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Mechanical models of musical instruments and sound synthesis: the case of gongs and cymbals

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Abstract

This paper summarizes some results obtained in the last few years for the modeling of nonlinear vibrating instruments such as gongs and cymbals. Linear, weakly nonlinear and chaotic regimes are successively examined. A theoretical mechanical model is presented, based on the nonlinear von Kármán equations for thin shallow spherical shells. Modal projection and Nonlinear Normal Mode (NNM) formulation leads to a subset of coupled nonlinear oscillators. Current developments are aimed at using this subset for sound synthesis purpose.

1. Introduction

Gongs and cymbals belong to that category of instruments for which there is no sufficiently elaborate mechanical description yet for thinking of a possible sound synthesis entirely based on a physical model. Experimentations performed on these instruments for many years have shown typical features of a nonlinear behavior: amplitude-dependent level of harmonics in the spectrum, hardening/softening effects on resonance backbone curves, bifurcations, combination of modes and chaotic regime [1], [2], [3]. More recently, similar experiments were made on structures of simpler geometry (circular plates, shallow spherical shells) which exhibit comparable effects [4]. The main advantage of these studies is to allow a comparison with known theoretical models of nonlinear vibrations for plates and shells. Using these models, the relevance of quadratic geometrical nonlinearity due to large amplitude of the structure was clearly established. Starting from the von Kármán nonlinear equations, a set of nonlinear coupled differential equations was obtained that govern the dynamics of the problem. These equations are the result of the projection of the solution on the linear modes of the structure. In these oscillator equations, the coefficients are directly connected to the geometry and elastic properties of the vibrating object. A truncation of this set of nonlinearly coupled oscillators is performed in order to investigate the essential properties of the combination of modes. Instability conditions and the threshold values for the excitation are derived from the resolution of these subsets, using the method of multiple scales. Using nonlinear normal modes (NNM), a subset composed of a limited number of equations is obtained which approximate the solution more accurately than a subset obtained from the linear modal projection [5].

2. Summary of experiments

In their normal use, cymbals and gongs are set into vibrations by means of a mallet impact. However, as stated in [6], the nonlinear vibrations are due to large amplitude motion and can be thus studied using sinusoidal excitation, which greatly simplifies the interpretation. In fact, one can notice that sounds produced with large harmonic excitation are very similar to those resulting from impulsively excited instruments.

Figure 1 shows the fundamental experiments performed on the investigated instruments and structures. The top of the figure shows the progressively increasing noncontact excitation force at frequency Ω close to one eigenfrequency of the structure. As a result, the transverse acceleration of one selected point exhibits first a periodic motion with increasing level of harmonics 2Ω, 3Ω,... A first bifurcation occurs suddenly for a given force threshold. This bifurcation is characterized by the apparition of new frequencies ω_i and ω_j whose values are governed by the following rules:

$$\begin{aligned} p\Omega &= a_i\omega_i + a_j\omega_j \\ \text{with } \ a_i, a_j &\in \mathbb{Z} \ \text{ and } \ |a_i| + |a_j| = 2 \end{aligned} \tag{1}$$

where p is the harmonicity order of the excitation and ω_i and ω_j are other eigenfrequencies of the structure. These rules are direct consequences of quadratic nonlinearity [7]. The first equation in (1) characterizes the phenomenon of *combination resonances*. It shows that combinations can occur under the condition that one eigenfrequency, or one of its multiple, has particular algebraic relationship with one or two other eigenfrequencies. This property is called *internal resonance*.

Between the first and second bifurcation, the motion is quasiperiodic. The spectrum is enriched by the various frequencies ω_i and ω_j, resulting from the combination rules, and by their harmonics. After the second bifurcation, the motion has been proven to be chaotic [3].

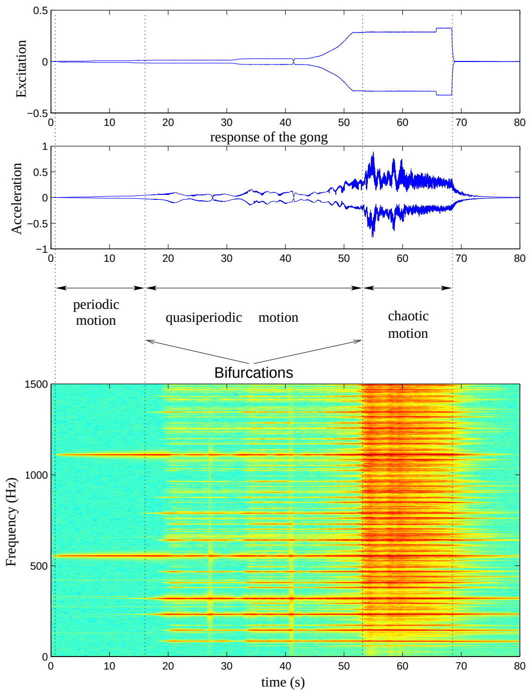
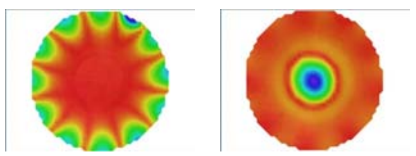


Figure 1: Harmonic excitation of a gong with increasing force amplitude.

3. Linear regime



(a) Mode (6,0) - f=111 Hz

(b) Mode (0,1) - f=224 Hz

Figure 2: Measured linear eigenmodes of the spherical cap linked by internal resonance of order two.

As shown in Sec. 2 the nonlinear behavior of the structure depends on its eigenfrequencies. This imposes to investigate first its linear vibrations. The linear motion of the structure is governed by an equation of the form:

$$\mathcal{L}(w) + \ddot{w} = 0 \tag{2}$$

where w is the transverse displacement of the thin shallow structure and $\mathcal{L}(w)$ a linear operator. A free edge boundary condition is added. Eq. (2) was solved analytically for a thin shallow shell, after confirmation that the main features of nonlinear vibrations for these structures were similar to those of gongs and cymbals [4]. This equation was also solved numerically, using a Finite Element modeling, for orchestral gongs (Chinese tam-tam), for cymbals, and for spherical caps with two different curvatures, successively. This latter case, in particular, shows that the values of the eigenfrequencies are highly sensitive to small changes of curvature. In addition, the modal shapes are strongly influenced by slope discontinuities. This property was mentioned in previous studies by Fletcher [2]. Finally, the linear analysis confirms the particularity of structures with symmetry of revolution which exhibit two modes with equal eigenfrequencies under the assumption of perfect homogeneity. For spherical caps, the experiments show some slight differences in frequency due to small imperfections (suspensions, holes, local defects,..). For the Chinese tam-tam, significant differences between the frequencies of the “twin modes” are observed, due to the loss of symmetry in the structure. The exact causes of this asymmetry are not yet fully understood.

4. Weakly nonlinear regime

4.1. Unimodal regime

As shown in a previous paper, the presence of harmonics in the shell motion is a consequence of quadratic nonlinearity due to large transverse displacement [8]. In fact, the equations of motion show together quadratic and cubic terms. However, the coefficients of the cubic terms are generally small (see Sec. 5) so that the effects of cubic nonlinearity are masked [9].

4.2. Nonperiodic regime

The presence of new frequencies in the nonperiodic motion can be explained by a stability study performed on a set of two nonlinearly coupled oscillators. We take here the example of a system with an internal resonance of order two. This corresponds to the two-modes subset of Fig. 2 for which we have $\omega_2 = 2\omega_1 + \varepsilon\sigma_1$, where σ_1 is the internal detuning parameter and $\varepsilon \ll 1$. The solution is obtained by applying the method of multiple scales [7] to the system:

$$\begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 &= \varepsilon\left[-\beta_{12}x_1x_2 - 2\mu_1\dot{x}_1\right] \\ \ddot{x}_2 + \omega_2^2 x_2 &= \varepsilon\left[-\beta_{21}x_1^2 - 2\mu_2\dot{x}_2 + P\cos\Omega t\right] \end{aligned} \tag{3}$$

where the forcing frequency $\Omega = \omega_2 + \varepsilon\sigma_2$ and where σ_2 is the external detuning parameter. The coefficients β_{12} and β_{22} depend on the geometry and material of the structure. The damping coefficients μ_1 and μ_2 are generally derived from experiments. It is assumed here that these coefficients are of the same order of magnitude than for circular plates of comparable geometry and material [10]. Solving Eq. (3) yields:

$$x_1 = a_1\cos\left(\frac{\Omega}{2}t + \gamma_1\right) \; ; \; x_2 = a_2\cos\left(\Omega t + \gamma_2\right) \tag{4}$$

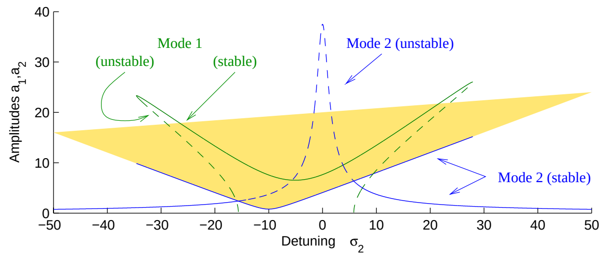


Figure 3: Internal resonance. The instability region corresponds to the shaded area.

Fig. (3) shows that, under certain conditions of both forcing amplitude and external detuning, a stable subharmonic of order 2 with amplitude a_1 can exist. This corresponds to the case observed on the spherical cap as the symmetric mode (0,1) forced at the center excites the asymmetric mode (6,0) through nonlinear coupling (see Fig. (2).

5. Nonlinear mechanical model

It is now examined how nonlinearly coupled oscillators of the form (3) are derived from the equations of motion of the structure. The general form of the field equations can be written:

$$\mathcal{L}(w) + \ddot{w} = \varepsilon \mathcal{N}(w) \tag{5}$$

where $\mathcal{N}(w)$ is a nonlinear operator and ε a "small" nondimensional parameter. We limit our study to the case of *geometrical* nonlinearity and we assume that the perturbating nonlinear terms are only due to large transverse displacement. The experiments (see Sec. 2) suggest that, for a sinusoidal excitation, only a limited number of modes are involved in the nonlinear vibration. Thus, the selected strategy for investigating the bifurcations is the following:

- Project Eq. (5) onto the basis of the eigenmodes of the linear operator $\mathcal{L}(w)$
- Truncate the obtained infinite series of nonlinearly coupled oscillators, keeping only the oscillators for which the linear eigenfrequencies are involved in a combination of modes (internal resonances).

5.1. Example of the thin shallow spherical shell

The nonlinear flexural motion of thin shallow spherical shells is well described by the von Kármán equations. Normalizing the displacement with respect to the quantity $\frac{h^2}{a}$, where a is the radius of the circle corresponding to the projection of the shell onto a plane perpendicular to its axis, and h its thickness, the equations of motion can be written in the following nondimensional form:

$$\begin{aligned} \Delta^2 w + \ddot{w} &= -\varepsilon_q \Delta F + \varepsilon_c \left[S(w,F) - 2\mu \dot{w} + p \right] \\ \Delta^2 F - \frac{a^3}{R h^2} \Delta w &= -\frac{1}{2} S(w,w) \end{aligned} \tag{6}$$

where F is the Airy stress function, S is a nonlinear function, μ is a damping coefficient, p is the forcing pressure and R the radius of curvature. The quadratic (ε_q) and cubic (ε_c) parameters are given by:

$$\varepsilon_q = 12(1-\nu^2)\frac{a}{R} \;\; ; \;\; \varepsilon_c = 12(1-\nu^2)\frac{h^2}{a^2} \tag{7}$$

where ν is the Poisson's ratio. For common geometries of thin shallow spherical shells, cymbals and gongs, we have $\varepsilon_c \ll \varepsilon_q$. Projection of Eq. (6) onto the eigenmodes of the structures with free edge yields the set of differential equations for the modal participation factors $q_i(t)$:

$$\begin{aligned} &\ddot{q}_i + \omega_i^2 q_i = -\varepsilon_q \sum_j \sum_k \alpha^i_{jk} \, q_j q_k \\ &+ \varepsilon_c \left[\sum_j \sum_k \sum_l \gamma^i_{jkl} \, q_j q_k q_l - 2\mu_i \dot{q}_i + P_i(t) \right] \end{aligned} \tag{8}$$

For the case presented in Fig. (2), we can assume that the nonlinear coupling only involves the asymmetric modes (6,0) and the symmetric mode (0,1), so that we can write approximately:

$$\begin{aligned} w(r,\theta,t) \approx & \Phi_{60}(r) \left[q_{11}(t) \cos 6\theta + q_{12}(t) \sin 6\theta \right] \\ & + \Phi_{01}(r) q_2(t) \end{aligned} \tag{9}$$

where q_{11} and q_{12} are the time histories of the two quadrature configurations of the asymmetric modes (6,0). With the additional assumption of perfect homogeneity, the coupling can be further reduced to only one one these two configurations interacting with the symmetric mode (0,1), so that we can limit our study to the system:

$$\begin{aligned} &\ddot{q}_1 + \omega_1^2 q_1 = \varepsilon_c \left[-\alpha_{12} q_1 q_2 - 2\mu_1 \dot{q}_1 \right] \\ &\ddot{q}_2 + \omega_2^2 q_2 = \\ &\varepsilon_c \left[-\alpha_{22} q_2^2 - \alpha_{21} q_1^2 - 2\mu_2 \dot{q}_2 + P_2(t) \right] \end{aligned} \tag{10}$$

where the cubic terms have been neglected, according to Eq. (7). Further simplifications are obtained by the use of NNM (see below). In practice, experiments show that the previous equations remain valid for $\|w\| \sim h$ and $R \sim a$, even if some of the nondimensional perturbating terms are not kept small compared to unity.

5.2. Nonlinear normal modes (NNM) formulation

NNM theory was developed in order to extend some of the properties of the linear normal modes to nonlinear systems. The leading idea is to decouple the nonlinear oscillators, in order to appropriately truncate the infinite series of differential equations derived from the nonlinear Partial Differential Equations that govern the motion. A NNM is defined as an invariant manifold in phase space, i.e. as a two-dimensional surface that is tangent to the linear modal eigenspace at the equilibrium point. As written in [11], the term "invariant" indicates that "any motion initiated on the manifold will remain on it for all time". In the context of sound synthesis, NNM are used for simplifying the dynamical system while keeping the essential features of the motion: hardening/softening behavior, dependence of mode shape with the amplitude. In practice, the reduction is obtained through nonlinear change of coordinates based on normal form theory [5]. From a physical point of view, this approach shows that the nonlinear dynamics is governed by a small number of *active* modes. However, thanks to the nonlinear change of coordinates, the relevance of the *slave* modes is retained thus improving the accuracy of the projection. In our example, the consequence of the nonlinear change of coordinates is that the term $\alpha_{22}q_2^2$ is removed in Eq. (10), thus leading exactly to Eq. (3).

6. Chaos

As the amplitude of the excitation increases, one can observe a transition from quasiperiodicity to chaos. The chaotic regime has been assessed and quantified through calculation of positive Lyapunov exponents [3]. The phase portrait shows that, during this transition, the two-torus trajectory breaks in favor of a strange attractor. Further analysis of the chaotic vibrations produced by gongs and cymbals with harmonic excitation shows that the dynamics of these systems are governed by a low number of state space coordinates. This important result, which confirms the mechanical approach (see Sec. 5), is based on the calculation of both correlation dimension and embedding dimension. As a consequence, the broadband spectrum shown in the analysis should not be confused with random noise since, it this case, the state space analysis shows an infinite number of coordinates.

7. Conclusion

In this paper, the nature of nonlinear coupling that leads to the particular sounds of cymbals and gongs has been described, by analogy with the similar behavior of thin shallow spherical shells. The same method can be now applied to more complex geometries, though, in this case, the coupling coefficients, which depend on the modal shapes, have to be calculated numerically. Is has been shown, both theoretically and experimentally, that the transfer of energy from low-order modes to higher frequencies is a consequence of quadratic nonlinearity due to curvature of the shell and is fully governed by frequency rules. The mechanical theory also shows that instability can occur with only a limited number of degrees of freedom, which confirms previous signal processing analysis on cymbals and gongs. Current development are thus aimed at obtaining relevant synthetic sounds from the numerical resolution of a small number of coupled nonlinear oscillators.

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