Vlavov and Poisson equations in the context of self-gravitating systems

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March 10, 2006

Abstract

We present Vlasov's equation and its association with Poisson's equation in the context of modelling self-gravitating systems such as Globular Clusters or Galaxies. We first review the classical hypotheses of the model. We continue with a presentation of the Hamilton-Morrison structure of Vlasov's equation to study the equilibrium and the stability of self-gravitating systems. Finally, we present some preliminary results concerning some properties of the time dependent solutions of the Vlasov-Poisson system.

1 Hypotheses

It is well known that Vlasov's equation can be used to describe the dynamics of a self gravitating system. We recall the usual descriptive hypotheses.

First, we consider a set of N point mass particles of individual masses $m_{i=1,..,N}$ and we consider the associated N particle distribution function in the phase space $f^{(N)}$ such that, at any time t

$$1 = \int f^{(N)}\left(\mathbf{\Gamma}_1, \cdots, \mathbf{\Gamma}_N, t\right) \prod_{i=1}^{N} d\mathbf{\Gamma}_i \quad , \tag{1}$$

where $\mathbf{\Gamma}_i = (\mathbf{p}_i, \mathbf{q}_i)$ is the 6-vector which indicates the position $\mathbf{q}_i$ and the impulsion $\mathbf{p}_i$ of the i^{th} particle. Provided that each $\mathbf{p}_i$ and $\mathbf{q}_i$ are Hamiltonian conjugated variables and all the forces are conservative, one can obtain directly from this normalization equation a conservation equation (Liouville's theorem)

$$\frac{\partial f^{(N)}}{\partial t}+\sum_{i=1}^{N}\left(\frac{\mathbf{p}_i}{m}.\frac{\partial f^{(N)}}{\partial \mathbf{q}_i}-\frac{\partial U_i}{\partial \mathbf{q}_i}.\frac{\partial f^{(N)}}{\partial \mathbf{p}}\right)=0 \tag{2}$$

where U_i is the potential energy at the origin of the force $\mathbf{f}_i$ seen by the i^{th} particle

$$\mathbf{f}_i=\frac{d\mathbf{p}_i}{dt}=-\frac{\partial U_i}{\partial \mathbf{q}_i} \tag{3}$$

If the only force considered is gravitation, we have

$$U_i=-G\sum_{i\neq j=1}^{N}\frac{m_i m_j}{|\mathbf{q}_i-\mathbf{q}_j|} \tag{4}$$

In order to obtain a gravitational Vlasov's equation, we first need to introduce marginal distribution functions $f^{(1)}$,and $f^{(2)}$ respectively for one and two particles as

$$f^{(1)}\left(\mathbf{\Gamma}_i,t\right)=\int f^{(N)}\left(\mathbf{\Gamma}_1,\cdots,\mathbf{\Gamma}_N,t\right)\prod_{j\neq i=1}^{N}d\mathbf{\Gamma}_j \tag{5}$$

$$f^{(2)}\left(\mathbf{\Gamma}_i,\mathbf{\Gamma}_j,t\right)=\int f^{(N)}\left(\mathbf{\Gamma}_1,\cdots,\mathbf{\Gamma}_N,t\right)\prod_{k\neq j\neq i=1}^{N}d\mathbf{\Gamma}_k \tag{6}$$

We then make use of the Boltzmann ansatz which assumes the existence of a two-particle correlation function $k\left(\Gamma_i,\Gamma_j,t\right)$ such that

$$\forall i\neq j=1,..,N \qquad f^{(2)}\left(\mathbf{\Gamma}_i,\mathbf{\Gamma}_j,t\right)=f^{(1)}\left(\mathbf{\Gamma}_i,t\right)f^{(1)}\left(\mathbf{\Gamma}_j,t\right)+k\left(\mathbf{\Gamma}_i,\mathbf{\Gamma}_j,t\right) \tag{7}$$

We now have suppose that the particles are indiscernible[1], and have the same probability law $f\left(\mathbf{\Gamma},t\right)=f^{(1)}\left(\mathbf{\Gamma},t\right)/N$. Some direct calculations then give

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}}{m}.\frac{\partial f}{\partial \mathbf{q}}-m\frac{\partial \psi}{\partial \mathbf{q}}.\frac{\partial f}{\partial \mathbf{p}}=N^2Gm\int\frac{\partial k\left(\mathbf{\Gamma},\mathbf{\Gamma}',t\right)}{\partial \mathbf{p}}\frac{\left(\mathbf{q}-\mathbf{q}'\right)}{|\mathbf{q}-\mathbf{q}'|^3}d\mathbf{\Gamma}' \tag{8}$$

where $\psi\left(\mathbf{q},t\right)$ is the mean field gravitational potential defined from f by the inverse Poisson equation

$$\psi\left(\mathbf{q},t\right)=-Gm\int\frac{f\left(\mathbf{\Gamma}',t\right)}{|\mathbf{q}-\mathbf{q}'|}d\mathbf{\Gamma}' \tag{9}$$

It is well known (see e.g. Binney and Tremaine[12]) that in the case of self-gravitating systems the time T_r taken by collisions to relax it, can be evaluated by the Chandrasekhar formula

$$T_r\approx\frac{N}{\ln N}T_d\approx\frac{N}{\ln N}\left(G\overline{\rho}\right)^{-1/2} \tag{10}$$

[1] In particular this means that all particles have the same mass m.

where T_d is the dynamical or crossing time of the system and $\overline{\rho}$ its mean mass density. The collisional relaxing time then ranges from $10^{9,10}yr$ for Globular Clusters $\left(N \sim 10^5\right)$ or $10^{11,15}yr$ for a typical galaxy $\left(N \sim 10^{10}\right)$ to $10^{13,17}yr$ for Galaxy Clusters $\left(N \sim 10^3\right)$. Hence, during a reasonable time of about a few hundreds dynamical times one can actually consider that two-particle correlations (namely gravitational collisions) are dynamically unimportant, and by consequence that

$$k\left(\mathbf{\Gamma},\mathbf{\Gamma}',t\right) \equiv 0 \tag{11}$$

During this long period the mean field dynamics of a self-gravitating system are governed by the Vlasov-Poisson system

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}.\frac{\partial f}{\partial \mathbf{q}} - m\frac{\partial \psi}{\partial \mathbf{q}}.\frac{\partial f}{\partial \mathbf{p}} = 0 \tag{12}$$

$$\psi\left(\mathbf{q},t\right) = -Gm\int \frac{f\left(\mathbf{\Gamma}',t\right)}{\left|\mathbf{q}-\mathbf{q}'\right|}d\mathbf{\Gamma}' \tag{13}$$

2 Vlasov-Poisson as a Hamiltonian system

It is often quoted that Vlasov's equation is a Hamiltonian system. It is true but the formulation of such a problem is generally unclear. Using classical Poisson brackets defined for any scalar fields $A\left(\mathbf{q},\mathbf{p},t\right)$ and $B\left(\mathbf{q},\mathbf{p},t\right)$ by

$$\left\{A,B\right\} := \frac{\partial A}{\partial \mathbf{q}}.\frac{\partial B}{\partial \mathbf{p}} - \frac{\partial B}{\partial \mathbf{q}}.\frac{\partial A}{\partial \mathbf{p}} \tag{14}$$

one can write Vlasov's equation in the form

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}.\frac{\partial f}{\partial \mathbf{q}} - m\frac{\partial \psi}{\partial \mathbf{q}}.\frac{\partial f}{\partial \mathbf{p}} = \frac{\partial f}{\partial t} + \left\{f,E\right\} = 0 \tag{15}$$

where

$$E = \frac{\mathbf{p}^2}{2m} + m\psi = \frac{\mathbf{p}^2}{2m} - Gm^2\int \frac{f\left(\mathbf{\Gamma}',t\right)}{\left|\mathbf{q}-\mathbf{q}'\right|}d\mathbf{\Gamma}' \tag{16}$$

is the mean field energy per particle. It is a fact that relation (15) is not a Hamiltonian equation. As a matter of fact, general Hamiltonian equations give the *total* derivative of a canonical variable from a lie bracket between this variable and a propagator. Nevertheless, a series of works initiated by Morrison[1] , have shown that a Hamiltonian formulation of non dissipative physics is generally possible. Let us recall these results in the context of Vlasov's equation.

We consider any physical functional of the distribution function, in a very general context this functional can be written

$$F\left[f\right] = \int \varphi\left(f,\mathbf{\Gamma}\right)d\mathbf{\Gamma} \tag{17}$$

where φ is a very general suitable function, for example if φ do not depend on $\mathbf{\Gamma}$, $F\left[f\right]$ is called a Casimir and it can be easily proven that this is a conserved quantity during the Vlasov dynamics. By direct calculation we have

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} = \int \frac{\partial f}{\partial t}\frac{\partial \varphi}{\partial f} d\mathbf{\Gamma} \quad . \tag{18}$$

It is important to note here that

$$\frac{\partial \varphi}{\partial f} = \frac{\delta F}{\delta f} \tag{19}$$

where $\delta / \delta f$ indicates the usual functional derivative defined in the case of (17) by the relation

$$F\left[f + \delta f\right] - F\left[f\right] = \int \left[\frac{\delta F}{\delta f}\delta f + \mathcal{O}\left(\delta f\right)\right] d\mathbf{\Gamma} \tag{20}$$

Using (18), (19) and (15) , one can then write

$$\frac{dF}{dt} = \int \left\{E, f\right\} \frac{\delta F}{\delta f} d\mathbf{\Gamma} \tag{21}$$

Using definition (20) , one can verify that the total energy contained in the system

$$H\left[f\right] := \int \frac{\mathbf{p}^2}{2m} f\left(\mathbf{\Gamma}, t\right) d\mathbf{\Gamma} - \frac{Gm^2}{2} \int \frac{f\left(\mathbf{\Gamma}, t\right) \;\; f\left(\mathbf{\Gamma}', t\right)}{\left|\mathbf{q} - \mathbf{q}'\right|} d\mathbf{\Gamma} d\mathbf{\Gamma}' \tag{22}$$

which is a functional of the form (17), is such that

$$\frac{\delta H}{\delta f} = E \quad . \tag{23}$$

Then, integrating the RHS of (21) by parts[2] and employing (23) one gets

$$\frac{dF}{dt} = \int \left\{\frac{\delta F}{\delta f}, \frac{\delta H}{\delta f}\right\} f d\mathbf{\Gamma} \tag{25}$$

which is the result introduced, perhaps more directly, by Morrison[1] . This last equation can be formulated as

$$\frac{dF}{dt} = \left\langle F, H\right\rangle \tag{26}$$

[2]This is actually a integration by parts. One can directly show indeed that for functions that decay sufficiently rapidly at $|\mathbf{q}|$ and $|\mathbf{p}|$ tend to ∞ we have

$$\int d\mathbf{\Gamma} A\left\{B, C\right\} = -\int d\mathbf{\Gamma} C\left\{B, A\right\} = \int d\mathbf{\Gamma}\left\{A, B\right\} C \tag{24}$$

In the case of equation (25) surface terms vanish as the distribution function vanishes at infinity.

where we have introduced the Morrison's bracket defined for any functional $A[f]$ and $B[f]$ of the form (17), by the relation

$$\langle A, B\rangle = \int \left\{ \frac{\delta A}{\delta f}, \frac{\delta B}{\delta f} \right\} f d\mathbf{\Gamma} \tag{27}$$

Some very simple calculations show that Morrison's bracket is skew-symmetric, verify Jacobi's identity and can be used to form a Lie algebra. Hence, equation (26) naturally defines the Hamiltonian structure of Vlasov's equation. We can note that in this formulation there is no conjugated variable to the canonical f.

3 Some uses ...

3.1 Generalities

It is well known that a regular Hamiltonian formulation of a problem provides room for much formal analysis. Let us see what is possible with Vlasov's equation. In a pure formal way, one can always write

$$\frac{dF}{dt} = \langle F, H\rangle = -\langle H, .\rangle F := \mathcal{T}(F) \quad . \tag{28}$$

Here, the functional operator $\mathcal{T}$ appearing in this last equation does not depend explicitly on time t, because the total energy of the system $H[f]$ is a conserved quantity. Hence, one can utilize the resolvant of the Ordinary Differential Equation (28) and we have

$$F[f] = \exp\left[-(t-t_o)\langle H, .\rangle\right] F[f_o] \tag{29}$$

$$= F[f_o] - \tfrac{(t-t_o)}{1!}\langle H, F[f_o]\rangle_{f=f_o} + \tfrac{(t-t_o)^2}{2!}\langle H, \langle H, F[f_o]\rangle\rangle_{f=f_o} + \mathcal{O}\left((t-t_o)^2\right) \tag{30}$$

where $f_o = f(\mathbf{\Gamma}, t_o)$. It is possible to compare this expansion with the general Taylor expansion obtained from equation (17) which has the form

$$F[f(\mathbf{\Gamma}, t)] = \int \varphi(f(\mathbf{\Gamma}, t), \mathbf{\Gamma})\, d\mathbf{\Gamma} \tag{31}$$

$$= \int \left[\varphi(f(\mathbf{\Gamma}, t_o), \mathbf{\Gamma}) + \tfrac{(t-t_o)}{1!}\frac{\partial\varphi(f(\mathbf{\Gamma}, t_o), \mathbf{\Gamma})}{\partial f} + \tfrac{(t-t_o)^2}{2!}\frac{\partial^2\varphi(f(\mathbf{\Gamma}, t_o), \mathbf{\Gamma})}{\partial f^2} + \mathcal{O}\left((t-t_o)^2\right)\right] d\mathbf{\Gamma} \tag{32}$$

Hence, we arrive at

$$\frac{\partial\varphi(f(\mathbf{\Gamma}, t_o), \mathbf{\Gamma})}{\partial f} = -\langle H, F[f_o]\rangle \tag{33}$$

$$\frac{\partial^2\varphi(f(\mathbf{\Gamma}, t_o), \mathbf{\Gamma})}{\partial f^2} = \langle H, \langle H, F[f_o]\rangle\rangle \tag{34}$$

etc ... It is well known from quantum field theory that this expansion allows us see that the total energy H is the time propagator of the system. In a more general way, Noether's Theorem says that any conserved quantity is associated to a symmetry of the system, hence invoking a general canonical transformation, the functional Vlasov's equation can be written (see Perez and Lachieze-Rey[10] for details)

$$\frac{dF}{d\lambda} = \langle F, K \rangle \tag{35}$$

where the pair $(K[f], \lambda)$ represents the conserved quantity (e.g. $H[f]$) and the variable related to system's symmetry (e.g. $t-$translation), respectively.

Below, we present two natural uses of these seemingly formal developments

3.2 Stability analysis

We now consider a steady state solution $f_o(\mathbf{\Gamma})$ of the Vlasov's equation. In a general way, Hamilton's theory of canonical transformations says us that any physical perturbation of this equilibrium is generated by some functional generator G, such that for any functional $F[f]$ of the distribution function we have

$$\frac{dF}{d\varepsilon} = \langle F, G \rangle \tag{36}$$

where ε controls the perturbation (see Perez and Lachieze-Rey [10] and reference therein). In particular, if the perturbed state is described by a distribution function $f(\mathbf{\Gamma}, t)$ written as

$$f(\mathbf{\Gamma}, t) = f_o(\mathbf{\Gamma}) + \varepsilon f_1(\mathbf{\Gamma}, t) + \varepsilon^2 f_2(\mathbf{\Gamma}, t) + O\left(\varepsilon^2\right) \tag{37}$$

Taking $F = Id$ in equation (36), one can then verify that

$$f_1(\mathbf{\Gamma}, t) = \{g, f_o\}, \qquad f_2(\mathbf{\Gamma}, t) = \frac{1}{2!}\{g, \{g, f_o\}\}, \qquad \text{etc...} \tag{38}$$

where the so-called generating function g is the functional derivative of G. This kind of theory says that the linearized perturbation $f_1 := \{g, f_o\}$ is generated from the unperturbed distribution $f_o(\mathbf{\Gamma})$ via a canonical transformation by some small perturbation $g(\mathbf{\Gamma}, t)$. And so on for high order non linear terms.

Considering such kinds of perturbation is not restrictive. As a matter of fact, only perturbations of this form can be generated via a deformation that enforces all the constraints associated with conservation of phase (Liouville's theorem).

Taking $F = H$ in equation (36) allows us to obtain an energy perturbation induced by equilibrium perturbation

$$\begin{aligned} H[f] &= H_o + \varepsilon H_1 + \varepsilon^2 H_2 + \mathcal{O}\left(\varepsilon^2\right) \\ &= H[f_o] - \varepsilon \langle G, H \rangle_{f=f_o} + \tfrac{1}{2}\varepsilon^2 \langle G, \langle G, H[f_o] \rangle\rangle_{f=f_o} + \mathcal{O}\left(\varepsilon^2\right) \end{aligned} \tag{39}$$

The first two terms of this expansion are very useful in stability analysis.

3.2.1 Equilibrium

It is a well known property that if a system is in an equilibrium state, the first order variation of its total energy vanishes. This property can be directly shown from the use of the $\varepsilon-$order term in the expansion (39). As a matter of fact,

$$H_1 = -\left\langle G, H \right\rangle_{f=f_o} = -\int \left\{ \frac{\delta G}{\delta f}, \frac{\delta H}{\delta f} \right\} f_o d\mathbf{\Gamma} = \int f_o \left\{ E, \frac{\delta G}{\delta f} \right\} d\mathbf{\Gamma} \tag{40}$$

an integration by parts and the use of Vlasov's equation then gives

$$H_1 = \int \left\{ f_o, E \right\} \frac{\delta G}{\delta f} d\mathbf{\Gamma} = \int \frac{\partial f_o}{\partial t} \frac{\delta G}{\delta f} d\mathbf{\Gamma} \tag{41}$$

which is evidently a vanishing term if f_o is an equilibrium state.

General equilibrium states are the following :

- $f_o = f_o\left(E\right)$ distribution functions represent all $\mathbf{q}-$spherical and $\mathbf{p}-$isotropic equilibria of collisionless self-gravitating systems ;
- $f_o = f_o\left(E, L^2\right)$ distribution functions where L^2 is the mean field squared angular momentum per particle, represent all $\mathbf{q}-$spherical and $\mathbf{p}-$anisotropic equilibria of collisionless self-gravitating systems ;
- $f_o = f_o\left(E, L_z\right)$ distribution functions where L_z is the mean field $z-$component of the angular momentum per particle, represent an equilibrium self-gravitating system with a $z-$symmetry in $\mathbf{q}-$space.

For detailed proofs of these affirmations and properties of these equilibria, we refer to Perez and Aly[8] .

3.2.2 Equilibrium stability

The stability of a given equilibrium described by the time-independent distribution function f_o is classically related to the sign of the ε^2-order term in the expansion (39)

$$H_2 = \tfrac{1}{2}\left\langle G, \left\langle G, H\left[f_o\right]\right\rangle\right\rangle_{f=f_o} \tag{42}$$

More precisely, one can prove that an equilibrium state described by a distribution function f_o is guaranteed to be linearly stable if the second variation H_2 is strictly positive for all perturbations. The positivity of H_2 thus provides a sufficient criterion for linear stability. If H_2 is of indeterminate sign, the situation is less obvious. In fact the existence of negative energy perturbations does not necessarily imply an instability (there exist some counterexample). However, one *does* at least expect that it guarantees non linear instability in the presence of dissipation. This idea goes back at least to Moser [5] . The implications for the Vlasov's equation were first considered by P.J. Morrison[6] in the context of plasma physics. That negative energy perturbations imply an instability towards dissipation was proved more recently in a fairly general context by Bloch et al [7] .

Let us indicates below the essential calculations which yields an explicit formula for H_2. From (27), taking $g := \delta G/\delta f$, we get

$$H^{(2)} = \frac{1}{2}\int f_o \left\{ g, \frac{\delta}{\delta f}\left[\int \{g', E'\} f' d\mathbf{\Gamma}'\right]\right\}_{f=f_o} d\mathbf{\Gamma} \tag{43}$$

from the differential properties of the Poisson bracket we have

$$H^{(2)} = \frac{1}{2}\int f_o \left\{ g, \int \left[\left\{\frac{\delta g'}{\delta f}, E'\right\} f'_o + \left\{g', \frac{\delta E'}{\delta f}\right\} f'_o + \{g', E'\}\,\delta_d\left(\mathbf{\Gamma} - \mathbf{\Gamma}'\right)\right] d\mathbf{\Gamma}'\right\} d\mathbf{\Gamma} \ . \tag{44}$$

An integration by parts of the two first terms in the middle sum and the integration of the δ_d Dirac mass term gives

$$H^{(2)} = \frac{1}{2}\int f_o \left\{ g, \left[\{g, E\} + \int \frac{\delta g'}{\delta f}\{E', f'_o\}\, d\mathbf{\Gamma}' - \int \{g', f'_o\}\frac{\delta E'}{\delta f} d\mathbf{\Gamma}'\right]\right\} d\mathbf{\Gamma} \ . \tag{45}$$

If f_o is an equilibrium then $\{E, f_o\}$ vanishes, we remark also that from (16)

$$\frac{\delta E'}{\delta f} = m\frac{\delta \psi'}{\delta f} = \frac{-Gm^2}{|\mathbf{q} - \mathbf{q}'|} \tag{46}$$

to arrive at

$$H^{(2)} = \frac{1}{2}\int f_o \{g, \{g, E\}\}\, d\mathbf{\Gamma} + \frac{Gm^2}{2}\int f_o \left\{ g, \int \frac{\{g', f'_o\}}{|\mathbf{q} - \mathbf{q}'|} d\mathbf{\Gamma}'\right\} d\mathbf{\Gamma} \ . \tag{47}$$

A last integration by parts on each term allows us to see that

$$H^{(2)} = -\frac{1}{2}\int \{g, f_o\}\{g, E\}\, d\mathbf{\Gamma} - \frac{Gm^2}{2}\int\int \frac{\{g, f_o\}\{g', f'_o\}}{|\mathbf{q} - \mathbf{q}'|} d\mathbf{\Gamma}' d\mathbf{\Gamma} \tag{48}$$

Studying of the sign of this quantity, one can study the stability of equilibrium associated to f_o against perturbations generated by g. Classical results are presented in Perez and Aly[8], Perez et al.[9] and Kandrup[2], [3], [4] . They can be summarized as follow :

- If $f_o = f_o(E)$ and $f_{oE} := \partial f_o/\partial E < 0$, then for all g we have $H^{(2)} > 0$. Hence, any spherical and isotropic stellar system with an outwardly decreasing distribution function is linearly stable against all perturbations.
- If $f_o = f_o\left(E, L^2\right)$, $f_{oE} < 0$, and g is such that $\left\{g, L^2\right\} = 0$ then $H^{(2)} > 0$, such perturbations are called preserving perturbations. All radial perturbations are preserving[3]. Hence, any spherical and anisotropic stellar system with an outwardly decreasing distribution function is linearly stable at least against all radial perturbations.

[3]This proposition is not reciprocal.

- If $f_o = f_o\left(E, L^2\right)$ (resp. $f_o\left(E, L_z\right)$), $f_{oE} < 0$, and $f_{oL^2} < 0$ (resp. $f_{oL_z} < 0$) then there always exists non radial (resp. nonaxisymetric) perturbations, corresponding to nonzero mass density perturbations, for which $H^{(2)} < 0$. It was conjectured[9] that when $f_o\left(E, L^2\right) \to f_o\left(E\right)\delta_d\left(L^2\right)$ (resp. $f_o\left(E, L_z\right) \to f_o\left(E\right)\delta_d\left(L_z\right)$) then $H^{(2)} < 0$ for all perturbation generator g. This may be associated to Radial Orbit Instability.

3.3 Study of the transient regime of gravitational Vlasov-Poisson dynamics

The transition to a steady state is a crucial problem in self-gravitating systems dynamics. As a matter of fact, observations suggest that almost all galaxies or globular clusters manifest overall similarities and that they do not deviate very much from a statistical equilibrium. We may also note that numerical simulation of collections of point masses tend to typically evidence a rapid, systematic evolution, on dynamical time scale T_d, towards a state, which is at least in a statistical sense, nearly time-independent (see Roy and Perez[13]).

In the toy model of one dimensional gravity, where one considers the interactions of infinite one dimensional-plane sheets, careful simulations can be performed (e.g. Mineau et al.[14]). The net result of such integrations is that typically such systems do not exhibit a pointwise evolution toward a steady state configuration. It is significant that solutions of $1D$ Vlasov-Poisson system for the same initial conditions also exhibit an evolution toward a final state which is not strictly time-independent.

In the unphysical 2D gravity model, a complete thermodynamical approach can be performed and show that there can exist an unique and well defined equilibrium state associated to a maximum entropy principle (see Aly and Perez[11]). This result should be soon confronted with numerical simulations of 2D Vlasov-Poisson system.

In the real 3D world, one can ask if there exists a correspondence between time-dependent Vlasov-Poisson solutions, numerical simulations of large collections of point masses and corresponding thermodynamical systems.

In order to progress toward this end, let us present now, some uses of Hamiltonian techniques presented above which perhaps could allow one to exhibit time-dependent solutions of 3D Vlasov-Poisson system.

It is a fact that, at a given position $\mathbf{q}$ and for a given time t the gravitational potential is a functional of the distribution function

$$\psi_{\mathbf{q},t}\left[f\right] = -Gm\int \frac{f\left(\mathbf{\Gamma}', t\right)}{\left|\mathbf{q}-\mathbf{q}'\right|}d\mathbf{\Gamma}' \tag{49}$$

one can then always take $F := \psi$ in the functional Vlasov's equation to have

$$\frac{d\psi}{dt} = \left\langle \psi, H\right\rangle = -\left\langle H, .\right\rangle \psi \tag{50}$$

hence, we can have a formal time development of the potential from which, applying Laplacian operator one can deduce the temporal evolution of the mass density. There is no problem in principle, rather only technical ones ...

As $-\langle H, .\rangle$ is a time-independent operator, we can employ the resolvant of equation (50) to write

$$\begin{aligned}\psi_{\mathbf{q},t}[f] &= \psi^{(0)}_{\mathbf{q},t_o} + \tfrac{(t-t_o)}{1!}\psi^{(1)}_{\mathbf{q},t_o} + \tfrac{(t-t_o)^2}{2!}\psi^{(2)}_{\mathbf{q},t_o} + \mathcal{O}\left((t-t_o)^2\right) \\ &= \psi_{\mathbf{q},t_o}[f_o] - \tfrac{(t-t_o)}{1!}\langle H, \psi_{\mathbf{q},t_o}[f_o]\rangle_{f=f_o} \\ &+ \tfrac{(t-t_o)^2}{2!}\langle H, \langle H, \psi_{\mathbf{q},t_o}[f_o]\rangle\rangle_{f=f_o} + \mathcal{O}\left((t-t_o)^2\right)\end{aligned} \tag{51}$$

The three first terms of this expansion are easily calculable.
The first term is obtained by definition

$$\psi_{\mathbf{q},t_o}[f_o] := -Gm\int \frac{f(\mathbf{\Gamma}', t_o)}{|\mathbf{q}-\mathbf{q}'|} d\mathbf{\Gamma}' \tag{52}$$

From this relation one can see that

$$\frac{\delta\psi_{\mathbf{q},t_o}}{\delta f} = \frac{-Gm}{|\mathbf{q}-\mathbf{q}'|} \tag{53}$$

and one can compute the second term

$$\begin{aligned}\psi^{(1)}_{\mathbf{q},t_o} &:= -\langle H, \psi_{\mathbf{q},t_o}[f_o]\rangle_{f=f_o} = -\int \left\{\frac{\delta H'}{\delta f}, \frac{\delta\psi_{\mathbf{q},t_o}}{\delta f}\right\} f(\mathbf{\Gamma}', t_o)\, d\mathbf{\Gamma}' \\ &= \int \left\{E', Gm\,|\mathbf{q}-\mathbf{q}'|^{-1}\right\} f(\mathbf{\Gamma}', t_o)\, d\mathbf{\Gamma}' \\ &= -G\int \mathbf{p}'.\frac{\partial\,|\mathbf{q}-\mathbf{q}'|^{-1}}{\partial\mathbf{q}'} f(\mathbf{\Gamma}', t_o)\, d\mathbf{\Gamma}'\end{aligned} \tag{54}$$

A first hypothesis on the initial distribution function can be now posed to simplify the calculus.

Hypothesis 1 : The initial state is symmetric in the velocity space. The consequence of this assumption is that for any vectorial space field $B(\mathbf{q})$ and any vectorial *odd* velocity field $A(\mathbf{p})$, we have

$$\int A(\mathbf{p}).B(\mathbf{q})\ f(\mathbf{\Gamma}, t_o)\, d\mathbf{\Gamma} = 0 \tag{55}$$

Under this non-restrictive assumption, one can easily show that

$$\psi^{(1)}_{\mathbf{q},t_o} = 0 \tag{56}$$

Moreover, in a general way one see also that

$$\frac{\delta\psi^{(1)}_{\mathbf{q},t_o}}{\delta f} = -G\mathbf{p}'.\frac{\partial\,|\mathbf{q}-\mathbf{q}'|^{-1}}{\partial\mathbf{q}'} \tag{57}$$

which allows the computation of the third term in the potential expansion.

$$\psi^{(2)}_{\mathbf{q},t_o} = \left\langle H, \psi^{(1)}_{\mathbf{q},t_o} \right\rangle_{f=f_o} = -G \int \left\{ E', \mathbf{p}'.\frac{\partial \left|\mathbf{q}-\mathbf{q}'\right|^{-1}}{\partial \mathbf{q}'} \right\} f\left(\mathbf{\Gamma}', t_o\right) d\mathbf{\Gamma}' \\ = \mathcal{A} + \mathcal{B} \tag{58}$$

where

$$\mathcal{A} = -Gm \int d\mathbf{\Gamma}' f\left(\mathbf{\Gamma}', t_o\right) \frac{\partial \psi_{\mathbf{q}',t_o}}{\partial \mathbf{q}'}.\frac{\partial \left|\mathbf{q}-\mathbf{q}'\right|^{-1}}{\partial \mathbf{q}'} \tag{59}$$

and

$$\mathcal{B} = +\frac{G}{m} \int d\mathbf{\Gamma}' f\left(\mathbf{\Gamma}', t_o\right) \mathbf{p}'.\frac{\partial}{\partial \mathbf{q}'} \left(\mathbf{p}'.\frac{\partial \left|\mathbf{q}-\mathbf{q}'\right|^{-1}}{\partial \mathbf{q}'} \right) \tag{60}$$

$\psi^{(2)}_{\mathbf{q},t_o}$ could be greatly simplified assuming some physical properties of the initial state $f\left(\mathbf{\Gamma}, t_o\right)$. First, we suppose that the initial density is constant on a bounded domain $\Omega \subset \mathbb{R}^3$. Then we may pose

Hypothesis 2

$$\int f\left(\mathbf{\Gamma}, t_o\right) d\mathbf{p} = \nu_o.\chi_\Omega = \frac{\rho_o}{m}.\chi_\Omega \tag{61}$$

where $\rho_o = m\nu_o$ is a strictly positive constant (ν_o is the value of the uniform initial number density whereas ρ_o represent the mass density) and χ_Ω denotes the characteristic function[4] on Ω.

This hypothesis which can appear very restrictive, is fairly adapted for the description of the initial state of the physical material before it collapse to form a self-gravitating structure. Under this hypothesis, we have

$$\mathcal{A} = -G\rho_o \int_{\mathbf{q}'\in\Omega} \frac{\partial \psi_{\mathbf{q}',t_o}}{\partial \mathbf{q}'}.\frac{\partial \left|\mathbf{q}-\mathbf{q}'\right|^{-1}}{\partial \mathbf{q}'} \quad , \tag{62}$$

an integration by parts in which all surface terms vanish then gives

$$\mathcal{A} = G\rho_o \int_{\mathbf{q}'\in\Omega} \psi_{\mathbf{q}',t_o}.\Delta' \left(\left|\mathbf{q}-\mathbf{q}'\right|^{-1} \right) \quad . \tag{63}$$

It is well known that $\left|\mathbf{q}-\mathbf{q}'\right|^{-1}$ is proportional to the Fundamental solution of the 3D radial Laplacian, more precisely we have

$$\Delta' \left(\left|\mathbf{q}-\mathbf{q}'\right|^{-1} \right) = -4\pi\delta_d\left(\mathbf{q}-\mathbf{q}'\right) \tag{64}$$

hence

$$\mathcal{A} = -\frac{\psi_{\mathbf{q},t_o}}{\tau_o^2} \tag{65}$$

[4] Classically, the characteristic function is defined to be unity on Ω, and is zero elsewhere.

where we have introduced the Jeans time $\tau_o := (4\pi G\rho_o)^{-1/2}$.

The second term is a little bit more complicated, in Cartesian coordinates we have

$$\mathcal{B} = \frac{G}{m}\sum_{i,j=1}^{3}\int d\mathbf{\Gamma}' f\left(\mathbf{\Gamma}', t_o\right) p_i' p_j' \frac{\partial^2 \left|\mathbf{q}-\mathbf{q}'\right|^{-1}}{\partial q_i' \partial q_j'} \tag{66}$$

We now make our last assumption.

Hypothesis 3 : The initial state has an isotropic and constant velocity dispersion σ^2

$$\forall i = 1,2,3 \quad \int d\mathbf{p}\ f\left(\mathbf{\Gamma}, t_o\right) p_i^2 = \ \nu_o \frac{m^2\sigma^2}{3}\chi_\Omega = \rho_o \frac{m\sigma^2}{3}\chi_\Omega \tag{67}$$

Physically this assumption means that the initial state is isothermal.

Coupling all the hypotheses, we have

$$\mathcal{B} = \sigma^2 G\rho_o \int_{\mathbf{q}'\in\Omega} \Delta'\left(\left|\mathbf{q}-\mathbf{q}'\right|^{-1}\right) = -\frac{\sigma^2}{\tau_o^2} \tag{68}$$

The beginning of the potential expansion may be written as

$$\psi_{\mathbf{q},t}\left[f\right] = \psi_{\mathbf{q},t_o}\left[f_o\right] - \tfrac{(t-t_o)^2}{2\tau_o^2}\left[\psi_{\mathbf{q},t_o}\left[f_o\right] + \sigma^2\right] + \mathcal{O}\left(\left(t-t_o\right)^2\right) \tag{69}$$

Attempting to go further in this expansion is extremely perilous. As a matter of fact, we see from the term $\mathcal{A}$ that, the functional derivatives of $\psi^{(2)}_{\mathbf{q},t_o}$ do not come directly due to the $f-$non linearity incoming in the term $\psi.f$. The calculus then becomes extremely tedious ...

At this stage, only some general remarks can be made using such a technique in a general way :

- Under hypothesis 2, all odd terms vanish in the expansion

$$\psi^{(2k+1)}_{\mathbf{q},t_o} = 0\ , \quad \forall k \in \mathbb{N} \tag{70}$$

- Keeping only linear terms, it seems[5] that only terms of the same nature of $\psi^{(2)}_{\mathbf{q},t_o}$ remains in the expansion, i.e.

$$\psi^{(2k)}_{\mathbf{q},t_o} \propto (-1)^k \frac{\psi_{\mathbf{q},t_o}\left[f_o\right]}{\tau_o^{2k}} + cst\ , \quad \forall k \in \mathbb{N} \tag{71}$$

  neglecting the constant, this gives

$$\psi_{\mathbf{q},t} = \psi_{\mathbf{q},t_o}\left[f_o\right]\sum_{k=0}^{\infty}(-1)^k \frac{(t-t_o)^{2k}}{(2k)!\tau_o^{2k}} = \psi_{\mathbf{q},t_o}\left[f_o\right]\cos\left(\frac{t-t_o}{\tau_o}\right) \tag{72}$$

  this maybe corroborates the τ_o-periodic damped regime which appear in generic gravitational collapse numerical experiments.

[5] It is more conjectured using preliminar results from high order terms, than strictly proved ...

We stress that there are preliminary results. We hope to give a more complete description of the transient regime of Vlasov-Poisson dynamics in an upcoming article.

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