Nonlinear mappings for reduced-order modeling of geometrically nonlinear structures: quadratic manifold and normal form theory

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Summary. Two different nonlinear mappings have been proposed recently in order to define reduced-order modeling strategies applicable to geometrically nonlinear structures: the normal form theory and the quadratic manifold obtained from static modal derivatives. The aim of this work is to compare the two methods. In particular, it is shown that the methods share similarities only if a slow/fast decomposition can be assumed.

Nonlinear mappings and reduced dynamics

In order to derive efficient reduced-order models for geometrically nonlinear structures, a key idea is to rely on a nonlinear change of coordinates in order to circumvent the limitations of the classical linear methods (Eigenmodes, Ritz basis or Proper Orthogonal Decomposition). One can then await that the manifolds produced by such nonlinear mappings should be more accurate as being closer to the important subspaces where the dynamical solutions lie. In the recent years, two different nonlinear mappings have been proposed independently.

The first one relies on the normal form theory as proposed in [1], where a full nonlinear change of coordinates, from the modal ones to newly introduced coordinates, is proposed, up to the third order, and reads :

$$X_p = R_p + \mathcal{P}_p(R_i, S_j), \tag{1a}$$

$$Y_p = S_p + \mathcal{Q}_p(R_i, S_j). \tag{1b}$$

In these equations, X_p is the modal displacement, $Y_p = \dot{X}_p$ the modal velocity, (R_p, S_p) are the *normal* coordinates and $\mathcal{P}_p$ and $\mathcal{Q}_p$ are analytic third-order polynomials. As shown in [1], the subspaces obtained when keeping only one normal coordinate is the invariant manifold also defined as the nonlinear normal mode (NNM). Hence Eqs. (1) allows one to pass from a phase space spanned by the linear modes to an invariant-based span where single contributions gives motions restricted to NNMs. One of the most interesting feature of the method is that it allows derivation of efficient reduced-order models (ROMs) thanks to truncation to invariant subspaces, a mandatory requirement in order to build ROMs that simulate trajectories of the original full system.

A second nonlinear mapping has been proposed recently based on the works on modal derivatives [2, 3]. It gives a quadratic relationship between physical coordinates $\mathbf{U}$ (generally obtained from a finite element (FE) discretization) and a newly introduced one $\mathbf{q}$ as:

$$\mathbf{U} = \mathbf{\Phi q} + \frac{1}{2}\mathbf{\Theta qq}\,, \tag{2}$$

where $\mathbf{\Phi}$ is the matrix of eigenvectors and $\mathbf{\Theta}$ the tensor of modal derivatives. One can note in particular that the linear term follows the linear mode basis, as in the case of normal form. The quadratic term is however different, and no cubic term is given. Also, the formula involves only displacements, which is restrictive as compared to normal form that uses both displacements and velocities. Finally, the invariance property is a built-in property in the normal form approach thanks to the connection to the NNMs seen as invariant manifold in phase space. However, no invariance property exist for the quadratic manifold method.

Single mode motion

The general equations for the two methods have been detailed term by term as well as the reduced-order dynamics in each case. Here only the simple case of a single mode motion (*i.e.* restricting the newly introduced variables $\mathbf{q}$ and $\mathbf{R}$ to a single component) is highlighted for the sake of brevity. The reduced order dynamics for the quadratic manifold reads:

$$\ddot{q}_p + \omega_p^2 q_p - 2g_{pp}^p q_p^2 - 2\frac{g_{pp}^p}{\omega_p^2}\dot{q}_p^2 - 4\frac{g_{pp}^p}{\omega_p^2} q_p \ddot{q}_p + \left(h_{ppp}^p - \sum_{s=1}^{n} g_{ps}^p \frac{g_{pp}^s}{\omega_s^2} \right) q_p^3 + \sum_{s=1}^{n} \left(\frac{2g_{pp}^s}{\omega_s^2} \right)^2 \left(q_p \dot{q}_p^2 + q_p^2 \ddot{q}_p \right) \,=\, 0, \tag{3}$$

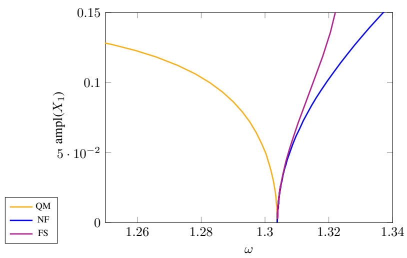
where g_{ij}^p and h_{ijk}^p are the tensors of quadratic and cubic nonlinear coupling coefficients from the original system. On the other hand, the reduction to a single NNM using the normal form theory reads:

$$\ddot{R}_p + \omega_p^2 R_p + \left(h_{ppp}^p - \sum_{s=1}^{n} \frac{\omega_s^2 - 2\omega_p^2}{\omega_s^2 - 4\omega_p^2}\, g_{ps}^p\, \frac{g_{pp}^s}{\omega_s^2} \right) R_p^3 + \sum_{s=1}^{n} \left(\frac{g_{ps}^p}{\omega_s^2} \frac{2g_{pp}^s}{\omega_s^2 - 4\omega_p^2} \right) R_p \dot{R}_p^{\,2} \,=\, 0. \tag{4}$$

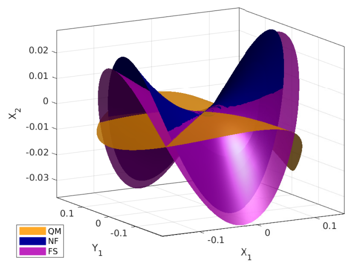
In particular, one can observe that the reduced dynamics with the quadratic manifold does not vanish the quadratic terms, even though the influence of the remaining terms on the hardening/softening behaviour cancels. Another important remark is that if one assumes a slow/fast decomposition of the dynamics, then the cubic coefficients tends to have the same values. Indeed, slow/fast decomposition induces that the eigenfrequency of the master mode p is smaller than the other ones, *i.e.* $\omega_s \gg \omega_p$. One can show that the cubic coefficients tends to have exactly the same values. This result confirms a general proposition given in [4]. It shows in particular that the quadratic manifold can be used safely only if this slow/fast decomposition is at hand in the system, which is often the case in thin structures where the slave modes are related to in-plane modes with very large frequencies as compared to those of the master flexural modes. However it cannot handle internal resonances and has to be used with care in other context.

Result on a the two-dofs system

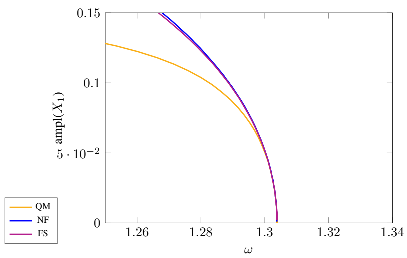
The general results have been applied to a simple two dofs system consisting of a mass connected to two nonlinear springs, already used in [1]. In Fig. 1 the backbone curves and the manifolds obtained with the two methods are compared to those obtained by solving the full system. In the first case reported in Fig. 1a-1b, the eigenvalues of the system are not well separated. The hardening effect predicted by the normal form approach is coherent with that of the full solution, whereas the quadratic manifold approach shows a softening behaviour. The shape of the numerical invariant manifold (full solution denoted as FS on the figure) displays a strong dependence with the velocity that cannot be captured with the quadratic manifold method. When the slow/fast decomposition is assumed with $\omega_2 \gg \omega_1$, Fig. 1c-1d show that the results between the two different appraoches are in much better agreement.



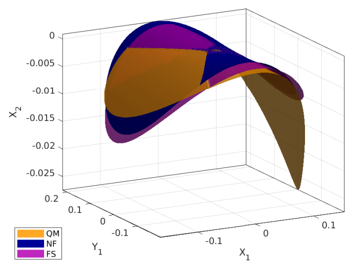
(a) Backbone curves of the first nonlinear mode for the case with $\omega_2 = \sqrt{6}$.



(b) Manifolds of the first nonlinear mode for the case with $\omega_2 = \sqrt{6}$.



(c) Backbone curves of the first nonlinear mode for the case with $\omega_2 = \sqrt{60}$.



(d) Manifolds of the first nonlinear mode for the case with $\omega_2 = \sqrt{60}$.

Figure 1: Backbone curves and manifolds evaluated with the quadratic manifold method (QM) and normal form approach (NF), compared to the numerical ones evaluated for the full system (FS). In both cases $\omega_1 = \sqrt{1.7}$.

Conclusions

A detailed, term by term comparison of quadratic manifold and normal form theory as nonlinear mappings for reduced-order dynamics, has been presented. It shows in particular that the quadratic manifold can be properly justified only if a slow/fast decomposition of the system exist. More examples and results on continuous structures will be presented at the conference.

References

[1] C. Touzé, O. Thomas and A. Chaigne (2004) Hardening/softening behaviour in non-linear oscillations of structural systems using non-linear normal modes, *J. Sound. Vib*, **273**(1-2): 77-101.

[2] S. Jain, P. Tiso, J. B. Rutzmoser and D. J. Rixen (2017) A quadratic manifold for model order reduction of nonlinear structural dynamics, *Computers and Structures*, **188**:80-94.

[3] J. B. Rutzmoser, D. J. Rixen, P. Tiso and S. Jain (2017) Generalization of quadratic manifolds for reduced order modeling of nonlinear structural dynamics, *Computers and Structures*, **192**:196-209, 2017.

[4] G. Haller and S. Ponsioen (2017) Exact model reduction by slow/fast decomposition of nonlinear mechanical systems, *Nonlinear Dynamics*, **90**:617-647.