

## REDUCTION OF GEOMETRICALLY NON-LINEAR MODELS OF SHELL VIBRATIONS INCLUDING IN-PLANE INERTIA

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### 1 INTRODUCTION

This article deals with the application of reduced-order models (ROMs), via the asymptotic NNM method proposed in [1], to thin shells large-amplitude vibrations. Two particular geometries are addressed: a doubly-curved shallow shell, simply supported on a rectangular base, and a circular cylindrical panel with simply supported, in-plane free edges. In both cases, the shell is subjected to a harmonic excitation, normal to its surface, and in the spectral neighbourhood of its fundamental frequency.

For both shells, the models use Donnell's non-linear strain-displacement relationships, with in-plane inertia retained. The discretized equations of motion are obtained by the Lagrangian approach described in [2, 3, 4], where the unknown displacements are expanded on an ad-hoc basis of approximation functions that are not the eigenmodes. As a consequence, a large number of degrees-of-freedom (dofs) is necessary in order to obtain convergence.

The reduction to a single NNM is shown for various excitation amplitude, and compared to a reference solution. Perfect results are obtained for vibration amplitude lower or equal to 1.5 times the thickness of the shell.

### 2 EQUATIONS OF MOTION

The main steps of the Lagrangian approach used for deriving the equations of motions are here only sketched, the interested reader can refer to [2, 3, 4] for a complete description. The two structures under study are: (i) a doubly curved shallow shell with rectangular base, (ii) a circular cylindrical panel. For case (i), a curvilinear coordinate system  $(O, x, y, z)$  is used, whereas a cylindrical coordinate system  $(O, x, r, \theta)$  describes case (ii), see Figure 1. A particular doubly curved shell is selected, such that  $R_x/R_y = -1$  (hyperbolic paraboloid panel), where  $R_x$  and  $R_y$  are the principal radii of curvature.

The Donnell's strain-displacement relationships for thin shells are used for deriving the elastic strain energy of the shells. Kinetic energy is expressed by neglecting rotary inertia and keeping in-plane inertia terms. An external transverse point excitation, located at  $(\tilde{x}, \tilde{y})$  for case (i), or at  $(\tilde{x}, \tilde{\theta})$  for case (ii) is considered. Its temporal content is harmonic with frequency  $\omega$  and magnitude  $\tilde{f}$  (in Newton). The boundary conditions are simply supported for the hyperbolic paraboloid panel, and with simply supported, in-plane free edges for the circular cylindrical panel. The equations of motions are discretized with a Lagrange formulation, by expanding the middle surface displacements  $(u, v, w)$  on a basis of trigonometric func-

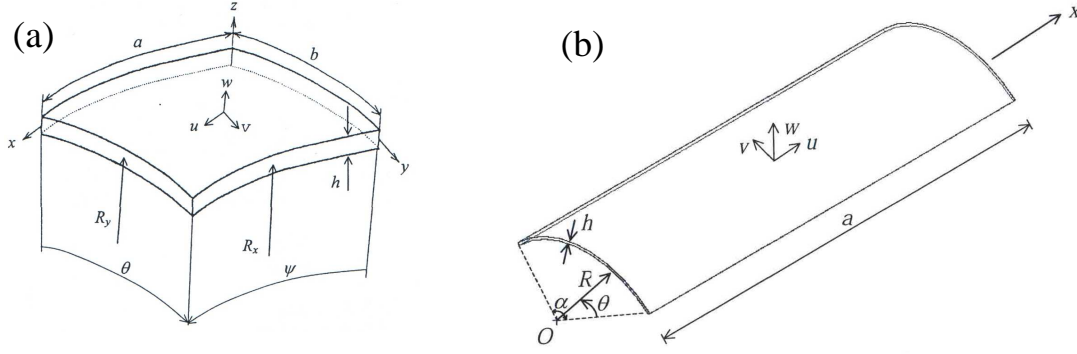


Figure 1. Geometry and coordinate system for (a) the doubly curved panel, from [3]; and (b) the circular cylindrical panel, from [4].

tions satisfying identically the geometric boundary conditions. For case (i), the expansion reads:

$$\begin{Bmatrix} w(x, y, t) \\ u(x, y, t) \\ v(x, y, t) \end{Bmatrix} = \sum_{m=1}^M \sum_{n=1}^N \begin{Bmatrix} w_{m,n}(t) \sin(m\pi x/a) \sin(n\pi y/b) \\ u_{m,n}(t) \cos(m\pi x/a) \sin(n\pi y/b) \\ v_{m,n}(t) \sin(m\pi x/a) \cos(n\pi y/b) \end{Bmatrix} \quad (1)$$

For case (ii),  $y/b$  must be replaced by  $\theta/\alpha$ , and the two sine functions appearing in the expansion of  $u$  and  $v$  must be replaced by cosine functions in order to satisfy boundary conditions. Let  $\mathbf{q}$  be the vector of generalized coordinates :

$$\mathbf{q} = [w_{m,n} \quad u_{m,n} \quad v_{m,n}]^T, \quad m = 1, \dots, M, \quad n = 1, \dots, N. \quad (2)$$

Let  $P$  be the dimension of  $\mathbf{q}$ , which generic element is referred to as  $q_p$ . The discretized Lagrange equations of motions then reads,  $\forall p = 1, \dots, P$ :

$$\ddot{q}_p + 2\zeta_p \omega_p \dot{q}_p + \sum_{i=1}^P z_i^p q_i + \sum_{i,j=1}^P z_{i,j}^p q_i q_j + \sum_{i,j,k=1}^P z_{i,j,k}^p q_i q_j q_k = f_p \cos(\omega t). \quad (3)$$

Modal damping has been added in Eq. (3), and  $\mathbf{f} = [f_1 \dots f_P]^T$  is the vector of the projected external forcing considered. The panels response to harmonic excitation in the vicinity of the first eigenfrequency is then numerically studied with the software AUTO. Numerical frequency-response curves are then obtained by continuation. By convention, the reponse given by Eq. (3) will be referred to as the reference solution. The number of dofs  $P$  is generally relatively large for attaining convergence. This reference solution will then be compared to frequency-response curves obtained by reduced-order models computed with NNMs, as explained in the next section.

### 3 REDUCED-ORDER MODELS

#### 3.1 Numerical computation of the eigenmodes

The first step for reducing the size of the system is to make the linear part diagonal. Let  $\mathbf{L} = [z_i^p]_{p,i}$  be the linear part (stiffness) of Eq. (3), and  $\mathbf{P}$  the matrix of eigenvectors (numerically computed) of  $\mathbf{L}$  such that:  $\mathbf{P}^{-1}\mathbf{L}\mathbf{P} = \mathbf{\Lambda}$ , with  $\mathbf{\Lambda} = \text{diag}[\omega_p^2]$ , with  $\omega_p$  the eigenfrequencies of the structure. A linear change of coordinates is computed,  $\mathbf{q} = \mathbf{P}\mathbf{X}$ , where  $\mathbf{X} = [X_1 \dots X_P]^T$  is, by definition, the vector of modal coordinates. Application of  $\mathbf{P}$  makes the linear part diagonal, so that the dynamics can now be expressed in the eigenmodes basis, and reads,  $\forall p = 1, \dots, P$ :

$$\ddot{X}_p + 2\zeta_p \omega_p \dot{X}_p + \omega_p^2 X_p + \sum_{i,j=1}^P g_{i,j}^p X_i X_j + \sum_{i,j,k=1}^P h_{i,j,k}^p X_i X_j X_k = F_p \cos(\omega t). \quad (4)$$

The application of  $\mathbf{P}$  let the damping unchanged, and  $\mathbf{F} = \mathbf{P}^{-1}\mathbf{f}$  is the new vector of modal forces. By increasing the number of linear modes retained in the modal vector  $\mathbf{X}$ , convergence with respect to the number of eigenmodes can be studied, see section 4 and 5.

### 3.2 Non-linear normal modes (NNMs)

Eq (4) is now transformed with the asymptotic NNM procedure described in [1]. A non-linear change of coordinate is computed:

$$\begin{pmatrix} X_p \\ Y_p \end{pmatrix} = \begin{pmatrix} R_p \\ S_p \end{pmatrix} + \begin{pmatrix} \mathcal{P}_p^{(3)}(R_i, S_i) \\ \mathcal{Q}_p^{(3)}(R_i, S_i) \end{pmatrix}, \quad (5)$$

where  $Y_p = \dot{X}_p$  is the velocity,  $(R_p, S_p)$  are the new variables that describe the dynamics in an invariant-based span of the phase space, and  $\mathcal{P}_p^{(3)}, \mathcal{Q}_p^{(3)}$  are polynomials of the third degree. The dynamics is then expressed with the introduced  $(R_p, S_p)$  variables, that describe the motion along NNMs (invariant manifolds), computed up to a third-order approximation (degree of the polynoms). Within this new, invariant and curved basis spanned by the NNMs, proper truncation of the system can now be realized, as already shown in [1]. In the remainder of the study, reference solution (with  $P$  basis functions) will be compared to simulations with an increasing number of *linear modes* (dimension of  $\mathbf{X}$ ), and to simulations with a single NNM, obtained by keeping in the dynamics only the directly excited NNM.

## 4 HYPERBOLIC PARABOLOID PANEL

The reference simulation is obtained with  $P = 22$  basis functions. Convergence has been checked in [3] for an excitation amplitude of  $\tilde{f} = 4.37$  N. The generalized coordinates retained are:  $w_{1,1}, w_{1,3}, w_{3,1}, w_{3,3}, u_{1,1}, u_{3,1}, u_{1,3}, u_{3,3}, u_{1,5}, u_{5,1}, u_{3,5}, u_{5,3}, u_{5,5}, v_{1,1}, v_{3,1}, v_{1,3}, v_{3,3}, v_{1,5}, v_{5,1}, v_{3,5}, v_{5,3}, v_{5,5}$ . Figure 2 shows the maximum of the panel response in the spectral neighbourhood of the first eigenfrequency  $\omega_1$ , numerically obtained by continuation with the software AUTO. Only the principal coordinate response,  $w_{1,1}$ , is represented. It is compared to the response obtained by two severely reduced-order models, obtained by keeping either only one linear mode (Only the first  $X_1$  coordinate is kept for  $\mathbf{X}$ ), or one NNM (only  $R_1$  is kept in this case).

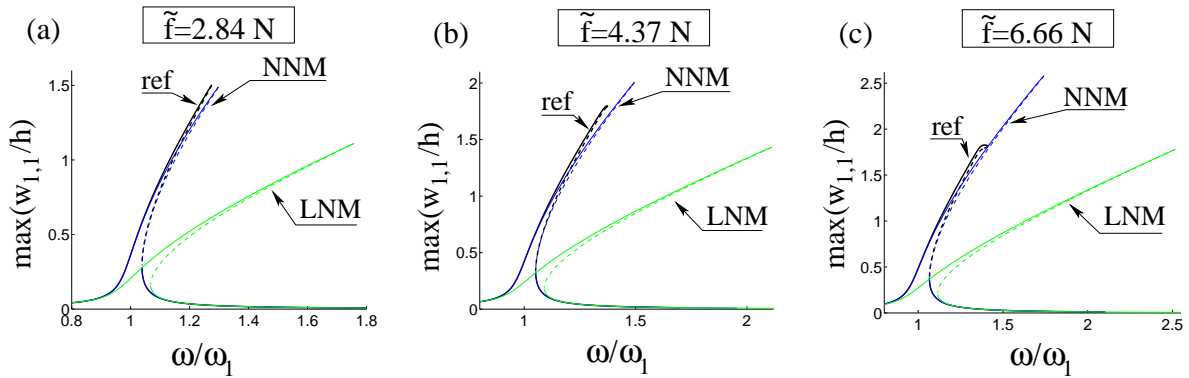


Figure 2. Comparison of reference simulation obtained with 22 dofs (ref) with reduced models consisting of a single oscillator equation: one linear mode (LNM) and one NNM. Three different excitation amplitude are shown: (a):  $\tilde{f} = 2.84$  N, (b):  $\tilde{f} = 4.37$  N, (c):  $\tilde{f} = 6.66$  N.

For  $\tilde{f} = 2.84$  N, it is observed that the NNM reduced model is very close to the reference solution, whereas the simulation with only one eigenmode gives erroneous result. For  $\tilde{f} = 4.37$  N, discrepancies between the reference solution and the NNM model are observed in the prediction of the maximum amplitude. This trend is confirmed for  $\tilde{f} = 6.66$  N, where it is observed that the reference simulation amplitude remains quite constant with a loop at its end. Observation of the other coordinates reveals that much of the energy is absorbed by the higher modes for this level of excitation.

Convergence of the solution with the number of linear modes has also been checked. It has been found that the convergence is very slow: 15 linear modes are necessary to recover the reference solution. Hence, the result provided by the reduction to a single NNM is very good, and can be considered as blindly reliable for amplitudes of vibration up to 1.5 times the thickness of the shell. This result is coherent with precedent studies, and points out the validity limits of the asymptotic approach.

## 5 CIRCULAR CYLINDRICAL PANEL

The reference simulation is obtained with  $P = 19$  basis functions in this case for an excitation amplitude of  $\tilde{f} = 2.2$  N. The generalized coordinates retained are:  $w_{1,1}, w_{1,3}, w_{3,1}, w_{3,3}, u_{1,0}, u_{1,2}, u_{1,4}, u_{3,0}, u_{3,2}, v_{0,1}, v_{2,1}, v_{4,1}, v_{0,3}, v_{2,3}, v_{4,3}, v_{0,5}, v_{2,5}, v_{4,5}$ . Figure 3(a) shows the convergence of the solution with an increasing number of linear modes retained. Once again, a very slow convergence is observed: 15 linear modes are necessary to recover the converged result with 19 basis functions.

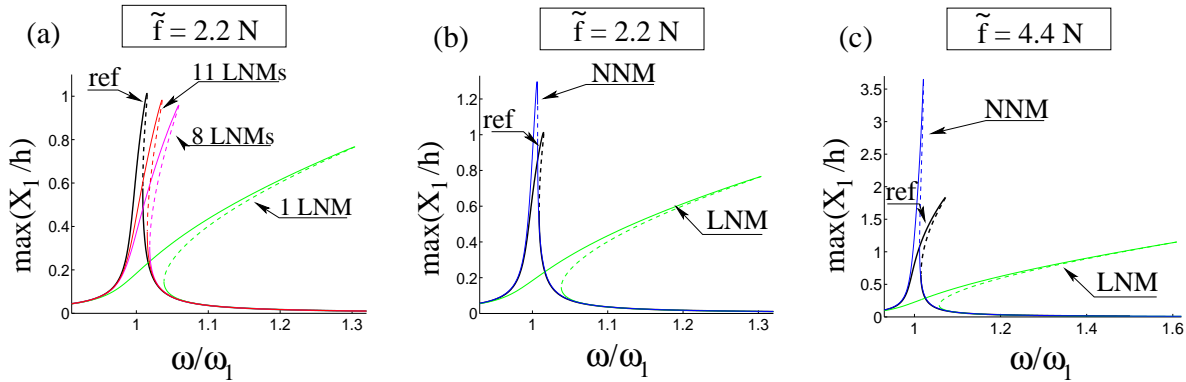


Figure 3. Maximum amplitude of the first modal coordinate  $X_1$ . (a): Convergence of the solution versus the number of linear modes retained, for  $\tilde{f} = 2.2$  N. (b)-(c): Comparison of reference solution with reduction to a single linear mode (LNM) and a single NNM, for  $\tilde{f} = 2.2$  N and  $\tilde{f} = 4.4$  N.

Figure 3(b)-(c) shows the results obtained by two reduced models of the same complexity (a single oscillator-equation). For  $\tilde{f} = 2.2$  N, the NNM solution is close to the reference but overpredict the maximum value, whereas the single linear mode truncation gives erroneous result. For  $\tilde{f} = 4.4$  N, the reduction to a single NNM deteriorates severely and is not anymore acceptable. In this case, the validity limit of the asymptotic NNM procedure used to reduce the system is thus estimated at 1.2 times the thickness of the shell.

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