Non-linear oscillations of continuous systems with quadratic and cubic non-linearities using non-linear normal modes

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Abstract

Non-linear normal modes (NNMs), defined as invariant manifolds, are introduced through Normal Form theory. In a conservative framework, it is shown that all NNMs, as well as the attendant dynamics onto the manifolds, are computed in a single operation. The general third-order approximation of the dynamics onto a single NNM is derived. It is underlined that single linear mode truncation can lead to erroneous results which are corrected when considering NNMs. These results are illustrated by studying the vibrations of a linear beam resting on a non-linear elastic foundation.

Keywords: Non-linear normal mode; Normal form theory; Hardening/softening behaviour

1. Introduction

Non-linear oscillations of continuous structures with curvature, such as arches and shells, are considered. The motion of these structure, at moderately large vibration amplitudes, are governed by PDEs which include quadratic and cubic non-linearities. Using non-linear normal modes defined as invariant manifolds in phase space — as proposed by Shaw and Pierre [1] — allows one to exhibit reduced-order models which capture the essential properties of the dynamics. Here, normal form theory is employed to define new coordinates (the *normal* coordinates) linked to the invariant manifolds, as well as the non-linear relation between these new coordinates and the initial (modal) ones. This idea has already been proposed by Jezequel and Lamarque [2]. As a result of this operation, the dynamics onto the manifolds is simply given by the normal form of the initial problem.

In what follows, the general equations governing the motion onto a single NNM (up to order three) are derived. The interest of this result is due to the fact that a physically observed single mode motion occurs on an invariant manifold. Comparisons can then be drawn with the usual single linear mode Galerkin truncation. It is shown that the latter method can lead to erroneous quantitative as well as qualitative results, which are corrected when considering the NNMs. A linear beam resting on a non-linear elastic foundation is studied to highlight the theoretical results and to compare them with those derived by Nayfeh and Lacarbonara [3].

2. Theory

2.1. General case

After projection onto the linear modes basis, the following temporal problem is considered, when dealing with non-linear oscillations of undamped continuous systems:

$$\forall\, p = 1,\ldots,+\infty:$$

$$\ddot{X}_p + \omega_p^2 X_p + \sum_{i=1}^{+\infty}\sum_{j\geq i}^{+\infty} g_{ij}^p X_i X_j + \sum_{i=1}^{+\infty}\sum_{j\geq i}^{+\infty}\sum_{k\geq j}^{+\infty} h_{ijk}^p X_i X_j X_k = 0. \qquad (1)$$

X_p is the *modal* coordinate, related to the pth linear mode. This system is truncated by considering N modes, with N sufficiently large. The coefficients g_{ij}^p and h_{ijk}^p arise from the projection of the non-linear terms of the PDE onto the linear modes. It is assumed that no internal resonance relationships are present.

A near-identity change of coordinates is defined by successive eliminations of the non-resonant terms (for a more complete presentation of Normal Form, see e.g. Arnold [4]). The velocity $Y_p = \dot{X}_p$ is used as second independent

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variable. It reads:

$$X_p = R_p + \sum_{i=1}^{N}\sum_{j\geq i}^{N}(a_{ij}^p R_i R_j + b_{ij}^p S_i S_j) + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N} r_{ijk}^p R_i R_j R_k + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\geq j}^{N} u_{ijk}^p R_i S_j S_k, \quad (2a)$$

$$Y_p = S_p + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}^p R_i S_j + \sum_{i=1}^{N}\sum_{j\geq i}^{N}\sum_{k\geq j}^{N} \mu_{ijk}^p S_i S_j S_k + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k\geq j}^{N} \nu_{ijk}^p S_i R_j R_k. \quad (2b)$$

R_p and S_p are the *normal* coordinates and are linked to a third-order approximation of the dynamics into a curved grid of the phase space spanned by third-order approximations of the NNMs. Complete expressions for the coefficients of the polynomial expressions in Eq. (2) are given in Touzé [5], as well as the normal dynamics, which is not reproduced here for the sake of brevity, and also because the normal form can be written with the knowledge of the linear eigenspectrum only [4].

2.2. Single-mode motion

Some important features are now discussed for the motion along a single NNM. It is obtained by simply cancelling all other coordinates. To study the pth NNM, one has just to set: $\forall k \neq p$, $R_k = S_k = 0$. It yields the third-order approximation of the geometry of the pth NNM, as well as the dynamics onto it, governed by:

$$\ddot{R}_p + \omega_p^2 R_p + (A_{ppp}^p + h_{ppp}^p)R_p^3 + B_{ppp}^p R_p \dot{R}_p^2 = 0, \quad (3)$$

where

$$A_{ppp}^p = \sum_{l\geq p}^{N} g_{pl}^p a_{pp}^l + \sum_{l\leq p} g_{lp}^p a_{pp}^l,$$

and $$B_{ppp}^p = \sum_{l\geq p}^{N} g_{pl}^p b_{pp}^l + \sum_{l\leq p} g_{lp}^p b_{pp}^l$$

(see Touzé [5]). One can notice the velocity-dependent terms in Eq. (3), whose presence is only due to the quadratic non-linearity. It is an important feature of the quadratic non-linearity, which does not generate trivially resonant terms, but has an influence on the behaviour of the oscillations, as shown next.

The backbone curve for Eq. (3) is derived by any of the perturbation methods available. At first order, it reads:

$$\omega_{\mathrm{NL}} = \omega_p(1+\Gamma_p a^2), \quad \text{with} \quad \Gamma_p = \frac{3(A_{ppp}^p + h_{ppp}^p) + \omega_p^2 B_{ppp}^p}{8\omega_p^2}, \quad (4)$$

where ω_{NL} is the non-linear angular frequency and a is the amplitude. This expression can be compared with the backbone curve given when approximating the motion with the projection along the pth linear mode, found by setting: $\forall k \neq p$: $X_k = 0$ in Eq. (1):

$$\tilde{\omega}_{\mathrm{NL}} = \omega_p(1+\tilde{\Gamma}_p a^2), \quad \text{with} \quad \tilde{\Gamma}_p = \frac{1}{8\omega_p^2}\left(3h_{ppp}^p - \frac{10(g_{pp}^p)^2}{3\omega_p^2}\right). \quad (5)$$

Eqs. (4) and (5) clearly shows that the reduced-equation, which gives the dynamics onto the pth NNM, effectively represents the dynamics of the retained mode and accounts for the nonlinearities of the non-modeled modes, through A_{ppp}^p and B_{ppp}^p terms. This is realized without increasing the complexity of the dynamical equation, since a single oscillator is still used. This will now be illustrated by studying a continuous system, showing that linear truncation can lead to erroneous results which are corrected with the NNM modeling.

3. Example: a continuous system

3.1. Equations of motion

A linear hinged–hinged beam resting on a non-linear elastic foundation with distributed quadratic and cubic non-linearities is considered. The results obtained will be systematically compared with those presented in Nayfeh et al. [3], where a perturbation technique (the method of multiple scales) is directly introduced into the PDE in order to overcome the difficulties encountered with the linear single-mode approximation.

In non-dimensional form, the undamped transverse vibrations are governed by [3]:

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + \alpha_2 w^2 + \alpha_3 w^3 = 0, \quad (6)$$

with boundary conditions:

$$w(x,t) = 0, \qquad \frac{\partial^2 w(x,t)}{\partial x^2} = 0 \qquad \text{for} \quad x = 0,1. \quad (7)$$

$w(x,t)$ is the transverse displacement, α_2 and α_3 are constants. The linear analysis provides the eigenmodes as well as the eigenfrequencies:

$$\Phi_n(x) = \sqrt{2}\sin(n\pi x), \qquad \omega_n = n^2\pi^2. \quad (8)$$

Projection onto the linear modes basis is performed via the development $w(x,t) = \sum X_p(t)\Phi_p(x)$, which is inserted into Eq. (6). This leads to the following temporal problem: $\forall p = 1,\ldots,N$:

$$\ddot{X}_p + \omega_p^2 X_p + \sum_{i,j=1}^{N} g_{ij}^p X_i X_j + \sum_{i,j,k=1}^{N} h_{ijk}^p X_i X_j X_k = 0. \quad (9)$$

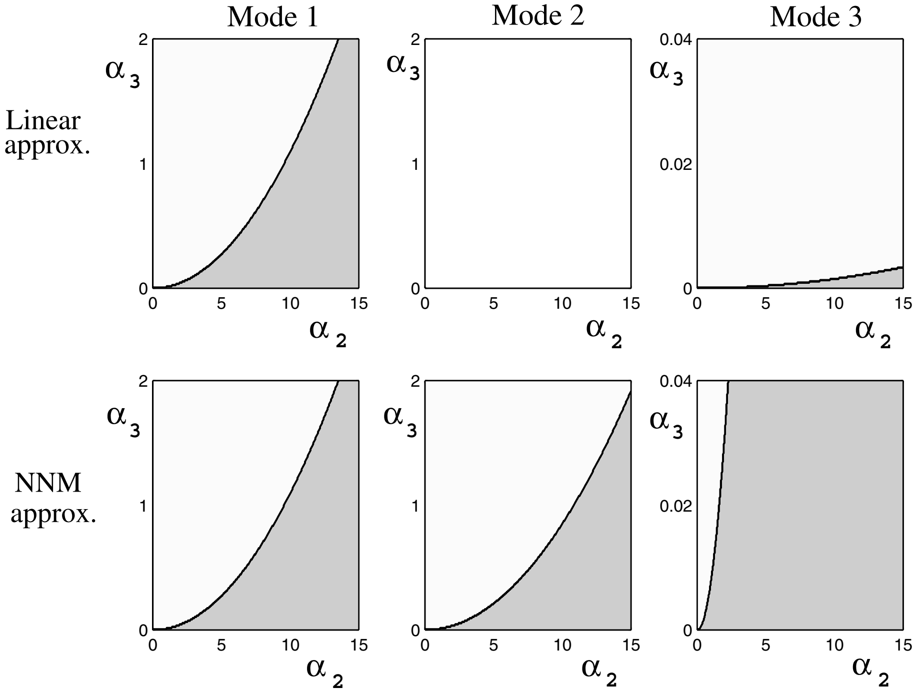

Fig. 1. Hardening and softening regions for the linear beam resting on a non-linear elastic foundation. White: hardening behaviour. Gray: softening behavior.

3.2. Regions of hardening and softening behaviour

The theoretical results of Section 2 are applied to Eq. (9). As discussed in Touzé et al. [6], single-mode motion on two-dimensional invariant manifolds exists for the beam problem in spite of the numerous internal resonance relationships. This is due to the particular form of the resonance relation set, implying only monoms which are not invariant-breaking.

Regions of hardening and softening behaviour are given by the sign of Γ_p when considering motions along the pth NNM (resp. by the sign of $\tilde{\Gamma}_p$ when considering motions along the pth linear eigenspace). Hardening behaviour regions, defined by a positive value for Γ_p and $\tilde{\Gamma}_p$, are represented in the parameter plane (α_2, α_3) as a white zone on Fig. 1. Softening behaviour regions are in gray. One can notice that for mode 1, considering the NNM does not change these regions. But this is not the case for all the other modes. In particular for mode 2, a linear-mode approximation predicts a hardening behaviour for every (α_2, α_3), since the quadratic coefficient vanishes: $g_{22}^2 = 0$. When one considers the motion along the second NNM, a softening region is in fact present. These results have been compared with those of Nayfeh and Lacarbonara [3]. They find exactly the same hardening/softening regions.

3.3. Mode shapes

A first-order perturbation method yields the oscillations along the pth NNM, the dynamics of which is given at third order by Eq. (3). It reads: $R_p = a_0 \cos(\omega_{\mathrm{NL}} t + \beta_0) + \cdots$, where ω_{NL} is given by Eq. (4).

Substituting this result into the non-linear change of coordinates, Eq. (2), gives the displacement of the structure expressed with the modal coordinates:

$$\forall k \neq p: \; X_k = a_{pp}^k R_p^2 + b_{pp}^k S_p^2 + r_{ppp}^k R_p^3 + u_{ppp}^k R_p S_p^2, \quad (10a)$$

$$k = p: \; X_p = R_p + a_{pp}^p R_p^2 + b_{pp}^p S_p^2, \quad (10b)$$

The complete expression for the displacement along NNM p, denoted here by $w_p(x,t)$, is given as usual by the decomposition: $w_p(x,t) = X_p(t)\Phi_p(x) + \sum_{k \neq p} X_k(t)\Phi_k(x)$.

A graphical representation of the maximum displacement, obtained for R_p maximum, is given in Fig. 2, first column, for the first three modes and two different amplitudes. The result is compared with the linear approximation, and with the method proposed in [3]. No difference is visible: the mode shapes match exactly (and thus the dash-dotted curve is not visible).

Finally, as a quadratic non-linearity is present, a constant term in the solution of the oscillations gives rise to a spatial drift in the response. It is represented in Fig. 2, second column. Once again, Normal form method is compared with perturbation method directly into the PDE [3], and the

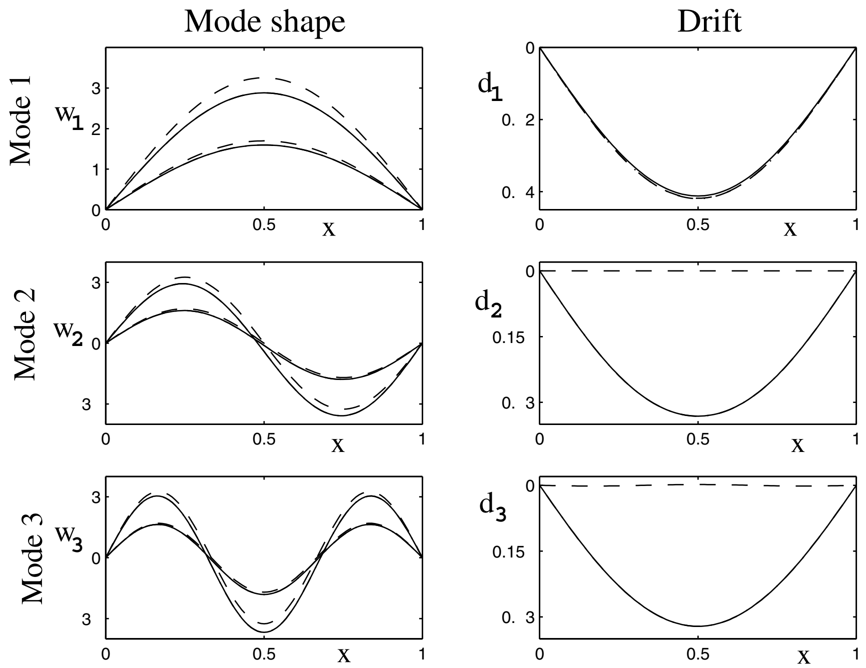

Fig. 2. Mode shapes and drifts, ($\alpha_2 = 12$, $\alpha_3 = 1$). Solid line (—): NNM approximation with normal form. Dashed line (– –): linear solution. Dash-dotted line (– ·): Nayfeh et al. solution [3].

linear result. Except for the first mode, where the result of Nayfeh et al. [3] fit with the linear drift, the results are found to be the same. The equivalence between the two approaches is thus explicitly shown on this example.

4. Conclusion

In this paper, NNM has been defined through Normal Form theory. It has been shown that single linear Galerkin truncation can lead to erroneous results which are corrected when considering the motion along invariant manifold. This has been realized through third-order asymptotic development for the dynamics expressed with *normal* coordinates. This method yields accurate results that accounts for the a priori non-modeled modes, without increasing the complexity of the dynamical problem treated. Hence reliable results are easily found. The effectiveness of the method has been illustrated on a continuous system.

References

[1] Shaw SW, Pierre C. Normal modes of vibration for non-linear continuous systems. J Sound Vib 1994;169(3):319–347.

[2] Jezequel L, Lamarque CH. Analysis of non-linear dynamical systems by the normal form theory. J Sound Vib 1991;149(3):429–459.

[3] Nayfeh AH, Lacarbonara W. On the discretization of distributed-parameter systems with quadratic and cubic non-linearities. Nonlinear Dynamics 1997;13:203–220.

[4] Arnold VI. Chapitres supplémentaires de la théorie des équations différentielles ordinaires, Mir-Librairie du Globe, 1980.

[5] Touzé C. A normal form approach for non-linear normal modes. Internal publication of the LMA, No. 156 (ISBN 2-909669-20-3), Marseille, 2003.

[6] Touzé C, Thomas O, Chaigne A. Hardening/softening behaviour in non-linear oscillations of structural systems using non-linear normal modes. J Sound Vib 2002; submitted.