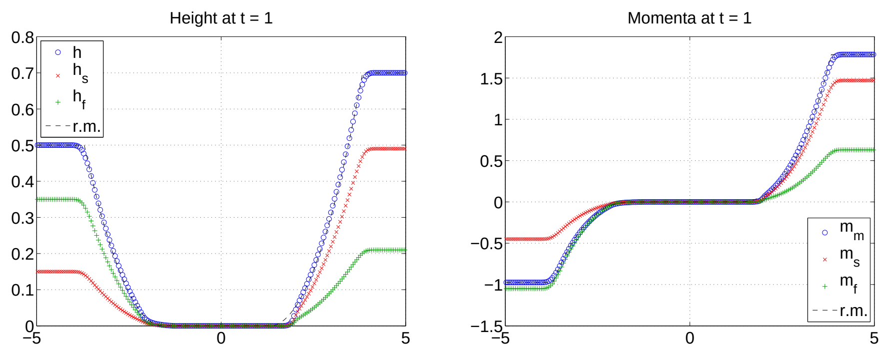

FIGURE 2. Opposite rarefactions with dry bed formation in the middle (infinitely large drag).

of unphysical states. The results for the flow height h and the mixture momentum m_{m} are also compared with the exact solution of the reduced model (dashed line) that can be obtained theoretically from the two-phase model (2.1) by assuming that drag forces are strong enough to drive instantaneously phase velocities to equilibrium. This model, presented in [**PBM08**], consists of conservative equations for the flow height h, for the mass $h\rho$, and for the mixture momentum $h\rho u$, where $\rho = \varphi + \gamma(1-\varphi)$, and u is the equilibrium velocity of the mixture. While for the full two-phase model exact solutions are not available (except trivial cases), this reduced model allows an easy derivation of exact Riemann solutions thanks to its simpler structure. Qualitative agreement is observed between the results of the two-phase model with instantaneous phase velocity equilibrium imposed numerically, and the analytical solution of the reduced model.

Additional numerical experiments involving formation of dry bed areas are reported in [**PB**], including examples with no drag forces and phase velocity disequilibrium.

7. Conclusions and Extensions

By means of a relaxation approach we have derived a new approximate Riemann solver for the numerical solution of a depth-averaged two-phase model of shallow flows made of solid grains and fluid. This new solver allows a more robust treatment of wet/dry fronts with respect to a Roe-type solver that we have introduced in previous work. Our current investigations focus on the extension of the new scheme to the more general model with bottom topography source terms studied in [**PBM08**]. While the technique (f-wave method [**BLMR02**]) employed in the latter work for the treatment of topography terms does not seem directly applicable to our relaxation solver, the well-balanced hydrostatic reconstruction method of [**ABB$^+$04**] could be used, and it appears a suitable choice for the preservation of the robustness of the scheme for the homogeneous system.

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Département de Mathématiques et Applications, École Normale Supérieure, 45 rue d'Ulm - 75230 Paris cedex 05, France.

E-mail address: `Marica.Pelanti@ens.fr`

CNRS & Département de Mathématiques et Applications, École Normale Supérieure, 45 rue d'Ulm - 75230 Paris cedex 05, France.

E-mail address: `Francois.Bouchut@ens.fr`