Examen of classical gravitation

Duration : 3H00

Calculators and documents are not allowed

The two body problem with geometry

We consider a mass m in the gravitational field of another mass M. We note $\vec{r} = \overrightarrow{Mm}$ the position vector between the two masses at each time. The Newton's laws of gravitation and dynamics write

$$m\dot{\vec{v}} = -\frac{\mu m}{r^2}\vec{e}_r \quad \text{with at each time } \vec{e}_r = \frac{\vec{r}}{\|\vec{r}\|} \text{ and } \vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt}$$

A Introduction

1. In which condition(s) can we consider that $\mu = GM$? We will consider this condition fullfilled from now.

2. Show that $\vec{r}$ stays in a plane. Caracterize this plane. We note (r, θ) the polar coordinates in this plane and $(\vec{e}_r, \vec{e}_\theta)$ the orthonormal local polar basis.

3. Write $\vec{e}_r$ as a function of $\dot{\vec{e}}_\theta$. Write r^2 in terms of the modulus L of the kinetic momentum of m relatively to M. Deduce that there exists a vector $\vec{u}$ which is proportional to one of the two vectors of the local polar basis such that

$$\frac{d}{dt}(\vec{v} - \vec{u}) = \vec{0}$$

   Write k, the coefficient of this proportionality, in terms of G, M, m and L.

4. Let $\vec{h} = \vec{v} - \vec{u}$, named the Hamilton vector, Caracterize geometrically the hodograph of velocities (The set of the extremity of the velocity $\vec{v}$ when m varies with time) in terms of $\vec{h}$ and $\vec{u}$.

5. Computing $\vec{u} \cdot \vec{h}$, show that the trajectory is conic (Kegelschnitt in german), precise it parameters.

6. The Lagrange vector $\vec{A}$ is defined by the relation $\vec{A} = \vec{h} \wedge \vec{L}$. Show that $\vec{A}$ give the direction of a symetry axis of the trajectory.

7. Determine the massic energy of m in terms of $\vec{h}$ and $\vec{u}$.

Often in astronomy, we only mesure the position and the velocity of a body at a given time of its orbit. We will see now how to construct geometrically the whole orbit only from this mesure assuming a newtonian force acting on m. For this we will use only a straightedge and a compass (Konstruktion mit Zirkel und Lineal in german) which was the only graphical instruments that astronomers has been able to use in the past.

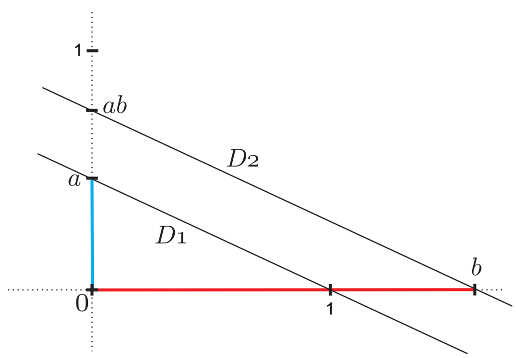
B Construction of the orbit with a straightedge and a compass

For all answers of this part use the paper given by the organiser and give it back. each step of the construction will be detailled by a written associated text in your copy. Figure 1 shows, at the initial time $t = 0$, position and velocity vectors of a mass m in the gravitational field of a mass M.

The units system used is such that $m = 1$ and $GM = 1$. This unit length is represented on the figure with a cartesian basis $(\vec{e}_x, \vec{e}_y)$. For this representation we have chosen the origin of velocities in M.

ais le vecteur $\vec{v}_0$ représente bien la vitesse de la particule de masse m.

8. Construct on figure 1, with only a straightedge and a compass, the length L of the kinetic momentum of m. One will verify after the construction that this length has a simple value. For eventual use, we recall below the construction with a straightedge and a compass of the product of two length a and b if one know the unit length.



Le repère (O,x,y) est orthonormé.

1) On construit la longueur b sur l 'axe Ox.
2) On construit la longueur a sur l 'axe Oy.
3) On construit la droite D_1 reliant a à l 'unité sur Ox
4) On construit la droite D_2 parallèle à D_1 passant par b.

D_2 coupe l 'axe Oy en un point ab, dont la distance à l 'origine est le produit de a par b.

9. Construct on figure 2, with only a straightedge and a compass, the Hamilton vector $\vec{h}$. Then deduce on the same figure the construction of the velocity hodograph.

10. Construct the Lagrange vector $\vec{A}$ on figure 2.

11. Starting from another point $\vec{v}_t$ chosen arbitrarily on the hodograph and using a reverse procedure, construct on figure 3, with only a straightedge and a compass, another point $m(t)$ of the orbit.

12. By constructing particular points of it, represent the whole orbit of the mass m on figure 4.

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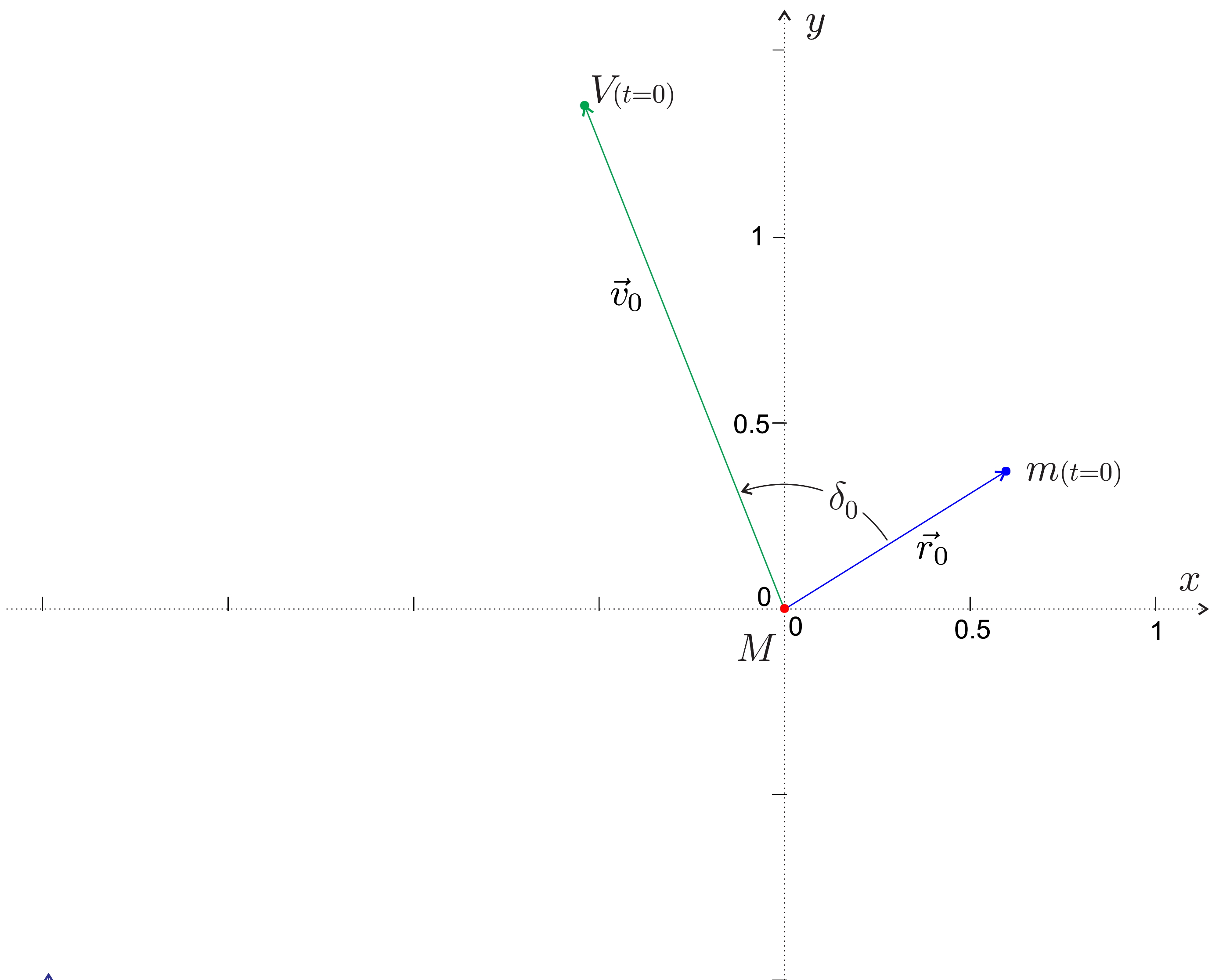

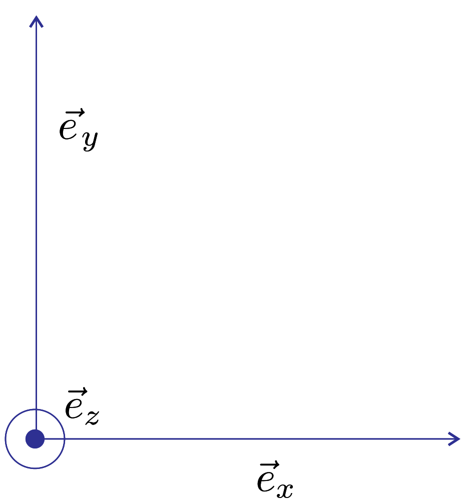

Figure 1 : Mesure de L

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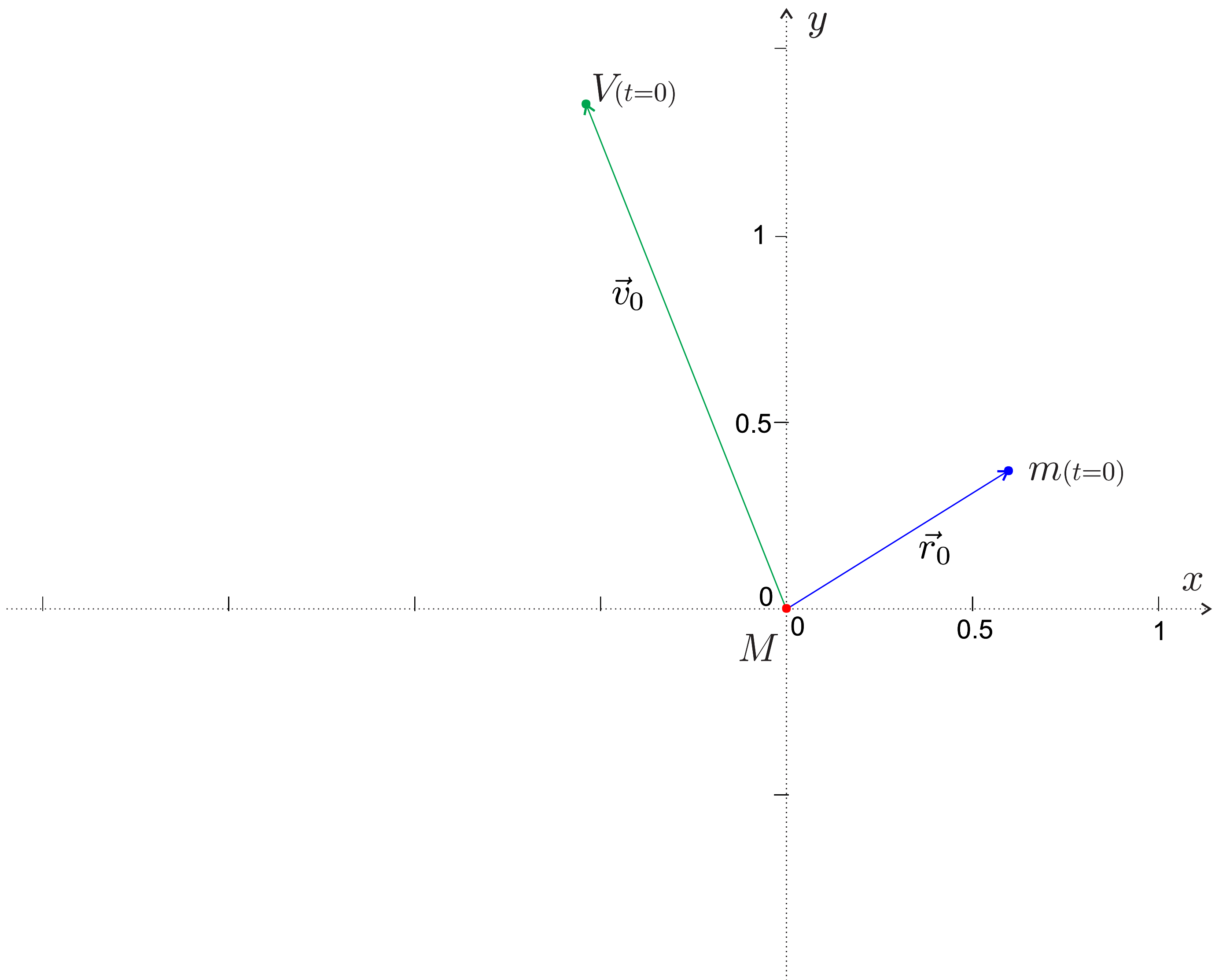

Figure 2 : Construction de $\vec{h}$ et de l'hodographe

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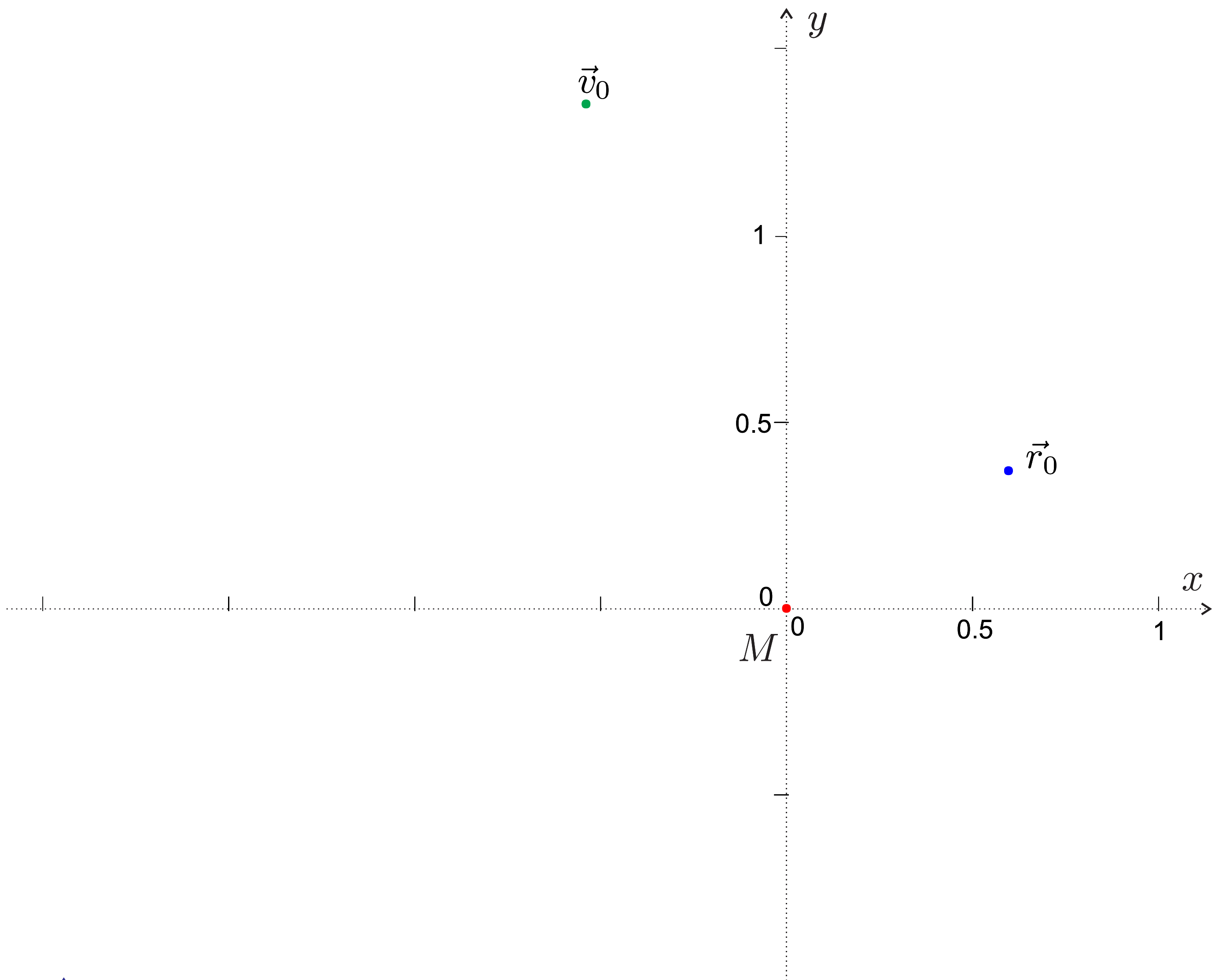

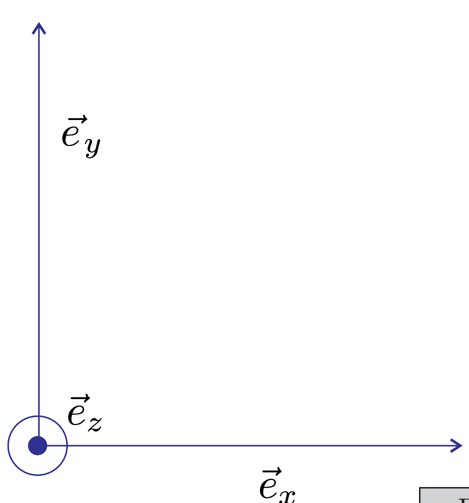

Figure 3 : Construction d'un autre point de l'orbite

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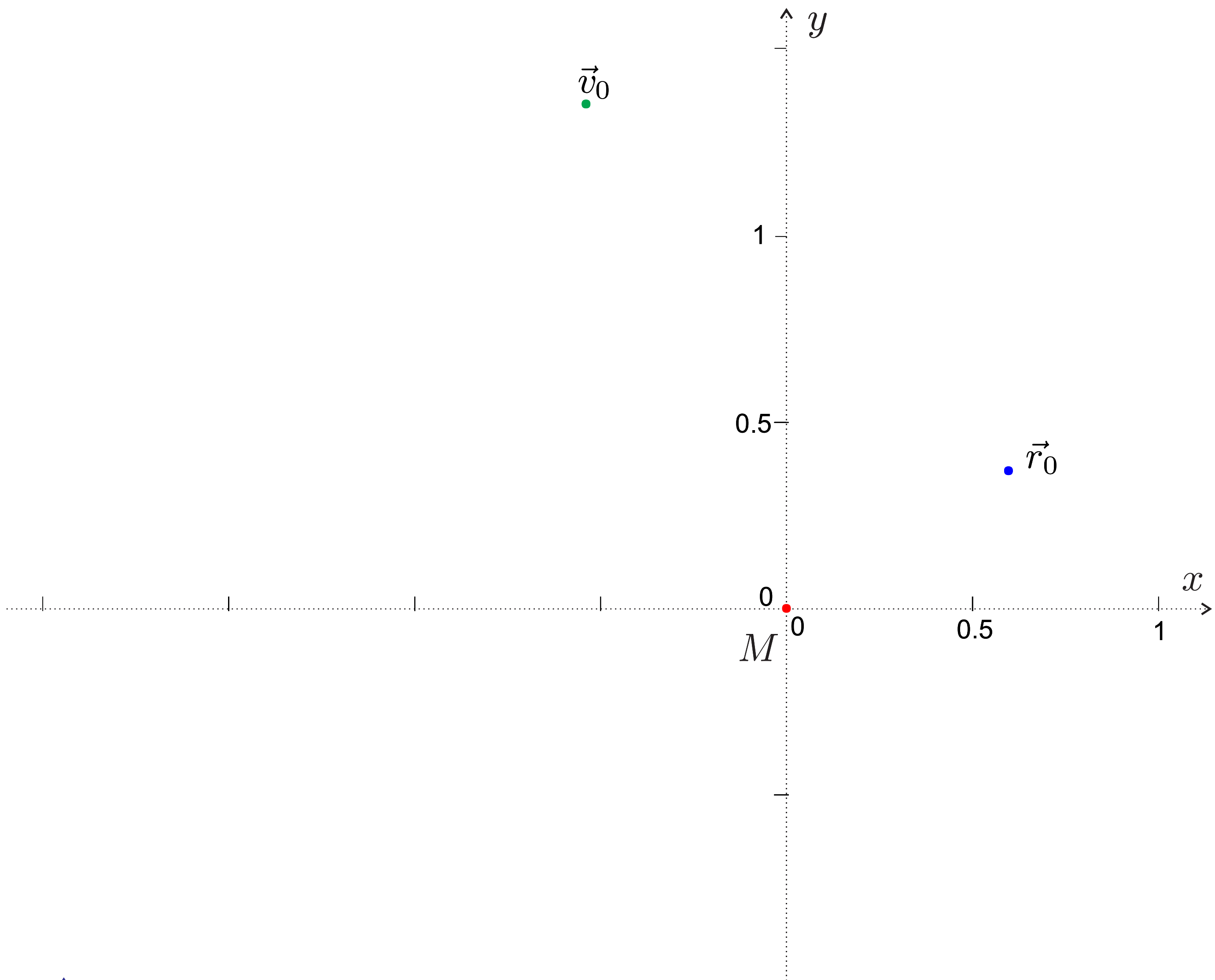

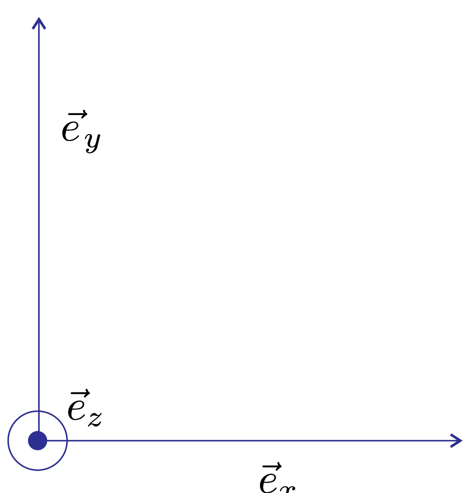

Figure 4 : Construction de l'orbite