Gravity, Dimension, Equilibrium, & Thermodynamics

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Abstract

Is it actually possible to interpret gravitation as space's property in a pure classical way. Then, we note that extended self-gravitating system equilibrium depends directly on the number of dimension of the space in which it evolves. Given those precisions, we review the principal thermodynamical knowledge in the context of classical gravity with arbitrary dimension of space. Stability analyses for bounded 3D systems, namely the Antonov instability paradigm, are then rapproched to some amazing properties of globular clusters and galaxies.

Key words: classical gravitation, thermodynamics, theory

1 Nature of classical gravitation

1.1 Gravitational field

In a classical way, gravitation is a force $F = -\mathrm{grad}U$ acting on a test mass m, deriving from a scalar potential energy U which at any time t depends only on the position $\mathbf{r} \in \mathbb{R}^3$. This potential is generated by a mass density field $\rho(\mathbf{r}, t)$ such that Poisson equation holds

$$U(\mathbf{r}, t) = m\psi(\mathbf{r}, t) \ \text{ and } \ \Delta_3\psi(\mathbf{r}, t) = 4\pi G\rho(\mathbf{r}, t) \tag{1}$$

Inverting the laplacian Δ_3, one writes the gravitational potential as

$$\psi(\mathbf{r}, t) = -G \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \tag{2}$$

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Using a convolution product, this last relation writes

$$\psi = 4\pi \ G \ g_3 * \rho \tag{3}$$

Where g_3 is a solution of the equation

$$\Delta_3 \ g_3 = \delta \ , \tag{4}$$

which is different from harmonic polynomials. In other words g_3 is the laplacian Green's function in $\mathbb{R}^3$

$$g_3 \left(\mathbf{r} \right) = -\frac{1}{4\pi} \frac{1}{|\mathbf{r}|} \tag{5}$$

Using information theory, interpretation of equation (3) is clear : Green's function is interpretable as impulsional response of the corresponding operator, gravitational potential $\psi \left(\mathbf{r}, t \right)$ is then the response from space when it is submitted to the presence of a mass density distribution $\rho \left(\mathbf{r}, t \right)$. Constant factors are also clearly interpretable : the Newton-Cavendish constant G fixes units and 4π is the value of the surface of unit radius sphere in $\mathbb{R}^3$, it is then attached to the radial nature of the gravitationnal interaction.

Let us note that the traditional link usually made between gravity and space in general relativity is already contained, using this formulation, in classical field gravity. Two main problems remain: the instantaneous character of the interaction and the fundamental restriction to isotropic spaces (laplacians acts equally in all directions). Two major fulfillments of Einstein theory of gravitation were to solve these problems.

In a classical context, one can directly generalize equation (3) to obtain the classical definition of the gravitational potential in a $n-$dimensionnal space, we have

$$\psi = G \ S_{n-1} \ g_n * \rho \tag{6}$$

where

$$S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{with} \quad \forall z \in \mathbb{R}_*^+ \quad \Gamma \left(z \right) := \int_0^{+\infty} e^{-s} s^{z-1} ds$$

represents the value of the surface of unit radius sphere in $\mathbb{R}^n$, and

$$\forall \mathbf{r} \in \mathbb{R}_*^n \qquad g_n \left(\mathbf{r} \right) = \begin{vmatrix} |\mathbf{r}| & \text{if } n = 1 \\ \\ \dfrac{1}{2\pi} \ln |\mathbf{r}| & \text{if } n = 2 \\ \\ -\dfrac{1}{(n-2) \ S_{n-1}} \dfrac{1}{|\mathbf{r}|^{n-2}} & \text{if } n > 2 \end{vmatrix}$$

is Green's function of the lapacian operator in $\mathbb{R}^n$.

Representation of the gravitational interaction in $\mathbb{R}^n$ via relation (6) is not original but it is presented here in a rational way.

1.2 Equilibrium

Let us describe global dynamical properties of a system of N particles of same[1] mass m represented by their positions and impulsions merged in the vector $\Gamma_{i=1\ldots N} := (\mathbf{r}_i(t), \mathbf{p}_i(t) = m\dot{\mathbf{r}}_i)$. Three tensors are of interest : The kinetic $\mathcal{K}$, potential $\mathcal{U}$ and inertial $\mathcal{I}$ one, which components are respectively

$$\forall i, j = 1, \cdots, N \qquad \mathcal{K}_{ij} := \frac{\mathbf{p}_i \mathbf{p}_j}{2m} \qquad \mathcal{U}_{ij} := mGS_{n-1} \sum_j g_n(\mathbf{r}_i) \qquad \mathcal{I}_{ij} := m\mathbf{r}_i\mathbf{r}_j \quad .$$

In the continuous case where $N \to \infty$, variable is no longer Γ_i but its probability density $f = f(\mathbf{\Gamma}_1, \cdots, \mathbf{\Gamma}_N, t)$. Dynamical tensors become usually

$$\mathcal{K}_{ij} := \int \frac{p_i p_j}{2m} \, f \, d\mathbf{\Gamma}^N \quad \mathcal{U}_{ij} := -m \int r_i \frac{\partial \psi}{\partial r_j} \, f \, d\mathbf{\Gamma}^N$$

$$\forall i, j = 1, \cdots, N$$

$$\mathcal{I}_{ij} := m \int r_i r_j \, f \, d\mathbf{\Gamma}^N \quad \text{where} \ \ d\mathbf{\Gamma}^N = \prod_{i=1}^N d\mathbf{\Gamma}_i$$

In the general conservative case f obeys Liouville equation, which reduces under generic symetry assuptions to Collisionless Boltzmann Equation for gravitating systems (see (5) for example).

It is not too long so, to prove[2] the fundamental virial theorem

Theorem 1 *If $\mathcal{U}$ is homogeneous of degree α, i.e. $\forall \lambda \in \mathbb{R}$, $\mathcal{U}(\lambda\mathbf{r}_1, \cdots, \lambda\mathbf{r}_N) = \lambda^\alpha \mathcal{U}(\mathbf{r}_1, \cdots, \mathbf{r}_N)$ then*

$$2\text{Tr}(\mathcal{K}) - \alpha\text{Tr}(\mathcal{U}) = \frac{1}{2} \frac{d^2\text{Tr}(\mathcal{I})}{dt^2}$$

It is quite natural to define an equilibrium state by

$$\frac{d^2\text{Tr}(\mathcal{I})}{dt^2} = 0$$

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[1] this assumption is not essential but it simplifies notably the notations, mainly in the continuous case where $N \to \infty$

[2] from Newton fundamental principle in the discrete case or from Liouville equation for the continuous case ...

in some mean sense. Hence, for a self-gravitating system in a $n-$dimensional space :

- If $n = 2, \mathcal{U}$ is not homogeneous : Virial theorem does not apply in this form;
- If $n \neq 2$, g_n then $\mathcal{U}$ is an homogeneous function of degree $(n-2)$ and equilibrium is characterized by the relation

$$2\mathrm{Tr}(\mathcal{K}) + (n-2)\mathrm{Tr}(\mathcal{U}) = 0$$

For extended self-gravitating systems like globular clusters or galaxies a 3 dimensional space in virial theorem seems compatible with observations.

2 Thermodynamics

If all particles have the same probability law, are independant and do not interact by pair but only globally through their whole mean gravitating field, one can reduce the dimensionality of the phase space : The system is statistically equivalent to a test particle of mass m at position $\mathbf{r} \in \mathbb{R}^n$, with impulsion $\mathbf{p} \in \mathbb{R}^n$, described at any time t by a distribution function $f(\mathbf{r}, \mathbf{p}, t)$ and evolving in a mean field $\psi(\mathbf{r}, t)$. These two functions are solutions of the Vlasov-Poisson system

$$\begin{cases} \dfrac{\partial f}{\partial t} - m\dfrac{\partial f}{\partial \mathbf{p}} \cdot \dfrac{\partial \psi}{\partial \mathbf{r}} + \dfrac{\mathbf{p}}{m} \cdot \dfrac{\partial f}{\partial \mathbf{r}} = 0 \\ \\ \psi = G \; S_{n-1} \; g_n * \left[m \displaystyle\int f \; \mathrm{d}\mathbf{p} \right] \end{cases}$$

2.1   Definitions

Several quantities are in general, used in thermodynamics :

- Phase space variable $\boldsymbol{\Gamma}$

$$\boldsymbol{\Gamma} = (\mathbf{r}, \mathbf{p}) \in \mathbb{R}^n \times \mathbb{R}^n$$

- Space (ν) or mass (ρ) density

$$\nu(\mathbf{r}, t) = \int f(\boldsymbol{\Gamma}, t) \, \mathrm{d}\mathbf{p} \; =: \rho/m$$

- "Number of particle" N or system's mass M

$$N[f] := \int f(\boldsymbol{\Gamma}, t) \, \mathrm{d}\boldsymbol{\Gamma} = \int \nu(\mathbf{r}, t) \, \mathrm{d}\mathbf{r} := M/m$$

- Energy

$$E[f] := K[f] + U[f] \quad \text{with} \quad \left| \begin{array}{l} K[f] := \dfrac{1}{2m} \int \mathbf{p}^2 f(\mathbf{\Gamma}, t) \, \mathrm{d}\mathbf{\Gamma} \\[2em] U[f] := \dfrac{m^2}{2} \int \nu(\mathbf{r}, t) \psi(\mathbf{r}, t) \, \mathrm{d}\mathbf{r} \end{array} \right.$$

- Angular Momentum [3]

$$J[f] = \int r p_\phi f(\mathbf{\Gamma}, t) \, \mathrm{d}\mathbf{\Gamma}$$

- Boltzmann Entropy

$$S[f] := - \int f(\mathbf{\Gamma}, t) \ln[f(\mathbf{\Gamma}, t)] \, \mathrm{d}\mathbf{\Gamma}$$

For precise considerations let us define some set :

(1) Unbounded Systems

$$\mathcal{G}_n(N, E) = \left\{ f \quad \text{s. t.} \quad \forall \mathbf{\Gamma} \; \left| \begin{array}{l} K < \infty \;,\; U < \infty \\[0.5em] E < \infty \;,\; N < \infty \end{array} \right. \right\}$$

$$\mathcal{G}_n(N, E, J) = \mathcal{G}(N, E) \cap \{f \quad \text{s. t.} \quad \forall \mathbf{\Gamma} \quad J < \infty\}$$

(2) Bounded Systems $D \subset R^n$

We designe by $\mathrm{Supp}(f)$ the support of the distribution function f , i.e. the complementary of the largest open set of $\mathbf{\Gamma} \in (\mathbb{R}^n \times \mathbb{R}^n)$ such that $f(\mathbf{\Gamma}) = 0$. We then call

$$\mathcal{G}_n(D, N, E) = \{f \in \mathcal{G}_n(N, E) \;;\; \mathrm{Supp}(f) \subset D\}$$

$$\mathcal{G}_n(D, N, E, J) = \{f \in \mathcal{G}_n(N, E, J) \;;\; \mathrm{Supp}(f) \subset D\}$$

2.2 Thermodynamical Equilibrium Problem

The classical thermodynamical equilibrium problem concerns the existence of an entropy extremalizer. Considering a set $\mathcal{G}_n$ of acceptable distribution functions, it can be posed as :

$$\exists? \;\; f^+ \in \mathcal{G}_n \quad \text{s.t.} \quad \forall f \in \mathcal{G}_n \;\; S[f] \;\leq\; S[f^+]$$

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[3] the quantity p_ϕ indicates the tangential components of the impulsion $\mathbf{p}$, the quantity r stands for the euclidian norm of the position $\mathbf{r}$ of the test particle.

For sets considered in the previous section, this problem corresponds to a classical Euler-Lagrange one, which solutions are the well-known isothermal spheres :

- For $f \in \mathcal{G}_n(E, N)$

$$f^+ = exp\left\{-\alpha - \beta(\frac{\mathbf{p}^2}{2m} + m\psi^+)\right\}$$

  Lagrange multiplier α and β correspond respectively to the conservation of N and E .

- For $f \in \mathcal{G}_n(E, N, J)$

$$f^+ = \exp\left\{-\alpha - \beta\left(\frac{p_r^2}{2m} + \frac{(p_\phi - mr\omega)^2}{2m} + m\psi^+ - \frac{m^2\omega^2r^2}{2}\right)\right\}$$

  The additional constraint of J conservation corresponds to the introduction of the ω multiplier.

2.3 2D Thermodynamics : $\mathcal{G}_2$

We reproduce here the classical results obtained in (1) and (2). The main one is twofold : We first produce a bound for entropy, we then study the existence of a distribution function which allow reaching that bound. Concerning the upper bound for the entropy, we prove that [4]

Theorem 2 $\forall f \in \mathcal{G}_2(E, N)$

$$S[f] \leq S^+(N, E) := \sup_{\mathcal{G}_2(N, E)} S[f] \leq \frac{2E}{N} + N\ln(e\pi^2)$$

In $2D$, entropy of unbounded self-gravitating systems for which $E = cst$, and $N = cst$ is bounded from bellow.

Theorem 3 *In the notations of Theorem 2*

$$S^+(N, E, J) = S^+(N, E)$$

Adding the angular momentum's constraint does not change the least upper bound on the entropy. The existence of a distribution function f^+ corresponding to this entropy maximizer, i.e. $S[f^+] = S^+$ is closely related to the set under consideration :

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[4] in the units of original papers

- In $\mathcal{G}_2(E, N)$: f^+ exists and is unique,

$$f^+ = \frac{e^{2(E-N^2)/N^2}}{\pi^2} \frac{e^{-p^2/N}}{\left(e^{2(E-N^2)/N^2} + r^2\right)^2}$$

it generates the potential

$$\psi^+ = N \ln\left(e^{2(E-N^2)/N^2} + r^2\right) .$$

in the units of papers (1) and (2).
- There is no $f^+ \in \mathcal{G}_2(E, N, J)$ for which $S[f^+] = S^+$.
- In $\mathcal{G}_2(D, E, N)$ and $\mathcal{G}_2(D, E, N, J)$: f^+ exists and is unique

2.4   3D Thermodynamics : $\mathcal{G}_3$

Introducing a new dimension changes drastically the situation concerning the classical equilibrium problem of gravitational thermodynamics. As a matter of fact, it is well known for a long time (see for example (5) p. 268) that

- Entropy has no global maximum on $\mathcal{G}_3(E, N)$, $\mathcal{G}_3(E, N, J)$, $\mathcal{G}_3(D, E, N)$ and $\mathcal{G}_3(D, E, N, J)$.
- Entropy has no local maximum on $\mathcal{G}_3(E, N)$ and $\mathcal{G}_3(E, N, J)$.

The existence of local maximum for entropy in $\mathcal{G}_3(D, E, N)$ corresponds to an extensive literature initiated by the works of V.A. Antonov (see (5) for a review) in the early 60's. It represents a beautifull problem of thermodynamics.

2.4.1   Entropy extremalizer in $\mathcal{G}_3(D, E, N)$

We reproduce below the main results obtained by T. Padmanabhan (see (3)) which clarify all previous works. If we denote by R the radius of the largest bowl contained in the spatial part of D. It is proven that any entropy extremalizer in $\mathcal{G}_3(D, E, N)$ must be of the form

$$f^+ = \left(\frac{2\pi}{\beta}\right)^{-3/2} \nu_o \, e^{-\beta E} \quad \text{with} \quad m\nu_o = \rho(0) \, e^{\beta\psi(0)} . \tag{7}$$

The associated potential verifies Poisson equation

$$\Delta_3 \, \psi = 4\pi G m \nu_o \, e^{-\beta\psi} \tag{8}$$

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with the limit condition $\psi(R) = -GM/R$. Introducing dimensionless variables

$$L_o = \sqrt{4\pi G\rho(0)\beta} \qquad M_o = 4\pi\rho(0) L_o^3 \qquad \psi_o = \beta^{-1} = \frac{GM_o}{L_o}$$

$$x = r/L_o \qquad n = \rho/\rho(0) \qquad \mu = M(r)/M_o \qquad y = \beta(\psi - \psi(0))$$

Poisson equation becomes

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = e^{-y} \quad \text{with} \quad y(0) = y'(0) = 0$$

Milne's functions, $v = \mu/x$ and $u = nx^3/\mu$, transform Poisson equation into

$$\frac{u}{v}\frac{dv}{du} = \frac{1-u}{u+v-3} \quad \text{with} \quad \left| \begin{array}{l} v = 0 \text{ when } u = 3 \\ \text{and} \\ \left.\dfrac{dv}{du}\right|_{(u,v)=(3,0)} = -5/3 \end{array} \right.$$

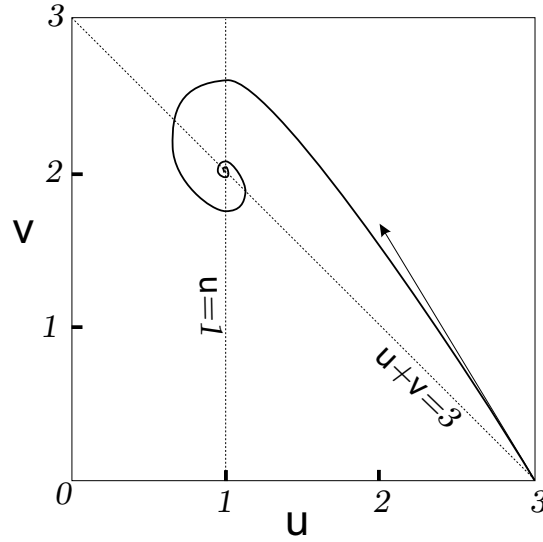Hence, Isothermal extremal spheres lie on a curve in the $u - v$ plane. This curve is plotted on Figure 1.

Fig. 1. Isothermal sphere in the Milne plane

2.4.2 Antonov Instability

In a meaningfull remark, Patmanabhan notes that dimensionless quantity

$$\lambda := \frac{RE}{GM^2} = \frac{1}{v}\left(u - \frac{3}{2}\right)$$

lies also on the same $u - v$ plane. He then asks the fundamental question : Can E, R and M be accomodated by a suitable choice of $\rho(0)$ and β ? As one can see on Figure 2, the answer is clearly no. There exists a critical value $\lambda_c \simeq -0.335$, associated with the possibility to put a given isothermal extremal sphere in a given box !
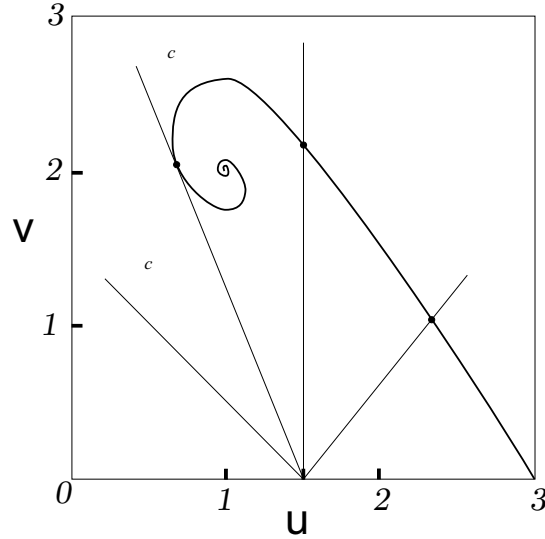

Fig. 2. Existence of an extremal isothermal sphere

- If $\lambda < \lambda_c$ isothermal sphere cannot exist, entropy extremum cannot exist;
- If $\lambda > \lambda_c$ isothermal sphere exist, entropy extremum exist !

More exciting is the fact that the nature of extremum depends on $\kappa = \rho(0)/\rho(R)$ the density concentration ratio of the isothermal sphere.

- If $\kappa > 709$: The extremum of the entropy is an unstable saddle point;
- If $\kappa < 709$: The extremum of the entropy is local maximum.

This last point is generally associated to the so-called gravothermal catastrophe, we prefer to call it Antonov Instability. As we will see in the next section, such an instability is certainly at the origin of some important characteristics of extended self gravitating systems.

3 Antonov instability in astrophysics

3.1 *Globular clusters in galaxies*

Since early 80's observations have shown that galactic globular clusters split in two categories which differ by some properties of their radial density profiles. On the one hand a large familly of about 120 clusters with a large constant density core which extends to almost the half mass radius (R_{50}) of the whole system. This large core is surrounded by a power law density decreasing halo. On the other hand, a small familly of about 20 core collapsed clusters with a very high central density which decreases monotonically outward with mainly two power law indexes. These two types of globular clusters are very well represented by two of their components, namely NGC 6388 for core halo cluster and Trz2 for core collapsed cluster, see Figure 1 of (6).

Such a behaviour of the radial density could be explained in a very simple way by Antonov Instability. As a matter of fact, if globular cluster formation results from the collapse of a small, hence homogeneous, region of some galaxy, the natural result is roughly an isothermal sphere (see (4)) with generally a contrast density κ less than the critical value. The evolution of the cluster in the galaxy produces a slow evaporation of the cluster (passing through the galalactic plane in spiral galaxies for example). Due to the negative specific heat of such gravitating systems, this evaporation makes the contrast density growing. When κ reaches the critical value, Antonov instability triggers and transforms the core halo density profile into a collapsed core one. On Figure 3 we represent radial density profile after collapse of an initially homogenenous system and of an initially inhomogeneous one (see (4)).
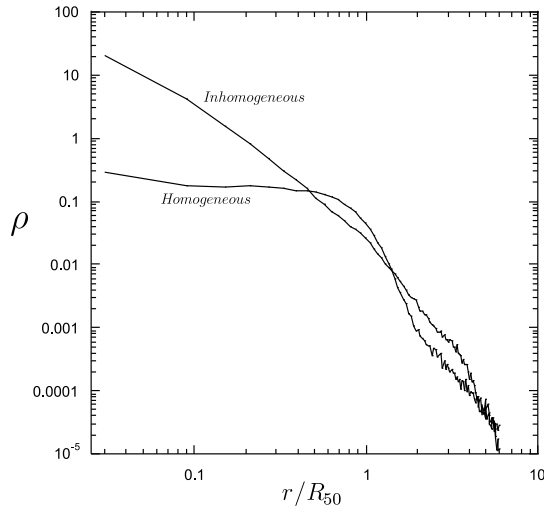

Fig. 3. Radial density profile obtained from the gravitational collapse of an homogeneous set of mass and an inhomogeneous one

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From the accumulation of observational data, it becomes necessary to put a Super Massive (from 10^6 to perhaps 10^9 $M_\odot$) Black Hole in the dynamical center of galaxies. Excepted the fact that such gravitational monsters are as older as their host, little is known about their formation process. In the context of hierachical galaxy formation scenario, Antonov instability could produce a good paradigm. As a matter of fact, if galaxies are the net result of successive collapse and merger of gravitational structures, it can be modelized generically by a general collapse of inhomogeneous media. As showed by (4), in such case, reaching the center small structures first collapse to form quasi-isothermal sphere surrounded by the rest of not yet collapsed large structures. Evaporating the smallest, largest's collapse could trigger Antonov instability. Progenitor of SMBH could then be formed. This process is up to now to be confirmed but seems correspond to all observed properties.

4   Conclusion

Classical gravity is an amazing topic. Although, $2D$ gravitating systems are well described by thermodynamics, their equilibrium is not well defined. By opposition, provided that $n \geq 3$, we possess a powerfull tool to describe gravitational equilibrium for systems in $\mathbb{R}^n$, but the corresponding thermodynamics is not too efficient. However, in the restricted case of bounded systems in $\mathbb{R}^3$, the message from gravitational thermodynamics and particulary Antonov instability, could be fundamental to explain some features of self gravitating systems.

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