# Comparison of nonlinear methods for reduced-order modeling of geometrically nonlinear structures

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*Summary*. The aim of this contribution is to review and compare three different methods that have been proposed in order to derive reduced-order models for geometrically nonlinear structures, and relying on a nonlinear technique to better take into account the nonlinearities of the initial problem. The three methods are: implicit condensation, quadratic manifold derived with modal derivatives, and projection onto an invariant manifold, tangent at the origin to the linear eigenspace of the master modes. The methods are briefly reviewed theoretically and then compared with dedicated examples.

## Nonlinear techniques for model-order reduction

In the realm of reduction methods for geometrically nonlinear structures, numerous methods have already been proposed in the past, and one can roughly divide the methods according to the fact that they use either a linear or a nonlinear change of coordinates. Among the linear methods, the POD is well-established and has been used for a long time. For the nonlinear techniques, the review paper [1] proposes an overview with applications to finite element (FE) problems.
When referring to vibrating structures discretized by the FE method, mainly three different nonlinear techniques have been proposed in the last years: implicit condensation (IC), quadratic manifold derived with modal derivatives (QM), and the direct parametrisation method for invariant manifolds, giving rise to either graph style or normal style solutions.
The implicit condensation is a non-intrusive method that can be used with any FE code. It relies on applying a series of static loads, having the shape of the selected master modes, to the structure; and retrieving the associated nonlinear displacements [2]. From this set that creates a stress manifold [3], a fitting procedure is derived in order to get the nonlinear restoring force of the master coordinates, which takes implicitely into account the non-resonant coupled modes in a static manner [2, 4].
The quadratic manifold method with modal derivatives have been introduced with the aim of proposing a nonlinear change of coordinates, directly applicable from the FE nodes of a mesh [5]. The main idea is to embed the modal derivatives, earlier defined in order to take into account the amplitude dependence of a mode when the nonlinearities are taken into account, as the quadratic component of the nonlinear mapping. The Galerkin projection is then used to compute the reduced dynamics from the nonlinear mapping.
The invariant manifold approach relies on firm theoretical results from dynamical systems theory. The main idea is to use an invariant manifold that is tangent to the master modes of interest at the origin, in order to obtain a rigorous computation of the amplitude dependence. Whereas the idea has been first introduced by Shaw and Pierre in the 90s [6], later embedded in the normal form theory in order to define nonlinear mappings [7], recent contributions tackled the problem of uniqueness by introducing spectral submanifolds (SSM) [8], as well as using the parametrisation method for invariant manifolds [9] in order to unify the previous derivations in the same settings, allowing one to select either a graph style solution or a normal form approach. However, until recently, the direct application of these methods to FE problems still remained difficult since the derivations assumed the problem expressed in the modal basis as a starting point. Recent contributions overcame this issue, by proposing a direct computation of the reduced dynamics, from the FE nodes to the invariant-based span of the phase space, thanks to dedicated nonlinear mappings, see *e.g.* [10, 11, 12].

## Comparisons

### Theoretical results

The three reduction methods have been compared in details in the following references [13, 4, 14]. In particular, it has been clearly demonstrated that both IC and QM methods needs a slow/fast assumption in order to propose accurate predictions. By slow/fast separation, it is meant that the eigenfrequencies of the slave modes needs to be larger than those of the master modes. This frequency gap has been estimated in [4, 14] to be around 4, based on the hardening/softening prediction of a single mode reduction. As a matter of fact, this limitation of these two methods comes from their derivation, which is static in nature and neglects important dynamical couplings. In particular, both IC and QM methods are neglecting the velocity dependence in the proposed nonlinear mappings, which then creates important limitations and the need to have this slow/fast separation [4, 14, 1]. As underlined for example in [1], when taking back the velocity dependence in the nonlinear mapping proposed in the QM method, then one recovers the more complete change of coordinates proposed with a direct normal form approach as proposed in [10].

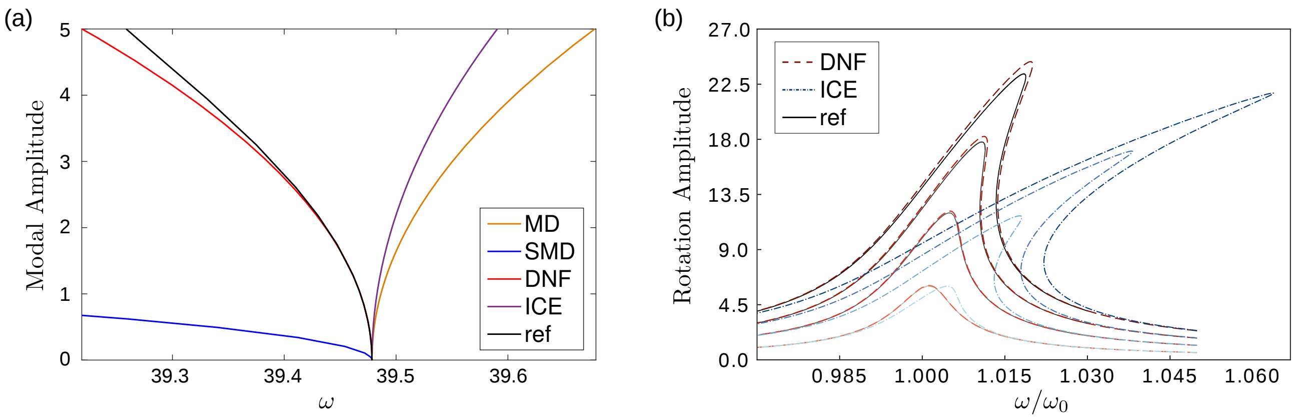


Figure 1: (a) Backbone curves for a linear beam resting on a nonlinear elastic foundation, comparison between QM methods with either full modal derivatives (MD) or static modal derivatives (SMD), Implicit condensation (ICE), and direct normal form (DNF). Reference solution in black. (b) Comparison between Implicit condensation (IC), direct normal form (DNF) and full-order solution for a MEMS micromirror.

**Numerical results**

Two different numerical results are reported for the sake of illustration. The first case consists in computing the backbone curve of a linear beam resting on a nonlinear elastic foundation with reduction to a single master mode. The results are shown in Fig. 1(a), highlighting that in such a case, only the solution based on invariant manifold theory is able to correctly reproduce the softening behaviour [15]. The second example is that of a MEMS (Micro Electro Mechanical System) micromirror, discretized with 3D block elements. The mesh has 15 341 nodes. Fig. 1(b) highlights that the IC method overpredicts the hardening behaviour of this structure [16].

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