TYPE OF NON-LINEARITY OF DAMPED IMPERFECT PLATES USING NON-LINEAR NORMAL MODES

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Abstract
Non-linear normal modes (NNMs) are used for deriving in a systematic manner the type of non-linearity of circular plates and shallow spherical shells with a free edge. A special attention is paid to geometric imperfections, that are unavoidable in real systems. It is quantitatively shown, for a number of different axisymmetric and asymmetric imperfections, how the hardening type non-linearity, that is typical of flat plate displaying only cubic non-linearity, is turned to a softening type behaviour for an imperfection amplitude being a fraction of the plate thickness. The role of 2:1 internal resonance in this process is underlined. When damping is included in the calculation, it is found that the softening behaviour is generally favoured, but its effect remains limited.

Key words
Hardening/softening behaviour, Non-linear modes.

1 Introduction

The type of non-linearity (*i.e.* the hardening or softening behaviour) of structures vibrating with large amplitude has been for a long time a subject of controversy, especially with the specific case of circular cylindrical shells, see *e.g.* (Amabili and Padoussis 2003). In fact, the problem is difficult to solve, due to the geometrical non-linearity, as well as the number of expansion functions one has to keep in the truncations for attaining convergence. Before the beginning of the 1990s, numerous studies were published where the type of non-linearity was predicted under the assumption of a single-mode vibration, see *e.g.* (Grossman *et al.* 1969, Hui 1983*b*, Yasuda and Kushida 1984) for shallow spherical shells, or (Hui 1983*a*) for imperfect circular plates. Unfortunately, it has been shown by a number of more recent investigations that too severe truncations lead to erroneous results in the prediction of the type of non-linearity, see for example (Nayfeh *et al.* 1992, Touzé *et al.* 2004). Recent papers are now available where a reliable prediction is realized, for the case of buckled beams (Rega *et al.* 2000), circular cylindrical shells (Pellicano *et al.* 2002), suspended cables (Arafat and Nayfeh 2003) and shallow spherical shells (Touzé and Thomas 2006).

The main problem for the computation is the number of linear modes that has to be retained in the truncation, so that the numerical burden associated to such predictions becomes rapidly important. In order to speed up the numerical computation, the modal equations are here reduced by using the Non-linear Normal Modes (NNMs) of the system, defined as invariant manifold of the phase space. The NNMs are computed via an asymptotic approach, within the framework of normal form theory. In particular, it has been shown in (Touzé *et al.* 2004) that this formalism allows an easy and accurate prediction of the correct type of non-linearity.

In this study, a special attention is paid to two complicating effects that are generally neglected in the prediction of the type of non-linearity. First, the effect of geometric imperfection, defined as a static displacement with zero inital stresses, on the non-linear behaviour of circular plates with free edge, is investigated. It is a well-known results that flat plates display a hardening behaviour. Here we will show quantitatively that the behaviour may change from hardening to softening type, for geometric imperfection with small amplitude. Secondly, the effect of the damping on the type of non-linearity is addressed. Whereas most studies neglects the damping, it has been shown recently in (Touzé and Amabili 2006) that the damping have an influence on the type of non-linearity. Consequently, the effect of a viscous damping on the non-linear behaviour of shallow spherical shell will be discussed.

2 Theoretical formulation

2.1 Local equations and boundary conditions

A thin plate of diameter $2a$ and uniform thickness h is considered, with $h << a$, and free-edge boundary condition. The local equations governing the large-amplitude displacement of a perfect plate, assuming the non-linear Von Kármán strain-displacement relationship, are used. An initial imperfection, denoted by $w_0(r,\theta)$ and associated with zero inital stresses is also considered. The shape of this imperfection is arbitrary, and its amplitude is small compared to the diameter (shallow assumption): $w_0(r,\theta) << a$. The local equations for an imperfect plate deduce from the perfect case. With $w(r,\theta,t)$ being the transverse displacement from the imperfect position at rest, the non-dimensional equations of motion write:

$$\Delta\Delta w + \ddot{w} = \varepsilon\left[L(w,F) + L(w_0,F) - c\dot{w} + p(r,\theta,t)\right], \tag{1a}$$

$$\Delta\Delta F = -\frac{1}{2}\left[L(w,w) + 2L(w,w_0)\right], \tag{1b}$$

where $w(r,\theta,t)$ and $w_0(r,\theta)$ have been made non-dimensional by dividing their value by the thickness h. Δ stands for the Laplacian operator, c accounts for structural damping of the viscous type, p denotes the external load, F is the Airy stress function, and L is a bilinear operator, whose expression is given *e.g.* in (Touzé *et al.* 2002). Finally $\varepsilon = 12(1-\nu^2)$.

The boundary conditions for the case of a free edge write, in non-dimensional form (Touzé *et al.* 2002), at $r = 1$:

$$F_{,r} + F_{,\theta\theta} = 0\,, \quad F_{,r\theta} + F_{,\theta} = 0\,, \tag{2a}$$

$$w_{,rr} + \nu w_{,r} + \nu w_{,\theta\theta} = 0\,, \tag{2b}$$

$$w_{,rrr} + w_{,rr} - w_{,r} + (2-\nu)w_{,r\theta\theta} - (3-\nu)w_{,\theta\theta} = 0\,. \tag{2c}$$

In order to discretize the PDEs, a Galerkin procedure is used. As the eigenmodes can not be computed analytically because the shape of the imperfection is arbitrary, the eigenmodes of the perfect plate $\Psi_p(r,\theta)$ are selected as basis functions. Analytical expressions of $\Psi_p(r,\theta)$ involve Bessel functions and can be found in (Touzé *et al.* 2002). The unknown displacement $w(r,\theta,t)$ and the static intial imperfection $w_0(r,\theta)$ are expanded with:

$$w(r,\theta,t) = \sum_{p=1}^{+\infty} q_p(t)\,\Psi_p(r,\theta), \tag{3}$$

$$w_0(r,\theta) = \sum_{p=1}^{+\infty} a_p \Psi_p(r,\theta), \tag{4}$$

where the time functions q_p are now the unknowns. In this expression, the subscript p refers to a specific mode of the perfect plate, defined by a couple $(k,\ n)$, where k is the number of nodal diameters and n the number of nodal circles. If $k \neq 0$, a binary variable is added, indicating the preferential configuration considered (*sine* or *cosine* companion mode). Inserting the expansion (4) into Eqs. (1), and using the orthogonality properties of the expansion functions, the dynamical equations are found to be:

$$\ddot{q}_p + 2\xi_p\omega_p\dot{q}_p + \varepsilon\left[\sum_{i=1}^{+\infty}\alpha_i^p q_i + \sum_{i,j=1}^{+\infty}\beta_{ij}^p q_i q_j + \sum_{i,j,k=1}^{+\infty}\Gamma_{ijk}^p q_i q_j q_k\right] = 0. \tag{5}$$

Linear coupling terms between the oscillator equations are present, as the natural modes have not been used for discretizing the PDEs. The cubic coefficients Γ_{ijk}^p appearing in Eqs (5) are those from the perfect plate. A major advantage of the present formulation is that the linear α_i^p and quadratic β_{ij}^p coefficients are expressed via simple expressions to the cubic plate coefficients:

$$\alpha_p^u = \sum_{r=1}^{N_p}\sum_{s=1}^{N_p} 2\Gamma_{rps}^u a_r a_s \tag{6a}$$

$$\beta_{pr}^u = \sum_{s=1}^{N_p}(\Gamma_{rps}^u + 2\Gamma_{srp}^u)a_s. \tag{6b}$$

In particular, it means that the linear and non-linear characteristics of a plate with arbitrary shape can be easily deduced from the analysis of the perfect plate only. Finally, the last step before deriving the type of non-linearity for this assembly of non-linear oscillator consists in making the linear part of the equations diagonal. Let $\mathbf{P}$ be the matrix of eigenvectors of the linear part $\mathbf{L} = [\alpha_i^p]_{p,i}$. A linear change of co-ordinates is processed, $\mathbf{q} = \mathbf{PX}$, where $\mathbf{X} = [X_1 \ldots X_N]^T$ is, by definition, the vector of modal co-ordinates, and N is the number of expansion function kept in practical application of the Galerkin's method. Finally, the discretized equations of motion writes, $\forall\ p = 1 \ldots N$:

$$\ddot{X}_p + 2\xi_p\omega_p\dot{X}_p + \omega_p^2 X_p + \varepsilon\left[\sum_{i,j=1}^{N} g_{ij}^p X_i X_j + \sum_{i,j,k=1}^{N} h_{ijk}^p X_i X_j X_k\right] = 0. \tag{7}$$

2.2 Type of non-linearity

Non-linear normal modes (NNMs), defined as invariant manifolds in phase space, have been defined with the objective of embedding the main dynamical features of a N-dof system into a single non-linear equation, hence providing accurate reduced-order models for non-linear analysis/synthesis. Proper truncations can be realized, as the motion is described in an invariant-based span of the phase space, and thus non-resonant coupling terms between oscillators have been cancelled. Keeping a single non-linear mode predicts the correct type of non-linearity, as it has been demonstrated and numerically verified, see *e.g.* (Touzé *et al.* 2004, Touzé and Thomas 2006).

The basic results are here briefly recalled. A first-order perturbative development of the amplitude-frequency relationship on the p^{th} NNM gives:

$$\omega_{NL} = \omega_p(1 + T_p a^2), \tag{8}$$

where a is the amplitude of the response of the p^{th} NNM and T_p the coefficient governing the type of non-linearity. If $T_p > 0$, then hardening behaviour occurs, whereas $T_p < 0$ implies softening behaviour. The analytical expression of T_p reads:

$$T_p = \frac{1}{8\omega_p^2}\left[3(A^p_{ppp} + \varepsilon_c \Gamma^p_{ppp}) + \omega_p^2 B^p_{ppp}\right], \tag{9}$$

where:

$$A^p_{ppp} = \varepsilon\left[\sum_{l \geq i}^{N} g^p_{pl} a^l_{pp} + \sum_{l \leq i} g^p_{lp} a^l_{pp}\right], \tag{10}$$

$$B^p_{ppp} = \varepsilon\left[\sum_{l \geq i}^{N} g^p_{pl} b^l_{pp} + \sum_{l \leq i} g^p_{lp} b^l_{pp}\right]. \tag{11}$$

In these last expressions, g^k_{ij} are the quadratic coupling coefficients from Eqs (7), and a^k_{ij} and b^k_{ij} are coefficients arising from the non-linear change of coordinates between the modal amplitudes X_k and the *normal* co-ordinates R_p that are associated to the NNMs, see (Touzé *et al.* 2004) for more details.

Finally, the method used for deriving the type of non-linearity can be summarized as follows. For a geometric imperfection of a given amplitude, the discretization leading to the non-linear oscillator equations (7) is first computed. The numerical effort associated to this operation is the most important but remains acceptable on a standard computer. Then the non-linear change of co-ordinates is computed, which allows derivation of the A^p_{ppp} and B^p_{ppp} terms occuring in Eq. (9), the sign of which determines the type of non-linearity. Numerical results are given in the next section for specific imperfections.

3 Effect of imperfections

3.1 Axisymmetric imperfection

In this section, the particular case of an axisymmetric imperfection having the shape of mode (0,1) (*i.e.* with one nodal circle and no nodal diameter), is considered. Fig. 1 shows the effect of the imperfection on the eigenfrequencies, for an imperfection amplitude from 0 (perfect plate) to $10h$. It is observed that the purely asymmetric modes $(k, 0)$, having no nodal circle and k nodal diameters, are marginally affected by the axisymmetric imperfection. The computation has been done by keeping 51 basis functions: purely asymmetric modes from (2,0) to (10,0), purely axisymmetric modes from (0,1) to (0,13) and mixed modes from (1,1) to (6,1), (1,2), (2,2), (3,2) and (1,3).

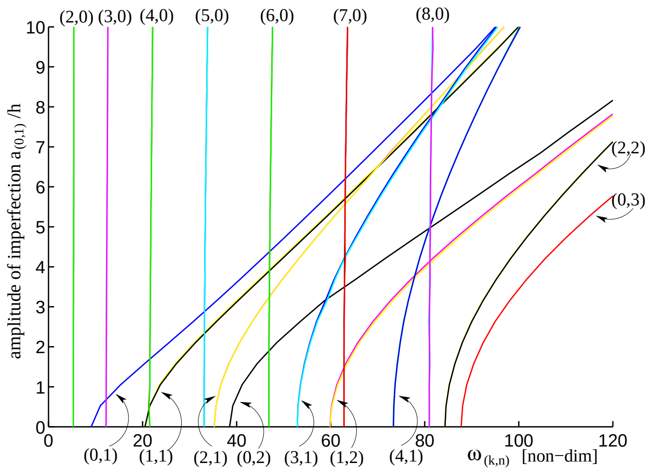

Figure 1. Non-dimensional natural frequencies $\omega_{(k,n)}$ of the imperfect plate versus the amplitude of the imperfection having the shape of mode (0,1).

The effect of the imperfection on the axisymmetric modes (0,1) is studied. In this case the problem is fully axisymmetric so that all the truncations can be limited to axisymmetric modes only, which drastically reduces the numerical burden. The result for mode (0,1) is shown in Fig. 2. It is observed that the huge variation

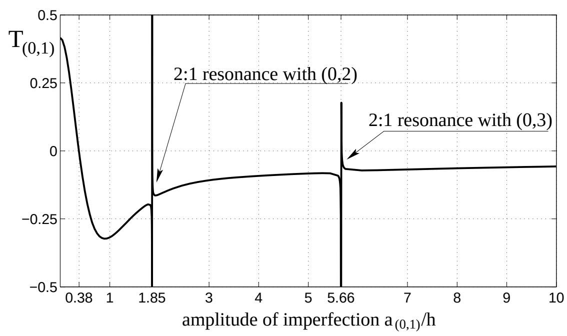

Figure 2. Type of non-linearity for mode (0,1) with an axisymmetric imperfection having the shape of mode (0,1).

of the eigenfrequency with respect to the amplitude of the imperfection results in a quick turn of the behaviour from the hardening to the softening type, occuring for an imperfection amplitude of $a_{(0,1)}$ = 0.38h. The second main observation inferred from Fig. 2 is the occurrence of 2:1 internal resonance between eigenfrequencies, leading to discontinuities in the coefficient $T_{(0,1)}$ dictating the type of non-linearity. This fact has already been observed and commented for the case of shallow spherical shells in (Touzé and Thomas 2006). It has also been observed for buckled beams and suspended cables (Rega *et al.* 2000, Arafat and Nayfeh 2003). This is a small denominator effect typical of internal resonance, *i.e.* when the frequency of the studied mode (0,1) exactly fulfills the relationship $2\omega_{(0,1)} = \omega_{(0,n)}$ with another axisymmetric mode. 2:1 resonance arises here with mode (0,2) at $1.85h$ and with mode (0,3) at $5.66h$.

Finally, the effect of the imperfection on asymmetric modes is shown in Fig. 3 for mode (2,0). The very slight variation of the eigenfrequencies of this modes versus the axisymmetric imperfection results in a very slight effect of the geometry. It is observed that before the first 2:1 internal resonance, the type of non-linearity shows small variations. The 2:1 internal resonance with mode (0,1) at $a_{(0,1)}$= $0.44h$ makes the behaviour change. Finally, for high amplitudes of imperfection, it tends to be neutral.

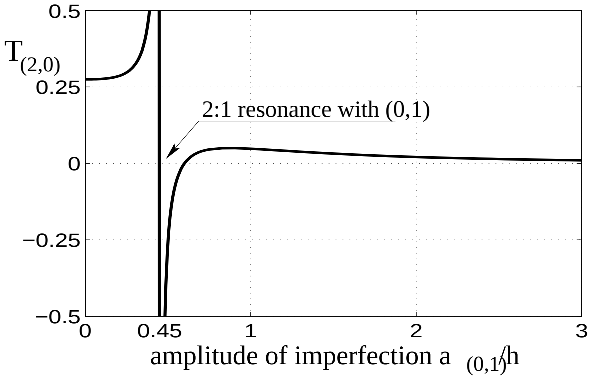

Figure 3. Type of non-linearity for mode (2,0) with an axisymmetric imperfection having the shape of mode (0,1).

3.2 Asymmetric imperfection

In this section, the effect of an imperfection having the shape of mode (2,0), is studied. Due to the loss of symmetry, degenerated modes are awaited to cease to exist : the equal eigenfrequencies of the *sine* and *cosine* configuration of degenerated modes split. The numerical results for type of non-linearity relative to the two configurations (2,0,C) and (2,0,S), are shown in Fig. 4 and 5. The natural frequency of mode (2,0,C) undergoes a huge variation, which result in a quick change of behaviour, occurring at $0.54h$. Then, a 2:1 internal resonance with (0,2) is noted, but without a noticeable change in the type of non-linearity, as the interval where the discontinuity is present is very narrow. In this case, the behaviour of $T_{(2,0,C)}$ looks like

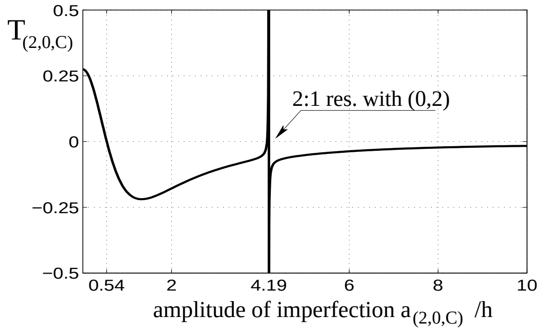

Figure 4. Type of non-linearity for mode (2,0,C) for an imperfection having the shape of mode (2,0,C).

the one observed in the precedent case, *i.e.* the variation of $T_{(0,1)}$ versus an imperfection having the same shape. On the other hand, the eigenfrequency of mode (2,0,S) remains quite unchanged, so that the behaviour of $T_{(2,0,S)}$ is not much affected by the imperfection, until the 2:1 internal resonance is encountered. In that case, the resonance occurs with the other configuration, *i.e.* mode (2,0,C).

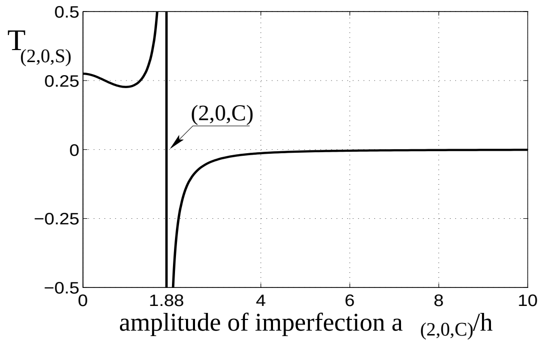

Figure 5. Type of non-linearity for mode (2,0,S) for an imperfection having the shape of mode (2,0,C).

4 Effect of damping

In this section, the effect of viscous damping on the type of non-linearity, is addressed. The particular case of a spherical imperfection is selected. With this case, the equations of motions turn out to be equivalent to those of the shallow spherical shell, under the assumption that the main term of the curvature is kept in a Taylor expansion (shallow assumption), as shown in (Camier *et al.* 2007). Besides, the equations of motion for the shallow spherical shell depends on one geometric parameter only: $\kappa = \frac{a^4}{R^2\,h^2}$, where R is the radius of curvature. Finally, the type of non-linearity for undamped shallow spherical shells have been already studied in (Touzé and Thomas 2006), so that the results shown here complement this earlier study.

First we study the effect of a damping factor that fulfills the following relationship: $\forall\ p = 1 \ldots N, \quad \xi_p = \xi \,/\, \omega_p$, which means that the rate of decay of each oscillator equation is the same. We will refer to this case as ”constant damping case”. The result is shown in Fig.6 for mode (0,1). The values of ξ that are selected for the computation are: ξ=0, 0.01, 0.1 and 0.3. The numerical result shows that for $\xi \leq 0.01$, the effect of damping is unnoticeable. Secondly, one can observe that the discontinuity ocurring at 2:1 internal resonance is smoothened. However, it happens for a quite large amount of damping in the structure. Moreover, outside the narrow intervals where 2:1 resonance occurs, the effect of damping is not visible. As a conclusion for this case, it appears that this kind of damping has a really marginal effect on the type of non-linearity, so that undamped results can be estimated as reliable for lightly damped structures with modal damping factor below 0.1.

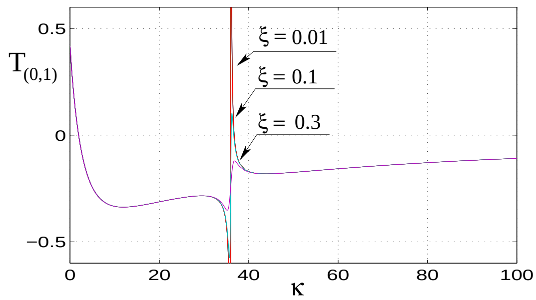

Figure 6. Type of non-linearity for mode (0,1) versus the aspect ratio κ of a shallow spherical shell. Increasing values of damping for case of ”constant damping” ($\forall\ p = 1 \ldots N, \quad \xi_p = \xi \,/\, \omega_p$), are shown, with ξ = 0 and 0.01 (red), 0.1 (cyan) and 0.3 (violet).

Secondly, the case of a damping law reading: $\forall\ p = 1 \ldots N, \quad \xi_p = \xi$, is selected. This case corresponds to a decay rate that is proportional to the eigenfrequency for each oscillator-equation, and thus is referred to as the ”proportional damping case”.

It is observed in Fig. 7 that the discontinuity is not smoothened at the 2:1 internal resonance. Inspecting back the analytical results show that this is a natural consequence of the expression of the coefficients of the non-linear change of co-ordinates for asymptotic NNMs. When the specific case of constant damping factors is selected, small denominators remain present. On the other hand, outside the regions of 2:1 resonance, the effect of damping is pronounced and enhances the softening behaviour. But once again, very large values of damping factors such as 0.3 must be reached to see a prominent influence.

Finally, the case of a rapidly increasing decay factor with respect to the frequency is investigated. The values: $\forall\ p = 1 \ldots N, \quad \xi_p = \xi\omega_p$ are selected. The decay factor is now proportional to the square value

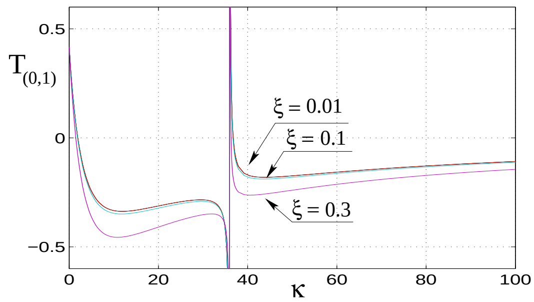

Figure 7. Type of non-linearity for mode (0,1) versus the aspect ratio κ of a shallow spherical shell. Increasing values of damping for case of ”proportional damping” ($\forall\ p = 1 \ldots N, \quad \xi_p = \xi$), are shown, with ξ = 0 and 0.01 (red), 0.1 (cyan) and 0.3 (violet).

of the eigenfrequeny, so that this case is referred to as the ”square damping”. As the overall damping in the structure is thus larger, smaller values of ξ have been selected, namely 1e-4, 1e-3 and 1e-2. ξ= 1e-4 gives quite coincident results with ξ= 0. But from ξ= 1e-3, the effect of the damping is very important: the discontinuity is smoothened. For ξ= 1e-2, 2:1 resonance is not visible anymore.

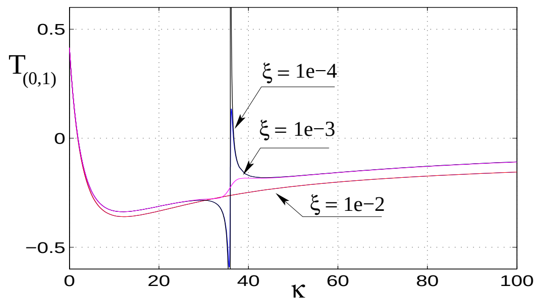

Figure 8. Type of non-linearity for mode (0,1) versus the aspect ratio κ of a shallow spherical shell. Increasing values of damping for case of ”square damping” ($\forall\ p = 1 \ldots N, \quad \xi_p = \xi\,\omega_p$), are shown, with ξ = 0 and 1e-4 (black and blue, collapsed), 1e-3 (magenta) and 1e-2 (red).

Previous results on the effect of the damping on the type of non-linearity were derived for a very simple model with two-degrees of freedom (Touzé and Amabili 2006). It was shown that the damping generally favours the softening behaviour. In some cases, it can turn a hardening-type non-linearity to a softening type. however, large values of the damping were needed. These preliminary results are here extended to the case of a shallow spherical shell. The selected values of damping are typical of those encountered for metallic shells where the internal losses are very small. The numerical results show that the effect of the damping, when considering realistic values, is small, and can generally be neglected, excpet in the case where the damping factor is proportional to the square of the eigenfrequency.

5 Conclusion

The effect of geometric imperfections on the hardening/softening behaviour of circular plates with a free edge have been studied. Thanks to the NNMs, quantitative results for the transition to hardening to softening behaviour has been documented, for an axisymmetric as well as for an asymmetric imperfection. It has been shown that when the eigenfrequency of a mode is strongly affected by the change in geometry, then the behaviour of this mode tends to go quickly from the hardening to the softening type. On the other hand, eigenfrequencies that are not much affected by the change of geometry generally change their type of non-linearity when encountering 2:1 internal resonance. Secondly, the effect of the damping has been studied. Numerical results shows that the effect is very slight and can be neglected for usual damping laws found in thin metallic structures.

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