Model order reduction for geometrically nonlinear beams featuring internal resonance and centrifugal effect

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Summary. The direct parametrisation of invariant manifold is used for model order reduction of large amplitude vibrations of clamped-clamped and rotating cantilever beams. A particular emphasis is set on the computation of the backbone curve in case of internal resonance. For the clamped beam, the 1:5 resonance between first and third mode occuring at large amplitude, is reproduced with the model. For the rotating cantilever, a Campbell diagram is first used to detect the appearance of a 1:5 resonance, which is then computed with the reduction method.

Introduction

The study of slenders structures is becoming a prominent issue in several industries as the mass gain becomes crucial. For instance, in the aeronautic industry, the development of more efficient engines by the use of ever larger fan is targeted. This leads to the design of large fan blades, i.e. more than 1 meter long. In order to compensate the dimension of the blade, ligther and softer materials are used, such as composite materials. The combination of the slenderness of the blade with the softeness of the material may induce large displacements of the structure when it is submitted to a dynamical loading. It is then important to predict such behaviours, which can lead to specific phenomena like mode coupling and internal resonances, and to the premature failure of the structure.
Model order reduction methods are often used to compute more easily the nonlinear dynamics of such structures [6]. For application to finite element (FE) problems, important steps have been made recently with the possibility of computing directly nonlinear mappings to go from the physical space to the reduced subspace where the dynamics is governed by a very small number of master modes. The Direct Normal Form (DNF) has been proposed in [4], elaborated on a third-order development based on previous works in modal space [5]. An arbitrary order expansion, fully relying on the parametrisation method of invariant manifolds [3], has also been proposed in [7].
In this contribution, the direct parametrisation of invariant manifold is applied to two different test cases featuring internal resonances. In the case of the clamped beam, it is known that a 1:5 internal resonance exists between the 1st and 3rd flexural modes [8]. Concerning the rotating cantilever beam, the internal resonance condition may appear with the rotation and the stiffening of the structure. We will thus show the appearance of a 1:5 internal resonance between the 2nd and the 4th flexural modes, and compute its backbone with the reduction method.

Reduction method with the direct parametrisation

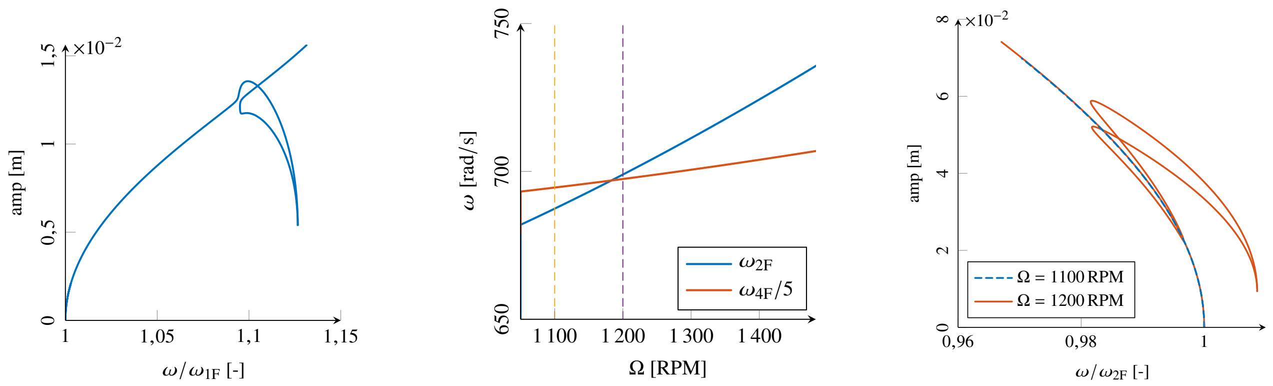
The Direct Parametrisation of Invariant Manifold (DPIM) is very briefly recalled here, relying on the developments shown in [7], and adapted in order to handle the effect brought by centrifugal force for a rotating system. The general equations of motion writes, using standard notations:

$$\mathbf{M}\ddot{\mathbf{x}} + [\mathbf{K}+\mathbf{N}]\mathbf{x} + \mathbf{g}(\mathbf{x},\mathbf{x}) + \mathbf{h}(\mathbf{x},\mathbf{x},\mathbf{x}) = \mathbf{f}_\Omega, \tag{1}$$

with $\mathbf{N}$ the spin softening matrix, $\mathbf{f}_\Omega$ the centrifugal effects, and where the Coriolis forces have not been taken into account [2], in contrary to geometric nonlinear terms expressed via $\mathbf{g}$ and $\mathbf{h}$. These equations can be rewritten around the static equilibrium position $\mathbf{x}_0$, depending on the rotation speed, by introducing $\mathbf{x} = \mathbf{x}_0 + \mathbf{u}$, leading to:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}_\mathrm{t}\mathbf{u} + \mathbf{g}(\mathbf{u},\mathbf{u}) + 3\mathbf{h}(\mathbf{x}_0,\mathbf{u},\mathbf{u}) + \mathbf{h}(\mathbf{u},\mathbf{u},\mathbf{u}) = \mathbf{0} \tag{2}$$

with $\mathbf{K}_\mathrm{t} = \mathbf{K} + \mathbf{N} + 2\mathbf{g}(\mathbf{x}_0,\mathbf{I}) + 3\mathbf{h}(\mathbf{x}_0,\mathbf{x}_0,\mathbf{I})$ the tangent stiffness matrix.
The direct parametrisation method introduces a nonlinear mapping between the original coordinates (displacement $\mathbf{u}$ and velocity $\mathbf{v}$), and a new, *normal coordinate* $\mathbf{z}$, which describes the motion on invariant manifold associated to the selected linear master modes, as $\mathbf{u} = \mathbf{\Psi}(\mathbf{z})$ and $\mathbf{v} = \mathbf{\Upsilon}(\mathbf{z})$, where $\mathbf{\Psi}$ and $\mathbf{\Upsilon}$ are unknowns to be determined. The reduced dynamics is also searched under the form $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$. All unknowns are expanded via polynomial expressions at arbitrary order, and solved for, by plugging the expansions in the invariance equation [7]. At each order p, the homological equation gathers the unknowns, which depends only on previous orders, while the first-order term leads to the known modal problem.
Importantly, different styles of solutions exist. Finally, the reduced dynamics, which contains very few equations, can be solved with numerical continuation. The results shown in the next section used MATCONT [1] for this step.



(a) backbone curve of the clamped beam

(b) Campbell diagram of the rotating beam

(c) backbone curves of the rotating beam

Figure 1: Internal resonances of the beam with 2 boundary conditions : clamped and rotating cantilever

Application case : nonlinear beam

The DPIM is applied here on a beam with dimensions $1\,\mathrm{m}$-$2\,\mathrm{cm}$-$3\,\mathrm{cm}$. The space is discretised with 27-nodes hexaedral elements with 50 elements in the length and 2×2 in the cross-section.
The first configuration considered is the clamped one. The backbone curve of the first mode is searched for, since it is known that at relatively large amplitudes, it meets a nonlinear internal resonance relationship with the 3$^{\text{rd}}$ flexural mode (their linear frequency ratio being $\omega_{3\mathrm{F}} = 5.36\,\omega_{1\mathrm{F}}$). Since a 1:5 resonance is at hand, the parametrisation is developed up to order five, using a complex normal form style. The backbone curve is shown in figure 1a, and is found to very well reproduce the tongue of internal resonance (see *e.g.* [8] for a reference, full-order solution).
The other configuration is the rotating cantilever beam. For this, we consider the 2$^{\text{nd}}$ and the 4$^{\text{th}}$ flexural modes. In this case, the internal resonance does not appear without rotation, even though the relation is $\omega_{4\mathrm{F}} = 5.45\,\omega_{2\mathrm{F}}$. As it is visible on the Campbell diagram on figure 1b, the 2 modes cross at a rotation speed around $1180\,\mathrm{RPM}$. In fig. 1c two backbone curves are shown, computed with the reduced order model method. Before the crossing, for a rotational speed of $1100\,\mathrm{RPM}$, the backbone shows a classical softening behaviour without internal resonance (blue dashed curve in fig. 1c). On the other hand, for $\Omega = 1200\,\mathrm{RPM}$, the backbone shows a clear tongue of internal resonance (orange curve in fig. 1c), highlighting that the 1:5 internal resonance is excited and revealed by the model.

Conclusion

The DPIM [7] has been applied to compute the reduced order model solution of a beam where 2 modes interact in an internal resonance with 2 different boundary conditions. The interesting case is the rotating beam where a condition for the internal resonance to occur has been found on the rotation speed. The method allows to predict those specific behaviors with a very small computation time. Those results need full order model resolution to be compared with, which will be achieved in the near future.

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